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Research article

Completion problems of partial N_0^1-matrices under directed 2-trees

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Abstract: A matrix completion problem asks whether a partial matrix has a completion to a conventional matrix with a desired property. C. Mendes Araǘjo and J. R. Torregrosa explored the completion problem of a combinatorially symmetric N0-matrix by applying an undirected graph. However, in practical applications such as seismic data reconstruction, data transmission, and engineering computation data are often incomplete and must be represented by a non-combinatorially symmetric matrix. In this paper, we discuss the completion problem of a non-combinatorially symmetric partial matrix by using a directed graph and prove that a non-combinatorially symmetric partial matrix under a directed 2-tree is completed as an N_0^1-matrix.

Keywords: partial matrix; matrix completion; N_0^1-matrix; directed 2-tree
Mathematics Subject Classification: 05C22, 05C50, 15A48, 15A57

1. Introduction

A partial matrix is an array in which some entries are specified, while others are free to be chosen from a certain set. A matrix completion problem asks whether a partial matrix can be completed to a conventional matrix with a desired property. Matrix completion problems arise in optimization and in the study of Euclidean distance matrices [1] and have also been extensively used in seismic data reconstruction [2], data transmission [3], image processing [2], signal processing [2], and engineering computation [2].

Matrix completion problems can be intuitively studied through graph theory, where undirected graphs represent combinatorially symmetric matrices, and directed graphs model non-combinatorially matrices. This graphical approach leverages the structural properties of graphs to simplify the analysis of incomplete matrices, enabling the principled recovery of missing entries via spectral graph theory and combinatorial optimization techniques. Matrix completion problems have been studied for many classes of matrices [4–10]. Concurrently, algebraic state space theory (ASST), based on the semi-tensor product (STP), provides a robust algebraic framework for modeling and analyzing finite state machines (FSMs) with diverse characteristics [11]. By integrating STP with graph structures, ASST not only enriches the theoretical foundation of FSM analysis but also enhances its practical applicability in systems with complex dynamics, such as networked control and signal processing. The synergy between matrix completion and ASST lies in their shared capability to handle incomplete information and complex system interactions. While matrix completion focuses on reconstructing missing data through graph-regularized optimization, ASST systematically analyzes and controls FSM transitions via algebraic state space representations. This duality is further strengthened by integrating STP with graph-theoretic regularization, enabling robust analysis of both FSM behavior and partial matrix reconstruction in real-world applications such as networked control systems and signal processing pipelines.

An $n \times n$ real matrix is called an N_0^1*-matrix* if all its principal minors are non-positive and each entry is non-positive (see, e.g., [12,13]). Obviously, the diagonal entries of N_0^1-matrix are non-positive. A partial matrix is said to be a *partial* N_0^1*-matrix* if every completely specified principal submatrix is an N_0^1-matrix. An $n \times n$ partial matrix $A = (a_{ij})$ it is said to be non-combinatorially symmetric when a_{ij} is specified if and only if a_{ij} is unknown. For a non-combinatorially symmetric partial matrix, all main diagonal entries are specified. A natural way to describe an $n \times n$ non-combinatorially symmetric partial matrix is by a digraph that has an arc if an entry is nonzero. The non-combinatorially symmetric matrix completion problems have been studied in [5, 8]. And the non-combinatorially symmetric N-matrix completion problem has been studied if the graph of its specified entries is an acyclic graph or a double cycle in [5]. The combinatorially symmetric N_0^1-matrix completion was studied in [9]. In [10], the authors have studied the combinatorial symmetric partial N_0-matrix completion problems under undirected graphs, and they obtain that a combinatorially symmetric partial N_0-matrix with no null main diagonal entries has an N_0-matrix completion. In this paper, our interest is in the N_0^1-matrix completion problems under directed graphs. The study of this problem is different from the previous one since some main diagonal entries can be zero, and each specified off-diagonal entry is negative.

The outline of this paper is as follows: we introduce the definition of linear directed 2-tree in Section 2. We obtain the completion of a partial N_0^1-matrix if the digraph of its specified entries is a linear directed 2-tree and discuss that the completion problem for a partial N_0^1-matrix under a nonlinear directed 2-tree in Section 3.

2. The directed 2-tree

The completion of partial non-combinatorially symmetric matrices by using digraph theory is one of the important research directions in combinatorial matrix theory. In this paper, we will study the completion problem of an N_0^1*-matrix* under a linear directed 2-tree. First, we will introduce the linear directed 2-tree.

Definition 2.1. [14] *A* digraph *G consists of a finite nonempty set V of objects called* vertices *and a set E of ordered pairs of distinct vertices, each element of E is called an* arc *or a* directed edge. *A digraph H is called a subdigraph of digraph G if* $V(H) \subseteq V(G)$ *and* $E(H) \subseteq E(G)$.

Definition 2.2. [14] *If a digraph G has the property that for each pair* u, v *of distinct vertices of G, at most one of* (u, v) *and* (v, u) *is an arc of G, then G is an* oriented graph.

Definition 2.3. [14] *A oriented graph G is* transitive *if whenever* (u, v) *and* (v, w) *are arcs of G, then*

(u, w) is also an arc of G. In each oriented graph G, for some $k(1 \leq k \leq n)$, there is a transitive oriented subgraph. A maximum transitive oriented subgraph is a transitive oriented subgraph with the maximum number of vertices among all transitive oriented subgraphs.

A natural way to describe an $n \times n$ partial matrix A is via a graph $G_A = (V, E)$, where the set of vertices V is $1, 2, ..., n$, and $i, j, i \neq j$, is an edge or arc when the (i, j) entry is specified. For a non-combinatorially symmetric partial matrix, a natural way to describe an $n \times n$ non-combinatorially symmetric partial matrix is via a digraph G_A that has the corresponding arc if the (i, j) entry is specified.

