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Research article

Stability analysis of linear systems with a periodical time-varying delay based on an improved non-continuous piecewise Lyapunov functional

Wei Wang[1], **Chang-Xin Li**[2], **Ao-Qian Luo**[1] **and Hui-Qin Xiao**[1,*]

[1] Department of Electrical and Information Engineering, Hunan University of Technology, Zhuzhou, 412007, Hunan, China
[2] Department of Transportation, Hunan University of Technology, Zhuzhou, 412007, Hunan, China

* **Correspondence:** Email: xiaohuiqin@hut.edu.cn; Tel: +8673122183270.

Abstract: This paper mainly studies the stability of linear systems with a periodical time-varying delay. An improved delay-segmentation-based non-continuous piecewise Lyapunov–Krasovskii (L–K) functional is proposed. In comparison with the currently available L–K functional, this functional incorporates more delay-segmentation-interval-related information. Additionally, it effectively eases the boundary constraints at each segment point. Consequently, a stability criterion with reduced conservativeness for linear time-delay systems is derived. Finally, two numerical examples and a single-area load frequency control system are given to validate the efficiency of the proposed approach.

Keywords: periodical time-varying delay; delay segmentation; Lyapunov–Krasovskii functional; stability criteria
Mathematics Subject Classification: 34D20, 34K20, 34K25

1. Introduction

The time-delay phenomenon is widely present in various engineering systems, and its existence will affect the stability and performance of the system. Therefore, the investigations into time-delay systems have been widely studied in the control field in recent years [1–3]. Many achievements have been obtained in the fields involving discrete-time systems [4–6], fuzzy systems [7–9], networked control systems [10, 11], load frequency control systems [12–15], Markovian jump systems [16, 17], Lur'e systems [18, 19], and H_∞ filtering control [20–22].

The Lyapunov–Razumikhin approach and the Lyapunov–Krasovskii (L–K) approach are the most widely used methods for addressing stability issues in time-delay systems. The Lyapunov–Razumikhin approach has achieved some advancements in deriving necessary and sufficient conditions for the stability of time-delay systems [23]. On the other hand, the L–K approach aims to construct an appropriate L–K functional to obtain less conservative stability criteria in the form of Linear Matrix Inequality (LMI) for time-delay systems, especially for time-varying delay systems [24–27]. It is worth mentioning that the L–K approach is frequently employed in combination with some integral inequality methods for bounding the integral terms in the derivatives delay-related L–K functionals [28–32]. Admittedly, recent research mainly focuses on reducing the conservativeness of the stability criteria by refining the structure of delay-related L–K functionals. An augmented L–K functional incorporating additional state information was presented in [21], enhancing the stability criteria's dependence on time delay. The augmented L–K functional constructed in [33] further considered the integrity of the information about delay intervals and gave less conservative results. An augmented L–K functional was also introduced in [27]. Unlike the L–K functionals in [21, 33], it avoids the occurrence of high-order terms of variables in the derivative of the functional, thus easing the hardship of the solving process.

In addition to various augmented L–K functionals mentioned above, the delay-segmentation-based piecewise L–K functionals have also been widely discussed in recent research. The delay segmenting method can augment the delay-interval-related information in the functional derivatives. Subsequently, it enables a more in-depth exploration of the functional decrease within each interval instead of merely considering its global decreasing property. Consequently, this functional effectively reduces the conservativeness of the system stability criterion. Delay-segmentation-based piecewise L–K functionals can be categorized into continuous and non-continuous piecewise L–K functionals. A time-delay-partitioning-based L–K functional was constructed in [34], which effectively relaxes the constraints on the stability criteria of the system. In [35], Han et al. established a non-continuous L–K functional and gave a delay-related stability criterion. Additionally, for time-varying delay systems, some quadratic terms with time-varying delay were introduced in [18] in constructing the L–K functional using an improved delay-segmentation method, proposing a novel delay-segmentation-based piecewise functional.

Based on the domain of time-varying delay and its derivatives, the definition of allowable delay set (ADS) was given in [1]. Building upon this, an improved ADS was presented in [31], which optimized the stability criterion for linear systems with time-varying delay. However, the ADS given in [1] and [31] exhibit the coverage areas in the form of polygons, as indicated in [36]. However, this issue was further explored in [37], which gave ADS covering a complete ellipsoidal field by introducing specific periodically varying delays. Subsequently, an ADS partitioning approach was introduced in [38], which further refines the ADS through the application of the delay-segmenting method. By integrating this with the non-continuous L–K functional method, a stability criterion with reduced conservativeness was developed. However, the non-continuous piecewise function in [38] does not account for information regarding the delay segmentation and hinders potential improvements. Consequently, the existing criterion remains rather conservative.

In recent years, the research for periodical time delay has aroused the interest of researchers, and it exists widely in some mechanical motions [39–41]. Combining delay-related L–K functional and a looped function, a stability criterion based on a periodically varying delay with monotone intervals was given in [37]. Based on this, a higher-order free-matrix-based integral inequality was introduced to optimize the stability criteria in [42], reducing the conservativeness of the stability criterion of the system. Furthermore, an exponential stability analysis of switching time-delay systems is presented in [43], which utilizes the symbolic transformation of delay derivatives as switching information. The stability of periodic time-delay systems is also studied in [38] by further partitioning the monotone intervals. In light of this, conducting in-depth research on periodic time-delay systems characterized by monotonic intervals on such a foundation holds significant and promising research prospects.

This paper proposes an innovative delay-segmentation-based non-continuous piecewise L–K functional. In different segments, the construction of different L–K functionals is presented according to the intervals where the delay is located, and information regarding the delay-interval-segmentation is fully exploited in the functional of each segment. Therefore, the derived stability condition is less conservative. Finally, two numerical examples and a single-area load frequency control system are given to demonstrate that the proposed approach significantly reduces the conservativeness of the presented stability criterion.

2. Preliminary

2.1. Notations

For brevity, the notations used in this paper are summarized in Table 1.

Table 1. Notations.

Symbol	Meaning
$\mathbb{N}$	Natural numbers: $\{1, 2, 3, \cdots\}$
$\mathbb{R}^n$	Euclidean space in n dimensions
$\mathbb{R}^{m\times n}$	All real matrices of dimension $m \times n$
$\mathbb{S}^n$	All symmetric matrices of dimension $n \times n$
$\mathbb{S}^n_+$	All positive definite symmetric matrices of dimension $n \times n$
$\aleph^{-1}$	Inverse of matrix $\aleph$
$\aleph^T$	Transpose of matrix $\aleph$
Sym$\{\aleph\}$	The sum of matrix $\aleph$ and its transpose
$*$	A symmetric component inside a symmetric matrix
diag{}	A matrix with a block diagonal structure

2.2. System description

Consider the following linear system with time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_r x(t - r(t)), \\ x(t) = \varphi(t), t \in [-r_2, 0], \end{cases} \tag{2.1}$$

where $A, A_r \in \mathbb{R}^{n\times n}$ are system matrices, $x(t) \in \mathbb{R}^n$ is state vector, and $r(t)$ represents the time-varying delay, which satisfies the following form derived from [37, 38, 41]:

$$r(t) = r_0 + \tilde{r} f(\Omega t), \tag{2.2}$$

where r_0 is a constant, $f : \mathbb{R} \to [-1, 1]$ is a differentiable periodic function satisfying $|f| \leqslant 1$, whose each period contains a monotone increasing interval and a monotone decreasing interval, and parameters $\tilde{r}$ and Ω represent its amplitude and frequency, respectively. For $r(t)$, r_1 is the lower bound, and r_2 is the upper bound. In addition, the derivative of $r(t)$ is defined as $\dot{r}(t)$, $-\mu, \mu$ are its lower and upper bounds. Therefore, $r(t)$ and $\dot{r}(t)$ are satisfied

$$r_1 \leqslant r(t) \leqslant r_2, \ \ |\dot{r}(t)| \leqslant \mu < 1, \tag{2.3}$$

with $r_1 = r_0 - \tilde{r}$, $r_2 = r_0 + \tilde{r}$ and $\mu = \tilde{r}\Omega$.

The ADS partitioning approach based on a periodical time-varying delay is discussed in [38]. There exists t_{2k-1}, t_{2k}, t_{2k+1}, $k \in \mathbb{N}$, in the periodic function $r(t)$, which are extreme values of $r(t)$ satisfying $r(t_{2k-1}) = r(t_{2k+1}) = r_1$ and $r(t_{2k}) = r_2$. $r(t)$ is monotone increasing in the intervals $t \in [t_{2k-1}, t_{2k})$ and monotone decreasing in the intervals $t \in [t_{2k}, t_{2k+1})$.

Then, we choose moving points $\varrho_{1k} \in [t_{2k-1}, t_{2k})$ and $\varrho_{2k} \in [t_{2k}, t_{2k+1})$, ensuring that $r(\varrho_{1k}) = r(\varrho_{2k}) = r_\varrho$. The uncertain delay value $r_\varrho = r_1 + \varrho(r_2 - r_1), \varrho \in [0, 1]$ that satisfies $r_1 \leqslant r_\varrho \leqslant r_2$, and its value changes with the change of parameter ϱ. Depending on the value range of $r(t)$, we obtain two delay-segmentation-based intervals: $r(t) \in [r_1, r_\varrho]$ and $r(t) \in [r_\varrho, r_2]$. Thus, the following ADS $\gimel \triangleq \gimel_{\alpha i} \cup \gimel_{\beta i}, i = 1, 2$ are obtained

$$\begin{cases} \gimel_{\alpha 1} \triangleq [r_1, r_\varrho] \times [0, \mu] & \gimel_{\beta 1} \triangleq [r_\varrho, r_2] \times [0, \mu], \\ \gimel_{\alpha 2} \triangleq [r_1, r_\varrho] \times [-\mu, 0] & \gimel_{\beta 2} \triangleq [r_\varrho, r_2] \times [-\mu, 0]. \end{cases} \tag{2.4}$$

Remark 2.1. The ADS described in formula (2.4) was initially presented in [38]. To underscore the efficacy of the enhanced functional put forward in this paper, we opt for the identical ADS to analyze the stability of the considered systems.

