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Research article

Some results on circulant matrices involving Fibonacci polynomials

Fatih Yılmaz*, **Aybüke Ertaş and Samet Arpacı**

Department of Mathematics, Ankara Hacı Bayram Veli University, Ankara, Turkey

* **Correspondence:** Email: fatih.yilmaz@hbv.edu.tr.

Abstract: In this study, we considered circulant matrices whose elements are Fibonacci polynomials. Then, we computed their determinants in two ways. In this content, we initially benefited from Chebyshev polynomials of the second kind. In the second way, we utilized some basic matrix operations. Moreover, we computed the inverse of these matrices in a general form. Furthermore, we found some kind of norms such as the Euclidean norm, upper and lower bounds for $||C_n||_2$. In addition, we added some illustrative examples to make the results clear for the readers. In addition to these, we provide a MATLAB-R2023a code that writes the circulant matrix with the Fibonacci polynomial inputs, as well as computes Euclidean norms and bounds for their spectral norms.

Keywords: circulant matrices; Fibonacci polynomials; determinants; matrix norm
Mathematics Subject Classification: 11C08, 11C20, 15A09, 15A15, 15A60

1. Introduction

1.1. A brief review on the development circulant matrices

A circulant matrix (see [1]) is represented as follows:

$$\mathbb{C}_n = \begin{pmatrix} c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \dots & c_0 & c_1 \\ c_1 & c_2 & \dots & c_{n-1} & c_0 \end{pmatrix}, \tag{1.1}$$

where c_n's are real numbers.

It is seen that any circulant matrix is a particular kind of Toeplitz matrix. Numerous papers (see references [2–10]) that support this argument show the importance of circulant matrices and their applications in a wide range of fields. The authors examined circulant matrices with Gaussian

Nickel Fibonacci number entries in [11]. Moreover, Solak presented norms of circulant matrices with Fibonacci and Lucas numbers in [12]. In addition, the authors computed the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers in [13]. In [14], Liu and Jiang presented the determinants and inverses of circulant-type matrices with Tribonacci entries.

Let $C = (c_{ij})$ and $D = (d_{ij})$ be $n \times n$ real matrices, then the Hadamard product of these matrices is defined by $C \circ D \equiv [c_{ij}d_{ij}]$ see [15]. For $n = 3$, we give the following example:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \circ \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = \begin{bmatrix} c_{11}d_{11} & c_{12}d_{12} & c_{13}d_{13} \\ c_{21}d_{21} & c_{22}d_{22} & c_{23}d_{23} \\ c_{31}d_{31} & c_{32}d_{32} & c_{33}d_{33} \end{bmatrix}.$$

A matrix's norm is a nonnegative real integer. There are multiple ways for calculating a matrix norm; however, all the methods have the same features. Let $A = (a_{ij})$ be a $n \times n$ matrix. Then, the maximum column length norm (shortly, $c_1(.)$) and the maximum row length norm (shortly, $r_1(.)$) are defined as follows:

$$c_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2}$$

and

$$r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2}.$$

The Euclidean norm of matrix A is

$$||A||_{\mathbb{E}} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$$

and the spectral norm of matrix A is

$$||A||_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^H A)},$$

where λ_i is the eigenvalue of matrix AA^H; here, A^H is a conjugate transpose of A. Thus, the following inequalities hold:

$$\begin{aligned} \frac{1}{\sqrt{n}}||A||_{\mathbb{E}} \leq ||A||_2 \leq ||A||_{\mathbb{E}}, \\ ||A||_2 \leq ||A||_{\mathbb{E}} \leq \sqrt{n}||A||_2. \end{aligned} \tag{1.2}$$

Let A, B, and C be $m \times n$ matrices. According to [16, 17], the following inequalities are well known:

- If $A = B \circ C$, then
$$||A||_2 \leq r_1(B)c_1(C).$$
- If $||.||$ is arbitrary norm on $n \times m$ matrices, then
$$||A \circ B|| \leq ||A||||B||.$$

1.2. Some notes on Fibonacci and Chebyshev polynomials

The Fibonacci polynomials have a wide range of applications in mathematics. In 1883, Eugene Charles Catalan and E. Jacobsthal studied the Fibonacci polynomials [18, 19]. The well-known Fibonacci polynomials have the following recurrence relation:

$$\mathcal{F}_{n+2}(x) = x\mathcal{F}_{n+1}(x) + \mathcal{F}_n(x),\ n \geq 1,$$

where $\mathcal{F}_1(x) = 1,\ \mathcal{F}_2(x) = x$. Moreover, we give some initial values for Fibonacci polynomials in Table 1.

Table 1. Some of the Fibonacci Polynomials.

n	$\mathcal{F}_n(x)$
1	1
2	x
3	$1 + x^2$
4	$2x + x^3$
5	$1 + 3x^2 + x^4$

A Fibonacci-like recurrence relation can be used to define large classes of polynomials, yielding Fibonacci numbers [20]. Note that $\mathcal{F}_n(1) = F_n$, which is the *nth* Fibonacci number. In [21], the generating function $G_F(\lambda)$ of the Fibonacci polynomials is defined as follows:

$$G_F(\lambda) = \sum_{n=0}^{\infty} \mathcal{F}_n(x)\lambda^n = \frac{\lambda}{1 - x\lambda - \lambda^2}.$$