Definition 2.4. [15] A clique *in an undirected graph G is simply a complete (all possible edges) induced subgraph. We also use clique to refer to a complete graph and use K_p to indicate a clique on p vertices. Recall that a* k-tree *is a graph sequentially constructed from $k+1$-cliques (K_{k+1}) via articulation along k-cliques (see [16]).* 2-trees *is a graph in which the building blocks are triangles ($K_3^{'}s$) and the articulation is along edges. A 2-tree is* linear *if there is a natural order to the building triangles that have precisely two vertices of degree two.*

K_3 is considered to be, linear 2-tree. For more on linear 2-trees, see [15].

Based on Definition 2.4, we will explore the analogous structure in directed graphs, namely the directed 2-tree. A oriented graph is called a directed k-tree if it is sequentially constructed from $k+1$-transitive oriented graphs via articulation along k-transitive oriented graphs. Thus, a directed 2-tree is an oriented graph sequentially constructed from transitive oriented graphs of order 3 via articulation along arcs. If there is a natural order to the building transitive digraphs of order 3, which have precisely two vertices of degree two, then a directed 2-tree is called a linear directed 2-tree. Otherwise, a directed 2-tree is called a nonlinear directed 2-tree.

Example 2.5. Assume a partial matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & ? & ? \\ ? & a_{22} & a_{23} & a_{24} & ? & ? \\ ? & ? & a_{33} & a_{34} & a_{35} & ? \\ ? & ? & ? & a_{44} & a_{45} & a_{46} \\ ? & a_{52} & ? & ? & a_{55} & a_{56} \\ ? & a_{62} & a_{63} & ? & ? & a_{66} \end{bmatrix},$$

whose graph is an oriented graph G_A with a loop at each vertex; see Figure 1,

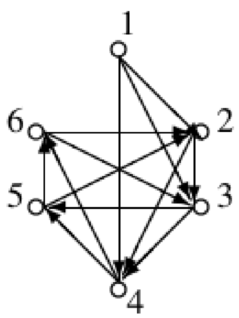

Figure 1. Oriented graph G_A.

For $G_A = (V, E)$, let $V = \{V_1, V_2, V_3, V_4\}$, where $V_1 = \{1, 2, 3\}, V_2 = \{2, 3, 4\}, V_3 = \{3, 4, 5\}, V_4 = \{4, 5, 6\}$.

A directed 2-tree Γ is sequentially constructed from these transitive oriented graphs of order 3 via articulation along arcs; see Figure 2.

Figure 2. Directed 2-tree Γ.

The completion of A is

$$A_C = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & c_{15} & c_{16} \\ c_{21} & a_{22} & a_{23} & a_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & a_{33} & a_{34} & a_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & a_{44} & a_{45} & a_{46} \\ c_{51} & a_{52} & c_{53} & c_{54} & a_{55} & a_{56} \\ c_{61} & a_{62} & a_{63} & c_{64} & c_{65} & a_{66} \end{bmatrix},$$

where c_{ij} may be different values, the digraph G_A of A_C is also different.

Remark. Throughout the paper, we denote the entries of a partial matrix A as follows: a_{ij} denotes an specified entry, and "?" denotes a unspecified entry. The entry c_{ij} denotes a value assigned to the unspecified entry during the process of completing a partial matrix. A_C is the completion of the partial matrix A.

Definition 2.6. [17] *Let the partitioned (block) matrix* $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$*, where the matrix* B *is nonsingular; the matrix* A *need not be square. Then,*

$$A/B = E - DB^{-1}C$$

is the *Schur complement* of B in the partitioned matrix A. In addition, we refer to the Guttman rank formula:

$$rank(A) = rank(B) + rank(A/B).$$

3. The N_0^1-matrix completion under the directed 2-tree

In this section, we will obtain the completion of a partial N_0^1-matrix if the digraph of its specified entries of is a linear directed 2-tree. In addition, we will discuss the completion problem for a partial N_0^1-matrix under a nonlinear directed 2-tree.

The submatrix of a matrix A, of size $n \times n$, lying in rows α and β, $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. Therefore, a real matrix A, of size $n \times n$, is an N_0^1-matrix only if $\det A_c[\alpha] \leq 0$ for any $\alpha \subseteq \{1, 2, ..., n\}$.

Proposition 3.1. Let A be an N_0^1-matrix. Then,

(1) If P is a permutation matrix, then PAP^T is an N_0^1-matrix;

(2) If D is a positive diagonal matrix, then DA, DA is an N_0^1-matrix;

(3) Any principal submatrix of A is an N_0^1-matrix.

We suppose that all main diagonal entries in a partial N_0^1-matrix are specified and may be 0 or non-zero.

Proposition 3.2. If A is 2×2 partial non-combinatorially symmetric matrix, whose digraph is a linear

directed 2-tree, then there exists an N_0^1-matrix completion for A.
Proof: We can assume an 2×2 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} \\ ? & -a_{22} \end{pmatrix},$$

with specified entries $a_{11}, a_{12}, a_{22} \geq 0$.

Our aim is to prove the existence of c_{21} such that the completion

$$A = \begin{pmatrix} -a_{11} & -a_{12} \\ -c_{21} & -a_{22} \end{pmatrix}.$$

We will consider the following two cases:

Case 1: $a_{11}a_{22} = 0$.

We may choose $c_{21} \geq 0$,then $\det A_c \leq 0$.

Case 2: $a_{11}a_{22} \neq 0$.

We may choose $c_{21} \geq a_{11}a_{22}/a_{12}$,then $\det A_c \leq 0$.

