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Research article

Multiperiod distributionally robust portfolio selection with regime-switching under CVaR risk measures

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Abstract: Optimal investment strategy selection has become a primary research focus in investment science and operations research. Key challenges in this field include identifying an appropriate risk measure to capture potential extreme losses, accurately modeling the impact of market volatility on investment decisions, and effectively balancing returns and risks. To handle uncertainty in return distributions, robust portfolio optimization is a more recent approach. In this study, we employ robust Conditional Value-at-Risk (CVaR) as the risk measure and propose a multi-stage robust portfolio selection model incorporating both risk-free and risky assets under a known first and second moment uncertainty set. By integrating a regime-switching framework, we derive an analytical optimal investment strategy using dynamic programming (DP) techniques. Our numerical analysis demonstrates that the optimal strategy determined by dynamic programming adjusts dynamically at each stage in response to regime switches.

Keywords: portfolio selection; robust optimization; CVaR/VaR; uncertainty set; optimal investment strategy
Mathematics Subject Classification: 9110, 91G10, 91G70

1. Introduction

The effectiveness of financial activities depends on the choice of investment decision-making. Recent research has increasingly focused on optimal investment strategy selection in finance and operations. Key factors drive effective solutions in this area include selecting an appropriate risk measure to capture potential extreme losses, assessing how market fluctuations impact investments, and balancing returns against risks. In classical Markowitz mean-variance models, risk is measured by deviations from the mean, such as variance or standard deviation. However, relying on variance as a risk measure presents serious limitations. Compared to variance, measuring the downside risk of a portfolio is more critical, an insight long recognized by both scholars and practitioners.

Until the late 1980s, the Basel Committee emphasized the importance of widely accepted risk standards and recommended introducing quantitative models based on mathematical and statistical principles. Value-at-Risk (VaR) was explicitly recommended for evaluating the capital adequacy and market risk of commercial banks for the first time. However, VaR has notable limitations, particularly its inability to adequately measure tail risk. When actual losses exceed the VaR threshold, their magnitude and acceptability remain unknown. To address these shortcomings, recent research has focused on coherent risk measures. Artzner et al. [1] were among the first to investigate this issue, introducing the concept of coherent risk measures. Following their work, research on coherent risk measures has attracted significant attention, leading to significant advancements, including Expected Shortfall (ES) [2], Conditional Value-at-Risk (CVaR) [3], spectral risk measures [4], and one-sided moment measures [5]. A substantial body of literature has explored methods for minimizing these coherent risk measures, with CVaR emerging as the most extensively studied and widely applied. Rockafellar and Uryasev demonstrated that solving a simple convex optimization problem enables the simultaneous computation of both CVaR and VaR for a portfolio. CVaR offers an efficient approach for solving portfolio optimization problems, facilitating large-scale computations that would otherwise be infeasible.

Corresponding to the research on risk measures, the practical applications of portfolio selection models have also expanded. Private and institutional investors are developing dynamic techniques and tools to improve security price predicting and enhance investment capital management. Numerous portfolio selection models have been proposed, employing diverse solution techniques and applications across different markets. However, a significant limitation in existing research is the assumption that the distribution of risk asset prices or returns is known in advance or fully specified. This assumption often renders many risk management methods and optimal investment strategy models impractical for real-world investment decision-making, as precise characterizations of security returns are often unavailable. In response, modern optimization methods for decision-making under uncertainty, such as robust optimization techniques, have gained prominence in risk management and portfolio selection. Recently, various robust risk measures and corresponding robust portfolio selection models have emerged, yielding several valuable results. Nonetheless, many issues remain unresolved or require further refinement. This paper, therefore, focuses on constructing and solving robust portfolio selection models within this framework.

Robust optimization has emerged as a powerful tool for addressing optimization problems under uncertainty. Soyster [6] initially introduced the method of robust optimization, and definitions for robust feasible solutions and optimal solutions were later provided by Ben-Tal and Nemirovski [7] and Ghaoui [8]. Garlappi et al. [9] addressed the mean-variance robust portfolio selection problem, assuming only the mean is uncertain and belonging to a box uncertainty set. Costa and Paiva [10], Goldfarb and Iyengar [11], and Lu [12] investigated robust portfolio selection within the mean-variance framework. Goldfarb and Iyengar [11] considered a factor model for asset stochastic returns, constructing uncertainty sets for the model parameters using statistical processes. Lu [12] studied the robust portfolio selection problem using a joint ellipsoidal uncertainty set to describe the model parameters, demonstrating that the problem can be reformulated as a cone programming problem. Halldórsson and Tütüncü [13] extended these results [10–12] by applying interior point algorithms to address the robust mean-variance portfolio selection problem with mean vectors and covariance matrix parameters modeled as box uncertainty sets. Popescu [14] examined the robust mean-variance (M-

V) portfolio selection problem when the moment information of a given return vector distribution is known. They demonstrated that, for a broad class of objective functions, finding a robust solution is equivalent to solving a parameterized quadratic program. Natarajan et al. [15] focused on the worst-case CVaR robust portfolio selection model when only partial moment information of the stochastic return variables is known. Zhu and Fukushima [16] introduced a different type of uncertainty, where instead of focusing on the first and second moments of the portfolio, the uncertainty is in the distribution of portfolio returns themselves. Distributionally robust optimization addresses the uncertainty in asset return distributions by considering a set of possible distributions rather than relying on a single estimated distribution. This approach is beneficial when the underlying distribution is unknown or subject to change. The Wasserstein metric is commonly employed to define ambiguity sets, allowing for a robust optimization framework adaptable to various scenarios [17, 18]. Subsequently, Huang et al. [19] applied the methods from [16] to portfolio selection problems with uncertain termination times. In practice, there are various methods to handle uncertainty in the covariance matrix of a model. Some approaches involve additional factors in the return model [20], while others consider confidence intervals for individual covariance matrices [14]. Even when the uncertainty set is defined simply as a collection of possible scenarios for the covariance matrix, the advantages of such approaches are well recognized [10, 21]. Best and Grauer [22] and Black and Litterman [23] studied the sensitivity of optimal portfolio estimates to uncertainty in average returns.

Dynamic risk measures play a crucial role in assessing the risk of financial portfolios over time. Large portfolios that use the CVaR measure often exhibit non-smooth characteristics. To address this, [24] proposed a derivative-free method for nonsmooth functions. Regime-switching models account for the nonstationarity of financial markets by allowing parameters to shift across different regimes. These models can capture phenomena such as volatility clustering and fat tails frequently observed in financial data. For instance, the Markov regime-switching GARCH model has been widely used to model asset returns under varying market conditions [25]. [26] introduced explicit CRRA equilibrium strategies for two-player stochastic investment games under Markovian regime switching, while [27] derived globally optimal solutions for incomplete regime-switching markets. [28] proposed a novel VIX-based candlestick predictor with market regime analysis. Quantum-inspired optimization [29] and robust genetic strategies [30] provide scalable frameworks for high-dimensional and dynamic challenges, aligning with regime-switching CVaR portfolios. Additionally, AI techniques for dynamic risk management have also been applied across various fields [31].

In financial portfolio management, optimizing asset allocation while dynamically managing risk remains a critical challenge. Traditional robust risk measurement and portfolio selection models often rely on worst-case scenarios, resulting in overly conservative investment decisions that fail to accurately reflect the impact of market changes on the uncertainty set. Existing studies on robust portfolio selection models with known matrix uncertainty either focus on single-period scenarios or lack analytical solutions. However, the real financial market is highly dynamic, particularly in medium and long-term investments, and multi-period risk models are essential for effectively managing risks over investment horizons. Single-period risk models have limitations in offering optimal long-term investment strategies. Therefore, extending single-period risk measures to a multi-period framework is highly significant. Furthermore, most portfolio selection models assume deterministic information, such as known distribution functions, but in practice, market parameters are inherently uncertain. Even minor parameter changes can significantly affect investment outcomes, potentially leading to suboptimal or infeasible solutions. Robust optimization provides a powerful method to address parameter uncertainty. Given these challenges, this study aims to bridge these gaps by developing a multi-period robust portfolio selection model that effectively integrates dynamic uncertainty considerations with robust optimization techniques.

This study focuses on a market consisting of multiple risky assets and one risk-free asset, extending previous research [16] by including the risk-free asset. The returns of the risky assets are characterized by a given mean vector and covariance matrix, forming an uncertainty set distinct from the Wasserstein ambiguity set [18] and the asymmetric distribution uncertainty set [32]. We consider a multi-period robust portfolio selection model that utilizes robust CVaR as the risk measure, contrasting it with the mean lower partial moment [32], and solve the problem via dynamic programming, which differs from the SOCP optimization approach [33].

The main contributions of the paper include:

- The proposed multi-period investment strategy is formulated from a dynamic perspective, allowing investors to adjust their strategies according to market conditions throughout the holding period to enhance returns.
- We utilize regime-switching techniques to capture the dynamic dependencies between consecutive periods, adjusting the uncertainty set based on the first and second moments to reflect these dynamic relationships.
- The constructed uncertainty set features a mean vector that follows a Markov process. We demonstrate that the optimal investment strategy, derived recursively, depends on this mean vector, ensuring that the optimal strategy adapts to the state of the uncertainty set. This dynamic investment strategy offers a more realistic alternative to static strategies.
- By using wealth dynamic equations as constraints and leveraging existing solution techniques, we derive an analytical optimal investment strategy based on dynamic programming principles.

This paper is structured as follows. Section 2 introduces the multi-stage robust portfolio selection model, incorporating a regime-switching technique to capture the dynamic correlations. Section 3 presents an approach to derive the analytical optimal solution using dynamic programming. Section 4 details our proposed method, which recursively breaks down the problem from the current stage back to the initial stage. Section 5 provides numerical analysis and results. Finally, Section 6 concludes the paper.

2. Multi-period robust portfolio selection model

We consider a security market consisting of n risky assets and one risk-free asset with return R. To maintain model tractability, we use CVaR as the fundamental risk measure when constructing the multi-period robust portfolio selection model.

To characterize the dynamic changes in the stochastic returns of risky assets, we define a probability space $(\Omega, \mathcal{F}, P)$. The sigma algebra $\mathcal{F}_k$ represents all available information up to time k, with the assumption that $\mathcal{F}_0 = \{\Omega, \varnothing\}$ and $\mathcal{F}_K = \mathcal{F}$, where K denotes the total investment period. Thus, since $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F}_K$, the collection $\{\mathcal{F}_k\}$ forms a filtration. At the beginning of each period, the current wealth is reallocated among all assets. We denote the proportion of wealth allocated to n risky assets at stage k by the vector $x_k = \left(x_k^1, \cdots, x_k^n\right)^T$.

Let $\xi_k = \left(\xi_k^1, \cdots, \xi_k^n\right)^T$ represent the vector of random returns for the n risky assets at time k. This vector is defined as a random variable on the probability space $(\Omega, \mathcal{F}_k, P)$ for $k = 1, 2, \cdots, K$. Moreover, for each k, ξ_k is $\mathcal{F}_k$-measurable, indicating that the stochastic process $\{\xi_k, k = 1, 2, \cdots, K\}$ is adapted to the filtration $\{\mathcal{F}_k, k = 1, 2, \cdots, K\}$.