2.3. Lemma

To derive the main results, the following lemma needs to be introduced.

Lemma 2.1. [31, 32] If $x(t) \in [\omega_1, \omega_2] \rightarrow \mathbb{R}^n$ and $\kappa \in \mathbb{R}^m$ are continuous and differentiable, for matrices $U \in \mathbb{S}_+^n$, $H \in \mathbb{R}^{m \times 2n}$, the following inequalities hold:

$$-\int_{\omega_1}^{\omega_2} \dot{x}^T(s) U \dot{x}(s) ds \leqslant -\frac{1}{\omega_2 - \omega_1} \kappa^T \vartheta^T \hat{U} \vartheta \kappa, \tag{2.5}$$

$$-\int_{\omega_1}^{\omega_2} \dot{x}^T(s) U \dot{x}(s) ds \leqslant (\omega_2 - \omega_1) \kappa^T H \hat{U}^{-1} H^T \kappa + 2\kappa^T H \vartheta \kappa, \tag{2.6}$$

where

$$\begin{aligned} &\kappa = col\left\{ x(\omega_2), x(\omega_1), \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} x(s) ds \right\}, \\ &\vartheta = [\tilde{e}_1^T - \tilde{e}_2^T \ \ \tilde{e}_1^T + \tilde{e}_2^T - 2\tilde{e}_3^T]^T, \\ &\hat{U} = diag\{U, 3U\}, \\ &\tilde{e}_\theta = [0_{n \times (\theta - 1)n} \ \ I_n \ \ 0_{n \times (3-i)n}], \theta = 1, \cdots, 3. \end{aligned}$$

3. Main results

This section discusses the stability criterion for system (2.1) based on ADS $\beth$. The subsequent notations are introduced to represent the vectors and matrices for convenience.

$$\begin{aligned}
\delta_1(t) =&[x^T(t) \quad x^T(t-r_1) \quad x^T(t-r(t)) \quad x^T(t-r_2) \quad x^T(t-r_\varrho)]^T,\\
\delta_2(t) =&[\dot{x}^T(t-r_1) \;\; \dot{x}^T(t-r(t)) \;\; \dot{x}^T(t-r_2) \;\; \dot{x}^T(t-r_\varrho)]^T,\\
\delta_3(t) =& \int_{t-r_1}^{t} x(s)ds,\ \delta_4(t) = \int_{t-r(t)}^{t-r_1} x(s)ds,\ \delta_5(t) = \int_{t-r_\varrho}^{t-r(t)} x(s)ds,\\
\delta_6(t) =& \int_{t-r_2}^{t-r(t)} x(s)ds,\ \delta_7(t) = \int_{t-r_\varrho}^{t-r_1} x(s)ds,\ \delta_8(t) = \int_{t-r_2}^{t-r_\varrho} x(s)ds,\\
\varpi_0(t) =&[\delta_1^T(t) \quad \delta_3^T(t)]^T,\ \ \varpi_1(s) = [x^T(s) \quad \dot{x}^T(s)],\\
\varpi_{2\alpha}(t) =&[\delta_1^T(t) \;\; \delta_3^T(t) \;\; \delta_4^T(t) \;\; \delta_5^T(t) \;\; \delta_8^T(t)]^T,\\
\varpi_{2\beta}(t) =&[\delta_1^T(t) \;\; \delta_3^T(t) \;\; \delta_7^T(t) \;\; -\delta_5^T(t) \;\; \delta_6^T(t)]^T,\\
\varpi(t) =&[\delta_1^T(t) \;\; \delta_2^T(t) \;\; \frac{1}{r_1}\delta_3^T(t) \;\; \frac{1}{r(t)-r_1}\delta_4^T(t) \;\; \frac{1}{r_\varrho-r(t)}\delta_5^T(t) \;\; \frac{1}{r_2-r(t)}\delta_6^T(t) \;\; \frac{1}{r_\varrho-r_1}\delta_7^T(t) \;\; \frac{1}{r_2-r_\varrho}\delta_8^T(t)]^T,\\
e_\rho =&[0_{n\times(\rho-1)n} \quad I_n \quad 0_{n\times(15-\rho)n}], \rho = 1,2,\cdots,15.
\end{aligned}$$

As shown below, Theorem 3.1 provides a stability criterion for the system (2.1) based on ADS $\beth$.

Theorem 3.1. For scalars $\varrho \in [0,1], r_2 \geqslant r_1 \geqslant 0, \mu < 1$, suppose that there exist matrices $P \in \mathbb{S}_+^{9n}$, $F, X_{\alpha i\varsigma}, X_{\beta i\varsigma} \in \mathbb{S}_+^{6n}$, $Q_\varsigma, S_{\alpha i\varsigma}, S_{\beta i\varsigma} \in \mathbb{S}_+^{2n}$, $Y_\varsigma, T_{\alpha i\varsigma}, T_{\beta i\varsigma} \in \mathbb{S}_+^{n}$ satisfying the condition (3.1), $W_{\alpha i1}, W_{\alpha i2}, W_{\beta i1}, W_{\beta i2} \in \mathbb{S}^n$, $G_{\alpha i\varsigma}, G_{\beta i\varsigma} \in \mathbb{R}^{15n\times 2n}$, $N_{\alpha i1}, N_{\alpha i2}, N_{\beta i1}, N_{\beta i2} \in \mathbb{R}^{15n\times n}, i = 1,2, \varsigma = 1,\cdots,3$. Then, system (2.1) is stable if LMIs (3.2)–(3.3) and (3.4)–(3.5) are satisfied at the vertices of $\beth_{\alpha i}$ and $\beth_{\beta i}$, respectively.

$$\begin{aligned}
&X_{\alpha 11} \geqslant X_{\beta 11} \geqslant X_{\beta 21} \geqslant X_{\alpha 21},\ X_{\beta 12} \geqslant X_{\beta 22},\\
&X_{\beta 23} \geqslant X_{\alpha 23} \geqslant X_{\alpha 13} \geqslant X_{\beta 13},\ X_{\alpha 22} \geqslant X_{\alpha 12},\\
&S_{\alpha 11} \geqslant S_{\beta 11} \geqslant S_{\beta 21} \geqslant S_{\alpha 21},\ S_{\beta 12} \geqslant S_{\beta 22},\\
&S_{\beta 23} \geqslant S_{\alpha 23} \geqslant S_{\alpha 13} \geqslant S_{\beta 13},\ S_{\alpha 22} \geqslant S_{\alpha 12},\\
&T_{\alpha 11} \geqslant T_{\beta 11} \geqslant T_{\beta 21} \geqslant T_{\alpha 21},\ T_{\beta 12} \geqslant T_{\beta 22},\\
&T_{\beta 23} \geqslant T_{\alpha 23} \geqslant T_{\alpha 13} \geqslant T_{\beta 13},\ T_{\alpha 22} \geqslant T_{\alpha 12},
\end{aligned} \tag{3.1}$$

$$W_{YT\alpha i1} > 0,\ W_{YT\alpha i2} > 0, \tag{3.2}$$

$$\begin{bmatrix} \Upsilon_{\alpha i}(r(t),\dot{r}(t)) & \sqrt{r(t)-r_1}G_{\alpha i1} & \sqrt{r_\varrho - r(t)}G_{\alpha i2} & \sqrt{r_2-r_1}G_{\alpha i3} \\ * & -\widehat{W}_{YT\alpha i1} & 0 & 0 \\ * & * & -\widehat{W}_{YT\alpha i2} & 0 \\ * & * & * & -\widehat{Y}_{T1} \end{bmatrix} < 0, \tag{3.3}$$

$$W_{YT\beta i1} > 0,\ W_{YT\beta i2} > 0, \tag{3.4}$$

$$\begin{bmatrix} \Upsilon_{\beta i}(r(t),\dot{r}(t)) & \sqrt{r_\varrho-r_1}G_{\beta i1} & \sqrt{r(t) - r_\varrho}G_{\beta i2} & \sqrt{r_2-r(t)}G_{\beta i3} \\ * & -\widehat{Y}_{T2} & 0 & 0 \\ * & * & -\widehat{W}_{YT\beta i1} & 0 \\ * & * & * & -\widehat{W}_{YT\beta i2} \end{bmatrix} < 0, \tag{3.5}$$