Proposition 3.3. If A is a 3×3 non-combinatorially symmetric partial N_0^1-matrix whose digraph is a linear directed 2-tree, then there exists an N_0^1-matrix completion for A.
Proof: We can assume an 3×3 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ ? & -a_{22} & -a_{23} \\ ? & ? & -a_{33} \end{pmatrix},$$

where each $a_{ij} > 0 (i > j, i, j = 1, 2, 3)$ and $a_{ii} \geq 0 (i = 1, 2, 3)$.

Our aim is to prove the existence of nonnegative c_{13}, c_{21}, and c_{32} such that the completion

$$A_C = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} \\ -c_{21} & -a_{22} & -a_{23} \\ -c_{31} & -c_{32} & -a_{33} \end{pmatrix},$$

is N_0^1.

We may choose $c_{13} = c_{21} = c_{32} = t \geq 0$, and show that $\det A_C[\alpha] \leq 0$ for any $\alpha \subseteq \{1, 2, 3\}$. By Proposition 3.2, we will consider the following four different cases:

Case 1: $a_{11} = a_{22} = a_{33} = 0$.

According to 3.2, all 2×2 principal minors are non-positive. $\det A_C = t(-a_{12}a_{23} - ta_{13}) \leq 0$.

Case 2: $a_{11}a_{22}a_{33} \neq 0$.

If we choose t large enough, then,

$$\begin{aligned}
&\det A_C\{1, 2\} = -ta_{12} + a_{11}a_{22} \leq 0; \\
&\det A_C\{1, 3\} = -ta_{13} + a_{11}a_{33} \leq 0; \\
&\det A_C\{2, 3\} = -ta_{23} + a_{22}a_{33} \leq 0; \\
&\det A_C = -a_{13}t^2 + (-a_{12}a_{23} + a_{13}a_{22} + a_{11}a_{23} + a_{12}a_{33})t - a_{11}a_{22}a_{33} \leq 0.
\end{aligned}$$

Case 3: $a_{11} = a_{22} = 0, a_{33} \neq 0$ with $a_{23} = a_{33}$.

According to Property 3.2, all 2×2 principal minors are non-positive. $\det A_C = t(-a_{12}(a_{23} - a_{33}) - ta_{13}) \leq 0$.

Case 4: $a_{11} = 0, a_{22} = a_{33} \neq 0$ with $a_{23} = a_{33}$.

We may choose $t \geq a_{22}$, then all 2×2 principal minors are non-positive, and $\det A_C = t(-a_{12}(a_{23} - a_{33}) - a_{13}(-t + a_{22})) = t(-t + a_{22}) \leq 0$. □

Proposition 3.4. Let A be an 4×4 non-combinatorially symmetric partial N_0^1-matrix whose digraph is a linear directed 2-tree Γ; there exists an N_0^1-matrix completion for A.

Proof: Let Γ be the 4-vertex linear directed 2-tree with $V = (V_1, V_2)$, $V_1 = \{1, 2, 3\}$, $V_2 = \{2, 3, 4\}$. There are the following two possibilities for the 4-vertex linear directed 2-tree; see Figure 3.

Figure 3. 4-Vertex linear directed 2-tree.

Case 1: Assume 4×4 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ ? & -a_{22} & -a_{23} & -a_{24} \\ ? & ? & -a_{33} & -a_{34} \\ ? & ? & ? & -a_{44} \end{pmatrix},$$

with each $a_{ij} \geq 0 (i, j = 1, 2, 3, 4)$, whose digraph is Γ_1.

Our aim is to prove the existence of nonnegative $c_{21}, c_{31}, c_{32}, c_{41}, c_{42}$, and c_{43} such that the completion

$$A_C = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -c_{21} & -a_{22} & -a_{23} & -a_{24} \\ -c_{31} & -c_{32} & -a_{33} & -a_{34} \\ -c_{41} & -c_{42} & -c_{43} & -a_{44} \end{pmatrix}$$

is an N_0^1-matrix.

We may choose $c_{ij} = t \geq 0$ and show that $\det A_C[\alpha] \leq 0$ for any $\alpha \subseteq \{1, 2, 3, 4\}$. By Propositions 3.2 and 3.3, we will consider the following four different cases:

(1) $a_{11} = a_{22} = a_{33} = a_{44} = 0$ with $a_{24} = a_{34} = a_{44}$.

It is easy to prove that $\det A_C[\alpha] \leq 0$ for any $\alpha \subset \{1, 2, 3, 4\}$ by Propositions 3.2 and 3.3. $\det A_C = -a_{14}t^2 - a_{13}a_{34}t - a_{12}a_{23}a_{34} \leq 0$.

(2) $a_{11} = a_{22} = a_{33} = 0, a_{44} \neq 0$ with $a_{24} = a_{34} = a_{44}$.

It is easy to prove that $\det A_C[\alpha] \leq 0$ for any $\alpha \subset \{1, 2, 3, 4\}$ by Propositions 3.2 and 3.3. $\det A_C = -a_{14}t^2 \leq 0$.

(3) $a_{11} = a_{22} = 0, a_{33}a_{44} \neq 0$ with $a_{24} = a_{34} = a_{44}$.

We may choose $t \geq a_{33}$ and can easily prove that $\det A_C[\alpha] \leq 0$ for any $\alpha \subset \{1, 2, 3, 4\}$ by Propositions 3.2 and 3.3. $\det A_C = -a_{14}t^2 + a_{14}a_{33}t \leq 0$.

(4) $a_{11}a_{22}a_{33}a_{44} \neq 0$

It is easy to prove that all 2×2 principal minors are non-positive.

