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Research article

Weak and strong law of large numbers for weakly negatively dependent random variables under sublinear expectations

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Abstract: In the framework of sublinear expectations, we prove the Marcinkiewicz-Zygmund type weak law of large numbers for an array of row-wise weakly negatively dependent (WND) random variables. Moreover, we obtain the strong law of large numbers for linear processes generated by WND random variables. Our theorems extend the existed achievements of the law of large numbers under sublinear expectations.

Keywords: weakly negatively dependent; weak law of large numbers; strong law of large numbers; linear processes
Mathematics Subject Classification: 60F15

1. Introduction

Since the 20th century, the probability theory has gained profound and extraordinary applications in the fields of mathematical statistics, information science, finance, and economics. The probability limit theory is an important branch of the probability theory. The probability limit theory has a broad range of applications. In the course of development, many important theorems and concepts have been proposed, such as the central limit theorem and the law of large numbers. These theorems are not only important in theory, but are also widely used in practical applications. Under the classical probability space, the mathematical expectation is additive, where one can solve many deterministic problems in real life. However, with the development of the society, many uncertainty phenomena have appeared in many new industries, such as insurance, finance, risk management, and other industries. In order to solve these uncertainty phenomena, Peng [1–4] broke away from the theoretical constraints of the classical probability space, constructed a sublinear expectation theoretical framework, and created a complete axiomatic system, which provides a new direction for solving these uncertainty problems.

Many important results and theorems in classical probability spaces can be proven and applied to the sublinear expectation spaces. Therefore, some important research directions in the classical probability space can also be extrapolated to the sublinear expectation space. More and more scholars have begun to study the related theoretical achievements under sublinear expectations. For example, Xu and Kong [5] proved the complete integral convergence and complete convergence of negatively dependent (ND) random variables under sublinear expectations. Hu and Wu [6] proved the complete convergence theorems for an array of row-wise extended negatively dependent (END) random variables utilizing truncated methods under sublinear expectations. Wang and Wu [7] used truncated methods to derive the complete convergence and complete integral convergence of the weighted sums of END random variables under sublinear expectations. In addition, many scholars have received numerous theoretical results about the law of large numbers and the law of iterated logarithms from their investigations, and have obtained many theoretical achievements under sublinear expectations. Chen [8], Hu [9, 10], Zhang [11], and Song [12] studied the strong law of large numbers for independent identically distributed (IID) random variables under different conditions. Wu et al. [13] established inequalities such as the exponential inequality, the Rosenthal inequality, and obtained the Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of m-widely acceptable random variables under sublinear expectations. Chen and Wu [14] established the weak and strong law of large numbers for Pareto-type random variables, so that the relevant conclusions in the traditional probability space were extended to the sublinear expectation space. Chen et al. [15] studied the properties associated with weakly negatively dependent (WND) random variables and established the strong law of large numbers for WND random variables under sublinear expectations. Zhang [16] studied the limit behavior of linear processes under sublinear expectations and obtained a strong law of large numbers for linear processes generated by independent random variables. Zhang [17] provided the sufficient and necessary conditions of the strong law of large numbers for IID random variables under the sub-linear expectation. Guo [18] introduced the concept of pseudo-independence under sublinear expectations and derived the weak and strong laws of large numbers. Zhang [19] established some general forms of the law of the iterated logarithms for independent random variables in a sublinear expectation space. Wu and Liu [20] studied the Chover-type law of iterated logarithms for IID random variables. Zhang [21] studied the law of iterated logarithms for sequences of END random variables with different conditions. Guo et al. [22] studied two types of Hartman-Wintner iterated logarithmic laws for pseudo-independent random variables with a finite quadratic Choquet expectation and extended the existed achievements.

The goal of this article is to prove the Marcinkiewicz-Zygmund type weak law of large numbers for an array of row-wise WND random variables, and the strong law of large numbers for linear processes generated by WND random variables under sublinear expectations. The rest of the paper is as follows: in Section 2, we recall some basic definitions, notations, and lemmas needed to prove the main theorems under sublinear expectations; in Section 3, we state our main results; in Section 4, the proofs of these theorems are given; in Section 5, we conclude the paper.

2. Preliminaries

We use the framework and notation of Peng [1–4]. Considering the following sublinear expectation space$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, if $X_1, X_2, \cdots, X_n \in \mathcal{H}$, then $\psi(X_1,\ X_2,\ \cdots,\ X_n) \in \mathcal{H}$ for each $\psi \in \mathrm{C}_{b,Lip}(\mathbb{R}^n)$, where $\mathrm{C}_{b,Lip}(\mathbb{R}^n)$ denotes the linear space of functions ψ satisfying the following bounded Lipschitz condition:

$$|\psi(x)| \leq \mathrm{C}, \quad |\psi(x) - \psi(y)| \leq \mathrm{C}\,|x - y|, \ \ \forall x,\ y \in \mathbb{R}^n,$$

where the constant C > 0 depending on ψ.

Definition 2.1. [4] A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}}$: $\mathcal{H} \to \mathbb{R}$ satisfying the following:

(a) Monotonicity: $\hat{\mathbb{E}}(X) \le \hat{\mathbb{E}}(Y)$ if $X \le Y$;

(b) Constant preserving: $\hat{\mathbb{E}}(c) = c$ for $c \in \mathbb{R}$;

(c) Sub-additivity: For each $X, Y \in \mathcal{H}$, $\hat{\mathbb{E}}(X+Y) \le \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$;

(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X)$, for $\lambda \ge 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Through a sublinear expectation $\hat{\mathbb{E}}$, we can use $\hat{\varepsilon}X = -\hat{\mathbb{E}}(-X), \forall X \in \mathcal{H}$ to define the conjugate expectation of $\hat{\mathbb{E}}$.

From the above definition, for any $X, Y \in \mathcal{H}$ we obtain the following:

$$\hat{\varepsilon}(X) \le \hat{\mathbb{E}}(X),\ \hat{\mathbb{E}}(X+c) = \hat{\mathbb{E}}(X) + c,\ \left|\hat{\mathbb{E}}(X-Y)\right| \le \hat{\mathbb{E}}|X-Y|,\ \hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y) \le \hat{\mathbb{E}}(X-Y).$$

Definition 2.2. [23] A function $\mathrm{V} : \mathcal{F} \to [0,\ 1]$ is said a capacity satisfying the following:

(a) $\mathrm{V}(\emptyset) = 0$, $\mathrm{V}(\Omega) = 1$;

(b) $\mathrm{V}(A) \le \mathrm{V}(B)$, $\forall A \subseteq B$, $A, B \in \mathcal{F}$.