We assume that the first and second moments of asset returns are known. Due to dynamic dependence, the mean vector at time k is conditionally dependent on the information available at time $k-1$, represented as $\mu_k = \mathrm{E}_{P_{k-1}}[\xi_k]$. To capture the dynamic correlations of return rates, we employ a regime-switching model. In this framework, the regime process follows a Markov chain, where the set of possible regimes is constructed by m regimes $U = \left\{\mu^1, \mu^2, \cdots, \mu^m\right\}$. The transition probability from regime μ^i at time k to regime μ^j at time $k+1$ is denoted as $P_{\mu^i\mu^j}(k, k+1) = P\left\{\mu_{k+1} = \mu^j \middle| \mu_k = \mu^i\right\}$. We assume the Markov chain is time-homogeneous with stationary transition probabilities. Therefore, the transition probability matrix at time k is denoted as

$$P_k = \begin{pmatrix} P_{\mu^1\mu^1} & P_{\mu^1\mu^2} & \cdots & P_{\mu^1\mu^m} \\ P_{\mu^2\mu^1} & P_{\mu^2\mu^2} & \cdots & P_{\mu^2\mu^m} \\ \vdots & \vdots & \cdots & \vdots \\ P_{\mu^m\mu^1} & P_{\mu^m\mu^2} & \cdots & P_{\mu^m\mu^m} \end{pmatrix}.$$

Consequently, the state of μ_k at time $k+1$ depends only on its state at time k, satisfying the Markov property. This implies that, throughout the investment process, the mean for the next period relies on the mean return of the previous period. Investors can adjust their strategies at each stage of the investment horizon in response to market fluctuations, thereby optimizing returns.

Furthermore, we define the covariance matrix of risky asset returns at stage k as $\Gamma_k = Cov[\xi_k]$. For simplicity, we follow the approach in [15] and assume that $\Gamma_k = \Gamma \succ 0$, $k = 1, \cdots, K$ across different periods of the investment horizon, where $\Gamma \succ 0$ denotes that Γ is positive definite.

Next, we describe the uncertainty in the distribution of return rates, assuming that ξ_k at stage k belongs to a given uncertainty set D_k defined by its first two moments,

$$D_k = \{\pi_k \mid \mathrm{E}[\xi_k] = \mu_k, \mathrm{Cov}[\xi_k] = \Gamma\}.$$

Suppose an investor joins the market at time 0 with an initial wealth of $w_0 = 1$. The investor plans to allocate this wealth in the securities market over K periods, where the cumulative return rate at stage k is

$$r_k = \frac{w_k - w_0}{w_0}, k = 1, \cdots, K.$$

We define $\underline{r}_k$ as the minimum required cumulative return rate for each period, ensuring $\underline{r}_K - r_{K-1} \geq R$ for $k = 1, \cdots, K$. We assume that the investment process is self-financing, leading to the dynamic equation $r_{k-1} + \mathrm{E}[\xi_k]^T x_k + R\left(1 - x_k^T e\right) = \underline{r}_k, k = 1, \cdots, K$, where $e = [1, \cdots, 1]^T$. Without constraints, excessive leverage or short positions in risky assets may arise. However, the inclusion of a risk-free asset along with a minimum return constraint addresses this issue, resulting in a more stable portfolio. Furthermore, the presence of a risk-free asset allows investors to allocate capital between risky and risk-free assets, enabling dynamic position adjustments under different market regimes. Throughout the investment process, the investor constantly reallocates their wealth among n risky assets and one risk-free asset at the beginning of each period. The terminal total wealth at the end of stage k is denoted by w_k, while $-w_k$ can be viewed as the potential loss at stage k.

The optimal investment strategy must be determined at the decision-making outset for investors. When selecting x_1, the actual return rate ξ_1 is unknown. Similarly, when formulating the investment strategy x_k $(k \geq 2)$, the return ξ_k remains uncertain. Therefore, x_k should depend on ξ_{k-1} rather than ξ_k, making x_k a variable influenced by uncertain data ξ_{k-1}. This implies that the investment decision x_{k+1} for stage $k+1$ is made based on the information from the previous period, without knowledge of the current return ξ_{k+1}, for $k = 1, \cdots, K$.

Investors usually aim to minimize risk while maximizing terminal wealth in multi-stage investment scenarios. We continue to use CVaR as the fundamental risk measure. Based on the aforementioned uncertainty set, we derive a robust CVaR measure to control total risk from any intermediate moment until the end of the investment period.

We define the loss function at time k as follows:

$$f(x_k, \xi_k) = -\left(x_k^{\mathrm{T}}\xi_k + R\left(1 - x_k^{\mathrm{T}}e\right)\right).$$

Rockafellar and Uryasev [3, 34] demonstrated that CVaR can be computed by minimizing the auxiliary function

$$F_{\beta_k}(x_k, \alpha_k) = \alpha_k + \frac{1}{1-\beta_k}\mathrm{E}\left[(f(x_k, \xi_k) - \alpha_k)_+\right],$$

where α_k represents the threshold for the loss function, and $\beta_k \in (0, 1)$ denotes the confidence level. Then,

$$\mathrm{CVaR}_{\beta_k}(x_k) = \min_{\alpha_k \in \mathbb{R}} F_{\beta_k}(x_k, \alpha_k).$$

We express the robust CVaR risk measure as follows:

$$\mathrm{RCVaR}_{\beta_k}(x_k) = \max_{\pi_k \in \mathrm{D}_k} \mathrm{CVaR}_{\beta_k}(x_k) = \max_{\pi_k \in \mathrm{D}_k} \min_{\alpha_k \in \mathbb{R}} F_{\beta_k}(x_k, \alpha_k).$$

Unlike existing literature, our approach ensures that investors minimize total risk while ensuring that the return rates do not drop below a pre-specified threshold in each period. Let $V_k(w_{k-1})$ represent the optimal target value at stage k. Under these stochastic market conditions, we formulate the multi-stage robust portfolio selection model using dynamic programming principles:

$$\begin{aligned}
V_K(r_{K-1}) &= \min_{x_K \in \mathbb{R}^n} \max_{\xi_K \sim D_{K(\mu_K, \Gamma)}} \min_{\alpha_K \in \mathbb{R}} \left\{\alpha_K + \frac{1}{1-\beta_K}\mathrm{E}\left[\left(-R - \alpha_K - x_K^T(\xi_K - Re)\right)_+\right]\right\} \\
&\quad \text{s.t.} \quad r_{K-1} + (\mathrm{E}[\xi_K])^T x_K + R\left(1 - x_K^T e\right) = \underline{r}_K; \\
V_{K-1}(r_{K-2}) &= \min_{x_{K-1}} \max_{\xi_{K-1} \sim D_{K-1(\mu_{K-1}, \Gamma)}} \min_{\alpha_{K-1}} \left\{\alpha_{K-1} + \frac{1}{1-\beta_{K-1}}\mathrm{E}\left[\left(-R - \alpha_{K-1} - x_{K-1}^T(\xi_{K-1} - Re)\right)_+\right]\right. \\
&\quad \left. + \mathrm{E}\left[V_K^*(r_{K-1})\right]\right\} \\
&\quad \text{s.t.} \quad r_{K-2} + (\mathrm{E}[\xi_{K-1}])^T x_{K-1} + R\left(1 - x_{K-1}^T e\right) = \underline{r}_{K-1}; \\
&\quad \vdots \\
V_1(1) &= \min_{x_1} \max_{\xi_1 \sim D_{1(\mu_1, \Gamma)}} \min_{\alpha_1} \left\{\alpha_1 + \frac{1}{1-\beta_1}\mathrm{E}\left[\left(-R - \alpha_1 - x_1^T(\xi_1 - Re)\right)_+\right] + \mathrm{E}\left[V_2^*(r_1)\right]\right\} \\
&\quad \text{s.t.} \quad (\mathrm{E}[\xi_1])^T x_1 + R\left(1 - x_1^T e\right) = \underline{r}_1.
\end{aligned} \tag{2.1}$$

It is important to emphasize that when making decisions at stage k, the variable x_k does not depend on the unknown ξ_k; instead, it relies on ξ_{k-1}. According to the wealth dynamic equation, x_k is also influenced by the return rate r_{k-1}. Therefore, we denote $x_k(r_{k-1}, \xi_{k-1})$ as a variable dependent on both r_{k-1} and ξ_{k-1}.

3. Approach to solving the multi-period robust portfolio selection problem

In this section, we derive the analytical optimal solution for the multi-period robust portfolio selection problem using dynamic programming principles.

First, at stage K, given the cumulative return r_{K-1} from time $K-1$, we define the objective function as the robust CVaR for stage K. The corresponding robust optimization model is formulated as follows:

$$V_K(r_{K-1}) = \min_{x_K\in\mathbb{R}^n} \max_{\xi_K\sim D_{K(\mu_K,\Gamma)}} \min_{\alpha_K\in\mathbb{R}} \left\{\alpha_K + \frac{1}{1-\beta_K}\mathrm{E}\left[\left(-R-\alpha_K - x_K^T(\xi_K - Re)\right)_+\right]\right\}$$
$$\text{s.t.} \quad r_{K-1} + (\mathrm{E}[\xi_K])^T x_K + R\left(1 - x_K^T e\right) = \underline{r}_K.$$

Since the set D_k is convex and closed, the function $F_{\beta_k}(x_k, \alpha_k)$ is also convex. By applying the minimax theorem (see Theorem 4.2 in [35]), we can interchange the order of the maximum and minimum operations. Therefore, we formulate the robust CVaR portfolio selection problem as follows:

$$\begin{aligned} RCVaR_{\beta_k}(x_k) &= \min_{\alpha_k\in\mathbb{R}} \max_{\pi_k\in D_k} F_{\beta_k}(x_k,\alpha_k) \\ &= \min_{\alpha_K\in\mathbb{R}} \alpha_K + \frac{1}{1-\beta_K} \max_{\xi_K\sim D_{K(\mu_K,\Gamma)}} \left[\mathrm{E}\left(-R-\alpha_K - x_K^T(\xi_K - Re)\right)_+\right]. \end{aligned}$$

Using Lemmas 2.2 and 2.4 from [36], we obtain

$$RCVaR_{\beta_k}(x_k) = \min_{\alpha_K\in\mathbb{R}} \alpha_K + \frac{R\left({x_K}^T e - 1\right) - \alpha_K - {x_K}^T\mu_K + \sqrt{{x_K}^T\Gamma x_K + \left(R\left({x_K}^T e - 1\right) - \alpha_K - {x_K}^T\mu_K\right)^2}}{2(1-\beta_K)}. \tag{3.1}$$

The first-order optimality condition for problem (3.1) is given by

$$1 - \frac{1}{2(1-\beta_K)} - \frac{1}{2(1-\beta_K)} \frac{2\left(R\left({x_K}^T e - 1\right) - \alpha_K - {x_K}^T\mu_K\right)}{2\sqrt{{x_K}^T\Gamma x_K + \left(R\left({x_K}^T e - 1\right) - \alpha_K - {x_K}^T\mu_K\right)^2}} = 0.$$

Consequently, we have

$$\alpha_K^* = \frac{2\beta_K - 1}{2\sqrt{\beta_K(1-\beta_K)}}\sqrt{{x_K}^T\Gamma x_K} - {x_K}^T\mu_K - R\left(1 - {x_K}^T e\right).$$

The robust CVaR portfolio selection problem is formulated as follows:

$$RCVaR_{\beta_K}(x) = \sqrt{\frac{\beta_K}{1-\beta_K}}\sqrt{{x_K}^T\Gamma x_K} - {x_K}^T\mu_K - R\left(1 - {x_K}^T e\right).$$

Let $\tilde{\mu}_K = \mu_K - Re$. Consequently, we formulate the following optimization problem:

$$\begin{aligned} V_K(r_{K-1}) = &\min_{x_K\in\mathbb{R}^n} \sqrt{\frac{\beta_K}{1-\beta_K}}\sqrt{x_K^T\Gamma x_K} - x_K^T\tilde{\mu}_K - R \\ &\text{s.t.} \quad r_{K-1} + (\mathrm{E}[\xi_K])^T x_K + R\left(1 - x_K^T e\right) = \underline{r}_K. \end{aligned} \tag{3.2}$$