If we may choose $t \geq 0$ and large enough, then,

$$\det A_C\{1,3,4\} = -a_{14}t^2 + (-a_{13}a_{34} + a_{11}a_{34} + a_{13}a_{44} + a_{14}a_{33})t - a_{11}a_{33}a_{44} \leq 0;$$
$$\det A_C\{1,2,4\} = -a_{14}t^2 + (-a_{12}a_{24} + a_{11}a_{24} + a_{12}a_{44} + a_{14}a_{22})t - a_{11}a_{22}a_{44} \leq 0;$$
$$\det A_C\{1,2,3\} = -a_{13}t^2 + (-a_{12}a_{23} + a_{12}a_{33} + a_{11}a_{23} + a_{13}a_{22})t - a_{11}a_{22}a_{33} \leq 0;$$
$$\det A_C\{2,3,4\} = -a_{24}t^2 + (-a_{23}a_{34} + a_{23}a_{44} + a_{22}a_{34} + a_{24}a_{33})t - a_{22}a_{33}a_{44} \leq 0.$$

$\det A_C$ is a polynomial of t with the term $-a_{14}t^3$. Thus, we may make t large enough such that $\det A_C \leq 0$.

Case 2: We can assume an 4×4 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & ? \\ ? & -a_{22} & -a_{23} & -a_{24} \\ ? & ? & -a_{33} & -a_{34} \\ -a_{41} & ? & ? & -a_{44} \end{pmatrix},$$

whose digraph is Γ_2.

We will show that $\det A_C[\alpha] \leq 0$ for any $\alpha \subseteq \{1,2,3,4\}$ with the following two different cases:

(1) $a_{11} = a_{22} = a_{33} = a_{44} = 0$ or $a_{11} \neq 0, a_{22} = a_{33} = a_{44} = 0$.

We may choose "?" $= 0$. It is easy to prove that the completion

$$A_C = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & 0 \\ 0 & -a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & -a_{33} & -a_{34} \\ -a_{41} & 0 & 0 & -a_{44} \end{pmatrix}$$

of A is an N_0^1-matrix.

(2) $a_{11}a_{22} \neq 0, a_{33} = a_{44} = 0$ or $a_{11}a_{22}a_{33} \neq 0, a_{44} = 0$ or $a_{11}a_{22}a_{33}a_{44} \neq 0$.

We may choose $t > 0$ and large enough. It is easy to prove that $\det A_C[\alpha] \leq 0$ for any $\alpha \subset \{1,2,3,4\}$ by Propositions 3.2 and 3.3; $\det A_C$ is a polynomial of t with the term $-t^4$. Thus, we may make t large enough such that $\det A_C \leq 0$. □

Proposition 3.5. Let A be 5×5 non-combinatorially symmetric partial N_0^1-matrix whose digraph is a linear directed 2-tree Γ; there exists an N_0^1-matrix completion for A.

Proof: Let Γ be the 5-vertex linear directed 2-tree for $V = (V_1, V_2, V_3)$, where $V_1 = \{1,2,3\}, V_2 = \{2,3,4\}, V_3 = \{3,4,5\}$; there are the following four possibilities for the 5-vertex linear directed 2-tree, see Figure 4.

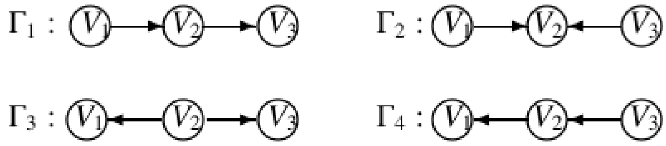

Figure 4. 5-Vertex linear directed 2-tree.

Case 1: Assume 5×5 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} & ? \\ ? & -a_{22} & -a_{23} & -a_{24} & -a_{25} \\ ? & ? & -a_{33} & -a_{34} & -a_{35} \\ ? & ? & ? & -a_{44} & -a_{45} \\ ? & ? & ? & ? & -a_{55} \end{pmatrix},$$

whose digraph is Γ_1, where each $a_{ij} > 0 (i > j, i, j = 1, 2, 3, 4, 5)$, and $a_{ii} \geq 0 (i = 1, 2, 3, 4, 5)$.

Our aim is to prove the existence of nonnegative $c_{15}, c_{21} c_{31}, c_{32}, c_{41}, c_{42}, c_{43}, c_{51}, c_{52}, c_{53}$, and c_{54} such that the completion

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} & -c_{15} \\ -c_{21} & -a_{22} & -a_{23} & -a_{24} & -a_{25} \\ -c_{31} & -c_{32} & -a_{33} & -a_{34} & -a_{35} \\ -c_{41} & -c_{42} & -c_{43} & -a_{44} & -a_{45} \\ -c_{51} & -c_{52} & -c_{53} & -c_{54} & -a_{55} \end{pmatrix}$$

is an N_0^1-matrix.

We may choose $c_{21} = c_{32} = c_{43} = c_{54} = c_{15} = t$ and large enough and another entry c_{ij}=0,

According to Proposition 3.2, all 2×2 principal minors are non-positive. According to Proposition 3.3, all 3×3 principal minors are non-positive.

Let $\alpha \subseteq \{1, 2, 3, 4, 5\}$ and $|\alpha| = k (k = 4, 5)$, $\det A_C[\alpha]$ is a polynomial of t with the term $-t^k$. Thus, we may make t large enough such that $\det A_C[\alpha] \leq 0$.

Case 2: Assume 5×5 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} & ? \\ ? & -a_{22} & -a_{23} & -a_{24} & ? \\ ? & ? & -a_{33} & -a_{34} & -a_{35} \\ ? & ? & ? & -a_{44} & -a_{45} \\ ? & -a_{52} & ? & ? & -a_{55} \end{pmatrix},$$

whose digraph is Γ_2, where each $a_{ij} > 0 (i > j, i, j = 1, 2, 3, 4, 5)$, and $a_{55} = 0$.

We may make the completion of A is

$$A_C = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} & -t \\ -t & -a_{22} & -a_{23} & -a_{24} & -t \\ -t & -t & -a_{33} & -a_{34} & -a_{35} \\ -t & -t & -t & -a_{44} & -a_{45} \\ 0 & -a_{52} & 0 & 0 & -a_{55} \end{pmatrix}.$$

If t is large enough, then it is easy to prove that $\det A_C[\alpha] \leq 0$ for any $\alpha \subset \{1, 2, 3, 4, 5\}$ by Propositions 3.2–3.4.