Also, the characteristic equation of the sequence $\mathcal{F}_n(x)$ is

$$\lambda^2 - x\lambda - 1 = 0, \tag{1.3}$$

where x is a real number. The roots of the characteristic Eq (1.3) are $\alpha = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta = \frac{x-\sqrt{x^2+4}}{2}$. Consider $\alpha+\beta = x$, $\alpha\beta = -1$ and $\alpha-\beta = \sqrt{x^2+4}$. Thus, Binet's formula for every integer n as follows:

$$\mathcal{F}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

In [22], Swamy studied the Fibonacci polynomials and investigated several other properties for these polynomials. The authors obtain more properties utilizing Fibonacci polynomials in [23]. A. Lupas presented many remarkable characteristics of Fibonacci polynomials in [24]. For more details, see references [25–28] and therein. In addition to these properties, in [29], Panwar et.al. presented the sum of Fibonacci polynomials as follows:

$$\sum_{i=0}^{j} \mathcal{F}_i(x) = \frac{\mathcal{F}_{j+1}(x) + \mathcal{F}_j(x) - 1}{x}. \tag{1.4}$$

In [30, 31], the authors investigated the sums of squares of the Fibonacci polynomials and the sums of products of consecutive Fibonacci polynomials with the help of the following formulas:

$$\sum_{i=1}^{j} \mathcal{F}_i^2(x) = \frac{\mathcal{F}_{j+1}(x)\mathcal{F}_j(x)}{x}. \tag{1.5}$$

The Chebyshev polynomials of the second kind, denoted by $\{U_k(x)\}_{k\geq 0}$, are defined by

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x), \quad k \geq 1, \tag{1.6}$$

where $U_0(x) = 1$ and $U_1(x) = 2x$, (see [32]). In the literature, there is an interesting application of Chebyshev polynomials in the determinant computation of tridiagonal matrices. In other words, it is well-known that

$$\det\begin{pmatrix} a & b & & \\ c & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{pmatrix}_{k\times k} = \left(\sqrt{bc}\right)^k U_k\left(\frac{a}{2\sqrt{bc}}\right),$$

for more details, see [33, 34].

Lemma 1. *[4] Consider the following matrix form*

$$E_n = \begin{pmatrix} e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n \\ x & y & & & & \\ z & x & y & & & \\ & z & x & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & z & x & y \end{pmatrix},$$

then

$$\det(E_n) = \sum_{j=1}^{n} e_j y^{n-j} \left(-\sqrt{yz}\right)^{j-1} U_{j-1}\left(\frac{x}{2\sqrt{yz}}\right), \tag{1.7}$$

where $U_j(x)$ is the j^{th} Chebyshev polynomial of the second kind.

The relationship between circulant matrices and Fibonacci polynomials has not been investigated, despite the fact that they are widely used in many applications. Our main motivation for this paper is to fill this gap in the literature. For this purpose, we consider the entries of circulant matrices as Fibonacci polynomials. It sounds interesting to consider circulant matrices with Fibonacci polynomials as entries. As a result, numerous important questions inevitably arise. In this paper, we answer basic questions about determinants, inverses, some linear algebraic characteristics, and matrix norms.

The following is scheduled for the rest of this paper: In Section 2, some matrix norms, determinants, and inverses for the circulant matrix associated with the Fibonacci polynomials are computed. Subsequently, some illustrative examples are added to help readers understand our paper in Section 3. Finally, MATLAB-R2023a code is provided to compute our results.

2. Main results

In this section, we present the n-square circulant matrix C_n associated with the Fibonacci polynomials. That is,

$$C_n = circ(\mathcal{F}_1(x), \mathcal{F}_2(x), ..., \mathcal{F}_n(x)) = \begin{pmatrix} \mathcal{F}_1(x) & \mathcal{F}_2(x) & \dots & \mathcal{F}_{n-1}(x) & \mathcal{F}_n(x) \\ \mathcal{F}_n(x) & \mathcal{F}_1(x) & \dots & \mathcal{F}_{n-2}(x) & \mathcal{F}_{n-1}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{F}_3(x) & \mathcal{F}_4(x) & \dots & \mathcal{F}_1(x) & \mathcal{F}_2(x) \\ \mathcal{F}_2(x) & \mathcal{F}_3(x) & \dots & \mathcal{F}_n(x) & \mathcal{F}_1(x) \end{pmatrix}, \tag{2.1}$$

where $\mathcal{F}_n(x)$ is the *nth* Fibonacci polynomial. We investigate some linear algebraic and combinatorial results for this type of matrix.

2.1. On circulant matrices involving the Fibonacci polynomials

We initially obtain the determinant of the matrix C_n by exploiting spectacular properties of Chebyshev polynomials of the second kind and some basic matrix operations. Furthermore, using the well-know property, which claims that the inverse of a circulant matrix is also a circulant matrix, we compute the inverse of C_n. In addition to these, we support the results by illustrative examples.

First, let us define n-square matrices $\mathcal{S}_n$ and M_n as below:

$$\mathcal{S}_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 1 & -x \\ 0 & 0 & 0 & \iddots & \iddots & \iddots & -1 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & -x & -1 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$M_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & & & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

In this case, we get the following lemma.