where

$$
\begin{aligned}
\Upsilon_{\alpha i}(r(t),\dot r(t)) &= \Lambda_1+\Lambda_{2\alpha i}+\Lambda_{3\alpha i}+\Lambda_{4\alpha i},\\
\Upsilon_{\beta i}(r(t),\dot r(t)) &= \Lambda_1+\Lambda_{2\beta i}+\Lambda_{3\beta i}+\Lambda_{4\beta i},\\
\Lambda_1 =& \mathrm{Sym}\{h_1\Pi_0^TF\Pi_1\}+\Pi_2^TQ_1\Pi_2+\Pi_3^T(Q_2-Q_1)\Pi_3+\Pi_4^T(Q_3-Q_2)\Pi_4\\
&-\Pi_5^TQ_3\Pi_5+r_1^2\Pi_{20}^TY_1\Pi_{20}+(r_\varrho-r_1)\Pi_{20}^TY_2\Pi_{20}+(r_2-r_\varrho)\Pi_{20}^TY_3\Pi_{20},\\
\Lambda_{2\alpha i} =& \mathrm{Sym}\{\Pi_0^T[(r(t)-r_1)X_{\alpha i1}+(r_\alpha-r(t))X_{\alpha i2}+(r_2-r_\varrho)X_{\alpha i3}]\Pi_1+\Pi_7^TP\Pi_8\}\\
&+\dot r(t)\Pi_0^T(X_{\alpha i1}-X_{\alpha i2})\Pi_0+\Pi_3^TS_{\alpha i1}\Pi_3+(1-\dot r(t))\Pi_6^T(S_{\alpha i2}-S_{\alpha i1})\Pi_6\\
&+\Pi_4^T(S_{\alpha i3}-S_{\alpha i2})\Pi_4-\Pi_5^TS_{\alpha i3}\Pi_5+(r_2-r_\varrho)e_9^T(T_{\alpha i3}-T_{\alpha i2})e_9\\
&+(r_2-r(t))(1-\dot r(t))e_7^T(T_{\alpha i2}-T_{\alpha i1})e_7+(r_2-r_1)e_6^TT_{\alpha i1}e_6,\\
\Lambda_{3\alpha i} =& (r(t)-r_1)((1-\dot r(t))e_7^TW_{\alpha i2}e_7-e_9^TW_{\alpha i2}e_9)\\
&-(r_\varrho-r(t))(e_6^TW_{\alpha i1}e_6-(1-\dot r(t))e_7^TW_{\alpha i1}e_7),\\
\Lambda_{2\beta i} =& \mathrm{Sym}\{\Pi_0^T[(r_\varrho-r_1)X_{\beta i1}+(r(t)-r_\alpha)X_{\beta i2}+(r_2-r(t))X_{\beta i3}]\Pi_1+\Pi_9^TP\Pi_{10}\}\\
&+\dot r(t)\Pi_0^T(X_{\beta i2}-X_{\beta i3})\Pi_0+\Pi_3^TS_{\beta i1}\Pi_3+(1-\dot r(t))\Pi_6^T(S_{\beta i3}-S_{\beta i2})\Pi_6\\
&+\Pi_4^T(S_{\beta i2}-S_{\beta i1})\Pi_4-\Pi_5^TS_{\beta i3}\Pi_5+(r_2-r(t))(1-\dot r(t))e_7^T(T_{\beta i3}-T_{\beta i2})e_7\\
&+(r_2-r_\varrho)e_9^T(T_{\beta i2}-T_{\beta i1})e_9+(r_2-r_1)e_6^TT_{\beta i1}e_6,\\
\Lambda_{3\beta i} =& (r(t)-r_\varrho)((1-\dot r(t))e_7^TW_{\beta i2}e_7-e_8^TW_{\beta i2}e_8)\\
&-(r_2-r(t))(e_9^TW_{\beta i1}e_9-(1-\dot r(t))e_7^TW_{\beta i1}e_7),\\
\Lambda_{4\alpha i} =& \mathrm{Sym}\{G_{\alpha i1}\Pi_{12}+G_{\alpha i2}\Pi_{13}+G_{\alpha i3}\Pi_{14}+N_{\alpha i1}\Pi_{18}+N_{\alpha i2}\Pi_{19}\}-\Pi_{11}^T\widehat{Y}_1\Pi_{11},\\
\Lambda_{4\beta i} =& \mathrm{Sym}\{G_{\beta i1}\Pi_{15}+G_{\beta i2}\Pi_{16}+G_{\beta i3}\Pi_{17}+N_{\beta i1}\Pi_{18}+N_{\beta i2}\Pi_{19}\}-\Pi_{11}^T\widehat{Y}_1\Pi_{11},\\
\widehat{Y}_{T1} =& diag\{Y_{T1},3Y_{T1}\},\quad \widehat{Y}_{T2}=diag\{Y_{T2},3Y_{T2}\},\\
\widehat{W}_{YT\alpha i1} =& diag\{W_{YT\alpha i1},3W_{YT\alpha i1}\},\quad \widehat{W}_{YT\alpha i2}=diag\{W_{YT\alpha i2},3W_{YT\alpha i2}\},\\
\widehat{W}_{YT\beta i1} =& diag\{W_{YT\beta i1},3W_{YT\beta i1}\},\quad \widehat{W}_{YT\beta i2}=diag\{W_{YT\beta i2},3W_{YT\beta i2}\},
\end{aligned}
$$

with

$$
\begin{aligned}
\Pi_0 =& [e_1^T\quad e_2^T\quad e_3^T\quad e_4^T\quad e_5^T\quad r_1e_{10}^T]^T,\quad \Pi_1=[\Pi_{20}^T\quad e_6^T\quad (1-\dot r(t))e_7^T\quad e_8^T\quad e_9^T\quad r_1e_{10}^T]^T,\\
\Pi_2 =& [e_1^T\quad \Pi_{20}^T]^T,\quad \Pi_3=[e_2^T\quad e_6^T]^T,\quad \Pi_4=[e_5^T\quad e_9^T]^T,\quad \Pi_5=[e_4^T\quad e_8^T]^T,\quad \Pi_6=[e_3^T\quad e_7^T]^T,\\
\Pi_7 =& [e_1^T\quad e_2^T\quad e_3^T\quad e_4^T\quad e_5^T\quad r_1e_{10}^T(r(t)-r_1)e_{11}^T\quad (r_\varrho-r(t))e_{12}^T\quad (r_2-r_\varrho)e_{15}^T]^T,\\
\Pi_8 =& [\Pi_{20}^T\quad e_6^T\quad (1-\dot r(t))e_7^T\quad e_8^T\quad e_9^T\quad e_1^T-e_2^T\quad e_2^T-(1-\dot r(t))e_3^T\quad (1-\dot r(t))e_3^T-e_5^T\quad e_5^T-e_4^T]^T,\\
\Pi_9 =& [e_1^T\quad e_2^T\quad e_3^T\quad e_4^T\quad e_5^T\quad r_1e_{10}^T\quad (r_\varrho-r_1)(t)e_{14}^T\quad (r(t)-r_\varrho)e_{12}^T\quad (r_2-r(t))e_{13}^T]^T,\\
\Pi_{10} =& [\Pi_{20}^T\quad e_6^T\quad (1-\dot r(t))e_7^T\quad e_8^T\quad e_9^T\quad e_1^T-e_2^T\quad e_2^T-e_5^T\quad e_5^T-(1-\dot r(t))e_3^T\quad (1-\dot r(t))e_3^T-e_4^T]^T,\\
\Pi_{11} =& [e_1^T-e_2^T\quad e_1^T+e_2^T-2e_{10}^T]^T,\quad \Pi_{12}=[e_2^T-e_3^T\quad e_2^T+e_3^T-2e_{11}^T]^T,\\
\Pi_{13} =& [e_3^T-e_5^T\quad e_3^T+e_5^T-2e_{12}^T]^T,\quad \Pi_{14}=[e_5^T-e_4^T\quad e_5^T+e_4^T-2e_{15}^T]^T,\\
\Pi_{15} =& [e_2^T-e_5^T\quad e_2^T+e_5^T-2e_{14}^T]^T,\quad \Pi_{16}=[e_5^T-e_3^T\quad e_5^T+e_3^T-2e_{12}^T]^T,\\
\Pi_{17} =& [e_3^T-e_4^T\quad e_3^T+e_4^T-2e_{13}^T]^T,\\
\Pi_{18} =& (r(t)-r_1)e_{11}+(r_\varrho-r(t))e_{12}-(r_\varrho-r_1)e_{14},\\
\Pi_{19} =& (r(t)-r_\varrho)e_{12}+(r_2-r(t))e_{13}-(r_2-r_\varrho)e_{15},
\end{aligned}
$$

$$\Pi_{20} = Ae_1 + A_r e_3,$$
$$Y_{T1} = Y_3 + T_{\alpha i3},\quad Y_{T2} = Y_2 + T_{\beta i1},\quad \widehat{Y_1} = diag\{Y_1, 3Y_1\},$$
$$W_{YT\alpha i1} = Y_2 + T_{\alpha i1} - \dot{r}(t)W_{\alpha i1},\quad W_{YT\alpha i2} = Y_2 + T_{\alpha i2} - \dot{r}(t)W_{\alpha i2},$$
$$W_{YT\beta i1} = Y_3 + T_{\beta i2} - \dot{r}(t)W_{\beta i1},\quad W_{YT\beta i2} = Y_3 + T_{\beta i3} - \dot{r}(t)W_{\beta i2}.$$

Proof. Choosing the non-continuous piecewise L–K functional defined below:

$$V(t) = \begin{cases} V_0(t) + V_{1\alpha 1}(t) + V_{2\alpha 1} + V_{l\alpha 1}(t),\ t \in [t_{2k-1}, \varrho_{1k}),\\ V_0(t) + V_{1\beta 1}(t) + V_{2\beta 1} + V_{l\beta 1}(t),\ t \in [\varrho_{1k}, t_{2k}),\\ V_0(t) + V_{1\beta 2}(t) + V_{2\beta 2} + V_{l\beta 2}(t),\ t \in [t_{2k},\ \varrho_{2k}),\\ V_0(t) + V_{1\alpha 2}(t) + V_{2\alpha 2} + V_{l\alpha 2}(t),\ t \in [\varrho_{2k},\ t_{2k+1}), \end{cases} \tag{3.6}$$