Let the partitioned (block) matrix $A_C = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where

$$A_1 = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -t & -a_{22} & -a_{23} & -a_{24} \\ -t & -t & -a_{33} & -a_{34} \\ -t & -t & -t & -a_{44} \end{pmatrix}$$

is nonsingular and $A_4 = -a_{55}$.

According to Definition 2.6, the Schur complement of A_1

$$A_C/A_1 = A_4 - A_3 A_1^{-1} A_2$$

is nonzearo. So,

$$rank(A_C) = rank(A_1) + rank(A_C/A_1) = 5$$

and $\det A_C[\alpha] < 0$.

Case 3: Assume 5×5 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & ? & ? \\ ? & -a_{22} & -a_{23} & -a_{24} & -a_{25} \\ ? & ? & -a_{33} & -a_{34} & -a_{35} \\ -a_{41} & ? & ? & -a_{44} & -a_{45} \\ ? & ? & ? & ? & -a_{55} \end{pmatrix},$$

whose digraph is Γ_3, where each $a_{ij} > 0 (i > j, i, j = 1, 2, 3, 4, 5)$ and $a_{55} = 0$.

We may make the completion of A is

$$A_C = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -t & -t \\ -t & -a_{22} & -a_{23} & -a_{24} & -a_{25} \\ -t & -t & -a_{33} & -a_{34} & -a_{35} \\ -a_{41} & -t & -t & -a_{44} & -a_{45} \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix}.$$

If t is large enough, then A_C is obviously N_0^1-matrix.

Case 4: Assume 5×5 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & ? & ? \\ ? & -a_{22} & -a_{23} & -a_{24} & ? \\ ? & ? & -a_{33} & -a_{34} & -a_{35} \\ -a_{41} & ? & ? & -a_{44} & -a_{45} \\ ? & -a_{52} & ? & ? & -a_{55} \end{pmatrix},$$

whose digraph is Γ_3, where each $a_{ij} > 0 (i > j, i, j = 1, 2, 3, 4, 5)$, and $a_{55} = 0$.

The proof is the same as Case 2.

Theorem 3.6. Let A be an $n \times n$ non-combinatorially symmetric partial N_0^1-matrix whose digraph is a linear directed 2-tree. Then, there exists an N_0^1-matrix completion for A.

Proof: A linear directed 2-tree Γ is sequentially constructed from these transitive oriented graphs of order 3 via articulation along arcs, see Figure 5.

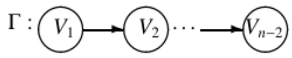

Figure 5. n-Vertex linear directed 2-tree.

We can assume an $n \times n$ partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & \cdots & ? & ? \\ -x_{21} & -a_{22} & -a_{23} & \cdots & ? & ? \\ ? & ? & -a_{33} & \cdots & ? & ? \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ? & ? & ? & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ ? & ? & ? & \cdots & ? & -a_{nn} \end{pmatrix}.$$

Our aim is to prove the existence of nonnegative c_{ij} such that the completion

$$A_C = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & \cdots & -c_{1,n-1} & -c_{1n} \\ -c_{21} & -a_{22} & -a_{23} & \cdots & -c_{2,n-1} & -c_{2n} \\ -c_{31} & -c_{32} & -a_{33} & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,1} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -c_{n1} & -c_{n2} & -c_{n3} & \cdots & -c_{n,n-1} & -a_{nn} \end{pmatrix}$$

is an N_0^1.

We may choose $c_{ij} = t$ and large enough. Our aim is to prove the existence of positive t such that the completion

$$A_C = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & \cdots & -t & -t \\ -t & -a_{22} & -a_{23} & \cdots & -t & -t \\ -t & -t & -a_{33} & \cdots & -t & -t \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -t & -t & -t & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -t & -t & -t & \cdots & -t & -a_{nn} \end{pmatrix}$$

is an N_0^1-matrix.

According to Property 3.2, all 2×2 principal minors are non-positive. According to Property 3.3, all 3×3 principal minors are non-positive.

Let $\alpha \subseteq \{1, 2, ..., n\}$ and $|\alpha| = k (k \geq 4)$; $\det A_C[\alpha]$ is a polynomial of t with the term $-t^k$. Thus, we may make t large enough such that $\det A_C[\alpha] \leq 0$. □

Example 3.7. Assume an 5×5 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -1 & -1 & -2 & -4 & ? \\ ? & -1 & -3 & -5 & -1 \\ ? & ? & -2 & -6 & -4 \\ ? & ? & ? & -4 & -3 \\ ? & ? & ? & ? & -3 \end{pmatrix},$$

whose digraph is a linear directed 2-tree. According to Theorem 3.6, we choose "?" $= -t$, and our aim is to find the positive t such that the completion of A

$$A_C = \begin{pmatrix} -1 & -1 & -2 & -4 & -10 \\ -t & -1 & -3 & -5 & -1 \\ -t & -t & -2 & -6 & -4 \\ -t & -t & -t & -4 & -3 \\ -t & -t & -t & -t & -3 \end{pmatrix}$$

is an N_0^1-matrix.

We may make $t = 10$ and obtain all the principal minors ($\det A_C[\alpha] \leq 0, \alpha \subseteq \{1,2,3,4,5\}$) of A_C.

Property 3.8. Let A be 4×4 partial N_0^1-matrix whose digraph is a nonlinear directed 2-tree with A satisfying the following conditions: $a_{11}a_{22}a_{33} = a_{13}a_{23}a_{41}$ and $a_{22}a_{33}a_{44} = a_{23}a_{24}a_{41}$. Then, there exists an N_0^1-matrix completion of A.