Let $\xi'_K = \mathrm{E}[\xi_K] - Re$, and set $s_K = x_K^T\tilde{\mu}_K$. Employing a transformation from the proof of Theorem 2.5 in [36], problem (3.2) is equivalent to

$$\begin{aligned} &\min_{s_K\in\mathbb{R}}\min_{x_K\in\mathbb{R}^n} \sqrt{\frac{\beta_K}{1-\beta_K}}\sqrt{x_K^T\Gamma x_K} - x_K^T\tilde{\mu}_K - R \\ &\text{s.t.} \quad {\xi'_K}^T x_K = \underline{r}_K - r_{K-1} - R, \\ &\qquad\ x_K^T\tilde{\mu}_K = s_K. \end{aligned} \tag{3.3}$$

To proceed, we first solve problem (3.4):

$$\begin{aligned} &\min_{x_K} \quad x_K^T\Gamma x_K \\ &\text{s.t.} \quad {\xi'_K}^T x_K = \underline{r}_K - r_{K-1} - R, \\ &\qquad\ x_K^T\tilde{\mu}_K = s_K. \end{aligned} \tag{3.4}$$

By obtaining the optimal solution $x_K^*(s_K)$ for (3.4), we can derive $x_K^*(s_K)^T\Gamma x_K^*(s_K)$ and substitute this into the objective function of problem (3.3), transforming it into an unconstrained optimization problem. This leads to the optimal strategy for period K. The Lagrangian function for problem (3.4) is therefore

$$L\left(x_K, \lambda_1^K, \lambda_2^K\right) = x_K^T\Gamma x_K + \lambda_1^K\left(s_K - x_K^T\tilde{\mu}_K\right) + \lambda_2^K\left(\underline{r}_K - r_{K-1} - R - x_K^T\xi'_K\right),$$

and applying the first-order optimality conditions yields the following equations:

$$\begin{cases} L_{x_K} = 2\Gamma x_K - \lambda_1^K\tilde{\mu}_K - \lambda_2^K\xi'_K = 0, & (3.5) \\ x_K^T\xi'_K - \left(\underline{r}_K - r_{K-1} - R\right) = 0, & (3.6) \\ x_K^T\tilde{\mu}_K - s_K = 0. & (3.7) \end{cases}$$

From (3.5), we obtain

$$x_K = \frac{1}{2}\Gamma^{-1}\left(\lambda_1^K\tilde{\mu}_K + \lambda_2^K\xi'_K\right), \tag{3.8}$$

and substituting (3.8) into (3.6) and (3.7) gives us

$$\begin{cases} \left(\lambda_1^K\tilde{\mu}_K + \lambda_2^K\xi'_K\right)^T\Gamma^{-1}\tilde{\mu}_K = 2s_K, \\ \left(\lambda_1^K\tilde{\mu}_K + \lambda_2^K\xi'_K\right)^T\Gamma^{-1}\xi'_K = 2\left(\underline{r}_K - r_{K-1} - R\right). \end{cases}$$

$$\begin{cases} \lambda_1^K\tilde{\mu}_K^T\Gamma^{-1}\tilde{\mu}_K + \lambda_2^K\left(\xi'_K\right)^T\Gamma^{-1}\tilde{\mu}_K = 2s_K, \\ \lambda_1^K\tilde{\mu}_K^T\Gamma^{-1}\xi'_K + \lambda_2^K\left(\xi'_K\right)^T\Gamma^{-1}\xi'_K = 2\left(\underline{r}_K - r_{K-1} - R\right). \end{cases}$$

We define the notation

$$a_0^K := (\xi'_K)^T \Gamma^{-1} \xi'_K, \quad a_1^K := (\xi'_K)^T \Gamma^{-1} \tilde{\mu}_K, \quad a_2^K := \tilde{\mu}_K^T \Gamma^{-1} \tilde{\mu}_K,$$

$$d_0^K := \frac{a_0^K}{a_0^K a_2^K - \left(a_1^K\right)^2}, \quad d_1^K := \frac{a_1^K}{a_0^K a_2^K - \left(a_1^K\right)^2}, \quad d_2^K := \frac{a_2^K}{a_0^K a_2^K - \left(a_1^K\right)^2}.$$

Then,

$$\begin{cases} a_2^K \lambda_1^K + a_1^K \lambda_2^K = 2 s_K, \\ a_1^K \lambda_1^K + a_0^K \lambda_2^K = 2\left(\underline{r}_K - r_{K-1} - R\right). \end{cases}$$

That is,

$$\begin{pmatrix} a_2^K & a_1^K \\ a_1^K & a_0^K \end{pmatrix} \begin{pmatrix} \lambda_1^K \\ \lambda_2^K \end{pmatrix} = 2 \begin{pmatrix} s_K \\ \underline{r}_K - r_{K-1} - R \end{pmatrix}.$$

We find the Lagrange multipliers

$$\begin{pmatrix} \lambda_1^K \\ \lambda_2^K \end{pmatrix} = \frac{2}{a_0^K a_2^K - \left(a_1^K\right)^2} \begin{pmatrix} a_0^K & -a_1^K \\ -a_1^K & a_2^K \end{pmatrix} \begin{pmatrix} s_K \\ \underline{r}_K - r_{K-1} - R \end{pmatrix} = 2 \begin{pmatrix} d_0^K & -d_1^K \\ -d_1^K & d_2^K \end{pmatrix} \begin{pmatrix} s_K \\ \underline{r}_K - r_{K-1} - R \end{pmatrix}.$$

Substituting λ_1^K, λ_2^K into (3.8), we derive the optimal solution for problem (3.4):

$$x_K^*(s_K) = \frac{1}{2} \Gamma^{-1} \begin{pmatrix} \tilde{\mu}_K & \xi'_K \end{pmatrix} \begin{pmatrix} \lambda_1^K \\ \lambda_2^K \end{pmatrix} = \begin{pmatrix} \Gamma^{-1} \tilde{\mu}_K & \Gamma^{-1} \xi'_K \end{pmatrix} \begin{pmatrix} d_0^K & -d_1^K \\ -d_1^K & d_2^K \end{pmatrix} \begin{pmatrix} s_K \\ \underline{r}_K - r_{K-1} - R \end{pmatrix}. \tag{3.9}$$

Thus,

$$\begin{aligned}
& x_K^*(s_K)^T \Gamma x_K^*(s_K) = \left(\begin{pmatrix} d_0^K s_K - d_1^K \left(\underline{r}_K - r_{K-1} - R\right) & d_2^K \left(\underline{r}_K - r_{K-1} - R\right) - d_1^K s_K \end{pmatrix} \begin{pmatrix} \tilde{\mu}_K^T \\ (\xi'_K)^T \end{pmatrix} \right) \Gamma^{-1} \\
& \left(\begin{pmatrix} \tilde{\mu}_K & \xi'_K \end{pmatrix} \begin{pmatrix} d_0^K s_K - d_1^K \left(\underline{r}_K - r_{K-1} - R\right) \\ d_2^K \left(\underline{r}_K - r_{K-1} - R\right) - d_1^K s_K \end{pmatrix} \right) \\
& = a_2^K \left(d_0^K s_K - d_1^K \left(\underline{r}_K - r_{K-1} - R\right) \right)^2 + a_0^K \left(d_2^K \left(\underline{r}_K - r_{K-1} - R\right) - d_1^K s_K \right)^2 \\
& + 2 a_1^K \left(-d_0^K d_1^K s_K^2 + \left(d_0^K d_2^K + \left(d_1^K\right)^2 \right) \left(\underline{r}_K - r_{K-1} - R\right) s_K - d_1^K d_2^K \left(\underline{r}_K - r_{K-1} - R\right)^2 \right) \\
& = \left(a_2^K \left(d_0^K\right)^2 - 2 a_1^K d_0^K d_1^K + a_0^K \left(d_1^K\right)^2 \right) (s_K)^2 \\
& + 2 \left(a_1^K \left(d_1^K\right)^2 - a_2^K d_0^K d_1^K + a_1^K d_0^K d_2^K - a_0^K d_1^K d_2^K \right) \left(\underline{r}_K - r_{K-1} - R\right) s_K \\
& + \left(a_2^K \left(d_1^K\right)^2 - 2 a_1^K d_1^K d_2^K + a_0^K \left(d_2^K\right)^2 \right) \left(\underline{r}_K - r_{K-1} - R\right)^2 \\
& = d_0^K (s_K)^2 - 2 d_1^K \left(\underline{r}_K - r_{K-1} - R\right) s_K + d_2^K \left(\underline{r}_K - r_{K-1} - R\right)^2.
\end{aligned} \tag{3.10}$$

Substituting (3.10) into the objective function of problem (3.3) transforms it into the following unconstrained optimization problem:

$$\min_{s_K \in \mathbb{R}} h_{\beta_K}(s_K) := \sqrt{\frac{\beta_K}{1 - \beta_K}} \sqrt{d_0^K (s_K)^2 - 2 d_1^K \left(\underline{r}_K - r_{K-1} - R\right) s_K + d_2^K \left(\underline{r}_K - r_{K-1} - R\right)^2} - s_K - R. \tag{3.11}$$

The first-order optimality condition for problem (3.11) is

$$h_{\beta_K}{}'(s_K) = \sqrt{\frac{\beta_K}{1-\beta_K}} \frac{d_0^K s_K - d_1^K\left(\underline{r}_K - r_{K-1} - R\right)}{\sqrt{d_0^K (s_K)^2 - 2d_1^K\left(\underline{r}_K - r_{K-1} - R\right)s_K + d_2^K\left(\underline{r}_K - r_{K-1} - R\right)^2}} - 1 = 0,$$

leading to

$$\begin{aligned}
&\beta_K\Big(d_0^K s_K - d_1^K\left(\underline{r}_K - r_{K-1} - R\right)\Big)^2 = (1-\beta_K)\Big(d_0^K (s_K)^2 - 2d_1^K\left(\underline{r}_K - r_{K-1} - R\right)s_K + d_2^K\left(\underline{r}_K - r_{K-1} - R\right)^2\Big)\\
&d_0^K\left(\beta_K d_0^K - (1-\beta_K)\right)(s_K)^2 - 2d_1^K\left(\beta_K d_0^K - (1-\beta_K)\right)\left(\underline{r}_K - r_{K-1} - R\right)s_K\\
&+\left(\beta_K\left(d_1^K\right)^2 - d_2^K(1-\beta_K)\right)\left(\underline{r}_K - r_{K-1} - R\right)^2 = 0.
\end{aligned}$$

This presents two scenarios:

(1) If $\frac{\beta_K}{1-\beta_K} \cdot d_0^K > 1$, the optimal solution for problem (3.11) is

$$\begin{aligned}
s_K^* &= \frac{d_1^K\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K} + \frac{\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K\left(\beta_K d_0^K - (1-\beta_K)\right)}\\
&\sqrt{\left[\beta_K d_0^K - (1-\beta_K)\right]\left[\left(d_1^K\right)^2\left(\beta_K d_0^K - (1-\beta_K)\right) - d_0^K\left(\beta_K\left(d_1^K\right)^2 - d_2^K(1-\beta_K)\right)\right]}\\
&= \frac{d_1^K\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K} + \frac{\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K\left(\beta_K d_0^K - (1-\beta_K)\right)}\sqrt{\left[\beta_K d_0^K - (1-\beta_K)\right]\left[\left(d_0^K d_2^K - \left(d_1^K\right)^2\right)(1-\beta_K)\right]}\\
&= \frac{d_1^K\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K} + \frac{\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K\sqrt{\beta_K d_0^K - (1-\beta_K)}}\sqrt{\left[\left(d_0^K d_2^K - \left(d_1^K\right)^2\right)(1-\beta_K)\right]}\\
&= \left(\underline{r}_K - r_{K-1} - R\right)\left(\frac{d_1^K}{d_0^K} + \frac{\sqrt{d_0^K d_2^K - \left(d_1^K\right)^2}}{d_0^K\sqrt{\frac{\beta_K d_0^K}{1-\beta_K} - 1}}\right).
\end{aligned}$$

(2) If $\frac{\beta_K}{1-\beta_K} \cdot d_0^K \leq 1$, the optimal solution for problem (3.11) is $s_K^* = +\infty$, indicating that problem (3.11) is unbounded.