It is called to be sub-additive if $\mathrm{V}(A \cup B) \le \mathrm{V}(A) + \mathrm{V}(B)$ for any A, $B \in \mathcal{F}$ with $A \cup B \in \mathcal{F}$. Let $\left(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}}\right)$ be a sub-linear expectation space; we define capacities of a pair $(\mathbb{V},\ \mathcal{V})$ by the following:

$$\mathbb{V}(A) := \inf\left\{\hat{\mathbb{E}}(\xi) : I_A \le \xi, \xi \in \mathcal{H}\right\}, \quad \mathcal{V}(A) = 1 - \mathbb{V}(A^c),\ \forall A \in \mathcal{F}.$$

From the above definition, we have the following:

$$\hat{\mathbb{E}}(f_1) \le \mathbb{V}(A) \le \hat{\mathbb{E}}(f_2),\ if\ f_1 \le I(A) \le f_2,\ f_1,\ f_2 \in \mathcal{H}. \tag{2.1}$$

Because $\mathbb{V}$ may be not countably sub-additive in general, we define another capacity $\mathbb{V}^*$.

Definition 2.3. [19] A countably sub-additive extension $\mathbb{V}^*$ of $\mathbb{V}$ is defined by the following:

$$\mathbb{V}^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mathbb{V}(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n\right\}, \mathcal{V}^*(A) = 1 - \mathbb{V}^*(A^c), A \in \mathcal{F}.$$

Then, $\mathbb{V}^*$ is a countably sub-additive capacity with $\mathbb{V}^*(A) \le \mathbb{V}(A)$ and the following properties:

(a) If $\mathbb{V}$ is countably sub-additive, then $\mathbb{V}^* \equiv \mathbb{V}$;

(b) If $I(A) \le g, g \in \mathcal{H}$, then $\mathbb{V}^*(A) \le \hat{\mathbb{E}}(g)$. Furthermore, if $\hat{\mathbb{E}}$ is countably sub-additive, then

$$\hat{\mathbb{E}}(f) \le \mathbb{V}^*(A) \le \mathbb{V}(A) \le \hat{\mathbb{E}}(g), \quad \forall f \le I(A) \le g, \quad f, g \in \mathcal{H};$$

(c) $\mathbb{V}^*$ is the largest countably sub-additive capacity satisfying the property that $\mathbb{V}^*(A) \le \hat{\mathbb{E}}(g)$whenever $I(A) \le g \in \mathcal{H}$ (i.e., if V is also a countably sub-additive capacity satisfying $\mathbb{V}(A) \le \hat{\mathbb{E}}(g)$ whenever $I(A) \le g \in \mathcal{H}$, then $\mathbb{V}(A) \le \mathbb{V}^*(A)$).

Definition 2.4. [24] In a sublinear expectation space $\left(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}}\right)$, let φ be a monotonically bounded function if for any $X, Y \in \mathcal{H}$ that satisfies

$$\hat{\mathbb{E}}[\varphi(X+Y)] \le \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\varphi(x+Y)]_{x=X}\right], \tag{2.2}$$

then the random variable Y is said to be WND on X under sublinear expectations. $\{X_i, i \in \mathbb{Z}\}$ is said to be a sequence of WND random variables if X_m is WND on $(X_{m-n}, X_{m-n+1}, \ldots, X_{m-1})$ for any $m \in \mathbb{Z}, n \in \mathbb{N}^+$.

Remark 2.1. By Chen [15], if $\{X_n, n \geq 1\}$ is a sequence of WND random variables under sublinear expectations, then for any $X_k \in \mathcal{H}, 1 \leq k \leq n$, we have the following:

$$\hat{\mathbb{E}}\left[\exp\left(\sum_{k=1}^{n} cX_k\right)\right] \leq \prod_{k=1}^{n} \hat{\mathbb{E}}\left[\exp\left(cX_k\right)\right], \quad \forall c \in \mathbb{R}. \tag{2.3}$$

Definition 2.5. [3] The Choquet integral of X with respect to V is defined as following:

$$C_{\mathrm{V}}(X) = \int_0^{\infty} \mathrm{V}(X \geq \mathrm{t})\mathrm{dt} + \int_{-\infty}^{0} \left[\mathrm{V}(X \geq t) - 1\right] \mathrm{dt}.$$

Usually, we denote the Choquet integral of $\mathbb{V}$ and $\mathcal{V}$ by $C_{\mathbb{V}}$ and $C_{\mathcal{V}}$, respectively.

Definition 2.6. [25] If a sublinear expectation $\hat{\mathbb{E}}$ satisfies $\hat{\mathbb{E}}[X] \leq \sum_{n=1}^{\infty} \hat{\mathbb{E}}[X_n] < \infty$, then $\hat{\mathbb{E}}$ is said to be countably sub-additive, where $X \leq \sum_{n=1}^{\infty} X_n < \infty$, $X, X_n \in \mathcal{H}$, and $X, X_n \geq 0, n \geq 1$.

Next, we need the following notations and lemmas. Let C be a positive constant that takes on different values in different places as needed. $I(\mathrm{A})$ stands for the indicator function of A. Given a capacity V, a set A is said to be a polar set if $V(A) = 0$. Additionally, we say a property holds "quasi-surely" (q.s.) if it holds outside a polar set. In this paper, the capacity $\mathbb{V}$ is countably sub-additive and lower continuous. Similar to Hu [10], we let Φ_c denote the set of nonnegative functions $\phi(x)$ defined on $[0, \infty)$, and $\phi(x)$ satisfies the following:

(1) Function $\phi(x)$ is positive and nondecreasing on $(0, \infty)$, and the series $\sum_{n=1}^{\infty} \frac{1}{n\phi(n)} < \infty$;

(2) For any $x > 0$ and fixed $a > 0$, there exists $C > 0$ such that $\phi(x + a) \leq C\phi(x)$.

For example, functions $(\ln(1 + x))^{1+\alpha}$ and $x^{\alpha}(\alpha > 0)$ belong to the Φ_c.

Lemma 2.1. [8] (Borel-Canteli's Lemma) Let $\{A_n, n \geq 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If $\sum_{n=1}^{\infty} \mathrm{V}(A_n) < \infty$, then

$$\mathrm{V}(A_n,\ i.o.) = 0,$$

where $\{A_n,\ i.o.\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$.

Lemma 2.2. Let $\{X, X_m, m \geq 1\}$ be a sequence of random variables under the sublinear expectations space.

(1) Chebyshev inequality [8]: Function $f(x)$ is positive and nondecreasing on $\mathbb{R}$; then

$$\mathbb{V}(X \geq x) \leq \frac{\hat{\mathbb{E}}[f(X)]}{f(x)}, \quad \mathcal{V}(X \geq x) \leq \frac{\hat{\varepsilon}[f(X)]}{f(x)}.$$

(2) $\mathrm{C_r}$ inequality [3]: Let $X_1, X_2, \cdots, X_m \in \mathcal{H}$ for $m \geq 1$; then

$$\hat{\mathbb{E}}|X_1 + X_2 + \cdots + X_m|^{\mathrm{r}} \leq \mathrm{C}_r\left[\hat{\mathbb{E}}|X_1|^{\mathrm{r}} + \hat{\mathbb{E}}|X_2|^{\mathrm{r}} + \cdots + \hat{\mathbb{E}}|X_m|^{\mathrm{r}}\right],$$

where

$$C_r = \begin{cases} 1, & 0 < r \le 1, \\ m^{r-1}, & r > 1. \end{cases}$$

(3) Markov inequality [8]: For any $\forall X \in \mathcal{H}$, we have

$$\mathbb{V}(|X| \ge x) \le \frac{\hat{\mathbb{E}}(|X|^p)}{x^p}, \quad \forall x > 0, p > 0.$$

Lemma 2.3. [26] Let $\{x_m, m \ge 1\}$ and $\{b_m, m \ge 1\}$ be sequences of real numbers with $0 < b_m \uparrow \infty$. If the series $\sum_{m=1}^{\infty} \frac{x_m}{b_m} < \infty$, then $\lim_{m\to\infty} \frac{1}{b_m} \sum_{i=1}^{m} x_i = 0$.