Proof: Suppose that the partial N_0^1-matrix is

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ ? & -a_{22} & -a_{23} & -a_{24} \\ ? & ? & -a_{33} & ? \\ -a_{41} & ? & ? & -a_{44} \end{pmatrix},$$

where each $a_{ij} (i, j = 1, 2, 3, 4)$ is nonnegative.

Our aim is to prove the existence of nonnegative $c_{21}, c_{31}, c_{32}, c_{34}, c_{42}$, and c_{43} such that the completion

$$A_c = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -c_{21} & -a_{22} & -a_{23} & -a_{24} \\ -c_{31} & -c_{32} & -a_{33} & -c_{34} \\ -a_{41} & -c_{42} & -c_{43} & -a_{44} \end{pmatrix}$$

is an N_0^1-matrix.

We will consider the following four cases:

Case 1: $a_{33} \neq 0, a_{44} \neq 0$.

We may choose $c_{31} = a_{11}a_{33}(a_{13})^{-1}$, $c_{21} = a_{11}a_{22}(a_{12})^{-1} > 0$, $c_{32} = a_{22}a_{33}(a_{23})^{-1} > 0, c_{42} = a_{22}a_{44}(a_{24})^{-1} > 0$, and $c_{34} = c_{43} = 0$; it is easy to prove that all 2×2 principal minors are non-positive. According to $a_{11}a_{22}a_{33} = a_{13}a_{23}a_{41}$ and $a_{22}a_{33}a_{44} = a_{23}a_{24}a_{41}$, we can prove $\det A_c\{1,2,3\} = 0, \det A_c\{1,2,4\} = 0, \det A_c\{2,3,4\} = 0$, and $\det A_c\{1,3,4\} = a_{11} \det A_c\{3,4\} \leq 0$, then $A_c\{2,3,4\}$ is an N_0^1-matrix. We can choose and easily prove A_c is an N_0^1-matrix.

Case 2: $a_{33} = 0, a_{44} \neq 0$.

We may choose $c_{32} = a_{24}a_{41}(a_{44})^{-1} > 0$, $c_{21} = c_{31} = c_{42} = 0$, and $c_{43}, c_{34} > 0$ and large enough. According to $a_{22}a_{33}a_{44} = a_{23}a_{34}a_{41}$ and Property 3.3, we can easily prove A_c is an N_0^1-matrix.

Case 3: $a_{33} \neq 0, a_{44} = 0$.

We may choose $c_{24} = a_{23}a_{24}(a_{33})^{-1} > 0$, $c_{21} = c_{31} = c_{42} = 0$, and $c_{43}, c_{34} > 0$ and large enough. According to $a_{22}a_{33}a_{44} = a_{23}a_{24}a_{41}$, and Property 3.3, we can easily prove A_c is an N_0^1-matrix.

Case 4: $a_{33} = 0, a_{44} = 0$.

We may choose $c_{ij} = t$ and large enough; we can easily prove A_c is an N_0^1-matrix.

Lemma 3.9. [12] Let A be an $n \times n$ matrix and D be a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$. Then, $|A + D| = |A| + \sum_i d_i A_i + \sum\sum_{i<j} d_i d_j A_{ij} + \sum\sum\sum_{i<j<k} d_i d_j d_k A_{ijk} + \cdots + d_1 d_2 \ldots d_n$. Where A_i is the determinant of the submatrix obtained by deleting the ith row and ith column. A_{ij} is the determinant obtained by the ith and jth rows and the ith and jth columns, and so on.

Theorem 3.10. Let A be an $n \times n$ $(n \geq 4)$ non-combinatorially symmetric partial N_0^1-matrix, whose digraph is a nonlinear directed 2-tree. Then, there exists an N_0^1-matrix completion of A.

Proof: The proof is by induction on n; the case in which $n = 4$ is shown in the proof of Property 3.8; assume true for $n - 1$. By permutation, we can assume that the partial N_0^1-matrix has the form

$$A = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & ? & ? & ? & \cdots & ? \\ ? & -a_{22} & \cdots & ? & ? & ? & \cdots & ? \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ ? & ? & \cdots & -a_{kk} & -a_{k,k+1} & ? & \cdots & ? \\ -a_{k+1,1} & ? & \cdots & ? & -a_{k+1,k+1} & -a_{k+1,k+2} & \cdots & ? \\ ? & ? & \cdots & ? & ? & -a_{k+2,k+2} & \cdots & ? \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & \cdots & -a_{nk} & ? & ? & \cdots & -a_{nn} \end{pmatrix}.$$

Our aim is to prove the existence of nonnegative c_{ij} such that the completion

$$A_c = \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -c_{1k} & -c_{1,k+1} & -c_{1,k+2} & \cdots & -c_{1n} \\ -c_{21} & -a_{22} & \cdots & -c_{2k} & -c_{2,k+1} & -c_{2,k+2} & \cdots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -c_{k1} & -c_{k2} & \cdots & -a_{kk} & -a_{k,k+1} & -c_{k,k+2} & \cdots & -c_{kn} \\ -a_{k+1,1} & -c_{k+1,2} & \cdots & -c_{k+1,k} & -a_{k+1,k+1} & -a_{k+1,k+2} & \cdots & -c_{k+1,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{n1} & -c_{n2} & \cdots & -a_{nk} & -c_{n,k+1} & -c_{n,k+2} & \cdots & -a_{nn} \end{pmatrix}$$

is an N_0^1 matrix.