where

$$\begin{aligned} V_0(t) =& r_1 \varpi_0(t)^T F \varpi_0(t) + \int_{t-r_1}^{t} \varpi_1^T(s) Q_1 \varpi_1(s) ds + \int_{t-r_\varrho}^{t-r_1} \varpi_1^T(s) Q_2 \varpi_1(s) ds\\ &+ \int_{t-r_2}^{t-r_\varrho} \varpi_1^T(s) Q_3 \varpi_1(s) ds + r_1 \int_{-r_1}^{0}\int_{t+u}^{t} \dot{x}^T(s) Y_1 \dot{x}(s) ds du\\ &+ \int_{-r_\varrho}^{-r_1}\int_{t+u}^{t} \dot{x}^T(s) Y_2 \dot{x}(s) ds du + \int_{-r_2}^{-r_\varrho}\int_{t+u}^{t} \dot{x}^T(s) Y_3 \dot{x}(s) ds du,\\ V_{1\alpha i}(t) =& \varpi_0^T(t)[(r(t) - r_1)X_{\alpha i1} + (r_\alpha - r(t))X_{\alpha i2} + (r_2 - r_\varrho)X_{\alpha i3}]\varpi_0(t) + \varpi_{2\alpha}^T(t) P \varpi_{2\alpha}(t)\\ &+ \int_{t-r(t)}^{t-r_1} \varpi_1^T(s) S_{\alpha i1} \varpi_1(s) ds + \int_{t-r_\varrho}^{t-r(t)} \varpi_1^T(s) S_{\alpha i2} \varpi_1(s) ds + \int_{t-r_2}^{t-r_\varrho} \varpi_1^T(s) S_{\alpha i3} \varpi_1(s) ds,\\ V_{1\beta i}(t) =& \varpi_0^T(t)[(r_\varrho - r_1)X_{\beta i1} + (r(t) - r_\alpha)X_{\beta i2} + (r_2 - r(t))X_{\beta i3}]\varpi_0(t) + \varpi_{2\beta}^T(t) P \varpi_{2\beta}(t)\\ &+ \int_{t-r_\varrho}^{t-r_1} \varpi_1^T(s) S_{\beta i1} \varpi_1(s) ds + \int_{t-r(t)}^{t-r_\varrho} \varpi_1^T(s) S_{\beta i2} \varpi_1(s) ds + \int_{t-r_2}^{t-r(t)} \varpi_1^T(s) S_{\beta i3} \varpi_1(s) ds,\\ V_{2\alpha i}(t) =& \int_{t-r(t)}^{t-r_1} (r_2 + s - t)\dot{x}^T(s) T_{\alpha i1} \dot{x}(s) ds + \int_{t-r_\varrho}^{t-r(t)} (r_2 + s - t)\dot{x}^T(s) T_{\alpha i2} \dot{x}(s) ds\\ &+ \int_{t-r_2}^{t-r_\varrho} (r_2 + s - t)\dot{x}^T(s) T_{\alpha i3} \dot{x}(s) ds,\\ V_{2\beta i}(t) =& \int_{t-r_\varrho}^{t-r_1} (r_2 + s - t)\dot{x}^T(s) T_{\beta i1} \dot{x}(s) ds + \int_{t-r(t)}^{t-r_\varrho} (r_2 + s - t)\dot{x}^T(s) T_{\beta i2} \dot{x}(s) ds\\ &+ \int_{t-r_2}^{t-r(t)} (r_2 + s - t)\dot{x}^T(s) T_{\beta i3} \dot{x}(s) ds,\\ V_{l\alpha i}(t) =& (r(t) - r_\varrho) \int_{t-r(t)}^{t-r_1} \dot{x}^T(s) W_{\alpha i1} \dot{x}(s) ds + (r(t) - r_1) \int_{t-r_\varrho}^{t-r(t)} \dot{x}^T(s) W_{\alpha i2} \dot{x}(s) ds,\\ V_{l\beta i}(t) =& (r(t) - r_2) \int_{t-r(t)}^{t-r_\varrho} \dot{x}^T(s) W_{\beta i1} \dot{x}(s) ds + (r(t) - r_\varrho) \int_{t-r_2}^{t-r(t)} \dot{x}^T(s) W_{\beta i2} \dot{x}(s) ds. \end{aligned}$$

Differentiating $V(t)$ yields

$$\dot{V}(t)=\begin{cases}\dot{V}_0(t)+\dot{V}_{1\alpha 1}(t)++\dot{V}_{2\alpha 1}(t)+\dot{V}_{l\alpha 1}(t),\ t\in[t_{2k-1},\varrho_{1k}),\\ \dot{V}_0(t)+\dot{V}_{1\beta 1}(t)+\dot{V}_{2\beta 1}(t)+\dot{V}_{l\beta 1}(t),\ t\in[\varrho_{1k},t_{2k}),\\ \dot{V}_0(t)+\dot{V}_{1\beta 2}(t)+\dot{V}_{2\beta 2}(t)+\dot{V}_{l\beta 2}(t),\ t\in[t_{2k},\ \varrho_{2k}),\\ \dot{V}_0(t)+\dot{V}_{1\alpha 2}(t)+\dot{V}_{2\alpha 2}(t)+\dot{V}_{l\alpha 2}(t),\ t\in[\varrho_{2k},\ t_{2k+1}),\end{cases} \tag{3.7}$$

where

$$\begin{aligned}
\dot{V}_0(t)=&2r_1\varpi_0^T(t)F\dot{\varpi}_0(t)+\varpi_1^T(t)Q_1\varpi_1(t)+\varpi_1^T(t-r_1)(Q_2-Q_1)\varpi_1(t-r_1)\\
&+\varpi_1^T(t-r_\varrho)(Q_3-Q_2)\varpi_1(t-r_\varrho)-\varpi_1^T(t-r_2)Q_3\varpi_1(t-r_2)+r_1^2\dot{x}^T(t)Y_1\dot{x}(t)\\
&+(r_\varrho-r_1)\dot{x}^T(t)Y_2\dot{x}(t)+(r_2-r_\varrho)\dot{x}^T(t)Y_3\dot{x}(t)+F_{c1}+F_{c2}+F_{c3},\\
\dot{V}_{1\alpha i}(t)=&2\varpi_0^T(t)[(r(t)-r_1)X_{\alpha i1}+(r_\alpha-r(t))X_{\alpha i2}+(r_2-r_\varrho)X_{\alpha i3}]\dot{\varpi}_0(t)\\
&+\dot{r}(t)\varpi_0^T(t)(X_{\alpha i1}-X_{\alpha i2})\varpi_0(t)+2\varpi_{2\alpha}^T(t)P\dot{\varpi}_{2\alpha}(t)+\varpi_1^T(t-r_1)S_{\alpha i1}\varpi_1(t-r_1)\\
&+(1-\dot{r}(t))\varpi_1^T(t-r(t))(S_{\alpha i2}-S_{\alpha i1})\varpi_1(t-r(t))+\varpi_1^T(t-r_\varrho)(S_{\alpha i3}-S_{\alpha i2})\varpi_1(t-r_\varrho)\\
&-\varpi_1^T(t-r_2)S_{\alpha i3}\varpi_1(t-r_2),\\
\dot{V}_{1\beta i}(t)=&2\varpi_0^T(t)[(r_\varrho-r_1)X_{\beta i1}+(r(t)-r_\alpha)X_{\beta i2}+(r_2-r(t))X_{\beta i3}]\dot{\varpi}_0(t)\\
&+\dot{r}(t)\varpi_0^T(t)(X_{\beta i2}-X_{\beta i3})\varpi_0(t)+2\varpi_{2\beta}^T(t)P\dot{\varpi}_{2\beta}(t)+\varpi_1^T(t-r_1)S_{\beta i1}\varpi_1(t-r_1)\\
&+(1-\dot{r}(t))\varpi_1^T(t-r(t))(S_{\beta i3}-S_{\beta i2})\varpi_1(t-r(t))+\varpi_1^T(t-r_\varrho)(S_{\beta i2}-S_{\beta i1})\varpi_1(t-r_\varrho)\\
&-\varpi_1^T(t-r_2)S_{\beta i3}\varpi_1(t-r_2),\\
\dot{V}_{2\alpha i}(t)=&(r_2-r_\varrho)\dot{x}^T(t-r_\varrho)(T_{\alpha i3}-T_{\alpha i2})\dot{x}(t-r_\varrho)\\
&+(r_2-r(t))(1-\dot{r}(t))\dot{x}^T(t-r(t))^T(T_{\alpha i2}-T_{\alpha i1})\dot{x}(t-r(t))\\
&+(r_2-r_1)\dot{x}^T(t-r_1)T_{\alpha i1}\dot{x}(t-r_1)+F_{T\alpha i1}+F_{T\alpha i2}+F_{T\alpha i3},\\
\dot{V}_{2\beta i}(t)=&(r_2-r(t))(1-\dot{r}(t))\dot{x}^T(t-r(t))(T_{\beta i3}-T_{\beta i2})\dot{x}(t-r(t))\\
&+(r_2-r_\varrho)\dot{x}^T(t-r_\varrho)(T_{\beta i2}-T_{\beta i1})\dot{x}(t-r_\varrho)+(r_2-r_1)\dot{x}^T(t-r_1)T_{\beta i1}\dot{x}(t-r_1)\\
&+F_{T\beta i1}+F_{T\beta i2}+F_{T\beta i3},\\
\dot{V}_{l\alpha i}(t)=&(r(t)-r_1)[(1-\dot{r}(t))\dot{x}^T(t-r(t))W_{\alpha i2}\dot{x}(t-r(t))-\dot{x}^T(t-r_\varrho)W_{\alpha i2}\dot{x}(t-r_\varrho)]\\
&+(r(t)-r_\varrho)[\dot{x}^T(t-r_1)W_{\alpha i1}\dot{x}(t-r_1)-(1-\dot{r}(t))\dot{x}^T(t-r(t))W_{\alpha i1}\dot{x}(t-r(t))]\\
&+F_{W\alpha i1}+F_{W\alpha i2},\\
\dot{V}_{l\beta i}(t)=&(r(t)-r_\varrho)[(1-\dot{r}(t))\dot{x}^T(t-r(t))W_{\beta i2}\dot{x}(t-r(t))-\dot{x}^T(t-r_2)W_{\beta i2}\dot{x}(t-r_2)]\\
&+(r(t)-r_2)[\dot{x}^T(t-r_\varrho)W_{\beta i1}\dot{x}(t-r_\varrho)-(1-\dot{r}(t))\dot{x}^T(t-r(t))W_{\beta i1}\dot{x}(t-r(t))]\\
&+F_{W\beta i1}+F_{W\beta i2},
\end{aligned}$$

with

$$F_{c1}=-r_1\int_{t-r_1}^{t}\dot{x}^T(s)Y_1\dot{x}(s)ds,\ \ F_{c2}=-\int_{t-r_\varrho}^{t-r_1}\dot{x}^T(s)Y_2\dot{x}(s)ds,\ \ F_{c3}=-\int_{t-r_2}^{t-r_\varrho}\dot{x}^T(s)Y_3\dot{x}(s)ds,$$