Lemma 2.4. [21] Suppose that $\hat{\mathbb{E}}$ is countably sub-additive; then, for any $X \in \mathcal{H}$, we have $\hat{\mathbb{E}}(|X|) \le C_{\mathbb{V}}(|X|)$.

Lemma 2.5. Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of row-wise random variables under sublinear expectation $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\sup_{i\ge1} C_{\mathbb{V}}\left((|X_{ni}|^p - c)^+\right) \to 0, c \to \infty, p \in (0, 2)$; if $\hat{\mathbb{E}}$ is countably sub-additive for any $X_{ni} \in \mathcal{H}$, then we have $\sup_{i\ge1} \hat{\mathbb{E}}\left[(|X_{ni}|^p - c)^+\right] \to 0, c \to \infty$.

Proof. From Lemma 2.4, we have $\hat{\mathbb{E}}(|X|) \le C_{\mathbb{V}}(|X|)$. Let $X = (|X_{ni}|^p - c)^+$; then, we have

$$\sup_{i\ge1} \hat{\mathbb{E}}\left[(|X_{ni}|^p - c)^+\right] \le \sup_{i\ge1} C_{\mathbb{V}}\left((|X_{ni}|^p - c)^+\right).$$

Thus, we get $\sup_{i\ge1} \hat{\mathbb{E}}\left[(|X_{ni}|^p - c)^+\right] \to 0, c \to \infty$.

Lemma 2.6. If $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ is an array of row-wise random variables under sublinear expectations, and $\sup_{i\ge1} C_{\mathbb{V}}\left((|X_{ni}|^p - c)^+\right) \to 0, c \to \infty, p \in (0, 2)$, then we have the following:

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}|^p \ge ak_n) = 0, \quad a > 0.$$

Proof. From the condition $\sup_{i\ge1} C_{\mathbb{V}}\left((|X_{ni}|^p - c)^+\right) \to 0, c \to \infty$ and the definition of a Choquet integral, it follows that for any $a > 0$, we have the following:

$$\begin{aligned}
\sum_{i=1}^{k_n} \mathbb{V}(|X_{ni}|^p \ge ak_n) &\le \frac{2}{k_n} \sum_{i=1}^{k_n} \int_{\frac{k_n}{2}}^{k_n} \mathbb{V}(|X_{ni}|^p \ge at)\, dt \\
&\le 2\sup_{i\ge1} \int_{\frac{k_n}{2}}^{k_n} \mathbb{V}(|X_{ni}|^p \ge at)\, dt \\
&\le 2\sup_{i\ge1} \int_{\frac{k_n}{2}}^{\infty} \mathbb{V}(|X_{ni}|^p \ge at)\, dt \\
&= 2\sup_{i\ge1} \int_{0}^{\infty} \mathbb{V}\left(\frac{1}{a}|X_{ni}|^p - \frac{k_n}{2} \ge t\right) dt \\
&= \frac{2}{a} \sup_{i\ge1} C_{\mathbb{V}}\left[\left(|X_{ni}|^p - \frac{ak_n}{2}\right)^+\right].
\end{aligned}$$

When $k_n \to \infty$, we obtain the following:

$$\sum_{i=1}^{k_n} \mathbb{V}\left(|X_{ni}|^{\mathrm{p}} \geq ak_n\right) \leq \frac{2}{a}\sup_{i\geq 1} C_{\mathbb{V}}\left[\left(|X_{ni}|^{\mathrm{p}} - \frac{ak_n}{2}\right)^{+}\right] \to 0.$$

Thus, the proof of $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{k_n} \mathbb{V}\left(|X_{ni}|^{\mathrm{p}} > ak_n\right) = 0$ is finished.

Lemma 2.7. [10] If $\hat{\mathbb{E}}\,|X| < \infty$, then $|X| < \infty$, q.s.$\mathbb{V}$.

Lemma 2.8. [10] Suppose $\phi(x) \in \Phi_c$; then, $\sum\limits_{n=1}^{\infty} \frac{1}{n\phi\left(\frac{n}{\ln(1+n)}\right)} < \infty$.

Proof. Since $\phi(x) \in \Phi_c$, we have $\phi\left(\frac{n}{\ln(1+n)}\right) \geq \phi(\sqrt{n})$; it is only necessary to show that $\sum\limits_{n=1}^{\infty} \frac{1}{n\phi(\sqrt{n})} < \infty$. From $\sum\limits_{n=1}^{\infty} \frac{1}{n\phi(n)} < \infty$, we obtian the following:

$$\sum_{n=1}^{\infty} \frac{1}{n\phi(\sqrt{n})} = \sum_{i=1}^{\infty} \sum_{i^2 \leq n < (i+1)^2} \frac{1}{n\phi(\sqrt{n})} \leq \sum_{i=1}^{\infty} \frac{2}{i\phi(i)} + \sum_{i=1}^{\infty} \frac{1}{i^2\phi(i)} < \infty.$$

Then, the Lemma 2.8 is proven.

Lemma 2.9. [10] If $\{\varepsilon_i, i \in \mathbb{Z}\}$ is a sequence of random variables, and there exists a constant $c > 0$ such that $|\varepsilon_n| \leq \frac{2cn}{\ln(1+n)}, \forall n \geq 1$, $\sup\limits_{i\in\mathbb{Z}} \hat{\mathbb{E}}\left[|\varepsilon_i|\phi(|\varepsilon_i|)\right] < \infty$, $\phi(x) \in \Phi_C$, and $\{\alpha_i, i \geq 0\}$ is a sequence of real numbers, $a_{n-i} = \sum\limits_{\mathrm{r}=0}^{n-i} \alpha_{\mathrm{r}}$, $\mathrm{T} = \sup\limits_{k\geq 0}|a_k| < \infty$, then for any $\mathrm{t} > 1$,

$$\sup_{1\leq i\leq n} \mathrm{t}\ln(1+n)|a_{n-i}|\hat{\mathbb{E}}\left[|\varepsilon_i| \ln\left(1 + \frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}||\varepsilon_i|\right)\right] \to 0, \quad n \to \infty. \tag{2.4}$$