In scenario (1), substituting s_K^* back into (3.9) provides the optimal investment strategy for period K, and substituting into (3.11) yields the optimal objective value for problem (3.2). Specifically, the optimal investment strategy for period K is

$$x_K^* = \left(\underline{r}_K - r_{K-1} - R\right)\begin{pmatrix}\Gamma^{-1}\tilde{\mu}_K & \Gamma^{-1}\xi_K{}'\end{pmatrix}\begin{pmatrix} d_0^K & -d_1^K \\ -d_1^K & d_2^K \end{pmatrix}\begin{pmatrix}\frac{\sqrt{d_0^K d_2^K - \left(d_0^K\right)^2}}{d_0^K\sqrt{\frac{\beta_K d_0^K}{1-\beta_K} - 1}} + \frac{d_1^K}{d_0^K} \\ 1\end{pmatrix}.$$

Substituting $\xi_K' = \mathrm{E}\left[\xi_K\right] - Re = \tilde{\mu}_K$ into the above expression yields

$$x_K^* = \left(\underline{r}_K - r_{K-1} - R\right)\Gamma^{-1}\tilde{\mu}_K\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix} d_0^K & -d_1^K \\ -d_1^K & d_2^K \end{pmatrix}\begin{pmatrix}\frac{\sqrt{d_0^K d_2^K - \left(d_1^K\right)^2}}{d_0^K\sqrt{\frac{\beta_K d_0^K}{1-\beta_K} - 1}} + \frac{d_1^K}{d_0^K} \\ 1\end{pmatrix}.$$

Remark 1. *A higher β_k implies greater risk aversion. The ratio $\frac{\beta_k}{1-\beta_k}$ scales the investor's risk aversion. d_0^k inversely measures diversification potential; a smaller d_0^k indicates higher diversification. When $\frac{\beta_k}{1-\beta_k} \cdot d_0^k \leq 1$, the level of risk aversion β_k is insufficient relative to the market's diversification potential d_0^k. This imbalance results in unbounded leverage in risky assets to minimize risk or maximize returns, causing the optimization problem to not have a finite solution (i.e., $s_K^* = +\infty$). When $\frac{\beta_k}{1-\beta_k} \cdot d_0^k > 1$, risk aversion dominates market conditions, ensuring the existence of a finite optimal portfolio.*

The expression for the optimal solution x_K^* involves $\tilde{\mu}_K$, which can be represented as $\tilde{\mu}_K = \mu_K - Re$. The random sequence $\{\mu_k, k = 1, \cdots, K\}$ forms a Markov chain with m possible states $\mu^1, \mu^2, \cdots, \mu^m$. Thus, the derived x_K^* varies with the state of μ_k, making this optimal solution a strategy that adapts to prior information. Suppose an investor finds that the sub-strategy from time k to K does not achieve optimality based on their initial decision at time k. In that case, it indicates that the initial investment choice is not the most effective across the entire investment period.

Compared to static investment strategies that fail to adapt to market changes, the adoption of a dynamic decision-making approach aligns more closely with real-world scenarios and represents a superior strategy. However, in practice, market information evolves, and the information available is continuously updated. If investors can swiftly adjust their strategies in response to market fluctuations, they tend to achieve greater returns than relying on a fixed approach. From the above conclusions, the optimal value for period K is given by

$$
\begin{aligned}
V_K^*(r_{K-1}) &= \sqrt{\frac{\beta_K}{1-\beta_K}} \sqrt{d_0^K\left(s_K^*\right)^2 - 2d_1^K\left(\underline{r}_K - r_{K-1} - R\right)s_K^* + d_2^K\left(\underline{r}_K - r_{K-1} - R\right)^2} - s_K^* - R \\
&= \sqrt{\frac{\beta_K}{1-\beta_K}} \sqrt{d_0^K\left(s_K^* - \frac{d_1^K}{d_0^K}\left(\underline{r}_K - r_{K-1} - R\right)\right)^2 + \frac{d_0^K d_2^K - \left(d_1^K\right)^2}{d_0^K}\left(\underline{r}_K - r_{K-1} - R\right)^2} - s_K^* - R \\
&= \sqrt{\frac{d_0^K\beta_K}{1-\beta_K}} \sqrt{\left(\frac{\sqrt{d_0^K d_2^K - \left(d_1^K\right)^2}\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K\sqrt{\frac{\beta_K d_0^K}{1-\beta_K} - 1}}\right)^2 + \frac{d_0^K d_2^K - \left(d_1^K\right)^2}{\left(d_0^K\right)^2}\left(\underline{r}_K - r_{K-1} - R\right)^2} - s_K^* - R \\
&= \sqrt{\frac{d_0^K\beta_K}{1-\beta_K}} \sqrt{\frac{\left(d_0^K d_2^K - \left(d_1^K\right)^2\right)\left(\underline{r}_K - r_{K-1} - R\right)^2}{\left(d_0^K\right)^2}\left(\frac{\beta_K d_0^K}{\beta_K d_0^K - (1-\beta_K)}\right)} - s_K^* - R \\
&= \frac{\left(\underline{r}_K - r_{K-1} - R\right)\beta_K}{1-\beta_K}\sqrt{\frac{d_0^K d_2^K - \left(d_1^K\right)^2}{\frac{\beta_K d_0^K}{1-\beta_K} - 1}} - \frac{d_1^K\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K} - \frac{\sqrt{d_0^K d_2^K - \left(d_1^K\right)^2}\left(\underline{r}_K - r_{K-1} - R\right)}{d_0^K\sqrt{\frac{\beta_K d_0^K}{1-\beta_K} - 1}} - R \\
&= \left(\underline{r}_K - r_{K-1} - R\right)\cdot\left(\frac{\sqrt{d_0^K d_2^K - \left(d_1^K\right)^2}\sqrt{\frac{\beta_K d_0^K}{1-\beta_K} - 1}}{d_0^K} - \frac{d_1^K}{d_0^K}\right) - R \\
&= q_{K-1}\cdot\left(\underline{r}_K - r_{K-1} - R\right) - R,
\end{aligned} \tag{3.12}
$$

where

$$q_{K-1} = \left(\frac{\sqrt{d_0^K d_2^K - \left(d_1^K\right)^2} \sqrt{\frac{\beta_K d_0^K}{1-\beta_K} - 1}}{d_0^K} - \frac{d_1^K}{d_0^K} \right).$$

This notation is convenient and aids in deriving recursive relationships during the solving process.

When the investor is in period $K-2$, for a given cumulative return rate r_{K-2}, substituting (3.12) into the objective function of problem (2.1) allows us to express the corresponding optimal investment decision problem as

$$\begin{aligned} V_{K-1}\left(r_{K-2}\right) = & \min_{x_{K-1}} \left\{ \sqrt{\frac{\beta_{K-1}}{1-\beta_{K-1}}} \sqrt{x_{K-1}^T \Gamma x_{K-1}} - \left(\mu_{K-1} - Re\right)^T x_{K-1} - R + \mathrm{E}\left[V_K^*\left(w_{K-1}\right)\right] \right\} \\ & \text{s.t.} \quad r_{K-2} + \left(\mathrm{E}\left[\xi_{K-1}\right]\right)^T x_{K-1} + R\left(1 - x_{K-1}^T e\right) = \underline{r}_{K-1}. \end{aligned} \tag{3.13}$$

Substituting the expression $V_K^*\left(r_{K-1}\right)$ into the objective function of problem (3.13) yields

$$\begin{aligned} V_{K-1}\left(r_{K-2}\right) = & \min_{x_{K-1}} \left\{ \sqrt{\frac{\beta_{K-1}}{1-\beta_{K-1}}} \sqrt{x_{K-1}^T \Gamma x_{K-1}} - \left(\mu_{K-1} - Re\right)^T x_{K-1} - R + q_{K-1} \cdot \mathrm{E}\left[\left(\underline{r}_K - r_{K-1} - R\right)\right] - R \right\} \\ & \text{s.t.} \quad r_{K-2} + \left(\mathrm{E}\left[\xi_{K-1}\right]\right)^T x_{K-1} + R\left(1 - x_{K-1}^T e\right) = \underline{r}_{K-1}. \end{aligned}$$

Incorporating the constraints into the above objective function, problem (3.13) becomes

$$\begin{aligned} V_{K-1}\left(r_{K-2}\right) = & \min_{x_{K-1}} \Bigg\{ \sqrt{\frac{\beta_{K-1}}{1-\beta_{K-1}}} \sqrt{x_{K-1}^T \Gamma x_{K-1}} - \left(\mu_{K-1} - Re\right)^T x_{K-1} - R \\ & + q_{K-1} \cdot \left(\underline{r}_K - \left(r_{K-2} + \left(\mathrm{E}\left[\xi_{K-1}\right]\right)^T x_{K-1} + R\left(1 - x_{K-1}^T e\right)\right) - R\right) - R \Bigg\} \\ = & \min_{x_{K-1}} \left\{ \sqrt{\frac{\beta_{K-1}}{1-\beta_{K-1}}} \sqrt{x_{K-1}^T \Gamma x_{K-1}} - \left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right)^T x_{K-1} + q_{K-1} \cdot \left(\underline{r}_K - r_{K-2}\right) - 2R\left(q_{K-1} + 1\right) \right\}. \end{aligned}$$

Letting $s_{K-1} = x_{K-1}^T \left(\tilde{\mu}_{K-1} + q_{K-1}\xi'{}_{K-1}\right)$ and employing a method similar to that used for the period K problem, we can equivalently represent problem (3.13) as

$$\begin{aligned} V_{K-1}\left(r_{K-2}\right) = & \min_{s_{K-1}\in\mathbb{R}} \min_{x_{K-1}\in\mathbb{R}^n} \left\{ \sqrt{\frac{\beta_{K-1}}{1-\beta_{K-1}}} \sqrt{x_{K-1}^T \Gamma x_{K-1}} - s_{K-1} + q_{K-1} \cdot \left(\underline{r}_K - r_{K-2}\right) - 2R\left(q_{K-1} + 1\right) \right\} \\ & \text{s.t.} \quad r_{K-2} + \left(\mathrm{E}\left[\xi_{K-1}\right]\right)^T x_{K-1} + R\left(1 - x_{K-1}^T e\right) = \underline{r}_{K-1}, \\ & \qquad x_{K-1}^T \left(\tilde{\mu}_{K-1} + q_{K-1}\xi'{}_{K-1}\right) = s_{K-1}. \end{aligned} \tag{3.14}$$

To this end, we first solve problem (3.15)

$$\begin{aligned} & \min_{x_{K-1}\in\mathbb{R}^n} \quad x_{K-1}^T \Gamma x_{K-1} \\ & \text{s.t.} \quad x_{K-1}^T \xi'{}_{K-1} = \underline{r}_{K-1} - r_{K-2} - R, \\ & \qquad x_{K-1}^T \left(\tilde{\mu}_{K-1} + q_{K-1}\xi'{}_{K-1}\right) = s_{K-1}. \end{aligned} \tag{3.15}$$

By finding the optimal solution $x^*_{K-1}(s_{K-1})$ for (3.15), we can derive $x^*_{K-1}(s_{K-1})^T \Gamma\, x^*_{K-1}(s_{K-1})$ and substitute it into the objective function of problem (3.14), thus transforming it into an unconstrained optimization problem to determine the optimal strategy for period $K-1$. The Lagrangian function for problem (3.15) is

$$L\left(x_{K-1}, \lambda_1^{K-1}, \lambda_2^{K-1}\right) = x_{K-1}^T \Gamma x_{K-1} + \lambda_1^{K-1}\left(s_{K-1} - x_{K-1}^T\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right)\right) + \lambda_2^{K-1}\left(\underline{r}_{K-1} - r_{K-2} - R - x_{K-1}^T \xi'_{K-1}\right).$$

