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Research article

Simultaneous characterizations of alternative partner-ruled surfaces

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Abstract: In this study, we present partner-ruled surfaces generated by the vectors of the alternative frame of a space curve in Euclidean 3-space. First, each pair of the alternative partner-ruled surfaces to be simultaneously developable and minimal is investigated based on the alternative frame by partial differential equations. Then, simultaneous characterizations of the coordinate curves of these surfaces to be asymptotic, geodesic, and lines of curvature are obtained and explicated. Finally, to illustrate the concepts, the study concludes with an example of alternative partner-ruled surfaces, featuring graphical representations.

Keywords: partner-ruled surfaces; alternative frame; developable and minimal surfaces; differential equations; geodesic curves; partial differential equations; asymptotic curves
Mathematics Subject Classification: 53A04, 53A05

1. Introduction

Ruled surfaces, initially explored by Gaspard Monge, arise by the translation of a straight line (referred to as a "ruling") along a prescribed path (referred to as a "base curve") in space. Despite extensive exploration over the years, ruled surfaces continue to be a captivating subject in differential geometry. Their straightforward and functional structure sets them apart among various surfaces, making them applicable in diverse fields such as geometric modeling, engineering, architecture, and computer-aided geometric design [1–3]. The first idea was to associate the rulings that generate the ruled surface with the tangent vectors of the base curve; in other words, tangent surfaces are formed by the tangent vectors of a base curve [4]. The exploration of their properties in differential geometry is both fascinating and essential for understanding surfaces. Smooth approximations of functions and their derivatives are considered a valuable tool in the differential geometry of curves and surfaces [5]. Subsequently, the principal normal and binormal surfaces were established and examined using the remaining elements of the Frenet frame of a curve [6, 7]. This study characterizes alternative partner-ruled surfaces obtained by the combinations of vectors in a new frame, including the unit of the Darboux vector of the base curve, since it is essential to examine the Darboux vector along a curve in order to determine how the curve rotates as it moves through space. This new frame, known as an alternative moving frame, was introduced for the exploration of a curve's differential geometric qualities in space [8] and used for finding the position vector of a general helix given its curvature and torsion [9]. This frame comprises orthonormal vectors: the principal normal vector N, the unit vector in the direction of the velocity of the principal normal C, and the unit Darboux vector W. Utilizing the alternative moving frame associated with a base curve makes it possible to derive new and distinctive ruled surfaces. The examination of ruled surfaces created by N and W was conducted [6]. Moreover, Tuncer defined the curves drawn by moment vectors of Frenet vectors and investigated the Frenet apparatus of these curves, their helical status, and whether they can be included in a fixed-width curve pair [10]. Kaya and Önder defined the alternative frame vectors. Later, they defined the CN^*-partner curve and gave some properties of this curve [11]. According to the alternative frame, Ouarab obtained the results, providing sufficient and necessary conditions to satisfy the developability and minimality of NC and $NW-$Smarandache-ruled surfaces [12]. Li et al. presented partner-ruled surfaces constructed by polynomial base curves via Frenet-like curve frame in Euclidean space and partner-ruled surfaces in Minkowski 3-space [13–15]. Also, Soukaina consulted the Darboux frame to study the simultaneous developability of partner-ruled surfaces [16].

In the comparison of our study with [16], which is the first study on forming special couples of surfaces with the same approach, the following commentaries can be expressed. In [16], these partner-ruled surfaces associated with Darboux frame vectors of a regular curve lying on an arbitrary regular surface were studied. Here the given curve has to lie on a surface. In our paper, any unit speed curve has been considered in 3-dimensional Euclidean space and does not need to lie on a surface. In addition to this difference, instead of the Darboux frame, we refer to the alternative frame consisting of the principal normal, the instantaneous unit velocity vector of the principal normal vector, and the Darboux vector. The advantage of the alternative frame is that the first and second curvatures of the curve with respect to the alternative frame represent a measure of the combined influence of both bending (curvature) and twisting (torsion) on the rotation of the frame and classify the curve as a helix, slant helix, or not. Even though the alternative invariants are more directly related to how the curve is "rotating" in space, the only potential limitation of this method is that the alternative frame of complicated regular curves might cause complexity to calculations. Regardless, consulting the alternative frame elements is worth considering for the problems on the curves and partner-ruled surfaces associated with them, emphasizing rotational. In that regard, by introducing alternative partner-ruled surfaces formed by the vectors of the alternative frame along a space curve in E^3 we establish conditions for each alternative partner-ruled surface to be simultaneously developable and minimal. Furthermore, the conditions for the coordinate curves to be simultaneously asymptotic, geodesic, and curvature lines are obtained. In the concluding part of the research, an illustrative example is presented, and the graphical representations of each alternative partner-ruled surface are demonstrated.

2. Preliminaries

Let $\alpha : I \subset R \to E^3$ be a unit speed curve in 3-dimensional Euclidean space. The unit tangent $T(u) = \alpha'(u)$, the principal normal $N(u) = \frac{\alpha''(u)}{\|\alpha''(u)\|}$, and the binormal vector $B(u) = T(u) \times N(u)$ along the curve α constitute the Frenet frame at each $u \in I$. Moreover, the functions $\kappa(u) = \|\alpha''(u)\|$ and $\tau(u) = \langle N'(u), B(u)\rangle$ are the curvature and torsion of α, respectively. As an alternative to this frame, the principal normal vector $N(u)$, the unit vector in the direction of the velocity of the principal normal vector function $C(u) = \frac{N'(u)}{\|N'(u)\|}$, and the unit Darboux vector $W(u) = \frac{\tau(u)T(u)+\kappa(u)B(u)}{\sqrt{\kappa(u)^2+\tau(u)^2}}$ of the space curves α are taken into consideration as elements of a new frame called the alternative moving frame $\{N, C, W\}$ [8]. It can be clearly expressed on the basis of this definition that the elements of the alternative frame $\{N, C, W\}$ correspond to the elements of the Frenet frame $\{T, N, B\}$ at each $u \in I$ by the relationships $T(u) = \frac{-\kappa(u)C(u)+\tau(u)W(u)}{\sqrt{\kappa(u)^2+\tau(u)^2}}$, $B(u) = \frac{\tau(u)C(u)+\kappa(u)W(u)}{\sqrt{\kappa(u)^2+\tau(u)^2}}$, and the principal normal vectors $N(u)$ are common in each frame. Furthermore, the following differential equations formulas provide the alternate moving frame derivatives:

$$\frac{dN(u)}{du} = \lambda(u) C(u), \quad \frac{dC(u)}{du} = -\lambda(u) N(u) + \mu(u) W(u), \quad \frac{dW(u)}{du} = -\mu(u) C(u), \tag{2.1}$$

such that $\lambda(u) = \sqrt{\kappa(u)^2 + \tau(u)^2}$ and $\mu(u) = \eta(u)\lambda(u)$ are the first and second curvatures of the curve, respectively, where $\eta(u) = \frac{\kappa(u)^2}{\left(\kappa(u)^2+\tau(u)^2\right)^{3/2}}\left(\frac{\tau(u)}{\kappa(u)}\right)'$.

Lemma 2.1. *Let $\{N, C, W, \lambda, \mu\}$ be the alternative apparatus of any unit speed curve α in Euclidean 3-space. Then, the curve α is a helix provided that the second curvature function $\mu(u)$ vanishes under the condition $\kappa(u) \neq 0$ at each $u \in I$. Furthermore, the curve α is a slant helix provided that the function $\eta(u)$ is a constant at each $u \in I$.*

The following list some definitions related to any surface $P(u, \upsilon)$ in Euclidean 3-space E^3:

i. The unit normal vector of $P(u, \upsilon)$ is defined by $\Delta(u, \upsilon) = \frac{P_u \times P_\upsilon}{\|P_u \times P_\upsilon\|}$, where the tangent vectors of $P(u, \upsilon)$ are $P_u = \frac{\partial P}{\partial u}$ and $P_\upsilon = \frac{\partial P}{\partial \upsilon}$.

ii. The coefficients of the first fundamental form $\mathbb{I}(u, \upsilon) = Eds^2 + 2Fdud\upsilon + Gd\upsilon^2$ are defined by partial differential equations

$$E(u, \upsilon) = \langle P_u, P_u \rangle, \; F(u, \upsilon) = \langle P_u, P_\upsilon \rangle, \; G(u, \upsilon) = \langle P_\upsilon, P_\upsilon \rangle. \tag{2.2}$$

iii. The coefficients of the second fundamental form $\mathbb{II}(u, \upsilon) = kds^2 + 2ldud\upsilon + md\upsilon^2$ are defined by partial differential equations

$$k(u, \upsilon) = \langle \Delta, P_{uu} \rangle, \; l(u, \upsilon) = \langle \Delta, P_{u\upsilon} \rangle, \; m(u, \upsilon) = \langle \Delta, P_{\upsilon\upsilon}, \rangle. \tag{2.3}$$

iv. The Gaussian and the mean curvatures of the surface $P(u, \upsilon)$ are defined by

$$K(u, \upsilon) = \frac{km - l^2}{EG - F^2} \quad \text{and} \quad H(u, \upsilon) = \frac{Em - 2El + Gk}{2(EG - F^2)}, \tag{2.4}$$

respectively. Additionally, a surface is considered minimal provided that its mean curvature H is vanishing, and developable provided that its Gaussian curvature K is vanishing at each point of the surface [12, 17].

3. Simultaneous characterizations of alternative partner-ruled surfaces

In this section, we present alternative partner-ruled surfaces formed by using pairwise two of the three orthonormal vectors N, C, and W of the alternative frame along a space curve in Euclidean 3-space.

Definition 3.1. *Let $\alpha : I \subset R \to E^3$ be a differentiable unit speed space curve and $\{N, C, W, \lambda, \mu\}$ be its alternative frame elements in E^3. The pair of ruled surfaces associated with α by N and C is parametrized by*

$$\begin{cases} P_C^N(u,\upsilon) = N(u) + \upsilon C(u), \\ P_N^C(u,\upsilon) = C(u) + \upsilon N(u), \end{cases} \tag{3.1}$$

and it is called a pair of $NC-$alternative partner-ruled surfaces in E^3.

Theorem 3.1. *A pair of $NC-$alternative partner-ruled surfaces P_C^N and P_N^C are simultaneously developable and minimal surfaces if and only if their associated curve α is a helix.*

Proof. Considering the alternative frame derivative formulas given in Eq (2.1), the partial differential equations of the first equation in the equation set (3.1) in terms of u and υ are found as

$$\left(P_C^N\right)_u = -\upsilon\lambda N + \lambda C + \upsilon\mu W, \quad \left(P_C^N\right)_\upsilon = C, \tag{3.2}$$

respectively. Then the vector product of the tangent vectors $\left(P_C^N\right)_u$ and $\left(P_C^N\right)_\upsilon$ allows us to identify the unit normal vector of P_C^N as

$$\Delta_C^N(u,\upsilon) = \frac{\left(P_C^N\right)_u \times \left(P_C^N\right)_\upsilon}{\left\|\left(P_C^N\right)_u \times \left(P_C^N\right)_\upsilon\right\|} = -\frac{\mu N + \lambda W}{\sqrt{\lambda^2 + \mu^2}}. \tag{3.3}$$

Here, it is obvious that $\lambda^2 + \mu^2 \neq 0$ at each point by the fact that $\kappa(u) \neq 0$ which guarantees that all points are singular.