Lemma 2. *The following equalities hold:*

$$\det(\mathcal{S}_n)=\det(M_n)=\begin{cases}-1, & \begin{array}{l}n\equiv 3(mod\ 4)\\ n\equiv 0(mod\ 4)\end{array}\\ \\ 1, & \begin{array}{l}n\equiv 1(mod\ 4)\\ n\equiv 2(mod\ 4)\end{array}\end{cases}.$$

Proof. By using Laplace expansion on the first row, the proof can be seen. □

In the following theorem, we characterize the determinant of the circulant matrix given in (2.1) by utilizing magnificent properties of Chebyshev polynomials. Moreover, we add an example to make it more comprehensible for the readers.

Theorem 1. *For $n\geq 3$, the following equality holds:*

$$\begin{aligned}det(C_n) &= -x\sum_{k=1}^{n-1}(\mathcal{F}_k(x)+\mathcal{F}_{k+1}(x))Z^{k-1}\left(-\sqrt{ZX}\right)^{n-k-1}U_{n-k-1}\left(\frac{Y}{2\sqrt{ZX}}\right)\\ &\quad+\sum_{k=2}^{n-1}(\mathcal{F}_k(x)+\mathcal{F}_{k+1}(x))Z^{k-2}\left(-\sqrt{ZX}\right)^{n-k}U_{n-k}\left(\frac{Y}{2\sqrt{ZX}}\right)+\left[1+\mathcal{F}_n(x)\right]Z^{n-2},\end{aligned}$$

where

$$\begin{aligned}Y &= -x\sum_{i=0}^{n-1}\mathcal{F}_{n-i}(x),\\ Z &= \mathcal{F}_{n-2}(x)-x\mathcal{F}_{n-3}(x)-\mathcal{F}_{n+1}(x),\\ X &= Y-Z.\end{aligned}$$

Proof. For $n\geq 3$, let us multiply the matrices as below:

$$T_n=\mathcal{S}_nC_nM_n.$$

Thus, we obtain T_n matrix, as below:

$$T_n=\begin{bmatrix}\mathcal{F}_1(x) & \mathcal{F}_{n-1}(x)+\mathcal{F}_n(x) & \mathcal{F}_{n-2}(x)+\mathcal{F}_{n-1}(x) & \cdots & \mathcal{F}_2(x)+\mathcal{F}_3(x) & \mathcal{F}_2(x)\\ \mathcal{F}_2(x) & \mathcal{F}_n(x)+\mathcal{F}_1(x) & \mathcal{F}_{n-1}(x)+\mathcal{F}_n(x) & \cdots & \mathcal{F}_3(x)+\mathcal{F}_4(x) & \mathcal{F}_3(x)\\ 0 & Y & Z & \cdots & 0 & 0\\ 0 & X & Y & \ddots & 0 & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & X & Y & Z\end{bmatrix}.$$

Then, by adding the first column to the nth column, we get the matrix shown below:

$$T_n^{(1)}=\begin{bmatrix}\mathcal{F}_1(x) & \mathcal{F}_{n-1}(x)+\mathcal{F}_n(x) & \mathcal{F}_{n-2}(x)+\mathcal{F}_{n-1}(x) & \cdots & \mathcal{F}_2(x)+\mathcal{F}_3(x) & \mathcal{F}_2(x)+\mathcal{F}_1(x)\\ \mathcal{F}_2(x) & \mathcal{F}_n(x)+\mathcal{F}_1(x) & \mathcal{F}_{n-1}(x)+\mathcal{F}_n(x) & \cdots & \mathcal{F}_3(x)+\mathcal{F}_4(x) & \mathcal{F}_3(x)+\mathcal{F}_2(x)\\ 0 & Y & Z & \cdots & 0 & 0\\ 0 & X & Y & \ddots & 0 & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & X & Y & Z\end{bmatrix}.$$

By adding a multiple of a column to another column, the determinant of the matrix is unchanged, and the determinant of a product of two matrices is just the product of the determinants, so we have

$$\begin{aligned} det(T_n) &= det(T_n^{(1)}) = det(\mathcal{S}_n C_n M_n) \\ &= det(\mathcal{S}_n) det(C_n) det(M_n). \end{aligned}$$

By exploiting Lemma 2, it is seen that

$$det(T_n) = det(C_n).$$

Keeping this equality in mind, we can characterize the determinant of the matrix C_n, as follows:

$$\begin{aligned} det(C_n) &= -x \sum_{k=1}^{n-1} [\mathcal{F}_k(x) + \mathcal{F}_{k+1}(x)] Z^{k-1} \left(-\sqrt{ZX}\right)^{n-k-1} U_{n-k-1}\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &\quad + \sum_{k=2}^{n-1} [\mathcal{F}_k(x) + \mathcal{F}_{k+1}(x)] Z^{k-2} \left(-\sqrt{ZX}\right)^{n-k} U_{n-k}\left(\frac{Y}{2\sqrt{ZX}}\right) + [1 + \mathcal{F}_n(x)] Z^{n-2}, \end{aligned}$$

where $U_k(x)$ is the kth Chebyshev polynomial of the second kind. □

Example 1. *The determinant of the matrix* C_5 *is*

$$\begin{aligned} \det(C_5) &= -x \sum_{k=1}^{4} [\mathcal{F}_k(x) + \mathcal{F}_{k+1}(x)] Z^{k-1} \left(-\sqrt{ZX}\right)^{4-k} U_{4-k}\left(\frac{Y}{2\sqrt{ZX}}\right) \\ &\quad + \sum_{k=2}^{4} [\mathcal{F}_k(x) + \mathcal{F}_{k+1}(x)] Z^{k-2} \left(-\sqrt{ZX}\right)^{5-k} U_{5-k}\left(\frac{Y}{2\sqrt{ZX}}\right) + [1 + \mathcal{F}_5(x)] Z^3 \\ &= 3 - 15x + 55x^2 - 95x^3 + 210x^4 - 312x^5 + 445x^6 - 520x^7 + 730x^8 - 445x^9 + 879x^{10} \\ &\quad -200x^{11} + 685x^{12} - 45x^{13} + 330x^{14} - 4x^{15} + 95x^{16} + 15x^{18} + x^{20} \\ &= \det(K_5). \end{aligned}$$

In Theorem 2, we compute the determinant of the matrix C_n in a simpler form by utilizing some fundamental matrix operations. In addition, we support it with an example to increase the intelligibility of this result.