$$F_{T\alpha i1}=-\int_{t-r(t)}^{t-r_1}\dot{x}^T(s)T_{\alpha i1}\dot{x}(s)ds,\quad F_{T\alpha i2}=-\int_{t-r_\varrho}^{t-r(t)}\dot{x}^T(s)T_{\alpha i2}\dot{x}(s)ds,$$

$$F_{T\alpha i3} = -\int_{t-r_2}^{t-r_\varrho} \dot{x}^T(s)T_{\alpha i3}\dot{x}(s)ds, \quad F_{T\beta i1} = -\int_{t-r_\varrho}^{t-r_1} \dot{x}^T(s)T_{\beta i1}\dot{x}(s)ds,$$
$$F_{T\beta i2} = -\int_{t-r(t)}^{t-r_\varrho} \dot{x}^T(s)T_{\beta i2}\dot{x}(s)ds, \quad F_{T\beta i3} = -\int_{t-r_2}^{t-r(t)} \dot{x}^T(s)T_{\beta i3}\dot{x}(s)ds,$$
$$F_{W\alpha i1} = \dot{r}(t)\int_{t-r(t)}^{t-r_1} \dot{x}^T(s)W_{\alpha i1}\dot{x}(s)ds, \quad F_{W\alpha i2} = \dot{r}(t)\int_{t-r_\varrho}^{t-r(t)} \dot{x}^T(s)W_{\alpha i2}\dot{x}(s)ds,$$
$$F_{W\beta i1} = \dot{r}(t)\int_{t-r(t)}^{t-r_\varrho} \dot{x}^T(s)W_{\beta i1}\dot{x}(s)ds, \quad F_{W\beta i2} = \dot{r}(t)\int_{t-r_2}^{t-r(t)} \dot{x}^T(s)W_{\beta i2}\dot{x}(s)ds.$$

Applying (2.5) in Lemma 2.1 to estimate F_{c1}, we obtain

$$F_{c1} \leqslant -\varpi^T(t)E_{24}^T Y_1 E_{24}\varpi(t). \tag{3.8}$$

Then, merging the integrals that have the same integral intervals and applying (2.6) in Lemma 2.1 to estimate the various combinations of $\sum_{\iota=2}^{3} F_{c\iota} + \sum_{\iota=1}^{3} F_{T\alpha i\iota} + \sum_{\iota=1}^{2} F_{W\alpha i\iota}$.

For $F_{c2} + F_{T\alpha i1} + F_{T\alpha i2} + F_{W\alpha i1} + F_{W\alpha i2}$ and $F_{c3} + F_{T\alpha i3}$, suppose that $Y_2 + T_{\alpha i1} + \dot{r}(t)W_{\alpha i1} > 0$ and $Y_2 + T_{\alpha i2} + \dot{r}(t)W_{\alpha i2} > 0$ at the vertices of $\gimel_{\alpha i}$, we obtain

$$\begin{aligned} F_{c3} + F_{T\alpha i3} &= -\int_{t-r_2}^{t-r_\varrho} \dot{x}^T(s)(Y_3 + T_{\alpha i3})\dot{x}(s)ds \\ &\leqslant \varpi^T(t)[(r_2 - r_\varrho)G_{\alpha i3}(Y_3 + T_{\alpha i3})^{-1}G_{\alpha i3}^T + \mathrm{Sym}\{G_{\alpha i3}E_{27}\}]\varpi(t), \end{aligned} \tag{3.9}$$

$$\begin{aligned} & F_{c2} + F_{T\alpha i1} + F_{T\alpha i2} + F_{W\alpha i1} + F_{W\alpha i2} \\ =& -\int_{t-r(t)}^{t-r_1} \dot{x}^T(s)(Y_2 + T_{\alpha i1} + \dot{r}(t)W_{\alpha i1})\dot{x}(s)ds - \int_{t-r_\varrho}^{t-r(t)} \dot{x}^T(s)(Y_2 + T_{\alpha i2} + \dot{r}(t)W_{\alpha i2})\dot{x}(s)ds \\ \leqslant& \varpi^T(t)[(r(t) - r_1)G_{\alpha i1}(Y_2 + T_{\alpha i1} + \dot{r}(t)W_{\alpha i1})^{-1}G_{\alpha i1}^T \\ &+ (r_\varrho - r(t))G_{\alpha i2}(Y_2 + T_{\alpha i2} + \dot{r}(t)W_{\alpha i2})^{-1}G_{\alpha i2}^T + \mathrm{Sym}\{G_{\alpha i1}E_{25} + G_{\alpha i2}E_{26}\}]\varpi(t). \end{aligned} \tag{3.10}$$

Similarly, suppose that $Y_3 + T_{\beta i2} + \dot{r}(t)W_{\beta i1} > 0$ and $Y_3 + T_{\beta i3} + \dot{r}(t)W_{\beta i2} > 0$ at the vertices of $\gimel_{\beta i}$. Then, $F_{c3} + F_{T\beta i2} + F_{T\beta i3} + F_{W\beta i1} + F_{W\beta i2}$ and $F_{c2} + F_{T\beta i1}$ satisfy the following inequalities:

$$\begin{aligned} F_{c2} + F_{T\beta i1} &= -\int_{t-r_\varrho}^{t-r_1} \dot{x}^T(s)(Y_2 + T_{\beta i1})\dot{x}(s)ds \\ &\leqslant \varpi^T(t)[(r_\varrho - r_1)G_{\beta i1}(Y_2 + T_{\beta i1})^{-1}G_{\beta i1}^T + \mathrm{Sym}\{G_{\beta i1}E_{28}\}]\varpi(t), \end{aligned} \tag{3.11}$$

$$\begin{aligned} & F_{c3} + F_{T\beta i2} + F_{T\beta i3} + F_{W\beta i1} + F_{W\beta i2} \\ =& -\int_{t-r(t)}^{t-r_\varrho} \dot{x}^T(s)(Y_3 + T_{\beta i2} + \dot{r}(t)W_{\beta i1})\dot{x}(s)ds - \int_{t-r_2}^{t-r(t)} \dot{x}^T(s)(Y_3 + T_{\beta i3} + \dot{r}(t)W_{\beta i2})\dot{x}(s)ds \\ \leqslant& \varpi^T(t)[(r(t) - r_\varrho)G_{\beta i2}(Y_3 + T_{\beta i2} + \dot{r}(t)W_{\beta i1})^{-1}G_{\beta i2}^T \\ &+ (r_2 - r(t))G_{\beta i3}(Y_3 + T_{\beta i3} + \dot{r}(t)W_{\beta i2})^{-1}G_{\beta i3}^T + \mathrm{Sym}\{G_{\beta i2}E_{29} + G_{\beta i3}E_{30}\}]\varpi(t). \end{aligned} \tag{3.12}$$

According to the relationship between the internal elements of the vector, the subsequent equations are valid.

$$
\begin{aligned}
0 &= 2\varpi(t)^T N_{\alpha i1}[\delta_4(t) + \delta_5(t) - \delta_7(t)],\\
0 &= 2\varpi(t)^T N_{\alpha i2}[\delta_6(t) - \delta_5(t) - \delta_8(t)],\\
0 &= 2\varpi(t)^T N_{\beta i1}[\delta_4(t) + \delta_5(t) - \delta_7(t)],\\
0 &= 2\varpi(t)^T N_{\beta i2}[\delta_6(t) - \delta_5(t) - \delta_8(t)].
\end{aligned}
\tag{3.13}
$$

Combining (3.7)–(3.10) and (3.13), for $t \in [t_{2k-1}, \varrho_{1k}) \cup [\varrho_{2k}, t_{2k+1})$, we derive

$$
\begin{aligned}
\dot{V}(t) \leqslant& \varpi^T(t)[\Upsilon_{\alpha i}(r(t), \dot{r}(t)) + (r(t) - r_1)G_{\alpha i1}(Y_2 + T_{\alpha i1} + \dot{r}(t)W_{\alpha i1})^{-1}\alpha i1^T\\
&+ (r_\varrho - r(t))G_{\alpha i2}(Y_2 + T_{\alpha i2} + \dot{r}(t)W_{\alpha i2})^{-1}G_{\alpha i2}^T\\
&+ (r_2 - r_\varrho)G_{\alpha i3}(Y_3 + T_{\alpha i3})^{-1}G_{\alpha i3}^T]\varpi(t)\\
=& \varpi^T(t)\Xi_{\alpha i}(r(t), \dot{r}(t))\varpi(t),
\end{aligned}
\tag{3.14}
$$

where $\Upsilon_{\alpha i}(r(t), \dot{r}(t))$ is defined in Theorem 3.1. From the Schur complement lemma, $\Xi_{\alpha i}(r(t), \dot{r}(t)) < 0$ is equivalent to LMIs (3.3) at the vertices of $\gimel_{\alpha i}$. Therefore, there exist scalars $\gamma_{\alpha i}$ satisfying $\dot{V}(t) < -\gamma_{\alpha i}|x(t)|^2$ for $t \in [t_{2k-1}, \varrho_{1k}) \cup [\varrho_{2k}, t_{2k+1})$ if (3.3) is satisfied.

Similarly, for $t \in [\varrho_{1k}, \varrho_{2k})$, combining (3.7), (3.8), (3.11)–(3.13), we derive

$$
\begin{aligned}
\dot{V}(t) \leqslant& \varpi^T(t)[\Upsilon_{\beta i}(r(t), \dot{r}(t)) + (r_\varrho - r_1)G_{\beta i1}(Y_2 + T_{\beta i1})^{-1}G_{b\beta i1}^T\\
&+ (r(t) - r_\varrho)G_{\beta i2}(Y_3 + T_{\beta i2} + \dot{r}(t)W_{\beta i1})^{-1}G_{\beta i2}^T\\
&+ (r_2 - r(t))G_{\beta i3}(Y_3 + T_{\beta i3} + \dot{r}(t)W_{\beta i2})^{-1}G_{\beta i3}^T]\varpi(t)\\
=& \varpi^T(t)\Xi_{\beta i}(r(t), \dot{r}(t))\varpi(t),
\end{aligned}
\tag{3.15}
$$

where $\varpi^T(t)\Xi_{\beta i}(r(t), \dot{r}(t))\varpi(t) < 0$ corresponds to LMIs (3.5) at the vertices of $\gimel_{\beta i}$ and there exist scalars $\gamma_{\beta i}$ satisfying $\dot{V}(t) < -\gamma_{\beta i}|x(t)|^2$ for $t \in [\varrho_{1k}, \varrho_{2k})$ if (3.5) are satisfied.