By the inner product of the vectors in Eq (3.2) with respect to the rules in Eq (2.2), the coefficients E_C^N, F_C^N, and G_C^N for the ruled surface P_C^N are determined as

$$E_C^N(u,\upsilon) = \left(1 + \upsilon^2\right)\lambda^2 + \upsilon^2\mu^2, \; F_C^N(u,\upsilon) = \lambda, \; G_C^N(u,\upsilon) = 1. \tag{3.4}$$

The subsequent partial differentials of Eq (3.2) in terms of u and υ are

$$\begin{aligned} \left(P_C^N\right)_{uu} &= -\left(\lambda^2 + \upsilon\lambda'\right)N - \left(\upsilon\lambda^2 + \upsilon\mu^2 - \lambda'\right) + \left(\lambda\mu + \upsilon\mu'\right)W, \\ \left(P_C^N\right)_{u\upsilon} &= -\lambda N + \mu W, \; \left(P_C^N\right)_{\upsilon\upsilon} = 0. \end{aligned}$$

It is evident that by referring to Eq (2.3), the coefficients k_C^N, l_C^N, and m_C^N for the ruled surface are derived with the inner product of the above with the normal vector given in Eq (3.3) as

$$k_C^N(u,\upsilon) = \frac{\upsilon\left(\mu\lambda' - \lambda\mu'\right)}{\sqrt{\lambda^2 + \mu^2}}, \; l_C^N(u,\upsilon) = 0, \; m_C^N(u,\upsilon) = 0. \tag{3.5}$$

If Eqs (3.4) and (3.5) are substituted into Eq (2.4), then the obtained expressions are simplified to determine the Gaussian and the mean curvatures of the ruled surface P_C^N as

$$K_C^N(u,\upsilon)=0 \text{ and } H_C^N(u,\upsilon)=\frac{\mu\lambda'-\lambda\mu'}{2\upsilon(\lambda^2+\mu^2)^{3/2}}. \tag{3.6}$$

If we repeat the above steps, we obtain tangent vectors of the surface P_N^C as

$$\left(P_N^C\right)_u=-\lambda N+\upsilon\lambda C+\mu W,\ \left(P_N^C\right)_\upsilon=N. \tag{3.7}$$

Thus, the normal vector of P_N^C is

$$\Delta_N^C(u,\upsilon)=\frac{\left(P_N^C\right)_u\times\left(P_N^C\right)_\upsilon}{\left\|\left(P_N^C\right)_u\times\left(P_N^C\right)_\upsilon\right\|}=\frac{\mu C-\upsilon\lambda W}{\sqrt{\upsilon^2\lambda^2+\mu^2}}. \tag{3.8}$$

Here, the condition $\upsilon^2\lambda^2+\mu^2\neq 0$ exists at each point.

From Eqs (2.2), (2.3), and (3.7), the coefficients $E_N^C, F_N^C,\ G_N^C$, and $k_N^C,\ l_N^C,\ m_N^C$ for P_N^C are found as

$$E_N^C(u,\upsilon)=\left(1+\upsilon^2\right)\lambda^2+\mu^2, F_N^C(u,\upsilon)=-\lambda,\ G_N^C(u,\upsilon)=1, \tag{3.9}$$

and

$$k_N^C(u,\upsilon)=\frac{\mu\left(\upsilon\lambda'-\left(1+\upsilon^2\right)\lambda^2-\mu^2\right)-\upsilon\lambda\mu'}{\sqrt{\upsilon^2\lambda^2+\mu^2}},\ l_N^C(u,\upsilon)=\frac{\lambda\mu}{\sqrt{\upsilon^2\lambda^2+\mu^2}},\ m_N^C(u,\upsilon)=0, \tag{3.10}$$

respectively. If Eqs (3.9) and (3.10) are substituted into Eq (2.4), then K_N^C and H_N^C of P_N^C satisfy the equalities

$$K_N^C(u,\upsilon)=-\frac{\lambda^2\mu^2}{\left(\upsilon^2\lambda^2+\mu^2\right)^2} \text{ and } H_N^C(u,\upsilon)=\frac{\left(1-\upsilon^2\right)\lambda^2\mu+\upsilon\left(\mu\lambda'-\lambda\mu'\right)-\mu^3}{2(\upsilon^2\lambda^2+\mu^2)^{3/2}}. \tag{3.11}$$

Finally, Eqs (3.6) and (3.11) require the necessity and sufficiency condition of $\mu(u)=0$ at each $u\in I$ for the simultaneous developability and minimality of alternative partner-ruled surfaces. □

Theorem 3.2. *In Euclidean 3-space E^3, the $u-$coordinate curves of any pair of $NC-$alternative partner-ruled surfaces $P_C^N(u,\upsilon)$ and $P_N^C(u,\upsilon)$ are simultaneously*

i. Not geodesics.
ii. Asymptotic curves if and only if the curve α is a helix.

Proof. Assume that P_C^N and P_N^C are the pair of the $NC-$alternative partner-ruled surfaces with the alternative frame in E^3.

i. The vector products of the tangent vectors of P_C^N and P_N^C with the normal vectors of the $NC-$alternative partner-ruled surfaces are

$$\left(P_C^N\right)_{uu}\times\Delta_C^N=\frac{\lambda\left(\upsilon\left(\lambda^2+\mu^2\right)-\lambda'\right)}{\sqrt{\lambda^2+\mu^2}}N-\frac{\lambda^3+\lambda\left(\mu^2+\upsilon\lambda'\right)+\upsilon\mu\mu'}{\sqrt{\lambda^2+\mu^2}}C+\frac{\mu\left(\lambda'-\upsilon\left(\lambda^2+\mu^2\right)\right)}{\sqrt{\lambda^2+\mu^2}}W$$

and

$$\left(P_N^C\right)_{uu} \times \Delta_N^C = \frac{\upsilon\lambda\left(\lambda^2 - \upsilon\lambda'\right) - \mu\mu'}{\sqrt{\upsilon^2\lambda^2 + \mu^2}} N - \frac{\upsilon\lambda\left(\upsilon\lambda^2 + \lambda'\right)}{\sqrt{u^2\lambda^2 + \mu^2}} C - \frac{\mu\left(\upsilon\lambda^2 + \lambda'\right)}{\sqrt{u^2\lambda^2 + \mu^2}} W,$$

respectively. By the facts that $\left(P_C^N\right)_{uu} \times \Delta_C^N \neq 0$ and $\left(P_N^C\right)_{uu} \times \Delta_N^C \neq 0$ since $\kappa(u) \neq 0$, i.e., $\lambda(u) \neq 0$ at each point, the υ and $u-$coordinate curves of the $NC-$alternative partner-ruled surfaces simultaneously cannot be geodesics.