Theorem 2. *For* $n \geq 3$*, the determinant of the matrix* C_n *is*

$$det(C_n) = \left(1 - x\mathcal{F}_n(x) + \sum_{k=1}^{n-2} \mathcal{F}_k(x) \left(\frac{\mathcal{F}_n(x)}{\mathcal{F}_1(x) - \mathcal{F}_{n+1}(x)}\right)^{n-(k+1)}\right) [1 - \mathcal{F}_{n+1}(x)]^{n-2}.$$

Proof. It can be seen that $det(C_3) = 2 - 3x + 3x^2 - 2x^3 + 3x^4 + x^6$ for $n = 3$. Let us suppose that $n \geq 3$. We obtain only a matrix with nonzero values on the main diagonal and subdiagonal of the first two rows if we multiply C_n by Q on the right side and J on the left. In other words,

$$J=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -x & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 1 & -x \\ 0 & 0 & 0 & \iddots & \iddots & \iddots & -1 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & -x & -1 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$Q=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \left(\frac{H_n}{P_n}\right)^{n-2} & 0 & \cdots & 0 & 0 & 1 \\ 0 & \left(\frac{H_n}{P_n}\right)^{n-3} & 0 & \cdots & 0 & 1 & 0 \\ 0 & \left(\frac{H_n}{P_n}\right)^{n-4} & 0 & \cdots & \iddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \iddots & \iddots & \vdots \\ 0 & \left(\frac{H_n}{P_n}\right) & 1 & \iddots & \iddots & \iddots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $H_n = \mathcal{F}_n(x)$ and $P_n = \mathcal{F}_1(x) - \mathcal{F}_{n+1}(x)$. By utilizing these matrices, we get the following equality:

$$\begin{aligned} T_n &= JC_nQ \\ &= \begin{bmatrix} \mathcal{F}_1(x) & W'_n & \mathcal{F}_{n-1}(x) & \mathcal{F}_{n-2}(x) & \mathcal{F}_{n-3}(x) & \cdots & \mathcal{F}_2(x) \\ 0 & W_n & \mathcal{F}_{n-2}(x) & \mathcal{F}_{n-3}(x) & \mathcal{F}_{n-4}(x) & \cdots & \mathcal{F}_1(x) \\ 0 & 0 & P_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & -H_n & P_n & 0 & & 0 \\ \vdots & \vdots & \ddots & -H_n & \ddots & \ddots & \vdots \\ & & & & \ddots & & 0 \\ 0 & 0 & \cdots & & 0 & -H_n & P_n \end{bmatrix}, \end{aligned}$$

where

$$W'_n = \sum_{k=1}^{n-1} \mathcal{F}_{k+1}(x) \left(\frac{H_n}{P_n}\right)^{n-(k+1)},$$

$$W_n = \mathcal{F}_1(x) - x\mathcal{F}_n(x) + \sum_{k=1}^{n-2} \mathcal{F}_k(x) \left(\frac{H_n}{P_n}\right)^{n-(k+1)}.$$

Thus, we get

$$
\begin{aligned}
det(C_n) &= det(J)det(T_n)det(Q) \\
&= det(JT_nQ) \\
&= W_n\mathcal{F}_1(x)[\mathcal{F}_1(x)-\mathcal{F}_{n+1}(x)]^{n-2} \\
&= \left(1 - x\mathcal{F}_n(x) + \sum_{k=1}^{n-2}\mathcal{F}_k(x)\left(\frac{\mathcal{F}_n(x)}{\mathcal{F}_1(x)-\mathcal{F}_{n+1}(x)}\right)^{n-(k+1)}\right)[1-\mathcal{F}_{n+1}(x)]^{n-2}.
\end{aligned}
$$

□

Example 2. *The determinant of matrix C_5 is*

$$
\begin{aligned}
\det(C_5) &= \left(1 - x\mathcal{F}_5(x) + \sum_{k=1}^{3}\mathcal{F}_k(x)\left(\frac{\mathcal{F}_5(x)}{1-\mathcal{F}_6(x)}\right)^{5-(k+1)}\right)[1-\mathcal{F}_6(x)]^3 \\
&= 3 - 15x + 55x^2 - 95x^3 + 210x^4 - 312x^5 + 445x^6 - 520x^7 + 730x^8 - 445x^9 + 879x^{10} \\
&\quad -200x^{11} + 685x^{12} - 45x^{13} + 330x^{14} - 4x^{15} + 95x^{16} + 15x^{18} + x^{20} \\
&= \det(T_5).
\end{aligned}
$$

In the following lemma, we show the characterization given for the invertibility of an arbitrary circulant matrix, which we use later.