Proof. Becase $|\varepsilon_n| \leq \frac{2cn}{\ln(1+n)}, \forall n \geq 1$, then

$$\begin{aligned}
&|\varepsilon_i| \ln\left(1 + \frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}||\varepsilon_i|\right)\\
=&|\varepsilon_i| \ln\left(1 + \frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}||\varepsilon_i|\right) I\left(|\varepsilon_i| \leq n^{\frac{1}{3}}\right) + |\varepsilon_i| \ln\left(1 + \frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}||\varepsilon_i|\right) I\left(n^{\frac{1}{3}} < |\varepsilon_i| \leq \frac{2cn}{\ln(1+n)}\right).
\end{aligned}$$

Let $I_1 = |\varepsilon_i| \ln\left(1 + \frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}||\varepsilon_i|\right) I\left(|\varepsilon_i| \leq n^{\frac{1}{3}}\right)$, since $\mathrm{T} = \sup\limits_{k\geq 0}|a_k| < \infty$, when $n \to \infty$, we have

$$\begin{aligned}
I_1 &\leq n^{\frac{1}{3}} \cdot \ln\left(1 + \frac{\mathrm{tT}\ln(1+n)}{n^{\frac{2}{3}}}\right)\\
&\leq \mathrm{tT}\frac{\ln(1+n)}{n^{\frac{1}{3}}}.
\end{aligned} \tag{2.5}$$

Let $I_2 = |\varepsilon_i| \ln\left(1 + \frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}||\varepsilon_i|\right) I\left(n^{\frac{1}{3}} < |\varepsilon_i| \leq \frac{2cn}{\ln(1+n)}\right)$, and $l(x) = \frac{\phi(x)}{\ln(1+x)}$; thus, we obtian the following:

$$\begin{aligned} I_2 &\leq |\varepsilon_i|\phi(|\varepsilon_i|)\frac{\ln\left(1 + \frac{\mathrm{tT}\ln(1+n)}{n} \cdot \frac{2cn}{\ln(1+n)}\right)}{\phi(n^{\frac{1}{3}})} \\ &\leq |\varepsilon_i|\phi(|\varepsilon_i|)\frac{\ln(1 + 2c\mathrm{tT})}{\phi(n^{\frac{1}{3}})} \\ &\leq |\varepsilon_i|\phi(|\varepsilon_i|)\frac{\ln(1 + 2c\mathrm{tT})}{\ln(1 + n^{\frac{1}{3}})l\left(n^{\frac{1}{3}}\right)}. \end{aligned} \tag{2.6}$$

Since $\phi(x) \in \Phi_c$, the function $l(x) = \frac{\phi(x)}{\ln(1+x)} \to \infty, x \to \infty$; then, combining (2.5) and (2.6), when $n \to \infty$, we have the following:

$$\begin{aligned} &\sup_{1\leq i\leq n} \mathrm{t}\ln(1+n)|a_{n-i}|\hat{\mathbb{E}}\left[|\varepsilon_i| \ln\left(1 + \frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}||\varepsilon_i|\right)\right] \\ \leq&(\mathrm{tT})^2\frac{(\ln(1+n))^2}{n^{\frac{1}{3}}} + \sup_{1\leq i\leq n}\hat{\mathbb{E}}\left[|\varepsilon_i|\phi(|\varepsilon_i|)\right]\frac{\mathrm{tT}\ln(1+n)\ln(1+2c\mathrm{tT})}{\ln(1+n^{\frac{1}{3}})l\left(n^{\frac{1}{3}}\right)} \\ \leq&(\mathrm{tT})^2\frac{(\ln(1+n))^2}{n^{\frac{1}{3}}} + \sup_{i\in\mathbb{Z}}\hat{\mathbb{E}}\left[|\varepsilon_i|\phi(|\varepsilon_i|)\right]\frac{\mathrm{tT}\ln(1+n)\ln(1+2c\mathrm{tT})}{\ln(1+n^{\frac{1}{3}})l\left(n^{\frac{1}{3}}\right)} \\ \to&0. \end{aligned}$$

Thus, the proof is finished.

Lemma 2.10. [16] Suppose that $\{\alpha_i, i \geq 0\}$ is a sequence of real numbers, $a_{n-i} = \sum_{\mathrm{r}=0}^{n-i} \alpha_{\mathrm{r}}$, $\mathrm{T} = \sup_{k\geq 0}|a_k| < \infty$. $\{\varepsilon_i, i \in \mathbb{Z}\}$ is a sequence of WND random variables under the sublinear expectation space $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$, $\hat{\mathbb{E}}[\varepsilon_i] = \bar{\mu}$, $\sup_{i\in\mathbb{Z}}\hat{\mathbb{E}}\left[|\varepsilon_i|\phi(|\varepsilon_i|)\right] < \infty$, $\phi(x) \in \Phi_C$, and there exists a constant $c > 0$ such that $|\varepsilon_i - \bar{\mu}| \leq \frac{2ci}{\ln(1+i)}$, $\forall i \geq 1$; then, for any $\mathrm{t} \geq 1$,

$$\sup_{n\geq 1}\hat{\mathbb{E}}\left[\exp\left(\frac{\mathrm{t}\ln(1+n)}{n}\sum_{i=1}^{n} a_{n-i}(\varepsilon_i - \bar{\mu})\right)\right] < \infty. \tag{2.7}$$

Proof. For any $x \in \mathbb{R}$, we have the inequality $\mathrm{e}^x \leq 1 + x + |x|\ln(1+|x|)\mathrm{e}^{2|x|}$. Let $x = \frac{\mathrm{t}\ln(1+n)}{n}a_{n-i}(\varepsilon_i - \bar{\mu})$; then,

$$\begin{aligned} &\exp\left(\frac{\mathrm{t}\ln(1+n)}{n}a_{n-i}(\varepsilon_i - \bar{\mu})\right) \\ \leq&1 + \frac{\mathrm{t}\ln(1+n)}{n}a_{n-i}(\varepsilon_i - \bar{\mu}) + \left|\frac{\mathrm{t}\ln(1+n)}{n}a_{n-i}(\varepsilon_i - \bar{\mu})\right| \\ &\ln\left(1 + \left|\frac{\mathrm{t}\ln(1+n)}{n}a_{n-i}(\varepsilon_i - \bar{\mu})\right|\right)\exp\left(\frac{2\mathrm{t}\ln(1+n)}{n}a_{n-i}(\varepsilon_i - \bar{\mu})\right). \end{aligned} \tag{2.8}$$

Since $\mathrm{T} = \sup_{k\geq 0}|a_k| < \infty$, for any $i \leq n$, we have the following:

$$\left|\frac{\mathrm{t}\ln(1+n)}{n}a_{n-i}\left(\varepsilon_i-\bar{\mu}\right)\right| \leq \frac{\mathrm{t}\ln(1+n)}{n}\cdot \mathrm{T}\frac{2ci}{\ln(1+i)} \leq 2c\mathrm{tT}. \tag{2.9}$$