ii. The inner products of the normal vector of the $NC-$alternative partner-ruled surfaces with the second-order partial differential equations of P_C^N and P_N^C are

$$\left\langle \left(P_C^N\right)_{uu}, \Delta_C^N \right\rangle = \frac{\upsilon\left(\mu\lambda' - \lambda\mu'\right)}{\sqrt{\lambda^2 + \mu^2}}$$

and

$$\left\langle \left(P_N^C\right)_{uu}, \Delta_N^C \right\rangle = \frac{-\mu\left(\left(1 + \upsilon^2\right)\lambda^2 + \mu^2 - \upsilon\lambda'\right) - \upsilon\lambda\mu'}{\sqrt{\upsilon^2\lambda^2 + \mu^2}}.$$

Under retainment of $\lambda(u) \neq 0$ it is obvious that, $\mu(u) = 0$ if and only if $\left\langle \left(P_C^N\right)_{uu}, \Delta_C^N \right\rangle = 0$ and $\left\langle \left(P_N^C\right)_{uu}, \Delta_N^C \right\rangle = 0$. So, we can say that $u-$coordinate curves of the $NC-$alternative partner-ruled surfaces are simultaneously asymptotic curves if and only if the curve α is a helix.

□

Theorem 3.3. *In Euclidean 3-space E^3, the $\upsilon-$coordinate curves of any pair of $NC-$alternative partner-ruled surfaces $P_C^N(u, \upsilon)$ and $P_N^C(u, \upsilon)$ are simultaneously*

i. Geodesics.
ii. Asymptotic curves.

Proof. Here, we directly find $\left(P_C^N\right)_{\upsilon\upsilon}, \Delta_C^N$ and $\left(P_N^C\right)_{\upsilon\upsilon}, \Delta_N^C$ as in the aforementioned steps for the pair of $NC-$alternative partner-ruled surfaces P_C^N and P_N^C.

i. Since $\left(P_C^N\right)_{\upsilon\upsilon} \times \Delta_C^N = 0$ and $\left(P_N^C\right)_{\upsilon\upsilon} \times \Delta_N^C = 0$, $\upsilon-$coordinate curves of the $NC-$alternative partner-ruled surfaces are simultaneously geodesics.
ii. Since $\left\langle \left(P_C^N\right)_{\upsilon\upsilon}, \Delta_C^N \right\rangle = 0$ and $\left\langle \left(P_N^C\right)_{\upsilon\upsilon}, \Delta_N^C \right\rangle = 0$, $\upsilon-$coordinate curves of the $NC-$alternative partner-ruled surfaces are simultaneously asymptotic curves.

□

Theorem 3.4. *The coordinate curves of a pair of $NC-$alternative partner-ruled surfaces P_C^N and P_N^C cannot be simultaneously lines of curvature.*

Proof. From Eqs (3.4), (3.5), (3.9), (3.10), and $\lambda(u) \neq 0$ at each $u \in I$, it is easy to say that u and $\upsilon-$coordinate curves of $NC-$alternative partner-ruled surfaces are not simultaneously lines of curvature. □

Definition 3.2. *Let $\alpha : I \subset R \to E^3$ be a differentiable unit speed space curve, and let $\{N, C, W, \lambda, \mu\}$ be its alternative frame elements in E^3. The pair of ruled surfaces associated with the alternative frame of the curve α and parametrized by*

$$\begin{cases} P_W^N (u, \upsilon) = N(u) + \upsilon W(u), \\ P_N^W (u, \upsilon) = W(u) + \upsilon N(u), \end{cases} \tag{3.12}$$

is called a pair of $NW-$alternative partner-ruled surfaces in E^3.

Theorem 3.5. *In Euclidean 3-space E^3, a pair $NW-$alternative partner-ruled surfaces P_W^N and P_N^W is simultaneously developable but not minimal.*

Proof. With the aid of Eq (2.1), the partial differentiations of the first equation of Eq (3.12) with respect to u and υ are, respectively, obtained as

$$\left(P_W^N\right)_u = (\lambda - \upsilon\mu) C, \ \left(P_W^N\right)_\upsilon = W. \tag{3.13}$$

Then, the vector product of the partial derivatives of the surface P_W^N given by Eq (3.13), the normal vector of the surface P_W^N is found as follows:

$$\Delta_W^N (u, \upsilon) = \frac{\left(P_W^N\right)_u \times \left(P_W^N\right)_\upsilon}{\left\| \left(P_W^N\right)_u \times \left(P_W^N\right)_\upsilon \right\|} = N. \tag{3.14}$$

It is well known that the points of a surface are singular where $\left(P_W^N\right)_u \times \left(P_W^N\right)_\upsilon = 0$. In the rest of the paper, we restrict the surface P_W^N with the condition $\lambda > \upsilon\mu$.

The inner products of each vector in Eq (3.13) allow us to determine the first fundamental form's coefficients E_W^N, F_W^N, and G_W^N for the ruled surface P_W^N as

$$E_W^N (u, \upsilon) = (\lambda - \upsilon\mu)^2, \ F_W^N (u, \upsilon) = 0, \ G_W^N (u, \upsilon) = 1. \tag{3.15}$$

Also, if we consider the following second-order partial differential equations of Eq (3.12) in terms of u and υ

$$\left(P_W^N\right)_{uu} = \left(\upsilon\lambda\mu - \lambda^2\right) T + (\lambda' - \upsilon\mu') C + \left(\lambda\mu - \upsilon\mu^2\right) W,$$
$$\left(P_W^N\right)_{u\upsilon} = -\mu C, \ \left(P_W^N\right)_{\upsilon\upsilon} = 0,$$

and the normal vector given in Eq (3.14), we obtain the coefficients k_W^N, l_W^N, and m_W^N of the second fundamental form of the ruled surface P_W^N as

$$k_W^N (u, \upsilon) = \lambda (\upsilon\mu - \lambda), \ l_W^N (u, \upsilon) = 0, \ m_W^N (u, \upsilon) = 0. \tag{3.16}$$