We will complete A to an N_0^1 matrix A_c in the following four steps:

Step 1: Choose c_{2n} and c_{n2} in an appropriate way so that $A_c[\{2, n\}]$ is an N_0^1-matrix. Then, the principal submatrix

$$C = \begin{pmatrix} -a_{22} & -a_{23} & -x_{24} & \cdots & -x_{2,n-1} & -c_{2n} \\ -a_{32} & -a_{33} & -a_{34} & \cdots & -x_{3,n-1} & -x_{3n} \\ -x_{24} & -a_{43} & -a_{44} & \cdots & -x_{4,n-1} & -x_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n-1,2} & -x_{n-1,3} & -x_{n-1,4} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -c_{n2} & -x_{n3} & -x_{n4} & \cdots & -a_{n,n-1} & -a_{nn} \end{pmatrix},$$

obtained by deleting row one and column one is a partial N_0^1-matrix that specified a pattern whose graph is a nonlinear directed 2-tree with a common arc. By the induction hypothesis, C can be completed to an N_0^1-matrix.

Without loss of generality, we assume that $a_{ii} = 0$ or 1 for all i by applying Proposition 3.1.

We may choose $c_{n2} = c_{2n} = t$, and large enough, we can easily prove $C[\{2, n\}]$ is an N_0^1-matrix.

Step 2: Using the induction hypothesis C can be completed to an N_0^1-matrix, denoted by $A_c[\{2, \ldots, n\}]$.

Step 3: For $2 < i, j < n$, choose $c_{i1} = c_{i2}$ and $c_{1j} = c_{2j}a_{12}$ to obtain the completion A_c of A.

Step 4: Show A_c is an N_0^1-matrix. We must show that $\det A_c[\alpha] \le 0$ for any $\alpha \subseteq \{1, 2, \ldots, n\}$. For $1 \notin \alpha$, $A_c[\alpha]$ is a principal submatrix of the N_0^1-matrix $A_c[\{2, \ldots, n\}]$, so $\det A_c[\alpha] \le 0$. Thus, assume $1 \in \alpha$. We will consider the following four different cases:

Case 1: $a_{11} = a_{22} = 1$ with $a_{12} \ge 1$.

$$A_c = \begin{pmatrix} -1 & -a_{12} & -a_{12}a_{23} & \cdots & -a_{12}c_{2,n-1} & -t \\ -a_{21} & -1 & -a_{23} & \cdots & -c_{2,n-1} & -t/a_{12} \\ -a_{32} & -a_{32} & -a_{33} & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -t & -t & -c_{n3} & \cdots & -a_{n,n-1} & -a_{nn} \end{pmatrix}.$$

For $2 \in \alpha$:

$$\det A_c[\alpha] = (a_{12} - 1) \det A_c[\alpha - \{1\}] \le 0.$$

For $2 \notin \alpha$: $A_c[\alpha]$ can be obtained from $A_c[(\alpha - \{1\}) \cup \{2\}]$ by multiplying the first row by $a_{12} \ge 1$ and adding diag $(a_{12} - 1,\ 0, \ldots, 0)$. According to Lemma 3.9,

$$\begin{aligned} \det A_c[\alpha] &= a_{12} \det A_c[(\alpha - \{1\}) \cup \{2\}] + (a_{12} - 1) \det A_c[\alpha - \{1, 2\}] \\ &\le a_{12} A_c[(\alpha - \{1\}) \cup \{2\}] \\ &\le 0. \end{aligned}$$

Case 2: $a_{11} = 0, a_{22} = 1$.

$$A_c = \begin{pmatrix} 0 & -a_{12} & -a_{12}a_{23} & \cdots & -a_{12}c_{2,n-1} & -t \\ -a_{21} & -1 & -a_{23} & \cdots & -c_{2,n-1} & -t/a_{12} \\ -a_{32} & -a_{32} & -a_{33} & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -t & -t & -c_{n3} & \cdots & -a_{n,n-1} & -a_{nn} \end{pmatrix}.$$

For $2 \in \alpha$:

$$\det A_c[\alpha] = a_{12} \det A_c[\alpha - \{1\}] \leq 0.$$

For $2 \notin \alpha$: $A_c[\alpha]$ can be obtained from $A_c[(\alpha - \{1\}) \cup \{2\}]$ by multiplying the first row by $a_{12} > 0$ and adding diag $(a_{12}, 0, \ldots, 0)$. According to Lemma 3.9,

$$\begin{aligned}\det A_c[\alpha] &= a_{12} \det A_c[(\alpha - \{1\}) \cup \{2\}] + a_{12} \det A_c[\alpha - \{1, 2\}] \\ &\leq a_{12} \det A_c[(\alpha - \{1\}) \cup \{2\}] \\ &\leq 0.\end{aligned}$$

Case 3: $a_{11} = a_{22} = 0$.

$$A_c = \begin{pmatrix} 0 & -a_{12} & -a_{23} & \cdots & -c_{2,n-1} & -t \\ -a_{21} & 0 & -a_{23} & \cdots & -c_{2,n-1} & -t \\ -a_{32} & -a_{32} & -a_{33} & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -t & -t & -c_{n3} & \cdots & -a_{n,n-1} & -a_{nn} \end{pmatrix}.$$

For $2 \in \alpha$:

$$\det A_c[\alpha] = a_{21} \det A_c[\alpha - \{1\}] + a_{12} \det A_c[\alpha - \{1\}|\alpha - \{2\}],$$

in which $A_c[\alpha - \{1\}|\alpha - \{2\}]$ can be obtained from $A_c[\alpha - \{1\}]$ by adding diag $(-a_{21}, 0, \ldots, 0)$. According to Lemma 3.9,

$$\det A_c[\alpha] = a_{21} \det A_c[\alpha - \{1\}] + a_{12}(-a_{21} \det A_c[\alpha - \{1, 2\}] + \det A_c[\alpha - \{1\}]).$$

If $\det A_c[\alpha - \{1, 2\}] = 0$, then,

$$\det A_c[\alpha] = (1 + a_{12}) \det A_c[\alpha - \{1\}] \leq 0.$$

If $\det A_c[\alpha - \{1, 2\}] \neq 0$, it is possible to choose $\det A_c[\alpha - \{1\}] \leq a_{21} \det A_c[\alpha - \{1, 2\}]$, then $\det A_c[\alpha] \leq 0$.