By $\sup_{i\in\mathbb{Z}}\hat{\mathbb{E}}\left[|\varepsilon_i|\phi(|\varepsilon_i|)\right] < \infty$ and $\phi(x+a) \leq C\phi(x)$, we have the following:

$$\sup_{i\in\mathbb{Z}}\hat{\mathbb{E}}\left[|\varepsilon_i-\bar{\mu}|\,\phi\left(|\varepsilon_i-\bar{\mu}|\right)\right] \leq \sup_{i\in\mathbb{Z}}\hat{\mathbb{E}}\left[\left(|\varepsilon_i|+|\bar{\mu}|\right)\phi\left(|\varepsilon_i|+|\bar{\mu}|\right)\right] \leq C\sup_{i\in\mathbb{Z}}\hat{\mathbb{E}}\left[\left(|\varepsilon_i|+|\bar{\mu}|\right)\phi(|\varepsilon_i|)\right] < \infty.$$

Thus, $\{\varepsilon_i-\bar{\mu}, i\in\mathbb{Z}\}$satisfies the conditions of Lemma 2.9; furthermore, we have

$$\sup_{1\leq i\leq n}\frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}|\hat{\mathbb{E}}\left[|\varepsilon_i-\bar{\mu}|\ln\left(1+\frac{\mathrm{t}\ln(1+n)}{n}|a_{n-i}||\varepsilon_i-\bar{\mu}|\right)\right] \leq \frac{C}{n}. \tag{2.10}$$

Taking $\hat{\mathbb{E}}$ for both sides of (2.8) and combining (2.9) and (2.10), we have the following:

$$\hat{\mathbb{E}}\left[\exp\left(\frac{\mathrm{t}\ln(1+n)}{n}a_{n-i}(\varepsilon_i-\bar{\mu})\right)\right] \leq 1+\frac{C}{n}\mathrm{e}^{4c\mathrm{tT}} \leq \mathrm{e}^{\frac{C}{n}\mathrm{e}^{4c\mathrm{tT}}}.$$

From (2.3), we obtain the following:

$$\begin{aligned}\hat{\mathbb{E}}\left[\exp\left(\frac{\mathrm{t}\ln(1+n)}{n}\sum_{i=1}^{n}a_{n-i}\left(\varepsilon_i-\bar{\mu}\right)\right)\right] &\leq \prod_{i=1}^{n}\hat{\mathbb{E}}\left[\exp\left(\frac{\mathrm{t}\ln(1+n)}{n}a_{n-i}(\varepsilon_i-\bar{\mu})\right)\right]\\ &\leq \left(\mathrm{e}^{\frac{C}{n}\mathrm{e}^{4c\mathrm{tT}}}\right)^n\\ &\leq \mathrm{e}^{C\mathrm{e}^{4c\mathrm{tT}}} < \infty.\end{aligned}$$

3. Main results

Theorem 3.1. Let $\{k_n, n\geq 1\}$ be a sequence of positive numbers, and $\lim_{n\to\infty}k_n = \infty$. Assume that $\hat{\mathbb{E}}$ is countably sub-additive. For any $i, n\geq 1$, $\hat{\mathbb{E}}[X_{ni}] = \bar{\mu}_{ni}$, $\hat{\mathcal{E}}[X_{ni}] = \underline{\mu}_{ni}$.

(1) Let $\{X_{ni}, 1\leq i\leq k_n, n\geq 1\}$ be an array of row-wise random variables under the sublinear expectation $(\Omega,\mathcal{H},\hat{\mathbb{E}})$. Suppose that $\sup_{i\geq 1}C_{\mathbb{V}}\left((|X_{ni}|^{\mathrm{p}}-c)^+\right)\to 0, c\to\infty$ for any $\mathrm{p}\in(0,1)$; then,

$$\lim_{n\to\infty}\mathbb{V}\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\left|\sum_{i=1}^{k_n}X_{ni}\right| \geq \varepsilon\right) = 0. \tag{3.1}$$

(2) Let $\{X_{ni}, 1\leq i\leq k_n, n\geq 1\}$ be an array of row-wise WND random variables under sublinear expectation $(\Omega,\mathcal{H},\hat{\mathbb{E}})$. Suppose that $\sup_{i\geq 1}C_{\mathbb{V}}\left((|X_{ni}|^{\mathrm{p}}-c)^+\right)\to 0, c\to\infty$ for any $\mathrm{p}\in[1,2)$; then,

$$\lim_{n\to\infty}\mathbb{V}\left(\left\{\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}X_{ni} \geq \frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}\bar{\mu}_{ni}+\varepsilon\right\}\bigcup\left\{\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}X_{ni} \leq \frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}\underline{\mu}_{ni}-\varepsilon\right\}\right) = 0. \tag{3.2}$$

For a fixed $n\geq 1$ in Theorem 3.1, we obtain the Corollary 3.1.

Corollary 3.1. Assume that $\hat{\mathbb{E}}$ is countably sub-additive.

(1) Let $\{X_i, i \geq 1\}$ be a sequence of random variables under the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Suppose that $\sup\limits_{i\geq 1} C_{\mathbb{V}}\left((|X_i|^{\mathrm{p}} - c)^+\right) \to 0, c \to \infty$ for any $\mathrm{p} \in (0,1)$; then,

$$\lim_{n\to\infty} \mathbb{V}\left(\frac{1}{n^{\frac{1}{\mathrm{p}}}}\left|\sum_{i=1}^{n} X_i\right| \geq \varepsilon\right) = 0. \tag{3.3}$$

(2) Let $\{X_i, i \geq 1\}$ be a sequence of WND random variables under the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and for any $i \geq 1, \hat{\mathbb{E}}[X_i] = \bar{\mu}_i, \hat{\mathcal{E}}[X_i] = \underline{\mu}_i$. Suppose that $\sup\limits_{i\geq 1} C_{\mathbb{V}}\left((|X_i|^{\mathrm{p}} - c)^+\right) \to 0, c \to \infty$ for any $\mathrm{p} \in [1,2)$; then,

$$\lim_{n\to\infty} \mathbb{V}\left(\left\{\frac{1}{n^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{n} X_i \geq \frac{1}{n^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{n}\bar{\mu}_i + \varepsilon\right\} \bigcup \left\{\frac{1}{n^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{n} X_i \leq \frac{1}{n^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{n}\underline{\mu}_i - \varepsilon\right\}\right) = 0. \tag{3.4}$$