Lemma 3. *[13] If we consider $A = circ(a_1, a_2, ..., a_n)$ to be a circulant matrix, we get*
(i) A is invertible if and only if $f(\omega^k) \neq 0$ $(k = 0, 1, 2, ..., n-1)$ where $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$ and $\omega = \exp(\frac{2\pi i}{n})$.
(ii) A^{-1} is a circulant matrix, if A is invertible.

In the following lemma, we express and prove the inverse of a lower triangular matrix denoted by $\mathcal{U}$. Thus, with the help of these lemmas we have presented, we prepare a background for the theory that we present in the rest of our paper.

Lemma 4. *Let the matrix $\mathcal{U} = [u_{ij}]_{i,j=1}^{n-2}$ be of the form*

$$
u_{ij} = \begin{cases} \mathcal{F}_1(x) - \mathcal{F}_{n+1}(x) &, \quad i = j \\ -\mathcal{F}_n(x) &, \quad i = j+1 \\ 0 &, \quad otherwise \end{cases},
$$

then the inverse $\mathcal{U}^{-1} = [u'_{ij}]_{i,j=1}^{n-2}$ of the matrix $\mathcal{U}$ is equal to

$$
u'_{ij} = \begin{cases} \frac{\mathcal{F}_n(x)^{i-j}}{[\mathcal{F}_1(x)-\mathcal{F}_{n+1}(x)]^{i-j+1}} &, \quad i \geq j \\ 0 &, \quad i < j \end{cases}.
$$

Proof. Assume that $\mathbb{P} := (p_{ij}) = \mathcal{U}\mathcal{U}^{-1}$. It is a well-known fact that $p_{ij} = \sum_{k=1}^{n-2} u_{ik}u'_{kj}$. The proof is split into all possible cases considering the relationships of the indices i and j.

Case 1. Let $i = j$, then

$$p_{ii} = u_{ii}u'_{ii} = [\mathcal{F}_1(x) - \mathcal{F}_{n+1}(x)].\frac{1}{[\mathcal{F}_1(x) - \mathcal{F}_{n+1}(x)]} = 1.$$

Case 2. Let $i > j$, then

$$\begin{aligned} p_{ij} &= \sum_{k=1}^{n-2} u_{ik}u'_{kj} = u_{i,i-1}u'_{i-1,j} + u_{ii}u'_{ij} \\ &= \frac{-\mathcal{F}_n(x)\mathcal{F}_n^{i-j-1}(x)}{[\mathcal{F}_1(x) - \mathcal{F}_{n+1}(x)]^{i-j}} + \frac{[\mathcal{F}_1(x) - \mathcal{F}_{n+1}(x)]\mathcal{F}_n^{i-j}(x)}{[\mathcal{F}_1(x) - \mathcal{F}_{n+1}(x)]^{i-j+1}} = 0. \end{aligned}$$

The proof of the case $i < j$ can be done from a similar point of view to Case 2. Therefore, we verify that $\mathcal{U}\mathcal{U}^{-1} = I_{n-2}$ where I_{n-2} is a unit matrix. Likewise, we may show that $\mathcal{U}^{-1}\mathcal{U} = I_{n-2}$. Thus, the proof is done. □

After giving the proof matrix C_n in (2.1) is an invertible matrix below, we answer the question of how to characterize the inverse of the circulant matrix whose entries are the Fibonacci polynomials for the case where $n \geq 2$ in Theorem 4.

Theorem 3. *Let $C_n = circ(\mathcal{F}_1(x), \mathcal{F}_2(x), ..., \mathcal{F}_n(x))$ be circulant matrix, if $n \geq 3$, then C_n is an invertible matrix.*

Proof. By Theorem 2, we know that $det(C_3) = 2 - 3x + 3x^2 - 2x^3 + 3x^4 + x^6 \neq 0$ and $det(C_4) = -x^{12} - 8x^{10} - 21x^8 - 16x^6 + 7x^4 + 4x^2 \neq 0$. Hence, C_n is an invertible matrix for $n = 3, 4$. Now, let us suppose that $n \geq 5$. The Binet formula for the Fibonacci polynomials is characterized by $\mathcal{F}_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. Keeping this in mind, the proof can be seen using Lemma 3 and the mathematical induction method. □

Theorem 4. *For $n \geq 2$, the inverse of C_n is given as below:*

$$C_n^{-1} = \begin{pmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ f_n & f_1 & \cdots & f_{n-2} & f_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_3 & f_4 & \cdots & f_1 & f_2 \\ f_2 & f_3 & \cdots & f_n & f_1 \end{pmatrix},$$

where

$$\begin{aligned} f_1 &= \frac{1}{W_n}\left(1 + \frac{x\mathcal{F}_n^{n-3}(x)}{(P_n)^{n-2}} + \sum_{k=1}^{n-3} \frac{\mathcal{F}_{n-k}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}\right), \\ f_2 &= \frac{1}{W_n}\left(-x + \sum_{k=1}^{n-2} \frac{\mathcal{F}_{n-k-1}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}\right), \\ f_i &= -\frac{\mathcal{F}_n^{i-3}(x)}{W_n(P_n)^{i-2}}, \quad i = 3, 4, ..., n, \end{aligned}$$

for

$$\begin{aligned} W_n &= \mathcal{F}_1(x) - x\mathcal{F}_n(x) + \sum_{k=1}^{n-2} \mathcal{F}_k(x) \left(\frac{H_n}{P_n}\right)^{n-(k+1)}, \\ H_n &= \mathcal{F}_n(x), \\ P_n &= \mathcal{F}_1(x) - \mathcal{F}_{n+1}(x). \end{aligned}$$