Let us put Eqs (3.15) and (3.16) into Eq (2.4) and find the Gaussian and the mean curvatures of the ruled surface P_W^N as

$$K_W^N (u, \upsilon) = 0 \quad \text{and} \quad H_W^N (u, \upsilon) = \frac{\lambda}{2\upsilon\mu - 2\lambda}. \tag{3.17}$$

By following a similar manner, we differentiate the second equation of Eq (3.12) with respect to u and υ as

$$\begin{aligned}&\left(P_N^W\right)_u = (\upsilon\lambda - \mu)\,C,\ \left(P_N^W\right)_\upsilon = N,\\&\left(P_N^W\right)_{uu} = \left(\lambda\mu - \upsilon\lambda^2\right)N + (\upsilon\lambda' - \mu')\,C + \left(\upsilon\lambda\mu - \mu^2\right)W,\\&\left(P_N^W\right)_{u\upsilon} = \lambda C,\ \left(P_N^W\right)_{\upsilon\upsilon} = 0.\end{aligned} \tag{3.18}$$

Then, the vector product of the tangents of the surface P_N^W given in Eq (3.18) gives the normal vector of the surface P_N^W as

$$\Delta_N^W(u,\upsilon) = \frac{\left(P_N^W\right)_u \times \left(P_N^W\right)_\upsilon}{\left\|\left(P_N^W\right)_u \times \left(P_N^W\right)_\upsilon\right\|} = W, \tag{3.19}$$

for $\mu > \upsilon\lambda$. The coefficients E_N^W, F_N^W, G_N^W, and k_N^W, l_N^W, m_N^W for the ruled surface P_N^W are

$$E_N^W(u,\upsilon) = (\mu - \upsilon\lambda)^2,\ F_N^W(u,\upsilon) = 0,\ G_N^W(u,\upsilon) = 1 \tag{3.20}$$

and

$$k_N^W(u,\upsilon) = (\upsilon\lambda - \mu)\,\mu,\ l_N^W(u,\upsilon) = 0,\ m_N^W(u,\upsilon) = 0, \tag{3.21}$$

respectively. So, by putting Eqs (3.20) and (3.21) into Eq (2.4), the Gaussian curvature K_N^W and the mean curvatures H_N^W of the ruled surface P_N^W are calculated as

$$K_N^W(u,\upsilon) = 0 \text{ and } H_N^W(u,\upsilon) = \frac{\mu}{2\upsilon\lambda - 2\mu}. \tag{3.22}$$

Therefore, using Eqs (3.17) and (3.22), it is simple to say that $NW-$alternative partner-ruled surfaces are simultaneously developable but not simultaneously minimal by the fact that $\lambda(u) \neq 0$ at each $u \in I$. □

Theorem 3.6. *In Euclidean 3-space E^3, the $u-$coordinate curves of $NW-$alternative partner-ruled surfaces P_W^N and P_N^W are not simultaneously geodesics and not simultaneously asymptotic curves.*

Proof. The proof is carried out similarly to the proof of Theorem 3.2. □

Theorem 3.7. *In Euclidean 3-space E^3, the $\upsilon-$coordinate curves of $NW-$alternative partner-ruled surfaces P_W^N and P_N^W are simultaneously geodesics and asymptotic curves.*

Proof. It is proved in a similar method to the proof of Theorem 3.3. □

Theorem 3.8. *In Euclidean 3-space E^3, the u and $\upsilon-$coordinate curves of $NW-$alternative partner-ruled surfaces P_W^N and P_N^W are simultaneously lines of curvature.*

Proof. The proof follows the same procedure as the proof of Theorem 3.4. □

Definition 3.3. *Let $\alpha : I \subset R \to E^3$ be a differentiable unit speed space curve, and let $\{N, C, W, \lambda, \mu\}$ be its alternative frame elements in E^3. The pair of ruled surfaces associated with α by C and W is parametrized by*

$$\begin{cases}P_W^C(u,\upsilon) = C(u) + \upsilon W(u),\\P_C^W(u,\upsilon) = W(u) + \upsilon C(u),\end{cases} \tag{3.23}$$

and it is called a pair of $CW-$alternative partner-ruled surfaces in E^3.

Theorem 3.9. *In Euclidean 3-space E^3, the $CW-$alternative partner-ruled surfaces P_W^C and P_C^W are simultaneously developable surfaces if and only if their associated curve α is a helix, but they are not minimal surfaces.*

Proof. If we differentiate subsequently the first equation of Eq (3.23) with respect to u and υ, respectively, considering the alternative frame derivative formulae, we obtain

$$\begin{aligned}
&\left(P_W^C\right)_u = -\lambda N - \upsilon\mu C + \mu W,\ \left(P_W^C\right)_\upsilon = W,\\
&\left(P_W^C\right)_{uu} = (\upsilon\lambda\mu - \lambda')\, N - \left(\lambda^2 + \mu^2 + \upsilon\mu'\right) C + \left(\mu' - \upsilon\mu^2\right) W,\\
&\left(P_W^C\right)_{u\upsilon} = -\mu C,\ \left(P_W^C\right)_{\upsilon\upsilon} = 0.
\end{aligned} \tag{3.24}$$

Then, the normal vector of the surface P_W^C is found by the vector product of tangent vectors of the surface P_W^C given by Eq (3.24) as:

$$\Delta_W^C(u,\upsilon) = \frac{\left(P_W^C\right)_u \times \left(P_W^C\right)_\upsilon}{\left\|\left(P_W^C\right)_u \times \left(P_W^C\right)_\upsilon\right\|} = \frac{-\upsilon\mu N + \lambda C}{\sqrt{\lambda^2 + \upsilon^2\mu^2}}. \tag{3.25}$$

Here, it is obvious that $\lambda^2 + \upsilon^2\mu^2 \neq 0$ at each point, which means that all points are singular.