For $2 \notin \alpha$:

$$\det A_c[\alpha] = \det A_c[(\alpha - \{1\}) \cup \{2\}] \leq 0.$$

Case 4: $a_{11} = 1, a_{22} = 0$ with $a_{12} \geq 1$.

$$A_c = \begin{pmatrix} -1 & -a_{12} & -a_{23} & \cdots & -c_{2,n-1} & -t \\ -a_{21} & 0 & -a_{23} & \cdots & -c_{2,n-1} & -t \\ -a_{32} & -a_{32} & -a_{33} & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -a_{n-1,n-1} & -a_{n-1,n} \\ -t & -t & -c_{n3} & \cdots & -a_{n,n-1} & -a_{nn} \end{pmatrix}.$$

For $2 \in \alpha$:

$$\det A_c[\alpha] = (a_{12} - 1)\det A_c[\alpha - \{1\}] + a_{21} \det A_c[\alpha - \{2\}|\alpha - \{1\}]$$

in which $A_c[\alpha-\{2\}|\alpha-\{1\}]$ can be obtained from $A_c[\alpha-\{1\}]$ by adding diag $(-a_{12}, 0, \ldots, 0)$. According to Lemma 3.9,

$$\det A_c[\alpha] \quad = \quad (a_{12} - 1)\det A_c[\alpha - \{1\}] + a_{21}(-a_{12}\det A_c[\alpha - \{1, 2\}] + \det A_c[\alpha - \{1\}]).$$

If $\det A_c[\alpha - \{1, 2\}] = 0$, then

$$\det A_c[\alpha] = ((a_{12} - 1)\det A_c[\alpha - \{1\}] + a_{12} \det A_c[\alpha - \{1\}] \leq 0.$$

If $\det A_c[\alpha - \{1, 2\}] \neq 0$, it is possible to choose

$$\det A_c[\alpha - \{1\}] \leq a_{21} \det A_c[\alpha - \{1, 2\}],$$

then $\det A_c[\alpha] \leq 0$.

For $2 \notin \alpha$:

$A_c[\alpha]$ can be obtained from $A_c[(\alpha - \{1\}) \cup \{2\}]$ by adding diag $(-1, 0, \ldots, 0)$, according to Lemma 3.9,

$$\det A_c[\alpha] = \det A_c[(\alpha - \{1\}) \cup \{2\}] - \det A_c[\alpha - \{1, 2\}].$$

If $\det A_c[\alpha - \{1, 2\}] = 0$, then $\det A_c[(\alpha - \{1\}) \cup \{2\}]$.

If $\det A_c[\alpha - \{1, 2\}] \neq 0$, it is possible to choose

$$\det A_c[(\alpha - \{1\}) \cup \{2\}] \leq \det A_c[\alpha - \{1, 2\}],$$

then $\det A_c[\alpha] \leq 0$.

Example 3.11. Assume an 5×5 partial non-combinatorially symmetric N_0^1-matrix

$$A = \begin{pmatrix} -1 & -2 & -3 & ? & ? \\ ? & -1 & -4 & -4 & -6 \\ ? & ? & -1 & -6 & -1 \\ -10 & ? & ? & -1 & -8 \\ ? & -10 & ? & ? & -1 \end{pmatrix},$$

whose digraph is a nonlinear directed 2-tree.

Our aim is to prove the existence of nonnegative c_{ij} such that the completion

$$A_C = \begin{pmatrix} -1 & -2 & -3 & -c_{14} & -c_{15} \\ -c_{21} & -1 & -4 & -4 & -6 \\ -c_{31} & -c_{32} & -1 & -6 & -1 \\ -10 & -c_{42} & -c_{43} & -1 & -8 \\ -c_{51} & -10 & -c_{53} & -c_{54} & -1 \end{pmatrix}$$

is an N_0^1-matrix.

First, we choose $c_{32} = c_{42} = c_{43} = c_{53} = c_{54} = 10$ so that the principal submatrix

$$A_C[\{2,3,4,5\}] = \begin{pmatrix} -1 & -4 & -4 & -6 \\ -10 & -1 & -6 & -1 \\ -10 & -10 & -1 & -8 \\ -10 & -10 & -10 & -1 \end{pmatrix}$$

is an N_0^1-matrix. $A_c[\{2, n\}]$ is an N_0^1-matrix.

According to Theorem 3.10, we choose $c_{i1} = c_{i2}, c_{1j} = c_{2j}a_{12}(i = 2, 3, 5, j = 4, 5)$, and make $c_{14} = 8, c_{15} = 12, c_{21} = 1, c_{31} = 10, c_{21} = 10$ to obtain the N_0^1-completion

$$A_C = \begin{pmatrix} -1 & -2 & -3 & -8 & -12 \\ -1 & -1 & -4 & -4 & -6 \\ -10 & -10 & -1 & -6 & -1 \\ -10 & -10 & -10 & -1 & -8 \\ -10 & -10 & -10 & -10 & -1 \end{pmatrix}$$

of A.

4. Conclusions

In this paper, we discussed the completion problem of a non-combinatorially symmetric partial N_0^1-matrix using directed graphs. We proved that a non-combinatorially incomplete matrix can be completed to an N_0^1-matrix if its specified off-diagonal entries are negative and the graph of these entries forms a directed 2-tree. This study extends previous research by considering the inherent asymmetry and missing diagonal entries in matrices, providing an approach for reconstructing complex matrices in practical applications.

Use of Generative-AI tools declaration

The author declares that she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no conflict of interest.

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