Proof. Let us take into account a matrix of the form

$$V = \begin{bmatrix} 1 & -W'_n & \frac{W'_n}{W_n}\mathcal{F}_{n-2}(x) - \mathcal{F}_{n-1}(x) & \frac{W'_n}{W_n}\mathcal{F}_{n-3}(x) - \mathcal{F}_{n-2}(x) & \cdots & \frac{W'_n}{W_n}\mathcal{F}_1(x) - \mathcal{F}_2(x) \\ 0 & 1 & \frac{-\mathcal{F}_{n-2}(x)}{W_n} & \frac{-\mathcal{F}_{n-3}(x)}{W_n} & \cdots & \frac{-\mathcal{F}_1(x)}{W_n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where

$$\begin{aligned} W_n &= \mathcal{F}_1(x) - x\mathcal{F}_n(x) + \sum_{k=1}^{n-2} \mathcal{F}_k(x) \left(\frac{H_n}{P_n}\right)^{n-(k+1)}, \\ W'_n &= \sum_{k=1}^{n-1} \mathcal{F}_{k+1}(x) \left(\frac{H_n}{P_n}\right)^{n-(k+1)}. \end{aligned}$$

Then, we can write

$$JC_nQV = E \oplus H,$$

where $E = diag(\mathcal{F}_1(x), W_n)$ is a diagonal matrix. Let us assume that $\mathbb{V} = QV$. Hence, we obtain

$$C_n^{-1} = \mathbb{V}(E^{-1} \oplus H^{-1})J.$$

Given that C_n is a circulant matrix, its inverse is a circulant from Lemma 2. Let

$$C_n^{-1} = circ(f_1, f_2, ..., f_n).$$

Because the last row of the matrix $\mathbb{V}$ is

$$\left(0, 1, -\frac{\mathcal{F}_{n-2}(x)}{W_n}, -\frac{\mathcal{F}_{n-3}(x)}{W_n}, ..., -\frac{\mathcal{F}_2(x)}{W_n}, -\frac{\mathcal{F}_1(x)}{W_n}\right).$$

The following equations, by means of Lemma 4, give the last row elements:

$$
\begin{aligned}
f_2 &= \frac{1}{W_n}\left(-x+\sum_{k=1}^{n-2}\frac{\mathcal{F}_{n-k-1}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}\right),\\
f_3 &= -\frac{1}{W_nP_n},\\
f_4 &= -\frac{\mathcal{F}_n(x)}{W_n(P_n)^2},\\
f_5 &= -\frac{1}{W_n}\left(\sum_{k=1}^{3}\frac{\mathcal{F}_{4-k}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}-x\sum_{k=1}^{2}\frac{\mathcal{F}_{3-k}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}-\frac{\mathcal{F}_1(x)}{P_n}\right),\\
&\vdots\\
f_n &= -\frac{1}{W_n}\left(\sum_{k=1}^{n-2}\frac{\mathcal{F}_{n-k-1}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}-x\sum_{k=1}^{n-3}\frac{\mathcal{F}_{n-k-2}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}-\sum_{k=1}^{n-4}\frac{\mathcal{F}_{n-k-3}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}\right),\\
f_1 &= \frac{1}{W_n}\left(1+\frac{x\mathcal{F}_n^{n-3}(x)}{(P_n)^{n-2}}+\sum_{k=1}^{n-3}\frac{\mathcal{F}_{n-k}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k}\right),
\end{aligned}
$$

where $P_n=\mathcal{F}_1(x)-\mathcal{F}_{n+1}(x)$. If

$$G_n^{(r)}=\sum_{k=1}^{r}\frac{\mathcal{F}_{r-k+1}(x)\mathcal{F}_n^{k-1}(x)}{(\mathcal{F}_1(x)-\mathcal{F}_{n+1}(x))^k}\quad (r=1,2,...,n-2),$$

then, we get

$$G_n^{(2)}-xG_n^{(1)}=\frac{\mathcal{F}_n(x)}{(\mathcal{F}_1(x)-\mathcal{F}_{n+1}(x))^2}$$

and

$$G_n^{(r+2)}-xG_n^{(r+1)}-G_n^{(r)}=\frac{\mathcal{F}_n^{r+1}(x)}{(\mathcal{F}_1(x)-\mathcal{F}_{n+1}(x))^{r+2}},\quad (r=1,2,...,n-4).$$

Hence, we have

$$
\begin{aligned}
C_n^{-1} &= \frac{1}{W_n}circ\Big(1+xG_n^{(n-2)}+G_n^{(n-3)},-x+G_n^{(n-2)},-G_n^{(1)},-G_n^{(2)}+xG_n^{(1)},G_n^{(3)}-xG_n^{(2)}-G_n^{(1)},\ldots,G_n^{(n-2)}\\
&\qquad -xG_n^{(n-3)}-G_n^{(n-4)}\Big)\\
&= \frac{1}{W_n}circ\left(1+\frac{x\mathcal{F}_n^{n-3}(x)}{(P_n)^{n-2}}+\sum_{k=1}^{n-3}\frac{\mathcal{F}_{n-k}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k},-x+\sum_{k=1}^{n-2}\frac{\mathcal{F}_{n-k-1}(x)\mathcal{F}_n^{k-1}(x)}{(P_n)^k},\right.\\
&\qquad \left.-\frac{1}{(P_n)},-\frac{\mathcal{F}_n(x)}{(P_n)^2},-\frac{\mathcal{F}_n^2(x)}{(P_n)^3},...,-\frac{\mathcal{F}_n^{n-3}(x)}{(P_n)^{n-2}}\right).
\end{aligned}
$$

Thus, the proof is completed. □

With the following instance, we shed light on the conclusions acquired above.