Applying the first-order optimality conditions yields the following equations:

$$\begin{cases} L_{x_{K-1}} = 2\Gamma x_{K-1} - \lambda_1^{K-1}\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right) - \lambda_2^{K-1}\xi'_{K-1} = 0, & (3.16)\\ x_{K-1}^T \xi'_{K-1} - \left(\underline{r}_{K-1} - r_{K-2} - R\right) = 0, & (3.17)\\ x_{K-1}^T\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right) - s_{K-1} = 0. & (3.18) \end{cases}$$

From (3.16), we obtain

$$x_{K-1} = \frac{1}{2}\Gamma^{-1}\left(\lambda_1^{K-1}\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right) + \lambda_2^{K-1}\xi'_{K-1}\right), \tag{3.19}$$

and substituting (3.19) into (3.17) and (3.18) yields

$$\begin{cases} \left(\lambda_1^{K-1}\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right) + \lambda_2^{K-1}\xi'_{K-1}\right)^T \Gamma^{-1}\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right) = 2s_{K-1},\\ \left(\lambda_1^{K-1}\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right) + \lambda_2^{K-1}\xi'_{K-1}\right)^T \Gamma^{-1}\xi'_{K-1} = 2\left(\underline{r}_{K-1} - r_{K-2} - R\right). \end{cases}$$

$$\begin{cases} \left(a_2^{K-1} + 2q_{K-1}a_1^{K-1} + q_{K-1}^2 a_0^{K-1}\right)\lambda_1^{K-1} + \left(a_1^{K-1} + q_{K-1}a_0^{K-1}\right)\lambda_2^{K-1} = 2s_{K-1},\\ \left(a_1^{K-1} + q_{K-1}a_0^{K-1}\right)\lambda_1^{K-1} + a_0^{K-1}\lambda_2^{K-1} = 2\left(\underline{r}_{K-1} - r_{K-2} - R\right). \end{cases}$$

Thus, $\begin{pmatrix} a_2^{K-1} + 2q_{K-1}a_1^{K-1} + q_{K-1}^2 a_0^{K-1} & a_1^{K-1} + q_{K-1}a_0^{K-1}\\ a_1^{K-1} + q_{K-1}a_0^{K-1} & a_0^{K-1} \end{pmatrix}\begin{pmatrix}\lambda_1^{K-1}\\ \lambda_2^{K-1}\end{pmatrix} = 2\begin{pmatrix} s_{K-1}\\ \underline{r}_{K-1} - r_{K-2} - R\end{pmatrix}$.

The Lagrange multipliers are

$$\begin{pmatrix}\lambda_1^{K-1}\\ \lambda_2^{K-1}\end{pmatrix} = \frac{2}{a_0^{K-1}a_2^{K-1} - \left(a_1^{K-1}\right)^2}\begin{pmatrix} a_0^{K-1} & -\left(a_1^{K-1} + q_{K-1}a_0^{K-1}\right)\\ -\left(a_1^{K-1} + q_{K-1}a_0^{K-1}\right) & a_2^{K-1} + 2q_{K-1}a_1^{K-1} + q_{K-1}^2 a_0^{K-1}\end{pmatrix}\begin{pmatrix} s_{K-1}\\ \underline{r}_{K-1} - r_{K-2} - R\end{pmatrix}.$$

Substituting into (3.19) and simplifying yields the optimal solution for problem (3.15):

$$\begin{aligned} x^*_{K-1}(s_{K-1}) &= \frac{1}{2}\Gamma^{-1}\begin{pmatrix}\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1} & \xi'_{K-1}\end{pmatrix}\begin{pmatrix}\lambda_1^{K-1}\\ \lambda_2^{K-1}\end{pmatrix}\\ &= \begin{pmatrix}\Gamma^{-1}\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right) & \Gamma^{-1}\xi'_{K-1}\end{pmatrix}\begin{pmatrix} d_0^{K-1} & -\left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right)\\ -\left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right) & d_2^{K-1} + 2q_{K-1}d_1^{K-1} + q_{K-1}^2 d_0^{K-1}\end{pmatrix}\\ &\quad \begin{pmatrix} s_{K-1}\\ \underline{r}_{K-1} - r_{K-2} - R\end{pmatrix}. \end{aligned} \tag{3.20}$$

Consequently, $\left(a_0^{K-1}a_2^{K-1}-\left(a_1^{K-1}\right)^2\right)^2 x_{K-1}^*\left(s_{K-1}\right)^T\Gamma x_{K-1}^*\left(s_{K-1}\right)=$

$$\begin{aligned}
&\Big(\Big(a_0^{K-1}s_{K-1}-\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)\\
&\left(a_2^{K-1}+2q_{K-1}a_1^{K-1}+q_{K-1}^2a_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)-\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)s_{K-1}\Big)\\
&\begin{pmatrix}\left(\tilde{\mu}_{K-1}+q_{K-1}\xi'_{K-1}\right)^T\\ \left(\xi'_{K-1}\right)^T\end{pmatrix}\Bigg)\Gamma^{-1}\left(\left(\tilde{\mu}_{K-1}+q_{K-1}\tilde{\xi}'_{K-1}\quad \xi'_{K-1}\right)\right.\\
&\left.\begin{pmatrix}a_0^{K-1}s_{K-1}-\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)\\ \left(a_2^{K-1}+2q_{K-1}a_1^{K-1}+q_{K-1}^2a_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)-\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)s_{K-1}\end{pmatrix}\right)\\
&=\left[\left(a_2^{K-1}+2q_{K-1}a_1^{K-1}+q_{K-1}^2a_0^{K-1}\right)\left(a_0^{K-1}\right)^2-a_0^{K-1}\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)^2\right]s_{K-1}^2\\
&+2\left[\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)^3-a_0^{K-1}\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)\left(a_2^{K-1}+2q_{K-1}a_1^{K-1}+q_{K-1}^2a_0^{K-1}\right)\right]\\
&\left(\underline{r}_{K-1}-r_{K-2}-R\right)s_{K-1}+\Big[a_0^{K-1}\left(a_2^{K-1}+2q_{K-1}a_1^{K-1}+q_{K-1}^2a_0^{K-1}\right)^2\\
&-\left(a_2^{K-1}+2q_{K-1}a_1^{K-1}+q_{K-1}^2a_0^{K-1}\right)\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)^2\Big]\left(\underline{r}_{K-1}-r_{K-2}-R\right)^2\\
&=\left[a_0^{K-1}a_2^{K-1}-\left(a_1^{K-1}\right)^2\right]\Big[a_0^{K-1}s_{K-1}^2-2\left(a_1^{K-1}+q_{K-1}a_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)s_{K-1}\\
&+\left(a_2^{K-1}+2q_{K-1}a_1^{K-1}+q_{K-1}^2a_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)^2\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
x_{K-1}^*(s_{K-1})^T\Gamma x_{K-1}^*\left(s_{K-1}\right)&=d_0^{K-1}s_{K-1}^2-2\left(d_1^{K-1}+q_{K-1}d_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)s_{K-1}\\
&+\left(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)^2.
\end{aligned}\tag{3.21}$$

Therefore, substituting (3.21) into the objective function of problem (3.14) reveals that problem (3.14) is equivalent to the unconstrained optimization problem

$$\begin{aligned}
\min_{s_{K-1}\in\mathbb{R}} h_{\beta_{K-1}}\left(s_{K-1}\right):=\sqrt{\frac{\beta_{K-1}}{1-\beta_{K-1}}}&\sqrt{d_0^{K-1}s_{K-1}^2-2\left(d_1^{K-1}+q_{K-1}d_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)s_{K-1}}\\
\overline{+\left(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)^2}&-s_{K-1}+q_{K-1}\left(\underline{r}_K-r_{K-2}\right)-2R\left(q_{K-1}+1\right).
\end{aligned}\tag{3.22}$$

The first-order optimality condition for problem (3.22) is

$$\begin{aligned}
&h_{\beta_{K-1}}{}'\left(s_{K-1}\right)=\sqrt{\frac{\beta_{K-1}}{1-\beta_{K-1}}}\cdot\\
&\frac{d_0^{K-1}s_{K-1}-\left(d_1^{K-1}+q_{K-1}d_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)}{\sqrt{d_0^{K-1}s_{K-1}^2-2\left(d_1^{K-1}+q_{K-1}d_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)s_{K-1}+\left(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\right)\left(\underline{r}_{K-1}-r_{K-2}-R\right)^2}}-1=0.
\end{aligned}\tag{3.23}$$

Thus, we have

$$\begin{aligned}
&\beta_{K-1}\Big(d_0^{K-1}s_{K-1}-\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)\Big)^2\\
&=(1-\beta_{K-1})\Big(d_0^{K-1}s_{K-1}^2-2\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)s_{K-1}\\
&+\Big(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\Big)\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)^2\Big)d_0^{K-1}\Big(\beta_{K-1}d_0^{K-1}-(1-\beta_{K-1})\Big)s_{K-1}^2\\
&-2\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)\Big(\beta_{K-1}d_0^{K-1}-(1-\beta_{K-1})\Big)\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)s_{K-1}\\
&+\Big(\beta_{K-1}\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)^2-\Big(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\Big)(1-\beta_{K-1})\Big)\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)^2=0.
\end{aligned}$$

We then consider two scenarios:

(1)When $\frac{\beta_{K-1}}{1-\beta_{K-1}}\cdot d_0^{K-1}>1$, the optimal solution for problem (3.22) is

$$\begin{aligned}
s_{K-1}^* &= \frac{\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)}{d_0^{K-1}}+\frac{\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)}{d_0^{K-1}\sqrt{\beta_{K-1}d_0^{K-1}-(1-\beta_{K-1})}}\\
&\sqrt{\Big(d_0^{K-1}\Big(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\Big)-\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)^2\Big)(1-\beta_{K-1})}\\
&=\frac{\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)}{d_0^{K-1}}\\
&+\frac{\sqrt{d_0^{K-1}\Big(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\Big)-\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)^2}}{d_0^{K-1}\sqrt{\frac{\beta_{K-1}d_0^{K-1}}{(1-\beta_{K-1})}-1}}\Big(\underline{r}_{K-1}-r_{K-2}-R\Big).
\end{aligned}$$

(2) When $\frac{\beta_{K-1}}{1-\beta_{K-1}}\cdot d_0^{K-1}\le 1$, the optimal solution is $s_{K-1}^*=+\infty$, indicating that problem (3.22) is unbounded.