To ensure the overall decrement of the functional, some boundary conditions shown below need to be satisfied at each segmented point:

$$
\begin{cases}
\lim\limits_{t \to \varrho_{1k}^-}[V_{1\alpha 1}(t) + V_{2\alpha 1}(t)] \geqslant [V_{1\beta 1}(\varrho_{1k}) + V_{2\beta 1}(\varrho_{1k})],\\
\lim\limits_{t \to t_{2k}^-}[V_{1\beta 1}(t) + V_{2\beta 1}(t)] \geqslant [V_{1\beta 2}(t_{2k}) + V_{2\beta 2}(t_{2k})],\\
\lim\limits_{t \to \varrho_{2k}^-}[V_{1\beta 2}(t) + V_{2\beta 2}(t)] \geqslant [V_{1\alpha 2}(\varrho_{2k}) + V_{2\alpha 2}(\varrho_{2k})],\\
\lim\limits_{t \to t_{2k+1}^-}[V_{1\alpha 2}(t) + V_{2\alpha 2}(t)] \geqslant [V_{1\alpha 1}(t_{2k-1}) + V_{2\alpha 1}(t_{2k-1})].
\end{cases}
\tag{3.16}
$$

From which we obtain the restriction in (3.1). Thus, the proof is completed. □

Remark 3.1. An ADS based on the delay segmenting method is proposed in [38]. However, the L–K functional established therein does not match the ADS appropriately, i.e., the delay-segmentation-related information is not directly reflected in the established L–K functional. Undoubtedly, this makes the stability criterion of the system have a relatively high conservativeness in an intuitive way. Therefore, an improved L–K functional is established in this paper, which enables the direct manifestation of the segmented delay intervals. Compared with the functional established in [38], this functional increases the weight of delay derivatives in the criterion, thereby enhancing the correlation between the system stability conditions and the delay-related information. Moreover, as shown in $V_{1\alpha i}$ and $V_{1\beta i}$, the number of relevant Lyapunov matrices it encompasses has increased by 50%, while the number of constraint conditions between Lyapunov matrices has only increased by 33.33%. As shown in constraint conditions (3.1) of Theorem 3.1, $X_{\alpha 12} \geqslant X_{\beta 12}$, $X_{\beta 22} \geqslant X_{\alpha 22}$, $S_{\alpha 12} \geqslant S_{\beta 12}$, and $S_{\beta 22} \geqslant S_{\alpha 22}$ are not required. Consequently, the function presented in this paper will further reduce the conservativeness of the results.

Remark 3.2. In [38], only two delay-product terms were incorporated in the established functional, aside from loop-like functions. This suggests that the delay-dependent stability conditions across different intervals will rely on a fixed set of Lyapunov matrices. Consequently, the stability criterion derived from [38] tends to produce overly conservative results. In contrast, the functional introduced in this paper incorporates $V_{2\alpha i}$ and $V_{2\beta i}$ based on the specific intervals associated with the delay. This approach allows for the Lyapunov matrices used in the directly delay-dependent stability conditions across different intervals to differ from one another. As a result, the proposed function grants the stability criteria of the system greater flexibility and significantly diminishes its conservativeness.

Remark 3.3. A loop function based on periodic time delay was originally proposed in [37]. This function exhibits the property of being identically zero at the vertices of the domain of the delay function. Consequently, when integrated with the L–K functional, it inherently satisfies the requirements of Lyapunov functions. Moreover, within distinct intervals of the functional, the Lyapunov matrices associated with this function are not required to be identical. However, the ADS partitioning approach introduced in [38] leads to the derivation of stability conditions of more meticulous segmentation from the proposed functional. As a result, constraints are imposed on the Lyapunov matrices within the loop functions. This imposition undermines the definition of the loop functions and increases the conservativeness of the system stability criteria. In contrast, in the functional presented herein, a set of loop functions corresponding to the delay-belonging intervals within each segment of the functional is established. These loop functions are valid on every individual segment. For $t \in [t_{2k-1}, \varrho_{1k})$, given that $V(t_{2k-1}) \geqslant 0$, $V(\varrho_{1k}) \geqslant 0$, and $\dot{V}(t) < -\gamma_{\alpha 1}|x(t)|^2$, then $V(t_{2k-1}) \geqslant V(t) \geqslant V(\varrho_{1k})$ holds true. For other intervals, similar conditions also hold. Therefore, there are no restrictive relationships among their respective Lyapunov matrices. Evidently, this configuration further diminishes the conservativeness of the derived stability conditions.

Remark 3.4. As evident from LMIs (3.3) and (3.5), the stability criterion derived from Theorem 3.1 encompasses a substantial number of variables (NoVs). To minimize computational complexity to the greatest extent possible, conserve computational resources, and investigate the correlation between the increment in computational complexity and the results enhancement, we reduce the number of free matrices in Eq (3.13). Subsequently, we substitute Eq (3.13) with the following equations and apply them to Theorem 3.1,

$$
\begin{aligned}
0 &= 2\varpi(t)^T N_\alpha[\delta_4(t) + \delta_5(t) - \delta_7(t)],\\
0 &= 2\varpi(t)^T N_\beta[\delta_6(t) - \delta_5(t) - \delta_8(t)].
\end{aligned}
\tag{3.17}
$$

The results obtained from these modifications will be compared and analyzed in the section dedicated to numerical examples.

Then, the following corollary is derived based on a continuous L–K functional simplified by (3.6).

Corollary 3.1. For scalars $\varrho \in [0,1], r_2 \geqslant r_1 \geqslant 0, \mu < 1$, if there exist matrices $P \in \mathbb{S}_+^{9n}$, $F, \bar{X}_\varsigma \in \mathbb{S}_+^{6n}$, $Q_\varsigma, \bar{S}_\varsigma \in \mathbb{S}_+^{2n}$, $Y_\varsigma, \bar{T}_\varsigma \in \mathbb{S}_+^{n}$, $W_{\alpha i1}, W_{\alpha i2}, W_{\beta i1}, W_{\beta i2} \in \mathbb{S}^n$, $G_{\alpha i\varsigma}, G_{\beta i\varsigma} \in \mathbb{R}^{15n\times 2n}$, $N_{\alpha i1}, N_{\alpha i2}, N_{\beta i1}, N_{\beta i2} \in \mathbb{R}^{15n\times n}$, $i = 1,2, \varsigma = 1,\cdots,3$, such that LMIs (3.18)–(3.19) and (3.20)–(3.21) are satisfied at the vertices of $\gimel_{\alpha i}$ and $\gimel_{\beta i}$, respectively, then system (2.1) is stable,

$$W_{Y\bar{T}\alpha i1} > 0, \; W_{Y\bar{T}\alpha i2} > 0, \tag{3.18}$$

$$\begin{bmatrix} \bar{\Upsilon}_{\alpha i}(r(t),\dot{r}(t)) & \sqrt{r(t)-r_1}G_{\alpha i1} & \sqrt{r_\varrho - r(t)}G_{\alpha i2} & \sqrt{r_2-r_1}G_{\alpha i3} \\ * & -\widehat{W}_{Y\bar{T}\alpha i1} & 0 & 0 \\ * & * & -\widehat{W}_{Y\bar{T}\alpha i2} & 0 \\ * & * & * & -\widehat{Y}_{\bar{T}1} \end{bmatrix} < 0, \tag{3.19}$$

$$W_{Y\bar{T}\beta i1} > 0, \; W_{Y\bar{T}\beta i2} > 0, \tag{3.20}$$

$$\begin{bmatrix} \bar{\Upsilon}_{\beta i}(r(t),\dot{r}(t)) & \sqrt{r_\varrho-r_1}G_{\beta i1} & \sqrt{r(t) - r_\varrho}G_{\beta i2} & \sqrt{r_2-r(t)}G_{\beta i3} \\ * & -\widehat{Y}_{\bar{T}2} & 0 & 0 \\ * & * & -\widehat{W}_{Y\bar{T}\beta i1} & 0 \\ * & * & * & -\widehat{W}_{Y\bar{T}\beta i2} \end{bmatrix} < 0, \tag{3.21}$$