Example 3. *The inverse of the matrix C_5 is*

$$\begin{aligned} C_5^{-1} &= \frac{1}{W_5} circ\left(1+\frac{x\mathcal{F}_5^2(x)}{(\mathcal{F}_1(x)-\mathcal{F}_6(x))^3}+\sum_{k=1}^{2}\frac{\mathcal{F}_{5-k}(x)\mathcal{F}_5^{k-1}(x)}{(\mathcal{F}_1(x)-\mathcal{F}_6(x))^k}, -x+\sum_{k=1}^{3}\frac{\mathcal{F}_{4-k}(x)\mathcal{F}_5^{k-1}(x)}{(\mathcal{F}_1(x)-\mathcal{F}_6(x))^k},\right.\\ &\left.-\frac{1}{(\mathcal{F}_1(x)-\mathcal{F}_6(x))}, -\frac{\mathcal{F}_5(x)}{(\mathcal{F}_1(x)-\mathcal{F}_6(x))^2}, -\frac{\mathcal{F}_5^2(x)}{(\mathcal{F}_1(x)-\mathcal{F}_6(x))^3}\right)\\ &= circ\left(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\right), \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= \frac{2-9x+19x^2-30x^3+54x^4-72x^5+55x^6-120x^7+22x^8-105x^9+3x^{10}-48x^{11}-11x^{13}-x^{15}}{det(C_5)},\\ \alpha_2 &= \frac{2-6x+22x^2-38x^3+70x^4-81x^5+147x^6-68x^7+195x^8-24x^9+144x^{10}-3x^{11}+58x^{12}+12x^{14}+x^{16}}{det(C_5)},\\ \alpha_3 &= \frac{-1+6x-9x^2+8x^3-24x^4+2x^5-22x^6-8x^8-x^{10}}{det(C_5)},\\ \alpha_4 &= \frac{-1+3x-3x^2+13x^3-x^4+16x^5+7x^7+x^9}{det(C_5)},\\ \alpha_5 &= \frac{-1-6x^2-11x^4-6x^6-x^8}{det(C_5)}. \end{aligned}$$

2.2. *Some norms of circulant matrix with the Fibonacci polynomials*

In this section, we present spectacular results for some matrix kid of norms such as the Euclidean norm and the bounds of the spectral norm.

Theorem 5. *Assume x is a real value and $n \geq 3$ is an integer, then C_n be an $n \times n$ circulant matrix. In the circumstances, the following expression is correct:*

(i) The Euclidean norm of C_n is

$$\frac{1}{\sqrt{n}}\|C_n\|_{\mathbb{E}} = \sqrt{\frac{\mathcal{F}_n(x)\mathcal{F}_{n+1}(x)}{x}}.$$

(ii) The bounds for the spectral norm of C_n are

$$\sqrt{\frac{\mathcal{F}_n(x)\mathcal{F}_{n+1}(x)}{x}} \leq \|C_n\|_2 \leq \frac{1}{x}\sqrt{\mathcal{F}_{n-1}(x)\mathcal{F}_n^2(x)\mathcal{F}_{n+1}(x)}.$$

Proof. (i) The matrix C_n is of the form

$$C_n = circ(\mathcal{F}_1(x), \mathcal{F}_2(x), \mathcal{F}_3(x), ..., \mathcal{F}_n(x)).$$

Hence, considering of the definition of the Euclidean norm, we obtain the desired equality as follows:

$$\|C_n\|_{\mathbb{E}}^2 = n\sum_{k=1}^{n} F_k^2(x) = n\left(\frac{\mathcal{F}_n(x)\mathcal{F}_{n+1}(x)}{x}\right) \Rightarrow \frac{1}{\sqrt{n}}\|C_n\|_{\mathbb{E}} = \sqrt{\frac{\mathcal{F}_n(x)\mathcal{F}_{n+1}(x)}{x}}.$$

(ii) We obtain the wanted lower bound for the matrix C_n using (i) and (1.2). That is,

$$\sqrt{\frac{\mathcal{F}_n(x)\mathcal{F}_{n+1}(x)}{x}} \leq \|C_n\|_2.$$

In addition to this, let matrices $\mathcal{A}$ and $\mathcal{D}$ be

$$\mathcal{A} = (a_{ij}) = \begin{cases} a_{ij} = \mathcal{F}_{(mod(j-i,n))}(x), & i \geq j \\ a_{ij} = 1, & i < j \end{cases}$$

and

$$\mathcal{D} = (d_{ij}) = \begin{cases} d_{ij} = \mathcal{F}_{(mod(j-i,n))}(x), & i < j \\ d_{ij} = 1, & i \geq j, \end{cases}$$

such that $C_n = \mathcal{A} \circ \mathcal{D}$. Hence, we get an upper bound for the spectral norm by exploiting (1.5) and (1.2), i.e.,

$$\|C_n\|_2 \leq \frac{1}{x}\sqrt{\mathcal{F}_{n-1}(x)\mathcal{F}_n^2(x)\mathcal{F}_{n+1}(x)}.$$

□

3. Numerical examples with coding applications for norms

In this section, we show a few illustrative numerical examples for the bounds of the spectral norm and Euclidean norm of C_n. In addition, we give a Matlab code to calculate the results we get in the section before more easily and more accurately obtaining faster results in the computer environment.