In scenario (1), substituting s_{K-1}^* into (3.19) yields the optimal investment strategy for period $K-1$, and substituting into (3.14) provides the optimal objective value for problem (3.13). Thus, the optimal value for period $K-1$ is

$$\begin{aligned}
&V_{K-1}^*(r_{K-2})=\\
&\frac{\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)\sqrt{d_0^{K-1}\Big(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\Big)-\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)^2}\sqrt{\frac{\beta_{K-1}d_0^{K-1}}{1-\beta_{K-1}}-1}}{d_0^{K-1}}\\
&-\frac{\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)}{d_0^{K-1}}+q_{K-1}\Big(\underline{r}_K-r_{K-2}\Big)-2R(q_{K-1}+1)\\
&=\Big(\underline{r}_{K-1}-r_{K-2}-R\Big)\cdot\\
&\left(\frac{\sqrt{d_0^{K-1}\Big(d_2^{K-1}+2q_{K-1}d_1^{K-1}+q_{K-1}^2d_0^{K-1}\Big)-\Big(d_1^{K-1}+q_{K-1}d_0^{K-1}\Big)^2}\sqrt{\frac{\beta_{K-1}d_0^{K-1}}{1-\beta_{K-1}}-1}-d_1^{K-1}-q_{K-1}d_0^{K-1}}{d_0^{K-1}}\right)\\
&+q_{K-1}\Big(\underline{r}_K-r_{K-2}\Big)-2R(q_{K-1}+1)
\end{aligned}$$

$$= q_{K-2} \cdot \left(\underline{r}_{K-1} - r_{K-2}\right) + q_{K-1}\left(\underline{r}_K - r_{K-2}\right) - 2R\left(\frac{1}{2}q_{K-2} + q_{K-1} + 1\right),$$

where $q_{K-2} =$

$$\frac{\sqrt{d_0^{K-1}\left(d_2^{K-1} + 2q_{K-1}d_1^{K-1} + q_{K-1}^2 d_0^{K-1}\right) - \left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right)^2}\sqrt{\frac{\beta_{K-1}d_0^{K-1}}{1-\beta_{K-1}} - 1} - d_1^{K-1} - q_{K-1}d_0^{K-1}}{d_0^{K-1}},$$

$$q_{K-1} = \frac{\sqrt{d_0^K d_2^K - \left(d_1^K\right)^2}\sqrt{\frac{\beta_K d_0^K}{1-\beta_K} - 1}}{d_0^K} - \frac{d_1^K}{d_0^K}.$$

The optimal investment strategy for period $K-1$ is

$$x_{K-1}^* = \left(\underline{r}_{K-1} - r_{K-2} - R\right)\left(\Gamma^{-1}\left(\tilde{\mu}_{K-1} + q_{K-1}\xi'_{K-1}\right) \quad \Gamma^{-1}\xi'_{K-1}\right)$$
$$\begin{pmatrix} d_0^{K-1} & -\left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right) \\ -\left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right) & d_2^{K-1} + 2q_{K-1}d_1^{K-1} + q_{K-1}^2 d_0^{K-1} \end{pmatrix}$$
$$\begin{pmatrix} \frac{\left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right)}{d_0^{K-1}} + \frac{\sqrt{d_0^{K-1}\left(d_2^{K-1} + 2q_{K-1}d_1^{K-1} + q_{K-1}^2 d_0^{K-1}\right) - \left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right)^2}}{d_0^{K-1}\sqrt{\frac{\beta_{K-1}d_0^{K-1}}{(1-\beta_{K-1})} - 1}} \\ 1 \end{pmatrix}.$$

By substituting $\xi'_K = \mathrm{E}\left[\xi_K\right] - Re = \tilde{\mu}_K$ into the expression, we find

$$x_{K-1}^* = \left(\underline{r}_{K-1} - r_{K-2} - R\right)\Gamma^{-1}\tilde{\mu}_{K-1}\left(1 + q_{K-1} \quad 1\right)$$
$$\begin{pmatrix} d_0^{K-1} & -\left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right) \\ -\left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right) & d_2^{K-1} + 2q_{K-1}d_1^{K-1} + q_{K-1}^2 d_0^{K-1} \end{pmatrix}$$
$$\begin{pmatrix} \frac{\left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right)}{d_0^{K-1}} + \frac{\sqrt{d_0^{K-1}\left(d_2^{K-1} + 2q_{K-1}d_1^{K-1} + q_{K-1}^2 d_0^{K-1}\right) - \left(d_1^{K-1} + q_{K-1}d_0^{K-1}\right)^2}}{d_0^{K-1}\sqrt{\frac{\beta_{K-1}d_0^{K-1}}{(1-\beta_{K-1})} - 1}} \\ 1 \end{pmatrix}.$$

4. Optimal strategies for the multi-period robust portfolio selection model

We apply the same method from period K down to period 1 to solve the optimal investment decision problems for each subsequent stage. Theorem 1 summarizes the resulting optimal strategies.

Theorem 1. *For all $k = 1, \ldots, K$, let $\xi'_k = \mathrm{E}\left[\xi_k\right] - Re$, $a_0^k := (\xi'_k)^T\Gamma^{-1}\xi'_k$, $a_1^k := (\xi'_k)^T\Gamma^{-1}\tilde{\mu}_k$, $a_2^k := \tilde{\mu}_k^T\Gamma^{-1}\tilde{\mu}_k$, $d_0^k := \frac{a_0^k}{a_0^k a_2^k - (a_1^k)^2}$, $d_1^k := \frac{a_1^k}{a_0^k a_2^k - (a_1^k)^2}$, $d_2^k := \frac{a_2^k}{a_0^k a_2^k - (a_1^k)^2}$,*

$$q_i = \frac{\sqrt{d_0^{i+1}\left(d_2^{i+1} + 2d_1^{i+1}\sum_{j=i+1}^{K-1} q_j + d_0^{i+1}\left(\sum_{j=i+1}^{K-1} q_j\right)^2\right) - \left(d_1^{i+1} + d_0^{i+1}\sum_{j=i+1}^{K-1} q_j\right)^2}\sqrt{\frac{\beta_{i+1}d_0^{i+1}}{1-\beta_{i+1}} - 1}}{d_0^{i+1}}$$

$$-\frac{d_1^{i+1}+\left(\sum\limits_{j=i+1}^{K-1} q_j\right) d_0^{i+1}}{d_0^{i+1}}.$$

If $\frac{\beta_k}{1-\beta_k} \cdot d_0^k > 1$ *for all* $k = 1, \ldots, K$*, then, given a cumulative return* r_{k-1}*, the optimal objective value and optimal investment strategy at stage k are*

$$V_k^*\left(r_{k-1}\right)=\sum_{i=k}^{K} q_{i-1}\left(\underline{r}_i-r_{k-1}\right)-R\left(\sum_{j=k}^{K}(j-k+1) q_{j-1}+K-k+1\right)$$

and

$$x_k^*=\left(\underline{r}_k-r_{k-1}-R\right) \Gamma^{-1} \tilde{\mu}_k\left(1+\sum_{j=k}^{K-1} q_j \quad 1\right)\begin{pmatrix} d_0^k & -\left(d_1^k+d_0^k \sum\limits_{j=k}^{K-1} q_j\right) \\ -\left(d_1^k+d_0^k \sum\limits_{j=k}^{K-1} q_j\right) & d_2^k+2 d_1^k \sum\limits_{j=k}^{K-1} q_j+d_0^k\left(\sum\limits_{j=k}^{K-1} q_j\right)^2 \end{pmatrix}$$
$$\begin{pmatrix} \frac{\left(d_1^k+d_0^k \sum\limits_{j=k}^{K-1} q_j\right)}{d_0^k}+\frac{\sqrt{d_0^k\left(d_2^k+2 d_1^k \sum\limits_{j=k}^{K-1} q_j+d_0^k\left(\sum\limits_{j=k}^{K-1} q_j\right)^2\right)-\left(d_1^k+d_0^k \sum\limits_{j=k}^{K-1} q_j\right)^2}}{d_0^k \sqrt{\frac{\beta_k d_0^k}{\left(1-\beta_k\right)}-1}} \\ 1 \end{pmatrix},$$

respectively. If $\frac{\beta_k}{1-\beta_k} \cdot d_0^k \leq 1$ *for some* k *(*$1 \leq k \leq K$*), the optimal solution at period k diverges to infinity, making problem (2.1) unbounded.*

Proof. Assume the optimal value and solution hold at stage $k+1$. At stage k, given the cumulative return r_{k-1}, the corresponding optimal decision problem is expressed as

$$\begin{aligned} V_k\left(r_{k-1}\right)= & \min_{x_k \in R^n} \sqrt{\frac{\beta_k}{1-\beta_k}} \sqrt{x_k^T \Gamma x_k}-\left(\mu_k-R e\right)^T x_k-R+q_k\left(\underline{r}_{k+1}-\mathrm{E}\left[r_k\right]\right)+\cdots \\ & +q_{K-2}\left(\underline{r}_{K-1}-\mathrm{E}\left[r_k\right]\right)+q_{K-1}\left(\underline{r}_K-\mathrm{E}\left[r_k\right]\right)-R\left(\sum_{j=k+1}^{K}(j-k) q_{j-1}+K-k\right) \\ & \text{s.t.} \quad r_{k-1}+\left(\mathrm{E}\left[\xi_k\right]\right)^T x_k+R\left(1-x_k^T e\right)=\underline{r}_k . \end{aligned} \tag{4.1}$$

Substituting the constraints into the objective function of the problem (4.1), we obtain

$$\begin{aligned} V_k\left(r_{k-1}\right)= & \min_{x_k \in \mathbb{R}^n} \sqrt{\frac{\beta_k}{1-\beta_k}} \sqrt{x_k^T \Gamma x_k}-\tilde{\mu}_k^T x_k+q_k\left(\underline{r}_{k+1}-\left(r_{k-1}+\left(\mathrm{E}\left[\xi_k\right]\right)^T x_k+R\left(1-x_k^T e\right)\right)\right)+\cdots \\ & +q_{K-2}\left(\underline{r}_{K-1}-\left(r_{k-1}+\left(\mathrm{E}\left[\xi_k\right]\right)^T x_k+R\left(1-x_k^T e\right)\right)\right) \\ & +q_{K-1}\left(\underline{r}_K-\left(r_{k-1}+\left(\mathrm{E}\left[\xi_k\right]\right)^T x_k+R\left(1-x_k^T e\right)\right)\right) \\ & -R\left(\sum_{j=k+1}^{K}(j-k) q_{j-1}+K-k+1\right) \\ = & \min_{x_k \in \mathbb{R}^n} \sqrt{\frac{\beta_k}{1-\beta_k}} \sqrt{x_k^T \Gamma x_k}-\left(\tilde{\mu}_k+\left(q_k+q_{k+1}+\cdots+q_{K-1}\right) \xi'_k\right)^T x_k+q_k\left(\underline{r}_{k+1}-r_{k-1}\right)+\cdots \end{aligned}$$

$$+ q_{K-2}\left(\underline{r}_{K-1} - r_{k-1}\right) + q_{K-1}\left(\underline{r}_K - r_{k-1}\right) - R\left(\sum_{j=k+1}^{K}(j-k)q_{j-1} + K - k + 1\right),$$

with $s_k = x_k^T\left(\tilde{\mu}_k + (q_k + q_{k+1} + \cdots + q_{K-1})\,\xi'_k\right)$ showing that problem (4.1) is equivalent to

$$\begin{aligned}
\min_{s_k\in\mathbb{R}}\min_{x_k\in\mathbb{R}^n}\ & \sqrt{\frac{\beta_k}{1-\beta_k}}\sqrt{x_k^T\Gamma x_k} - s_k + q_k\left(\underline{r}_{k+1} - r_{k-1}\right) + \cdots \\
& + q_{K-2}\left(\underline{r}_{K-1} - r_{k-1}\right) + q_{K-1}\left(\underline{r}_K - r_{k-1}\right) - R\left(\sum_{j=k+1}^{K}(j-k)q_{j-1} + K - k + 1\right) \\
\text{s.t.}\quad & x_k^T\left(\tilde{\mu}_k + (q_k + q_{k+1} + \cdots + q_{K-1})\,\xi'_k\right) = s_k.
\end{aligned} \tag{4.2}$$

To this end, we consider the following problem:

$$\begin{aligned}
\min_{x_k\in\mathbb{R}^n}\quad & x_k^T\Gamma x_k \\
\text{s.t.}\quad & x_k^T\xi'_k = \underline{r}_k - r_{k-1} - R, \\
& x_k^T\left(\tilde{\mu}_k + \sum_{j=k}^{K-1} q_j\xi'_k\right) = s_k.
\end{aligned} \tag{4.3}$$

The Lagrangian function for problem (4.3) is

$$L\left(x_k, \lambda_1^k, \lambda_2^k\right) = x_k^T\Gamma x_k + \lambda_1^k\left(s_k - x_k^T\left(\tilde{\mu}_k + \sum_{j=k}^{K-1} q_j\xi'_k\right)\right) + \lambda_2^k\left(\underline{r}_k - r_{k-1} - R - x_k^T\tilde{\xi}_k\right).$$

Using the first-order optimality conditions, we derive the following equations:

$$\begin{cases}
L_{x_k} = 2\Gamma x_k - \lambda_1^k\left(\tilde{\mu}_k + \sum_{j=k}^{K-1} q_j\xi'_k\right) - \lambda_2^k\xi'_k = 0, & (4.4)\\
x_k^T\xi'_k - \left(\underline{r}_k - r_{k-1} - R\right) = 0, & (4.5)\\
x_k^T\left(\tilde{\mu}_k + \sum_{j=k}^{K-1} q_j\xi'_k\right) - s_k = 0. & (4.6)
\end{cases}$$