where

$$\begin{aligned}
\bar{\Upsilon}_{\alpha i}(r(t),\dot{r}(t)) &= \Lambda_1 + \bar{\Lambda}_{2\alpha} + \Lambda_{3\alpha i} + \Lambda_{4\alpha i},\\
\bar{\Upsilon}_{\beta i}(r(t),\dot{r}(t)) &= \Lambda_1 + \bar{\Lambda}_{2\beta} + \Lambda_{3\beta i} + \Lambda_{4\beta i},\\
\Lambda_1 =& \mathrm{Sym}\{h_1\Pi_0^T F\Pi_1\} + \Pi_2^T Q_1\Pi_2 + \Pi_3^T(Q_2-Q_1)\Pi_3 + \Pi_4^T(Q_3-Q_2)\Pi_4\\
&- \Pi_5^T Q_3\Pi_5 + r_1^2\Pi_{20}^T Y_1\Pi_{20} + (r_\varrho - r_1)\Pi_{20}^T Y_2\Pi_{20} + (r_2 - r_\varrho)\Pi_{20}^T Y_3\Pi_{20},\\
\bar{\Lambda}_{2\alpha} =& \mathrm{Sym}\{\Pi_0^T[(r(t)-r_1)\bar{X}_1 + (r_\alpha - r(t))\bar{X}_2 + (r_2-r_\varrho)\bar{X}_3]\Pi_1 + \Pi_7^T P\Pi_8\}\\
&+ \dot{r}(t)\Pi_0^T(\bar{X}_1 - \bar{X}_2)\Pi_0 + \Pi_3^T\bar{S}_1\Pi_3 + (1-\dot{r}(t))\Pi_6^T(\bar{S}_2 - \bar{S}_1)\Pi_6\\
&+ \Pi_4^T(\bar{S}_3 - \bar{S}_2)\Pi_4 - \Pi_5^T\bar{S}_3\Pi_5 + (r_2 - r_\varrho)e_9^T(\bar{T}_3 - \bar{T}_2)e_9\\
&+ (r_2 - r(t))(1-\dot{r}(t))e_7^T(\bar{T}_2 - \bar{T}_1)e_7 + (r_2 - r_1)e_6^T\bar{T}_1 e_6,\\
\bar{\Lambda}_{2\beta} =& \mathrm{Sym}\{\Pi_0^T[(r_\varrho-r_1)\bar{X}_1 + (r(t) - r_\alpha)\bar{X}_2 + (r_2-r(t))\bar{X}_3]\Pi_1 + \Pi_9^T P\Pi_{10}\}\\
&+ \dot{r}(t)\Pi_0^T(\bar{X}_2 - \bar{X}_3)\Pi_0 + \Pi_3^T\bar{S}_1\Pi_3 + (1-\dot{r}(t))\Pi_6^T(\bar{S}_3 - \bar{S}_2)\Pi_6\\
&+ \Pi_4^T(\bar{S}_2 - \bar{S}_1)\Pi_4 - \Pi_5^T\bar{S}_3\Pi_5 + (r_2 - r(t))(1-\dot{r}(t))e_7^T(\bar{T}_3 - \bar{T}_2)e_7\\
&+ (r_2 - r_\varrho)e_9^T(\bar{T}_2 - \bar{T}_1)e_9 + (r_2 - r_1)e_6^T\bar{T}_1 e_6,\\
\Lambda_{3\alpha i} =& (r(t)-r_1)((1-\dot{r}(t))e_7^T W_{\alpha i2}e_7 - e_9^T W_{\alpha i2}e_9) - (r_\varrho - r(t))(e_6^T W_{\alpha i1}e_6 - (1-\dot{r}(t))e_7^T W_{\alpha i1}e_7),\\
\Lambda_{3\beta i} =& (r(t)-r_\varrho)((1-\dot{r}(t))e_7^T W_{\beta i2}e_7 - e_8^T W_{\beta i2}e_8) - (r_2 - r(t))(e_9^T W_{\beta i1}e_9 - (1-\dot{r}(t))e_7^T W_{\beta i1}e_7),\\
\Lambda_{4\alpha i} =& \mathrm{Sym}\{G_{\alpha i1}\Pi_{12} + G_{\alpha i2}\Pi_{13} + G_{\alpha i3}\Pi_{14} + N_{\alpha i1}\Pi_{18} + N_{\alpha i2}\Pi_{19}\} - \Pi_{11}^T\widehat{Y}_1\Pi_{11},\\
\Lambda_{4\beta i} =& \mathrm{Sym}\{G_{\beta i1}\Pi_{15} + G_{\beta i2}\Pi_{16} + G_{\beta i3}\Pi_{17} + N_{\beta i1}\Pi_{18} + N_{\beta i2}\Pi_{19}\} - \Pi_{11}^T\widehat{Y}_1\Pi_{11},\\
\widehat{Y}_{\bar{T}1} =& diag\{Y_{\bar{T}1}, 3Y_{\bar{T}1}\}, \quad \widehat{Y}_{\bar{T}2} = diag\{Y_{\bar{T}2}, 3Y_{\bar{T}2}\},\\
\widehat{W}_{Y\bar{T}\alpha i1} &= diag\{W_{Y\bar{T}\alpha i1}, 3W_{Y\bar{T}\alpha i1}\}, \quad \widehat{W}_{Y\bar{T}\alpha i2} = diag\{W_{Y\bar{T}\alpha i2}, 3W_{Y\bar{T}\alpha i2}\},\\
\widehat{W}_{Y\bar{T}\beta i1} &= diag\{W_{Y\bar{T}\beta i1}, 3W_{Y\bar{T}\beta i1}\}, \quad \widehat{W}_{Y\bar{T}\beta i2} = diag\{W_{Y\bar{T}\beta i2}, 3W_{Y\bar{T}\beta i2}\},
\end{aligned}$$

with

$$\begin{aligned}
Y_{\bar{T}1} =& Y_3 + \bar{T}_3, \quad Y_{\bar{T}2} = Y_2 + \bar{T}_1, \\
W_{Y\bar{T}\alpha i1} =& Y_2 + \bar{T}_1 - \dot{r}(t)W_{\alpha i1}, \quad W_{Y\bar{T}\alpha i2} = Y_2 + \bar{T}_2 - \dot{r}(t)W_{\alpha i2}, \\
W_{Y\bar{T}\beta i1} =& Y_3 + \bar{T}_2 - \dot{r}(t)W_{\beta i1}, \quad W_{Y\bar{T}\beta i2} = Y_3 + \bar{T}_3 - \dot{r}(t)W_{\beta i2},
\end{aligned}$$

and $\Pi_\lambda(\lambda = 0, \cdots, 20)$, $\widehat{Y_1}$ are defined in Theorem 3.1.

Proof. Setting $\bar{X}_\varsigma = X_{\alpha i\varsigma} = X_{\beta i\varsigma}$, $\bar{S}_\varsigma = S_{\alpha i\varsigma} = S_{\beta i\varsigma}$ and $\bar{T}_\varsigma = T_{\alpha i\varsigma} = T_{\beta i\varsigma}$ in L–K functional (3.6), we obtain a continuous piecewise L–K functional satisfying the following equations:

$$\begin{cases}
\lim\limits_{t\to\varrho_{1k}^-} V_{l\alpha 1}(t) = V_{l\beta 1}(\varrho_{1k}), \\
\lim\limits_{t\to t_{2k}^-} V_{l\beta 1}(t) = V_{l\beta 2}(t_{2k}), \\
\lim\limits_{t\to \varrho_{2k}^-} V_{l\beta 2}(t) = V_{l\alpha 2}(\varrho_{2k}), \\
\lim\limits_{t\to t_{2k+1}^-} V_{l\alpha 2}(t) = V_{l\alpha 1}(t_{2k-1}).
\end{cases} \tag{3.22}$$

The proof employs a procedure analogous to that in Theorem 3.1, and the detail will not be repeated here. □

Remark 3.5. To show the advantage of the non-continuous functional (3.6), Corollary 3.1 provides a stability condition that is derived by using the continuous functional (3.22). It will be shown in the section of numerical examples that the non-continuous functional (3.6) plays an important role in the reduction of conservativeness.

4. Numerical examples

In this section, we will verify the validity and superiority of the new results obtained in the previous section through two numerical examples and a single-area load frequency control system provided.

Example 4.1. Consider the system (2.1) with

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, \quad A_r = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}.$$

To highlight the effectiveness of our novel function for improving the results, we choose [37, 38] to compare with our results in this example. Both studies adopt a similar bounding method to ours, utilizing the first-order Bessel-Legendre inequality along with the free-matrix-based inequality, and we adopt the optimal solutions for the results of [38] under the β changes. From Table 2, for various μ with $r_1 = 0$, we obtain larger allowable upper bounds (AUBs) of r_2 under the appropriate ϱ (the value of ϱ is accurate to the percentile) by applying Theorem 3.1 and corresponding NoVs.

Table 2. The AUBs of r_2 for various μ with $r_1 = 0$.

Methods	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.5$	$\mu = 0.8$	NoVs
[37] Theorem 1	5.10	4.57	3.78	3.38	$131.5n^2 + 9.5n$
[38] Corollary 7	4.82	4.13	3.13	2.70	$76.5n^2 + 11.5n$
[38] Theorem 3	5.44	5.00	4.18	3.66	$259.5n^2 + 32.5n$
Corollary 3.1	4.95($\varrho = 0.21$)	4.30($\varrho = 0.05$)	3.37($\varrho = 0.95$)	2.98($\varrho = 0.89$)	$317.5n^2 + 25.5n$
Theorem 3.1	5.61($\varrho = 0.55$)	5.29($\varrho = 0.56$)	4.40($\varrho = 0.40$)	3.86($\varrho = 0.18$)	$556n^2 + 70n$

Further, when $\mu = 0.2$, the result of Theorem 3.1 exhibits a 5.8% improvement compared to the result of Theorem 3 in [38]. Additionally, when $\mu = 0.5$, the result of Theorem 3.1 shows a 16.4% enhancement over the result reported in [37]. On the other hand, we note that the result of Theorem 3.1 yields a larger NoVs than those presented in the listed literature. Indeed, it is inescapable that the computational complexity escalates as the conservativeness of the results diminishes. To further emphasize the reduction in the conservativeness of the results yielded by our proposed method, we will conduct verifications using other examples in the subsequent contents.

In addition, compared with the results of Corollary 7 in [38] obtained by a continuous functional, Corollary 1 in this paper also gives higher AUBs in Table 2. When $\mu = 0.8$, Corollary 3.1 presents a 10.3% improvement compared to the result of Corollary 7 in [38].

Next, we assign the initial value of the system state as $x_0(t) = [1 \quad -1]^T$ and choose $r(t) = 5.29/2 + (5.29/2)cos(0.4t/5.29)$, which satisfies $r(t) \in [0, 5.29]$ and $\dot{r}(t) \in [-0.2, 0.2]$. As shown in Figure 1, the state response depiction of the system illustrated in Example 4.1 demonstrates that each state component converges towards zero. This outcome further reinforces the validity of the derived conclusions.