We have the coefficients of the first and second fundamental forms of the ruled surface P_W^C as follows:

$$E_W^C(u,\upsilon) = \lambda^2 + \left(1 + \upsilon^2\right)\mu^2,\ F_W^C(u,\upsilon) = \mu,\ G_W^C(u,\upsilon) = 1, \tag{3.26}$$

and

$$\begin{aligned}
k_W^C(u,\upsilon) &= \frac{\upsilon\mu\lambda' - \lambda^3 - \lambda\left(\left(1 + \upsilon^2\right)\mu^2 + \upsilon\mu'\right)}{\sqrt{\lambda^2 + \upsilon^2\mu^2}},\\
l_W^C(u,\upsilon) &= -\frac{\lambda\mu}{\sqrt{\lambda^2 + \upsilon^2\mu^2}},\ m_W^C(u,\upsilon) = 0.
\end{aligned} \tag{3.27}$$

Putting Eqs (3.26) and (3.27) into Eq (2.4) and ordinary calculations give the Gaussian curvature K_W^C and the mean curvature H_W^C of the ruled surface P_W^C as

$$K_W^C(u,\upsilon) = -\frac{\lambda^2\mu^2}{\left(\lambda^2 + \upsilon^2\mu^2\right)^2} \text{ and } H_B^N(u,\upsilon) = \frac{\upsilon\mu\lambda' - \lambda^3 + \lambda\left(\left(1 - \upsilon^2\right)\mu^2 - \upsilon\mu'\right)}{2\left(\lambda^2 + \upsilon^2\mu^2\right)^{3/2}}. \tag{3.28}$$

On the other hand, by using the alternative frame derivative formulae, the partial differentiations of the second equation of Eq (3.23) with respect to u and υ are found as

$$\left(P_C^W\right)_u = -\upsilon\lambda N - \mu C + \upsilon\mu W,\ \left(P_C^W\right)_\upsilon = C. \tag{3.29}$$

Then, the normal vector of the surface P_C^W is

$$\Delta_C^W(u,\upsilon) = \frac{\left(P_C^W\right)_u \times \left(P_C^W\right)_\upsilon}{\left\|\left(P_C^W\right)_u \times \left(P_C^W\right)_\upsilon\right\|} = -\frac{\mu N + \lambda W}{\sqrt{\lambda^2 + \mu^2}} \tag{3.30}$$

such that $\lambda^2 + \mu^2 \neq 0$. In addition, the coefficients E_C^W, F_C^W, and G_C^W for the ruled surface P_C^W are

$$E_C^W(u,\upsilon) = \mu^2 + \upsilon^2\left(\lambda^2 + \mu^2\right),\ F_C^W(u,\upsilon) = -\mu,\ G_C^W(u,\upsilon) = 1. \tag{3.31}$$

The second-order partial differential equations of P_C^W in terms of u and υ

$$\begin{aligned}\left(P_C^W\right)_{uu} &= (\lambda\mu - \upsilon\lambda')N - \left(\upsilon\lambda^2 + \upsilon\mu^2 + \mu'\right)C + \left(\upsilon\mu' - \mu^2\right)W,\\ \left(P_C^W\right)_{u\upsilon} &= -\lambda N + \mu W,\ \left(P_C^W\right)_{\upsilon\upsilon} = 0\end{aligned}$$

give the coefficients k_C^W, l_C^W, and m_C^W as

$$k_C^W(u,\upsilon) = \frac{\upsilon(\mu\lambda' - \lambda\mu')}{\sqrt{\lambda^2+\mu^2}},\ l_C^W(u,\upsilon) = 0,\ m_C^W(u,\upsilon) = 0. \tag{3.32}$$

Thus, K_C^W and H_C^W of the ruled surface P_C^W are found by substituting Eqs (3.31) and (3.32) into Eq (2.4) as follows:

$$K_C^W(u,\upsilon) = 0 \text{ and } H_C^W(u,\upsilon) = \frac{\mu\lambda' - \lambda\mu'}{2\upsilon(\lambda^2+\mu^2)^{3/2}}. \tag{3.33}$$

Consequently, from Eqs (3.28) and (3.33), it can easily be expressed that $CW-$alternative partner-ruled surfaces are simultaneously developable under the necessary and sufficient condition stated in the hypothesis, such as the curve α is a helix. Moreover, it is obvious that they are not simultaneously minimal surfaces since $\lambda(u) \neq 0$ at each $u \in I$. □

Theorem 3.10. *In Euclidean 3-space E^3, the $u-$coordinate curves of $CW-$alternative partner-ruled surfaces P_W^C and P_C^W are simultaneously neither geodesics nor asymptotic curves.*

Proof. It is proved similarly to the proof of Theorem 3.2. □

Theorem 3.11. *In Euclidean 3-space E^3, the $\upsilon-$coordinate curves of $CW-$alternative partner-ruled surfaces P_W^C and P_C^W are simultaneously geodesics and asymptotic curves.*

Proof. The proof follows the same procedure as the proof of Theorem 3.3. □

Theorem 3.12. *In Euclidean 3-space E^3, the u and $\upsilon-$coordinate curves of $CW-$alternative partner-ruled surfaces P_W^C and P_C^W are simultaneously lines of curvature if and only if their associated curve α is a helix.*

Proof. The proof follows the same procedure as the proof of Theorem 3.4. □

Example 3.1. *Let us consider the curve defined by the parametric equation*

$$\alpha(u) = \left(\frac{3}{\sqrt{2}}\cos u \sin\sqrt{2}u - 2\sin u\cos\sqrt{2}u, \frac{3}{\sqrt{2}}\cos u\cos\sqrt{2}u + 2\sin u\sin\sqrt{2}u, \frac{-1}{\sqrt{2}}\cos u\right),$$

see Figure 1.