From (4.4), we have

$$x_k = \frac{1}{2}\Gamma^{-1}\left(\lambda_1^k\left(\tilde{\mu}_k + \sum_{j=k}^{K-1} q_j\xi'_k\right) + \lambda_2^k\xi'_k\right), \tag{4.7}$$

substituting (4.7) into (4.5) and (4.6) yields

$$\begin{pmatrix}
a_2^k + 2a_1^k\sum_{j=k}^{K-1} q_j + a_0^k\left(\sum_{j=k}^{K-1} q_j\right)^2 & a_1^k + a_0^k\sum_{j=k}^{K-1} q_j \\
a_1^k + a_0^k\sum_{j=k}^{K-1} q_j & a_0^k
\end{pmatrix}\begin{pmatrix}\lambda_1^k \\ \lambda_2^k\end{pmatrix} = 2\begin{pmatrix} s_k \\ \underline{r}_k - r_{k-1} - R\end{pmatrix}.$$

Thus, the Lagrange multipliers are

$$\begin{pmatrix}\lambda_1^k\\ \lambda_2^k\end{pmatrix}=\begin{pmatrix} d_0^k & -\left(d_1^k+d_0^k\sum\limits_{j=k}^{K-1}q_j\right)\\ -\left(d_1^k+d_0^k\sum\limits_{j=k}^{K-1}q_j\right) & d_2^k+2d_1^k\sum\limits_{j=k}^{K-1}q_j+d_0^k\left(\sum\limits_{j=k}^{K-1}q_j\right)^2\end{pmatrix}\begin{pmatrix}s_k\\ \underline{r}_k-r_{k-1}-R\end{pmatrix}.$$

Substituting λ_1^k, λ_2^k into (4.7) and simplifying gives the optimal solution for problem (4.3):

$$x_k^*(s_k)=$$

$$\begin{pmatrix}\Gamma^{-1}\left(\tilde{\mu}_k+\sum\limits_{j=k}^{K-1}q_j\xi'_k\right) & \Gamma^{-1}\xi'_k\end{pmatrix}\begin{pmatrix} d_0^k & -\left(d_1^k+d_0^k\sum\limits_{j=k}^{K-1}q_j\right)\\ -\left(d_1^k+d_0^k\sum\limits_{j=k}^{K-1}q_j\right) & d_2^k+2d_1^k\sum\limits_{j=k}^{K-1}q_j+d_0^k\left(\sum\limits_{j=k}^{K-1}q_j\right)^2\end{pmatrix}\begin{pmatrix}s_k\\ \underline{r}_k-r_{k-1}-R\end{pmatrix}. \tag{4.8}$$

Therefore,

$$\begin{aligned}(x_k^*(s_k))^T\Gamma x_k^*(s_k)&=d_0^ks_k^2-2\left(d_1^k+d_0^k\sum_{j=k}^{K-1}q_j\right)\left(\underline{r}_k-r_{k-1}-R\right)s_k\\&+\left(d_2^k+2d_1^k\sum_{j=k}^{K-1}q_j+d_0^k\left(\sum_{j=k}^{K-1}q_j\right)^2\right)\left(\underline{r}_k-r_{k-1}-R\right)^2.\end{aligned} \tag{4.9}$$

Substituting (4.9) into the objective function of problem (4.2) transforms the problem into an unconstrained optimization problem

$$\begin{aligned}\min_{s_k\in R}h_{\beta_k}(s_k):=&\sqrt{\frac{\beta_k}{1-\beta_k}}\\&\sqrt{d_0^ks_k^2-2\left(d_1^k+d_0^k\sum_{j=k}^{K-1}q_j\right)\left(\underline{r}_k-r_{k-1}-R\right)s_k+\left(d_2^k+2d_1^k\sum_{j=k}^{K-1}q_j+d_0^k\left(\sum_{j=k}^{K-1}q_j\right)^2\right)\left(\underline{r}_k-r_{k-1}-R\right)^2}-s_k\\&+q_k\left(\underline{r}_{k+1}-r_{k-1}\right)+\cdots+q_{K-2}\left(\underline{r}_{K-1}-r_{k-1}\right)+q_{K-1}\left(\underline{r}_K-r_{k-1}\right)-R\left(\sum_{j=k+1}^{K}(j-k)q_{j-1}+K-k+1\right).\end{aligned} \tag{4.10}$$

At this point, we consider two cases: (1) When $\frac{\beta_k}{1-\beta_k}\cdot d_0^k>1$,

$$\begin{aligned}s_k^*=&\frac{\left(d_1^k+d_0^k\sum\limits_{j=k}^{K-1}q_j\right)\left(\underline{r}_k-r_{k-1}-R\right)}{d_0^k}+\frac{\left(\underline{r}_k-r_{k-1}-R\right)}{d_0^k\sqrt{\beta_kd_0^k-(1-\beta_k)}}\\&\sqrt{\left(d_0^k\left(d_2^k+2d_1^k\sum_{j=k}^{K-1}q_j+d_0^k\left(\sum_{j=k}^{K-1}q_j\right)^2\right)-\left(d_1^k+d_0^k\sum_{j=k}^{K-1}q_j\right)^2\right)(1-\beta_k)},\end{aligned}$$

that is

$$s_k^* = \frac{\left(d_1^k + d_0^k \sum_{j=k}^{K-1} q_j\right)\left(\underline{r}_k - r_{k-1} - R\right)}{d_0^k} + \frac{\sqrt{d_0^k\left(d_2^k + 2d_1^k \sum_{j=k}^{K-1} q_j + d_0^k\left(\sum_{j=k}^{K-1} q_j\right)^2\right) - \left(d_1^k + d_0^k \sum_{j=k}^{K-1} q_j\right)^2}}{d_0^k \sqrt{\frac{\beta_k d_0^k}{(1-\beta_k)} - 1}} \left(\underline{r}_k - r_{k-1} - R\right).$$

(2) When $\frac{\beta_k}{1-\beta_k} \cdot d_0^k \leq 1$, the optimal solution for problem (4.10) is $s_k^* = +\infty$, indicating it is unbounded. In case (1), substituting s_k^* back into (4.8) provides the optimal investment strategy for period k, and substituting into (4.2) gives the optimal objective value for problem (4.1). Thus, the optimal value for period k is

$$\begin{aligned}
V_k^* &= \left(\underline{r}_k - r_{k-1} - R\right) \cdot \\
&\left(\frac{\sqrt{d_0^k\left(d_2^k + 2d_1^k \sum_{j=k}^{K-1} q_j + d_0^k\left(\sum_{j=k}^{K-1} q_j\right)^2\right) - \left(d_1^k + d_0^k \sum_{j=k}^{K-1} q_j\right)^2} \sqrt{\frac{\beta_k d_0^k}{1-\beta_k} - 1}}{d_0^k} - \frac{d_1^k + d_0^k \sum_{j=k}^{K-1} q_j}{d_0^k}\right) \\
&+ q_k\left(\underline{r}_{k+1} - r_{k-1}\right) + \cdots + q_{K-2}\left(\underline{r}_{K-1} - r_{k-1}\right) + q_{K-1}\left(\underline{r}_K - r_{k-1}\right) - R\left(\sum_{j=k+1}^{K} (j-k) q_{j-1} + K - k + 1\right) \\
&= q_{k-1}\left(\underline{r}_k - r_{k-1} - R\right) + q_k\left(\underline{r}_{k+1} - r_{k-1}\right) + \cdots + q_{K-2}\left(\underline{r}_{K-1} - r_{k-1}\right) \\
&+ q_{K-1}\left(\underline{r}_K - r_{k-1}\right) - R\left(\sum_{j=k+1}^{K} (j-k) q_{j-1} + K - k + 1\right),
\end{aligned}$$

where $q_{k-1} =$

$$\left(\frac{\sqrt{d_0^k\left(d_2^k + 2d_1^k \sum_{j=k}^{K-1} q_j + d_0^k\left(\sum_{j=k}^{K-1} q_j\right)^2\right) - \left(d_1^k + d_0^k \sum_{j=k}^{K-1} q_j\right)^2} \sqrt{\frac{\beta_k d_0^k}{1-\beta_k} - 1}}{d_0^k} - \frac{d_1^k + R d_0^k \sum_{j=k}^{K-1} q_j R^{j-k}}{d_0^k}\right).$$

The optimal investment strategy for period k is

$$x_k^* = \left(\underline{r}_k - r_{k-1} - R\right)\Gamma^{-1}\tilde{\mu}_k \begin{pmatrix} 1 + \sum_{j=k}^{K-1} q_j & 1 \end{pmatrix} \begin{pmatrix} d_0^k & -\left(d_1^k + d_0^k \sum_{j=k}^{K-1} q_j\right) \\ -\left(d_1^k + d_0^k \sum_{j=k}^{K-1} q_j\right) & d_2^k + 2d_1^k \sum_{j=k}^{K-1} q_j + d_0^k\left(\sum_{j=k}^{K-1} q_j\right)^2 \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{\left(d_1^k+d_0^k \sum\limits_{j=k}^{K-1} q_j\right)}{d_0^k} + \dfrac{\sqrt{d_0^k\left(d_2^k+2d_1^k \sum\limits_{j=k}^{K-1} q_j+d_0^k\left(\sum\limits_{j=k}^{K-1} q_j\right)^2\right)-\left(d_1^k+d_0^k \sum\limits_{j=k}^{K-1} q_j\right)^2}}{d_0^k \sqrt{\frac{\beta_k d_0^k}{(1-\beta_k)}-1}} \\ 1 \end{pmatrix}.$$

Thus, by employing dynamic programming, we can recursively solve problem (2.1) to obtain the optimal objective values and strategies for period k (where $k = 1, \ldots, K$) as

$$V_k^*(w_{k-1}) = \sum_{i=k}^{K} q_{i-1}\left(\underline{r}_i - r_{k-1}\right) - R\left(\sum_{j=k}^{K} (j-k+1)q_{j-1} + K - k + 1\right).$$

Here,

$$q_i = \frac{\sqrt{d_0^{i+1}\left(d_2^{i+1} + 2d_1^{i+1} \sum\limits_{j=i+1}^{K-1} q_j + d_0^{i+1}\left(\sum\limits_{j=i+1}^{K-1} q_j\right)^2\right) - \left(d_1^{i+1} + d_0^{i+1} \sum\limits_{j=i+1}^{K-1} q_j\right)^2} \sqrt{\frac{\beta_{i+1} d_0^{i+1}}{1-\beta_{i+1}} - 1}}{d_0^{i+1}}$$
$$- \frac{d_1^{i+1} + \left(\sum\limits_{j=i+1}^{K-1} q_j\right) d_0^{i+1}}{d_0^{i+1}},$$

and

$$x_k^* = \left(\underline{r}_k - r_{k-1} - R\right)\Gamma^{-1}\tilde{\mu}_k \begin{pmatrix} 1 + \sum\limits_{j=k}^{K-1} q_j & 1 \end{pmatrix} \begin{pmatrix} d_0^k & -\left(d_1^k + d_0^k \sum\limits_{j=k}^{K-1} q_j\right) \\ -\left(d_1^k + d_0^k \sum\limits_{j=k}^{K-1} q_j\right) & d_2^k + 2d_1^k \sum\limits_{j=k}^{K-1} q_j + d_0^k\left(\sum\limits_{j=k}^{K-1} q_j\right)^2 \end{pmatrix}$$
$$\begin{pmatrix} \dfrac{\left(d_1^k+d_0^k \sum\limits_{j=k}^{K-1} q_j\right)}{d_0^k} + \dfrac{\sqrt{d_0^k\left(d_2^k+2d_1^k \sum\limits_{j=k}^{K-1} q_j+d_0^k\left(\sum\limits_{j=k}^{K-1} q_j\right)^2\right)-\left(d_1^k+d_0^k \sum\limits_{j=k}^{K-1} q_j\right)^2}}{d_0^k \sqrt{\frac{\beta_k d_0^k}{(1-\beta_k)}-1}} \\ 1 \end{pmatrix}.$$

□

With the analytical solution provided in Theorem 1, the optimal investment strategies can be directly determined for portfolio selection at each stage within a robust optimization framework.