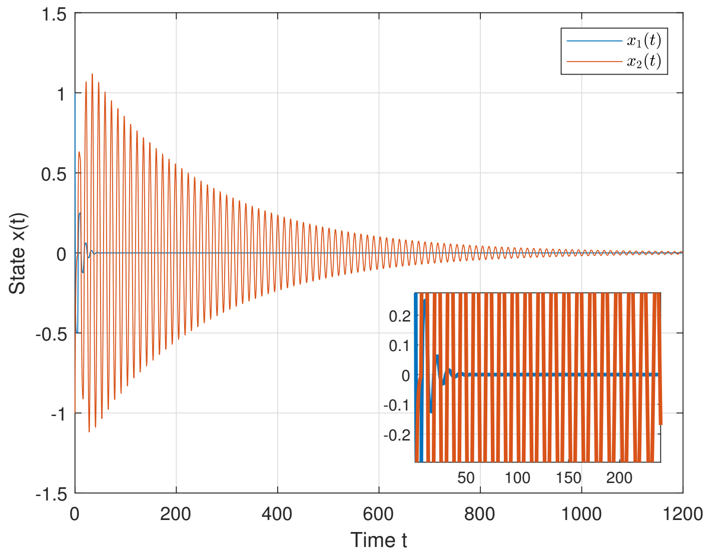

Figure 1. State responses of the single-area load frequency control system.

Example 4.2. Consider the system (2.1) with

$$A = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -2.0 \end{bmatrix}, \quad A_r = \begin{bmatrix} 0.0 & 0.0 \\ -1.0 & 1.0 \end{bmatrix}.$$

In this example, we further present the simulation results of Theorem 3.1 and Remark 3.4 for various μ and r_1 in Table 3 and compare them with [38, 43]. Generally speaking, it is readily apparent that the results derived from Theorem 3.1 are notably superior to those presented in the existing literature, showcasing a substantial improvement. For $r_1 = 0$, the result of Theorem 3.1 is 92% better than that of Theorem 3 in [38] for the case of slow delay ($\mu = 0.1$), although the NoVs is nearly doubled. In the case of rapid delay ($\mu = 0.8$), the result of Theorem 3.1 also outperforms that of Theorem 3 in [38] by 27%. When $r_1 = 1$, it is observable that although the improvement effect starts to diminish as the lower bound of the delay increases, the improvement can still reach 20% when $\mu = 0.5$. Meanwhile, compared with Theorem 4 in [43], Theorem 3.1 has comparable NoVs, yet its results are evidently less conservative.

Furthermore, the results presented in Remark 3.4 are tabulated in Table 3. Evidently, Remark 3.4 involves fewer NoVs compared to Theorem 3.1. Consequently, it exhibits a relatively lower computational complexity. Nevertheless, the results derived from Remark 3.4 are comparable to those obtained from Theorem 3.1.

Table 3. The AUBs of r_2 for various μ and r_1.

	Methods	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.5$	$\mu = 0.8$	NoVs
$r_1 = 0$	[43] Theorem 4	9.94	–	4.46	3.33	$521n^2 + 27n$
	[38] Theorem 3	13.67($\varrho = 0.49$)	9.18($\varrho = 0.49$)	5.02($\varrho = 0.53$)	3.18 ($\varrho = 0.51$)	$259.5n^2 + 32.5n$
	Theorem 3.1	26.49($\varrho = 0.01$)	13.16($\varrho = 0.04$)	5.93($\varrho = 0.27$)	4.05($\varrho = 0.21$)	$556n^2 + 70n$
	Remark 3.4	26.33($\varrho = 0.01$)	13.14($\varrho = 0.03$)	5.93($\varrho = 0.25$)	4.05($\varrho = 0.21$)	$466n^2 + 70n$
$r_1 = 1$	[38] Theorem 3	13.48($\varrho = 0.43$)	8.74($\varrho = 0.45$)	4.18($\varrho = 0.57$)	2.37($\varrho = 0.55$)	$259.5n^2 + 32.5n$
	Theorem 3.1	15.55($\varrho = 0.05$)	9.05($\varrho = 0.13$)	5.03($\varrho = 0.87$)	2.81($\varrho = 0.85$)	$556n^2 + 70n$
	Remark 3.4	15.55($\varrho = 0.05$)	9.04($\varrho = 0.13$)	5.03($\varrho = 0.88$)	2.81($\varrho = 0.84$)	$466n^2 + 70n$

Subsequently, we set the initial state as $x_0(t) = [1 \;\; 0]^T$ and choose the delay function as $r(t) = 13.16/2 + (13.16/2)cos(0.4t/13.16)$. This function satisfies the conditions $r(t) \in [0, 13.16]$ and $\dot{r}(t) \in [-0.2, 0.2]$. The state response is depicted in Figure 2. It is shown in the figure that the system is stable.

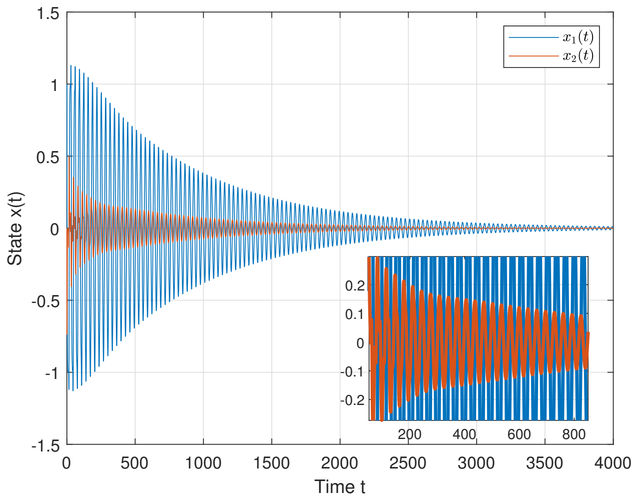

Figure 2. State responses of the single-area load frequency control system.

Example 4.3. Consider the single-area load frequency control system, which can be expressed as the system (2.1) with

$$A = \begin{bmatrix} -\frac{D}{M} & \frac{1}{M} & 0 & 0 \\ 0 & -\frac{1}{T_t} & \frac{1}{T_t} & 0 \\ -\frac{1}{T_g R} & 0 & -\frac{1}{T_g} & 0 \\ \nu & 0 & 0 & 0 \end{bmatrix}, \quad A_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\nu K_p}{T_g} & 0 & 0 & -\frac{K_i}{T_g} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where D stands for the generator's damping coefficient, M is its moment of inertia, T_t and T_g represent the time constants corresponding to the turbine and governor, respectively, R refers to the speed drop, and ν represents the frequency bias factor. K_p and K_i denote the proportional and integral gains of the *PI* controller, respectively. Let $D = 1.0, M = 10, T_t = 0.3, T_g = 0.1, R = 0.05, \nu = 21, K_p = 0.8, K_i = \{0.1, 0.2, 0.3\}$ and $r_1 = 0, \mu = 0.1$. The AUBs of r_2 under the appropriate ϱ are shown in Table 4, demonstrating results that are evidently less conservative than those of Theorem 1 in [37] as well as Corollary 7 and Theorem 3 in [38]. Especially when integral gains $K_i = 0.1$, the result shows a 30% improvement in the AUBs compared to Theorem 3 in [38].

Table 4. The AUBs of r_2 for various K_i.

Methods	$K_i = 0.1$	$K_i = 0.2$	$K_i = 0.3$
[37] Theorem 1	9.55	5.31	3.74
[38] Corollary 7	9.04	5.12	3.39
[38] Theorem 3	9.77	5.61	3.84
Corollary 3.1	11.01($\varrho = 0.07$)	5.95($\varrho = 0.11$)	3.85($\varrho = 0.13$)
Theorem 3.1	12.74($\varrho = 0.17$)	6.38($\varrho = 0.17$)	4.05($\varrho = 0.15$)

To further confirm the validity of our results, setting $K_i = 0.1$ and the initial state $x_0(t) = [1 \;\; 0 \;\; 0.5 \;\; -1]^T$, and choosing $r(t) = 12.74/2 + (12.74/2)cos(0.2t/12.74)$, which satisfies $r(t) \in [0, 12.74]$ and

$\dot{r}(t) \in [-0.1, 0.1]$, respectively, the state responses of the single-area load frequency control system are shown in Figure 3. Obviously, they tend to be stable.

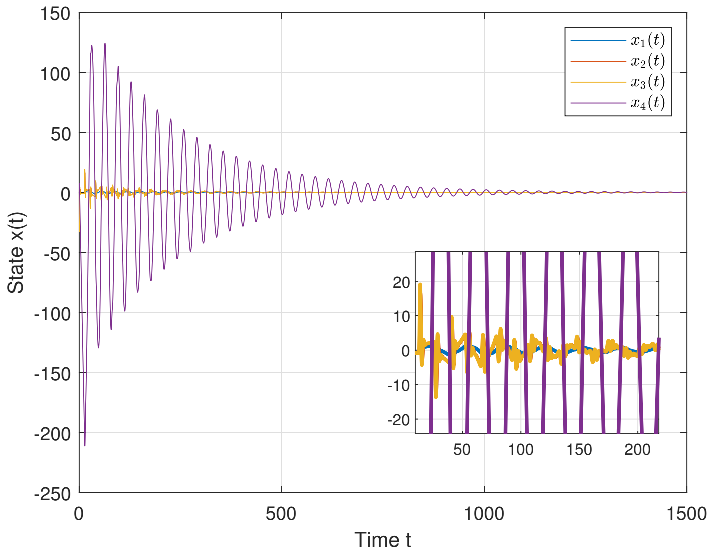

Figure 3. State responses of the single-area load frequency control system.

5. Conclusions

In this paper, an improved delay-segmentation-based non-continuous piecewise L–K functional is established. Then, a low-conservativeness stability criterion for linear systems with a periodical time-varying delay is presented based on the functional. Finally, we provide two simulation examples alongside an application to a single-area load frequency control system to demonstrate the effectiveness of the proposed approach.

Author contributions

Wei Wang: Writing-review & editing, formal analysis, validation, conceptualization, funding acquisition; Chang-Xin Li: Writing-original draft, software, methodology, investigation; Ao-Qian Luo: Writing-review & editing; Hui-Qin Xiao: Writing-review & editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article.

Conflict of interest

The authors declare no conflict of interest.

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