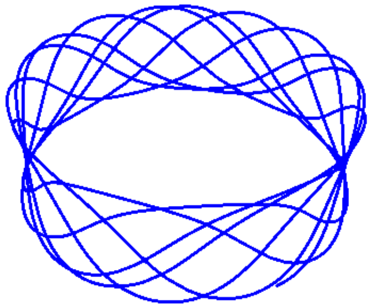


Figure 1. The curve $\alpha(u)$ for $u = (-16\pi, 16\pi)$.

The alternative frame elements of the curve $\alpha(u)$ are determined as

$$N(u) = \left(\frac{-1}{\sqrt{2}} \sin \sqrt{2}u, \frac{-1}{\sqrt{2}} \cos \sqrt{2}u, \frac{1}{\sqrt{2}}\right),$$
$$C(u) = \left(-\cos \sqrt{2}u, \sin \sqrt{2}u, 0\right),$$
$$W(u) = \left(\frac{-1}{\sqrt{2}} \sin \sqrt{2}u, \frac{-1}{\sqrt{2}} \cos \sqrt{2}u, \frac{-1}{\sqrt{2}}\right),$$
$$\lambda(u) = 1, \text{ and } \mu(u) = -1.$$

The parametric representations of the NC−alternative partner-ruled surfaces associated with the aforementioned curve α are expressed in

$$P_C^N = \left(-\upsilon \cos \sqrt{2}u - \frac{\sin \sqrt{2}u}{\sqrt{2}}, \upsilon \sin \sqrt{2}u - \frac{\cos \sqrt{2}u}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$
$$P_N^C = \left(-\cos \sqrt{2}u - \frac{\upsilon \sin \sqrt{2}u}{\sqrt{2}}, \sin \sqrt{2}u - \frac{\upsilon \cos \sqrt{2}u}{\sqrt{2}}, \frac{\upsilon}{\sqrt{2}}\right),$$

see Figure 2.

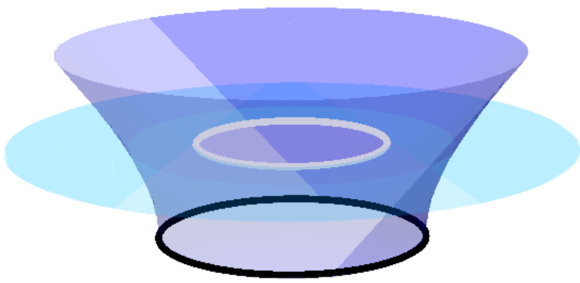


Figure 2. $NC-$alternative partner-ruled surfaces $P_C^N(u,\upsilon)$ (cyan) and $P_N^C(u,\upsilon)$ (blue), $N(u)$ (white) and $C(u)$ (black) for $u = (-\pi, \pi)$ and $\upsilon = (-2, 2)$.

Second, we give the parametric representations of the NW−alternative partner-ruled surfaces below:

$$P_W^N = \left(-\frac{(1+\upsilon)\sin \sqrt{2}u}{\sqrt{2}}, -\frac{(1+\upsilon)\cos \sqrt{2}u}{\sqrt{2}}, \frac{1-\upsilon}{\sqrt{2}}\right),$$
$$P_N^W = \left(-\frac{(1+\upsilon)\sin \sqrt{2}u}{\sqrt{2}}, -\frac{(1+\upsilon)\cos \sqrt{2}u}{\sqrt{2}}, \frac{\upsilon-1}{\sqrt{2}}\right),$$

see Figure 3.

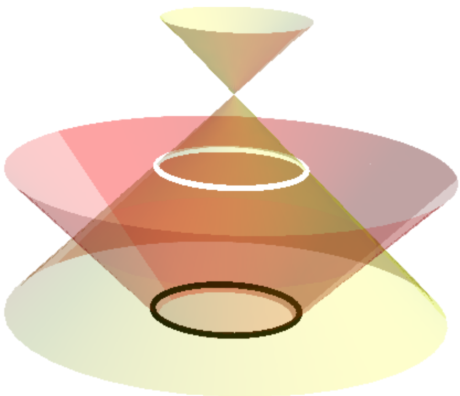


Figure 3. $NW-$alternative partner-ruled surfaces $P_W^N(u,\upsilon)$ (yellow) and $P_N^W(u,\upsilon)$ (red), $N(u)$ (white) and $W(u)$ (black) for $u=(-\pi,\ \pi)$ and $\upsilon=(-2,2)$.

Finally, we give the parametric representations of the $CW-$alternative partner-ruled surfaces as follows:

$$P_W^C=\left(-\cos\sqrt{2}u-\frac{\upsilon\sin\sqrt{2}u}{\sqrt{2}},\sin\sqrt{2}u-\frac{\upsilon\cos\sqrt{2}u}{\sqrt{2}},-\frac{\upsilon}{\sqrt{2}}\right),$$
$$P_C^W=\left(-\upsilon\cos\sqrt{2}u-\frac{\sin\sqrt{2}u}{\sqrt{2}},\upsilon\sin\sqrt{2}u-\frac{\cos\sqrt{2}u}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right),$$

see Figure 4.

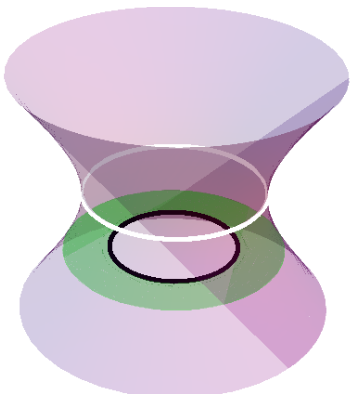


Figure 4. $CW-$alternative partner-ruled surfaces $P_W^C(u,\upsilon)$ (purple) and $P_C^W(u,\upsilon)$ (green), $C(u)$ (white) and $W(u)$ (black) for $u=(-\pi,\ \pi)$ and $\upsilon=(-2,2)$.

Example 3.2. *Now, we visualize the alternative partner-ruled surfaces of another unit speed curve with the parametric equation:*

$$\beta(u)=\left(\frac{1}{20}\cos(10u)-\frac{1}{52}\cos(26u),\ -\frac{1}{20}\sin(10u)+\frac{1}{52}\sin(26u),\ \frac{1}{8}\sin(8u)\right),$$

see Figure 5.