5. Numerical results

Our dataset consists of five stocks (NVDA, DFS, MTCH, WBA, and GOOGL) from `https://finance.yahoo.com`. These stocks were selected for sectoral diversity across technology, finance, healthcare, and consumer discretionary sector while also exhibiting high liquidity and non-Gaussian return characteristics, as validated by the Shapiro-Wilk test in Table 1. It includes the weekly closing price of these stocks from January 5, 2014, to January 7, 2024, covering diverse market regimes, such as bull markets and the COVID-19 crash. This selection ensures rigorous validation of the regime-switching CVaR model and aligns with our focus on distributionally robust optimization under uncertain return distributions. Figure 1 presents the time series plots of the original price data. This study adopts the robust CVaR as the risk measure to solve the robust optimization problem, aiming to minimize risk while satisfying the minimum target return rate constraint.

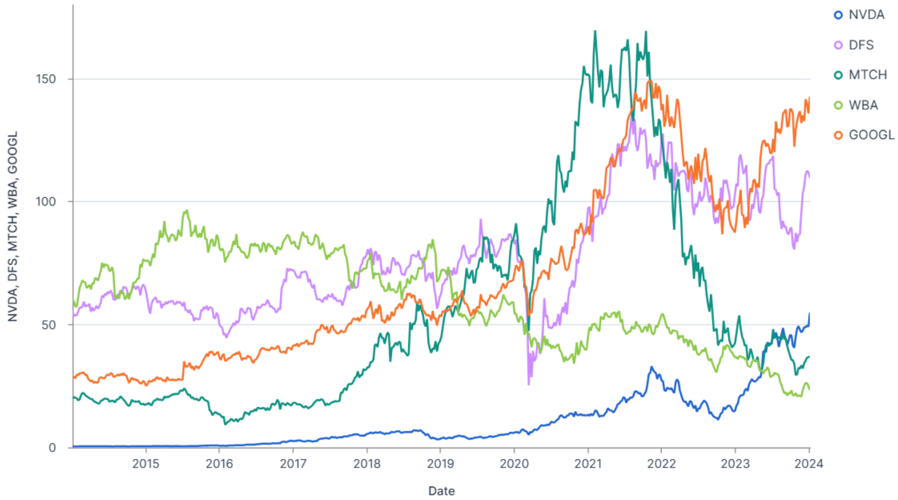

Figure 1. Time series plots of the dataset.

We computed the logarithmic return rates for all five stocks. Table 1 presents the central moments and Shapiro-Wilk test results for the selected assets. Based on the S-W test results, the P-values in the last column of Table 1 were all below 0.01. Thus, the null hypothesis of normality is rejected, indicating that the data do not follow a normal distribution.

Table 1. Moments and KS test statistic.

	mean	Var	skew	kurt	S-W test	P-value
NVDA	0.0113	0.0036	0.4364	2.005	0.9771	0.0000
DFS	0.0031	0.0034	0.5131	27.1993	0.7765	0.0000
MTCH	0.0032	0.0040	0.3137	3.7597	0.9564	0.0000
WBA	-0.0010	0.0016	-0.0633	1.3410	0.9842	0.0000
GOOGL	0.0038	0.0014	0.6776	4.5066	0.9609	0.0000

We aim to dynamically adjust asset allocation based on capital market conditions. The underlying regimes are predicted by a Markov chain with two regimes, which are commonly interpreted as a bear market or a bull market, denoted as $U = \{0, 1\}$. By incorporating regime switching, the model can more accurately reflect market dynamics.

Using historical data, we identify market regimes based on a threshold criterion. Specifically, we set a threshold of 0.01, meaning a return above 1% signifies a bull market. Asset returns, volatility, and other parameters fluctuate between the two regimes. We estimate the expected mean returns and covariance matrices for each regime based on historical data, as shown in Eq (5.1) to (5.4). The confidence level β_k is set to 0.95, which is assumed to be the same across regimes. The minimum required return $\underline{r}_k$ is set to 0.03, and the risk-free rate R is set to 0.02.

$$\mu^0 = 10^{-2}(0.7843 \quad -0.5138 \quad -1.1134 \quad -0.8089 \quad 0.2239)^T, \tag{5.1}$$

$$\mu^1 = 10^{-2}(0.9948 \quad 0.3196 \quad 0.4803 \quad 0.0012 \quad 0.3353)^T, \tag{5.2}$$

$$\Gamma^0 = 10^{-2}\begin{pmatrix} 0.6154 & 0.3471 & 0.3945 & 0.0384 & 0.2318 \\ 0.3471 & 1.3260 & 0.4550 & 0.2238 & 0.2317 \\ 0.3945 & 0.4550 & 0.9793 & 0.1117 & 0.2020 \\ 0.0384 & 0.2238 & 0.1117 & 0.3897 & 0.0634 \\ 0.2318 & 0.2317 & 0.2020 & 0.0634 & 0.2938 \end{pmatrix}, \tag{5.3}$$

$$\Gamma^1 = 10^{-3}\begin{pmatrix} 2.7641 & 0.5327 & 0.8717 & 0.4835 & 0.7327 \\ 0.5327 & 0.8597 & 0.2803 & 0.4115 & 0.3851 \\ 0.8717 & 0.2803 & 2.2574 & 0.3398 & 0.5778 \\ 0.4835 & 0.4115 & 0.3398 & 0.9886 & 0.3319 \\ 0.7327 & 0.3851 & 0.5778 & 0.3319 & 0.9563 \end{pmatrix}. \tag{5.4}$$

By analyzing regime transition frequencies, we compute the transition probability matrix

$$P = \begin{pmatrix} 0.5823 & 0.4177 \\ 0.6390 & 0.3610 \end{pmatrix}.$$

The optimal investment strategy can be determined using dynamic programming, which recursively simplifies the problem to reduce computational complexity, particularly when state transitions exhibit Markov properties. The state variables include the current market condition and current wealth levels. The decision variable is the portfolio adjustments at each stage. State transitions are determined by the transition probability matrix of the market states, while the objective function represents the minimum risk from the current stage to the final stage.

A robust optimization model that minimizes risk under the worst-case scenario is applied at each stage. Based on the current state and decision-making, the optimal portfolios for the next stage are computed while considering the state transition probabilities. Thus, embedding the robust optimization problem within the dynamic programming framework is necessary.

By recursively decomposing the multi-stage problem through dynamic programming, the robust optimization subproblem is solved at each stage by calculating backward from the final stage $k = K$ to the initial time $k = 1$. The proposed framework's time complexity is $O(K \cdot m^2 \cdot n^3)$, where K is the number of periods, m is the number of regimes, and n is the number of risky assets. Dynamic programming avoids the exponential explosion of the scenario tree, enhancing the computational efficiency of the multi-stage robust portfolio optimization problem.

Table 2. Optimal portfolios under DP.

Period	**Regime**	**NVDA**	**DFS**	**MTCH**	**WBA**	**GOOGL**	**Risk Asset Weight**
0	1	0.2000	0.2000	0.2000	0.2000	1.6912×10^{-9}	0.8000
1	0	0.2000	0.2000	1.1299×10^{-10}	1.2979×10^{-10}	2.3682×10^{-10}	0.4000
2	0	3.2500×10^{-11}	0.2000	6.0090×10^{-11}	0.2000	0.2000	0.6000
3	0	1.2931×10^{-7}	4.6059×10^{-10}	0.2000	9.7313×10^{-10}	0.2000	0.4000
4	1	0.2000	1.0664×10^{-9}	9.9738×10^{-10}	0.2000	0.2000	0.6000
5	1	0.2000	0.2000	0.2000	1.7060×10^{-11}	3.3000×10^{-11}	0.6000
6	0	0.2000	4.8124×10^{-10}	3.5599×10^{-10}	1.1482×10^{-9}	3.9763×10^{-10}	0.2000
7	1	2.5804×10^{-10}	0.2000	0.2000	0.2000	9.1840×10^{-10}	0.6000
8	0	0.2000	0.2000	1.8773×10^{-9}	0.2000	0.2000	0.8000
9	0	0.2000	1.1065×10^{-10}	9.4690×10^{-10}	3.4756×10^{-10}	0.2000	0.4000

We develop an investment strategy that dynamically adjusts asset allocation in response to regime transitions. Table 2 presents the optimal portfolios across different stages, demonstrating a dynamic adjustment strategy that minimizes risk while satisfying return constraints. The initial market state is assumed to be a bull market with an initial wealth of 10.0. At each stage, μ_k is dynamically updated, and the target value V_k and strategy are adjusted accordingly. The total number of stages is set to $K = 10$. The sum of portfolio weights in each row is less than 1.0 due to the inclusion of the risk-free asset. The changes in portfolio allocations across different stages are examined, with allocations to certain assets increasing while others decrease, reflecting the model's expectations of regime switches.

Building upon the optimal portfolios derived from the dynamic programming framework as detailed in Table 2, we further evaluate the strategy's efficacy and robustness. Key performance indicators, including the Sharpe ratio (SR), compound annual growth rate (CAGR), maximum drawdown (MDD), and turnover ratio (TR), are reported in Table 3.

Table 3. Performance measures of the portfolios.

Periods	**SR**	**CAGR**	**MDD**	**TR**
10	1.1235	3.2659%	12.7198%	37.1429%

The strategy exhibits robust performance over a 10-period investment horizon, achieving a Sharpe ratio of 1.12 and an annualized CAGR of 3.27%, indicating consistently superior risk-adjusted returns and stable growth. The maximum drawdown of 12.72% demonstrates effective downside risk management, while the turnover rate of 37.14% indicates that a balanced portfolio rebalancing corresponds with transaction costs. These performance measures provide empirical validation that our model effectively addresses the multi-stage robust portfolio selection problem under regime switching.

6. Conclusions

In this study, we propose a dynamic multi-period robust portfolio selection framework that integrates regime-switching techniques and distributionally robust optimization under CVaR risk measures. By refining the uncertainty set with known first and second moments, we construct a dynamic model that captures the dependencies between consecutive periods. Our approach employs dynamic programming to address the robust optimization problem, ensuring that the resulting optimal investment strategies adapt dynamically based on the state of the uncertainty set. Compared to static strategies that fail to respond to market fluctuations, our approach better aligns with real-world conditions and offers a more effective solution. Leveraging convex duality and dynamic programming, we derive analytical optimal investment strategies that dynamically adjust allocations based on regimes. This framework develops multi-period portfolio optimization by integrating dynamic uncertainty modeling with robust risk management.

Although we have analytically solved the multi-stage robust portfolio selection problem, the complexity of the problem increases with the number of stages. We assume the Markov transition matrices are time-homogeneous, while structural breaks on asset returns vary across markets. Developing a robust adaptive transition estimator will be a potential research direction. There remain numerous questions for further study. This study considers the case where the distribution of asset returns is uncertain. Future research could extend this framework to scenarios where both the distribution and mean of the return are uncertain. Approaches employing CVaR and VaR as risk measures would be valuable, especially in scenarios where the mean belongs to an ellipsoidal uncertainty set or a Wasserstein ambiguity set. Identifying multi-period risk measure models that not only have a strong financial and economic foundation but also ensure practical applicability remains a critical research area. Future work will apply Dempster-Shafer [37] and multi-scale fusion [38] to optimize dynamic CVaR in regime-switching portfolios. Furthermore, enhancing computational efficiency in solving multi-period portfolio selection problems through stochastic programming and other relevant methodologies, such as PMCTNN [39], will be crucial for advancing multi-period investment strategies.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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