Theorem 3.2. Suppose that $\hat{\mathbb{E}}$ is countably sub-additive. Let $\{\alpha_i, i \geq 0\}$ be a sequence of real numbers satisfying $\sum\limits_{i=0}^{\infty} i\,|\alpha_i| < \infty, \sum\limits_{i=0}^{\infty} \alpha_i = \mathrm{A} > 0$, and $\{\varepsilon_i, i \in \mathbb{Z}\}$ be a sequence of WND random variables under sublinear expectations satisfying $\hat{\mathbb{E}}[\varepsilon_i] = \bar{\mu}, \hat{\mathcal{E}}[\varepsilon_i] = \underline{\mu}$, $\sup\limits_{i\in\mathbb{Z}} \hat{\mathbb{E}}\left[|\varepsilon_i|\phi(|\varepsilon_i|)\right] < \infty, \phi \in \Phi_C$. $\{X_{\mathrm{t}}, \mathrm{t} \geq 1\}$ is a sequence of linear processes satisfying $X_{\mathrm{t}} = \sum\limits_{i=0}^{\infty} \alpha_i \varepsilon_{\mathrm{t}-i}$. Note that $\mathrm{T}_n = \sum\limits_{\mathrm{t}=1}^{n} X_{\mathrm{t}}$; then,

$$\mathbb{V}\left(\left\{\liminf_{n\to\infty} \frac{\mathrm{T}_n}{n} < \mathrm{A}\underline{\mu}\right\} \bigcup \left\{\limsup_{n\to\infty} \frac{\mathrm{T}_n}{n} > \mathrm{A}\bar{\mu}\right\}\right) = 0. \tag{3.5}$$

Remark 3.1. Under the sub-linear expectations, the main purpose of Theorem 3.1 is to extend the range of p and improve the result of Fu [24] from the Kolmogorov type weak law of large numbers to the Marcinkiewicz-Zygmund type weak law of large numbers.

Remark 3.2. Under the sub-linear expectations, the main purpose of Theorem 3.2 is to improve the result of Zhang [16] from IID random variables to WND random variables under a more general moment condition.

4. Proof

The proof of Theorem 3.1. (1) For a fixed constant c, let $Y_{ni} = ((-c) \bigvee X_{ni}) \bigwedge c$ and $Z_{ni} = X_{ni} - Y_{ni}$. Using the C_r inequality and the Markov inequality in Lemma 2.2, we obtain the following:

$$\begin{aligned}
\mathbb{V}\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) &\leq \mathbb{V}\left(\frac{\sum\limits_{i=1}^{k_n}|Y_{ni}|}{(k_n)^{\frac{1}{\mathrm{p}}}} \geq \frac{\varepsilon}{2}\right) + \mathbb{V}\left(\frac{\sum\limits_{i=1}^{k_n}|Z_{ni}|}{(k_n)^{\frac{1}{\mathrm{p}}}} \geq \frac{\varepsilon}{2}\right) \\
&\leq \mathbb{V}\left(\frac{c}{(k_n)^{\frac{1}{\mathrm{p}}-1}} \geq \frac{\varepsilon}{2}\right) + \frac{2^{\mathrm{p}}}{k_n\varepsilon^{\mathrm{p}}}\hat{\mathbb{E}}\left[\left(\sum_{i=1}^{k_n}|Z_{ni}|\right)^{\mathrm{p}}\right]
\end{aligned}$$

$$\leq \mathbb{V}\left(\frac{c}{(k_n)^{\frac{1}{\mathrm{p}}-1}} \geq \frac{\varepsilon}{2}\right) + \frac{2^{\mathrm{p}}}{k_n \varepsilon^{\mathrm{p}}} \sum_{i=1}^{k_n} \hat{\mathbb{E}}\left[|Z_{ni}|^{\mathrm{p}}\right]$$
$$\leq \mathbb{V}\left(\frac{c}{(k_n)^{\frac{1}{\mathrm{p}}-1}} \geq \frac{\varepsilon}{2}\right) + \frac{2^{\mathrm{p}}}{\varepsilon^{\mathrm{p}}} \sup_{i\geq 1} \hat{\mathbb{E}}\left[|Z_{ni}|^{\mathrm{p}}\right].$$

Thus,

$$\lim_{n\to\infty} \mathbb{V}\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) \leq \frac{2^{\mathrm{p}}}{\varepsilon^{\mathrm{p}}} \sup_{i\geq 1} \hat{\mathbb{E}}\left[|Z_{ni}|^{\mathrm{p}}\right]. \tag{4.1}$$

Therefore,

$$\begin{aligned}
|Z_{ni}|^{p} &= |Z_{ni}|^{\mathrm{p}} I\left(|X_{ni}| \leq c\right) + |Z_{ni}|^{\mathrm{p}} I\left(|X_{ni}| \geq c\right) \\
&= |Z_{ni}|^{\mathrm{p}} I\left(X_{ni} > c\right) + |Z_{ni}|^{p} I\left(X_{ni} < -c\right) \\
&= |X_{ni} - c|^{\mathrm{p}} I\left(X_{ni} > c\right) + |X_{ni} + c|^{\mathrm{p}} I\left(X_{ni} < -c\right) \\
&\leq \left(|X_{ni}| - c\right)^{\mathrm{p}} I\left(|X_{ni}| > c\right) \\
&\leq C\left(|X_{ni}|^{\mathrm{p}} - c\right)^{+}.
\end{aligned}$$

Taking $\hat{\mathbb{E}}$ for both sides of the above inequality, when $c \to \infty$, we have the following:

$$\sup_{i\geq 1} \hat{\mathbb{E}}\left[|Z_{ni}|^{\mathrm{p}}\right] \leq C \sup_{i\geq 1} \hat{\mathbb{E}}\left(\left(|X_{ni}|^{\mathrm{p}} - c\right)^{+}\right) \leq C \sup_{i\geq 1} C_{\mathbb{V}}\left(\left(|X_{ni}|^{\mathrm{p}} - c\right)^{+}\right) \to 0. \tag{4.2}$$

Substituting (4.2) into (4.1), we get that (3.1) holds.

(2) When $1 \leq \mathrm{p} < 2$, we construct a function $\Psi(y) \in C_b^2(\mathbb{R})$; for any $\varepsilon > 0$, we have $\Psi(y) = 0$ when $y \leq 0$, $0 < \Psi(y) < 1$ when $0 < y < \varepsilon$, and $\Psi(y) = 1$ when $y \geq \varepsilon$. It is obvious that $I(y \geq \varepsilon) \leq \Psi(y)$. Let $Y_{ni} = X_{ni} - \bar{\mu}_{ni}$; then, we have the following:

$$\begin{aligned}
\mathbb{V}\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \sum_{i=1}^{k_n} Y_{ni} \geq \varepsilon\right) &\leq \hat{\mathbb{E}}\left[\Psi\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \sum_{i=1}^{k_n} Y_{ni}\right)\right] \\
&= \sum_{m=1}^{k_n} \left\{\hat{\mathbb{E}}\left[\Psi\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \sum_{i=1}^{m} Y_{ni}\right)\right] - \hat{\mathbb{E}}\left[\Psi\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \sum_{i=1}^{m-1} Y_{ni}\right)\right]\right\}.
\end{aligned} \tag{4.3}$$

Let $h(y) = \hat{\mathbb{E}}\left[\Psi\left(y + \frac{Y_{nm}}{(k_n)^{\frac{1}{\mathrm{p}}}}\right)\right]$; by Definition 2.4 and the sub-additivity of $\hat{\mathbb{E}}$, then we obtain the following:

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\Psi\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \sum_{i=1}^{m} Y_{ni}\right)\right] - \hat{\mathbb{E}}\left[\Psi\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \sum_{i=1}^{m-1} Y_{ni}\right)\right] \\
\leq &\hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[\Psi\left(y + \frac{Y_{nm}}{(k_n)^{\frac{1}{\mathrm{p}}}}\right)\right]_{y=\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \sum_{i=1}^{m-1} Y_{ni}}\right] - \hat{\mathbb{E}}\left[\Psi\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}} \sum_{i=1}^{m-1} Y_{ni}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&=\hat{\mathbb{E}}\left[h\left(\frac{1}{(k_n)^{\frac{1}{p}}}\sum_{i=1}^{m-1}Y_{ni}\right)\right]-\hat{\mathbb{E}}\left[\Psi\left(\frac{1}{(k_n)^{\frac{1}{p}}}\sum_{i=1}^{m-1}Y_{ni}\right)\right]\\
&\leq\hat{\mathbb{E}}\left[h\left(\frac{1}{(k_n)^{\frac{1}{p}}}\sum_{i=1}^{m-1}Y_{ni}\right)-\Psi\left(\frac{1}{(k_n)^{\frac{1}{p}}}\sum_{i=1}^{m-1}Y_{ni}\right)\right]\\
&\leq\sup_{y\in\mathbb{R}}\{h(y)-\Psi(y)\}\\
&=\sup_{y\in\mathbb{R}}\left\{\hat{\mathbb{E}}\left[\Psi\left(y+\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)\right]-\Psi(y)\right\}\\
&=\sup_{y\in\mathbb{R}}\hat{\mathbb{E}}\left[\Psi\left(y+\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)-\Psi(y)\right].
\end{aligned}
\tag{4.4}
$$

Let $g(x)\in C_{l,Lip}(\mathbb{R})$; for any x, we have $0\leq g(x)\leq 1$, $g(x)=1$ when $|x|\leq\mu$, and $g(x)=0$ when $|x|>1$. Then, we have the following:

$$
\begin{aligned}
I(|x|\leq\mu)\leq g(x)\leq I(|x|\leq 1),\\
I(|x|>1)\leq 1-g(x)\leq I(|x|>\mu).
\end{aligned}
\tag{4.5}
$$

For any $1\leq m\leq k_n$, there exist $\lambda_{nm},\bar{\lambda}_{nm}\in[0,1]$ such that

$$
\begin{aligned}
\Psi\left(y+\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)-\Psi(y)&=\Psi'(y)\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}+\left(\Psi'\left(y+\lambda_{nm}\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)-\Psi'(y)\right)\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}},\\
\Psi'\left(y+\lambda_{nm}\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)-\Psi'(y)&=\Psi''\left(y+\lambda_{nm}\bar{\lambda}_{nm}\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)\cdot\lambda_{nm}\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}.
\end{aligned}
\tag{4.6}
$$

Since $\Psi(y)\in C_b^2(\mathbb{R})$, then we have $|\Psi(y)|\leq\sup_{y\in\mathbb{R}}|\Psi(y)|\leq C$ $|\Psi'(y)|\leq\sup_{y\in\mathbb{R}}|\Psi'(y)|\leq C$ and $|\Psi''(y)|\leq\sup_{y\in\mathbb{R}}|\Psi''(y)|\leq C$. Combining (4.5), (4.6), and the Cr-inequality in Lemma 2.2, then for any $\delta>0$, we have the following:

$$
\begin{aligned}
&\Psi\left(y+\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)-\Psi(y)\\
\leq&\Psi'(y)\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}+\left|\Psi'\left(y+\lambda_{nm}\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)-\Psi'(y)\right|\frac{|Y_{nm}|}{(k_n)^{\frac{1}{p}}}\\
\leq&C\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}+\left|\Psi'\left(y+\lambda_{nm}\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)-\Psi'(y)\right|\cdot\frac{|Y_{nm}|}{(k_n)^{\frac{1}{p}}}I\left(|X_{nm}|>\delta(k_n)^{\frac{1}{p}}\right)\\
&+\left|\Psi''\left(y+\lambda_{nm}\bar{\lambda}_{nm}\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}\right)\right|\cdot|\lambda_{nm}|\frac{|Y_{nm}|^2}{(k_n)^{\frac{2}{p}}}I\left(|X_{nm}|\leq\delta(k_n)^{\frac{1}{p}}\right)\\
\leq&C\frac{Y_{nm}}{(k_n)^{\frac{1}{p}}}+\frac{2C}{(k_n)^{\frac{1}{p}}}\cdot|X_{nm}|I\left(|X_{nm}|>\delta(k_n)^{\frac{1}{p}}\right)+\frac{2C}{(k_n)^{\frac{1}{p}}}\cdot|\bar{\mu}_{nm}|I\left(|X_{nm}|>\delta(k_n)^{\frac{1}{p}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2C}{(k_n)^{\frac{2}{\mathrm{p}}}}\cdot|X_{nm}|^2 I\left(|X_{nm}|\le\delta(k_n)^{\frac{1}{\mathrm{p}}}\right)+\frac{2C}{(k_n)^{\frac{2}{\mathrm{p}}}}\cdot|\bar{\mu}_{nm}|^2 I\left(|X_{nm}|\le\delta(k_n)^{\frac{1}{\mathrm{p}}}\right)\\
\le& C\frac{Y_{nm}}{(k_n)^{\frac{1}{\mathrm{p}}}}+\frac{2C}{k_n\delta^{\mathrm{p}-1}}\cdot|X_{nm}|^{\mathrm{p}} I\left(|X_{nm}|>\delta(k_n)^{\frac{1}{\mathrm{p}}}\right)+\frac{2C|\bar{\mu}_{nm}|}{(k_n)^{\frac{1}{\mathrm{p}}+1}\delta^{\mathrm{p}}}\cdot|X_{nm}|^{\mathrm{p}}\\
&+\frac{2C\delta^{2-\mathrm{p}}}{k_n}\cdot|X_{nm}|^{\mathrm{p}}+\frac{2C}{(k_n)^{\frac{2}{\mathrm{p}}}}\cdot|\bar{\mu}_{nm}|^2\\
\le& C\frac{Y_{nm}}{(k_n)^{\frac{1}{\mathrm{p}}}}+\frac{2C}{k_n\delta^{\mathrm{p}-1}}\left[(|X_{nm}|^{\mathrm{p}}-k_n)^+ + k_n I\left(|X_{nm}|>\delta(k_n)^{\frac{1}{\mathrm{p}}}\right)\right]\\
&+\frac{2C|\bar{\mu}_{nm}|}{(k_n)^{\frac{1}{\mathrm{p}}+1}\delta^{\mathrm{p}}}\cdot|X_{nm}|^{\mathrm{p}}+\frac{2C\delta^{2-\mathrm{p}}}{k_n}\cdot|X_{nm}|^{\mathrm{p}}+\frac{2C}{(k_n)^{\frac{2}{\mathrm{p}}}}\cdot|\bar{\mu}_{nm}|^2\\
\le& C\frac{Y_{nm}}{(k_n)^{\frac{1}{\mathrm{p}}}}+\frac{2C}{k_n\delta^{\mathrm{p}-1}}(|X_{nm}|^{\mathrm{p}}-k_n)^+ + \frac{2C}{\delta^{\mathrm{p}-1}} I\left(|X_{nm}|>\delta(k_n)^{\frac{1}{\mathrm{p}}}\right)\\
&+\frac{2C|\bar{\mu}_{nm}|}{(k_n)^{\frac{1}{\mathrm{p}}+1}\delta^{\mathrm{p}}}\cdot|X_{nm}|^{\mathrm{p}}+\frac{2C\delta^{2-\mathrm{p}}}{k_n}\cdot|X_{nm}|^{\mathrm{p}}+\frac{2C}{(k_n)^{\frac{2}{\mathrm{p}}}}\cdot|\bar{\mu}_{nm}|^2\\
\le& C\frac{Y_{nm}}{(k_n)^{\frac{1}{\mathrm{p}}}}+\frac{2C}{k_n\delta^{\mathrm{p}-1}}(|X_{nm}|^{\mathrm{p}}-k_n)^+ + \frac{2C}{\delta^{\mathrm{p}-1}}\left(1-g\left(\frac{X_{nm}}{\delta(k_n)^{\frac{1}{\mathrm{p}}}}\right)\right)\\
&+\frac{2C|\bar{\mu}_{nm}|}{(k_n)^{\frac{1}{\mathrm{p}}+1}\delta^{\mathrm{p}}}\cdot|X_{n,m}|^{\mathrm{p}}+\frac{2C\delta^{2-\mathrm{p}}}{k_n}\cdot|X_{nm}|^{\mathrm{p}}+\frac{2C}{(k_n)^{\frac{2}{\mathrm{p}}}}\cdot|\bar{\mu}_{nm}|^2.
\end{aligned}
\tag{4.7}
$$