In Table 2, we give some results obtained manually. These results may be easily validated for any desired n number utilizing the Matlab code shown in Table 3.

Table 2. Some upper and lower bounds for $\|C_n\|_2$.

n	***Lower bounds***	***Upper bounds***
3	$\sqrt{2 + 3x^2 + x^4}$	$2 + 3x^2 + x^4$
4	$\sqrt{2 + 7x^2 + 5x^4 + x^6}$	$2 + 7x^2 + 5x^4 + x^6$
5	$\sqrt{(1 + x^2)(3 + x^2)(1 + 3x^2 + x^4)}$	$(1 + x^2)(3 + x^2)(1 + 3x^2 + x^4)$
6	$\sqrt{3 + 22x^2 + 40x^4 + 29x^6 + 9x^8 + x^{10}}$	$3 + 22x^2 + 40x^4 + 29x^6 + 9x^8 + x^{10}$
7	$\sqrt{4 + 34x^2 + 86x^4 + 91x^6 + 46x^8 + 11x^{10} + x^{12}}$	$4 + 34x^2 + 86x^4 + 91x^6 + 46x^8 + 11x^{10} + x^{12}$

4. Conclusions

In this study, we consider the circulant matrices C_n whose components are the Fibonacci polynomials. Then, we examine a few linear algebraic properties of them. To sum up, we investigate the following properties:

(1) We compute its determinant in two ways. For this purpose, we benefit from the Chebyshev polynomials and exploit some matrix operations.

(2) We characterize the inverse of the matrix in a general form.
(3) We present some bounds for some norm of them.

Furthermore, we provide a MATLAB-R2023a code for the matrix C_n as well as norm computations for this matrix (refer to Table 3). Thus, we are building a novel method in MATLAB-R2023a code that is not available in the regular Matlab libraries. For the provided value n, the algorithm generates the matrix, which aids in the verification of the following norm types:

i. To calculate the row norm $r_1(\mathcal{A})$.
ii. To calculate the column norm $c_1(\mathcal{D})$.
iii. To compute an upper bound for the spectral norm $||C_n||_2$.
iv. To compute a lower bound for the spectral norm $||C_n||_2$.
v. To compute $||C_n||_E$.

We anticipate that all this will help shed light on future research of the circulant matrix and Fibonacci polynomials. In this content, we expect applications of our results in several branches of mathematics. Moreover, these results can be extended to the generalizations of some kind of circulant matrices such as g-circulant, geometric circulant, RFMLR, RLMFL, RFPrLrR, and RLPrFrL circulant matrices.

Author contributions

The authors F.Y., A.E. and S.A. contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

Appendix

Table 3. MATLAB-R2023a code for the matrix C_n and for the norm computations of this.

```
clc; clear all;
n=input('Enter the value of n='); % n is the input for the size of circulant matrix C_n
syms x y;
F(1) = y;  % Define the initial values for the Fibonacci polynomials
F(2) = x;
for i = 3: n % Generate the all values for the Fibonacci polynomials
   F(i)=subs(x*F(i-1) + F(i-2),y,1);
   F(i);
end
%-----------------------------------------------------------------------
 % Write the entries of the circulant matrix with Fibonacci polynomials
 for i=1:n
     for j=1:n
         if i==j
            a(i,j)=subs(F(1),F(1),1);
         elseif i<j
            a(i,j)=subs(F(mod(j-i,n)+1),F(1),1);
         elseif i>j
            a(i,j)=subs(F(mod(j-i,n)+1),F(1),1);
         end
     end
 end
 disp(a, 'Circulant matrix involving Fibonacci polynomials for n')
%-----------------------------------------------------------------------
 b = subs(simplify(F(1:n)),F(1),1); % Create a subvector for the row norm
 c = simplify(b.^2); % Square each component of the vector b
 rownorm_1 = cumsum(c); % Compute the cumulative sum of the vector c
 row_norm = simplify((rownorm_1(n))^(1/2)) % Write the row norm
%-----------------------------------------------------------------------
 d = simplify(F(2:n));  % Create a subvector for the column norm
 e = simplify(d.^2); % Square each component of the vector d
 columnnorm_2 = cumsum(e)+1;  % Compute the cumulative sum of the vector e
 column_norm = simplify((columnnorm_2(n-1))^(1/2)) % Write the column norm
%-----------------------------------------------------------------------
 % Compute upper and lower bounds for the spectral norm of C_n
 spectral_norm_less_than = simplify(row_norm*column_norm)
 spectral_norm_greater_than = simplify(row_norm)
 Euclidean_norm = simplify(sqrt(n)*row_norm) % Compute the Euclidean norm

%Example usage in command window
%-----------------------------------------------------------------------
 Enter the value of n=3
 %Circulant matrix involving Fibonacci polynomials for n =
 %[      1,        x, x^2 + 1]
 %[x^2 + 1,        1,       x]
 %[      x, x^2 + 1,        1]
 %spectral_norm_less_than = x^4 + 3*x^2 + 2
 %spectral_norm_greater_than = (x^4 + 3*x^2 + %2)^(1/2)
```

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