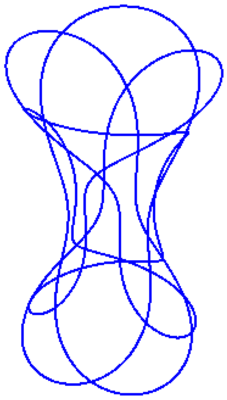


Figure 5. The curve $\beta(u)$ for $u = (-\pi, \pi)$.

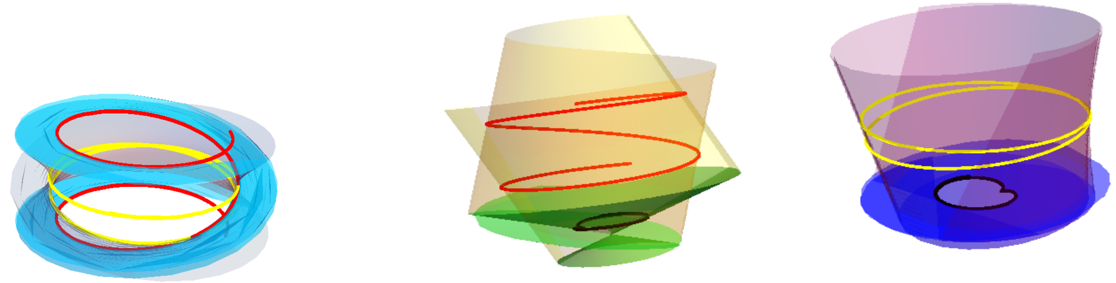
The alternative frame vectors of the curve $\beta(u)$ are found as

$$N(u) = \left(\frac{-5\cos(10u) + 13\cos(26u)}{\sqrt{226 - 162\cos(16u)}}, \frac{5\sin(10u) - 13\sin(26u)}{\sqrt{226 - 162\cos(16u)}}, -\frac{8\sin(8u)}{\sqrt{226 - 162\cos(16u)}} \right),$$

$$C(u) = \left(\frac{3645\sin(6u) + 23551\sin(10u) - 38599\sin(26u) + 9477\sin(42u)}{2\sqrt{(113 - 81\cos(16u))(3572387 - 3302532\cos(16u) + 426465\cos(32u))}}, -\frac{3645\cos(6u) - 23551\cos(10u) + 38599\cos(26u) - 9477\cos(42u)}{2(113 - 81\cos(16u))\sqrt{\frac{3572387 - 3302532\cos(16u) + 426465\cos(32u)}{113 - 81\cos(16u)}}}, -\frac{1024\cos(8u)}{\sqrt{(113 - 81\cos(16u))(3572387 - 3302532\cos(16s) + 426465\cos(32s))}} \right),$$

$$W(u) = \left(-\frac{2\sqrt{2}(45\sin(2u) - 226\sin(18u) + 117\sin(34u))}{\sqrt{3572387 - 3302532\cos(16u) + 426465\cos(32u)}}, -\frac{2\sqrt{2}(45\cos(2u) - 226\cos(18u) + 117\cos(34u))}{\sqrt{3572387 - 3302532\cos(16u) + 426465\cos(32u)}}, \frac{9\sqrt{2}(-129 + 65\cos(16u))}{\sqrt{3572387 - 3302532\cos(16u) + 426465\cos(32u)}} \right).$$

Subsequently, the parametric representations of NC, NW, and CW—alternative partner-ruled surfaces associated with β can easily be determined with respect to their definitions, such as in Example 3.1. In this manner, we plot these alternative partner-ruled surfaces for $u = \left(-\frac{\pi}{16}, \frac{\pi}{16}\right)$ and $v = (-1, 1)$ in Figures 6(a), 6(b), and 6(c), respectively.



(a) $NC-$alternative partner-ruled surfaces $P_C^N(u,\upsilon)$ (cyan) and $P_N^C(u,\upsilon)$ (gray).

(b) $NW-$alternative partner-ruled surfaces $P_W^N(u,\upsilon)$ (yellow) and $P_N^W(u,\upsilon)$ (green).

(c) $CW-$alternative partner-ruled surfaces $P_W^C(u,\upsilon)$ (purple) and $P_C^W(u,\upsilon)$ (blue).

Figure 6. Alternative partner-ruled surfaces associated with $N(u)$ (red), $C(u)$ (yellow), and $W(u)$ (black) of the curve $\beta(u)$ for $u = \left(-\frac{\pi}{16}, \frac{\pi}{16}\right)$ and $\upsilon = (-1, 1)$.

4. Conclusions

This study presents the invariants of alternative partner-ruled surfaces in Euclidean 3-space that are simultaneously produced by the vectors of any space curve in terms of the alternative frame. The coordinate curves of all alternative partner-ruled surfaces with the alternative frame are also characterized. The alternative partner-ruled surfaces' parametric equations and corresponding graphics are provided to illustrate the concepts. With the benefit of considering together ruled surfaces pairwise formed by the principal normal, the instantaneous unit velocity vector of a principal normal, and Darboux vectors and being able to comprehend the rotational properties of their associated space curve, the alternative partner-ruled surfaces and their invariants may be used in several domains where geometry, modeling, and mathematical theories, such as architectural designs or engineering of structural analysis or mechanism design, are used. For instance, in architecture, a partner-ruled surface may describe where support structures or roofs sit for curved buildings, or in the realm of mechanisms, a tool following a complementary motion may be represented by the partner-ruled surfaces. For duties like welding, painting, or scanning, this can be helpful. The results related to alternative partner-ruled surfaces may clarify the provision of complementary or conjugate motions in robotic tasks in the case of the necessity of instantaneous rotation properties.

Author contributions

Kemal Eren: Conceptualization, methodology, investigation, writing—original draft, writing-review & editing; Soley Ersoy: Conceptualization, methodology, investigation, supervision, writing-review & editing; Mohammad N. I. Khan: Conceptualization, methodology, investigation, writing-review & editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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