Substituting (4.4), (4.7), into (4.3), then combining (2.1) and (4.5), we obtain the following:

$$
\begin{aligned}
&\mathbb{V}\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}Y_{ni}\ge\varepsilon\right)\\
\le&\frac{2C}{\delta^{\mathrm{p}-1}}\sup_{m\ge1}\hat{\mathbb{E}}(|X_{nm}|^{\mathrm{p}}-k_n)^+ + \frac{2C}{\delta^{\mathrm{p}-1}}\sum_{m=1}^{k_n}\mathbb{V}(|X_{nm}|^{\mathrm{p}}>\mu^{\mathrm{p}}\delta^{\mathrm{p}}k_n)\\
&+\frac{2C|\bar{\mu}_{nm}|}{(k_n)^{\frac{1}{\mathrm{p}}}\delta^{\mathrm{p}}}\cdot\sup_{m\ge1}C_{\mathbb{V}}(|X_{nm}|^{\mathrm{p}})+2C\delta^{2-\mathrm{p}}\cdot\sup_{m\ge1}C_{\mathbb{V}}(|X_{nm}|^{\mathrm{p}})+\frac{2C}{(k_n)^{\frac{2}{\mathrm{p}}-1}}\cdot|\bar{\mu}_{nm}|^2.
\end{aligned}
$$

Taking the limit of the above inequality at both sides, then by Lemma 2.6, we obtain

$$
\lim_{n\to\infty}\mathbb{V}\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}Y_{ni}\ge\varepsilon\right)=2C\delta^{2-\mathrm{p}}\sup_{m\ge1}C_{\mathbb{V}}(|X_{nm}|^{\mathrm{p}}).
$$

Because $\sup_{m\ge1}C_{\mathbb{V}}((|X_{nm}|-c)^+)\to0, c\to\infty$ means $\sup_{m\ge1}C_{\mathbb{V}}(|X_{nm}|^{\mathrm{p}})<\infty$, and from the arbitrariness of δ, we obtain the following:

$$
\lim_{n\to\infty}\mathbb{V}\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}X_{ni}\ge\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}\bar{\mu}_{ni}+\varepsilon\right)=0. \tag{4.8}
$$

Similarly, for $\{-X_{ni}, 1\le i\le k_n, n\ge1\}$, we obtain the following:

$$
\lim_{n\to\infty}\mathbb{V}\left(\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}X_{ni}\le\frac{1}{(k_n)^{\frac{1}{\mathrm{p}}}}\sum_{i=1}^{k_n}\underline{\mu}_{ni}-\varepsilon\right)=0. \tag{4.9}
$$

Using the sub-additivity of $\mathbb{V}$ and combining (4.8) and (4.9), we obtain the following:

$$\lim_{n\to\infty}\mathbb{V}\left(\left\{\frac{1}{(k_n)^{\frac{1}{p}}}\sum_{i=1}^{k_n}X_{ni}\geq\frac{1}{(k_n)^{\frac{1}{p}}}\sum_{i=1}^{k_n}\bar{\mu}_{ni}+\varepsilon\right\}\bigcup\left\{\frac{1}{(k_n)^{\frac{1}{p}}}\sum_{i=1}^{k_n}X_{ni}\leq\frac{1}{(k_n)^{\frac{1}{p}}}\sum_{i=1}^{k_n}\underline{\mu}_{ni}-\varepsilon\right\}\right)=0.$$

The proof of Theorem 3.1 is completed.

The proof of Theorem 3.2. To prove Theorem 3.2, we only need to show that

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{\mathrm{T}_n}{n}>\mathrm{A}\bar{\mu}\right)=0, \tag{4.10}$$

and

$$\mathbb{V}\left(\liminf_{n\to\infty}\frac{\mathrm{T}_n}{n}<\mathrm{A}\underline{\mu}\right)=0. \tag{4.11}$$

First, we prove Eq (4.10); then, we need to show that

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{\mathrm{T}_n}{n}>\mathrm{A}\bar{\mu}+\epsilon\right)=0,\quad\forall\epsilon>0.$$

It is obvious that

$$\begin{aligned}\mathrm{T}_n&=\sum_{\mathrm{t}=1}^{n}X_{\mathrm{t}}\\&=\sum_{\mathrm{t}=1}^{n}\sum_{i=0}^{\infty}\alpha_i\varepsilon_{\mathrm{t}-i}\\&=\sum_{\mathrm{t}=1}^{n}\sum_{i=\mathrm{t}}^{\infty}\alpha_i\varepsilon_{\mathrm{t}-i}+\sum_{i=1}^{n}\varepsilon_i\sum_{\mathrm{t}=0}^{n-i}\alpha_{\mathrm{t}}:=N_n+M_n.\end{aligned}$$

It is only necessary to show that

$$\lim_{n\to\infty}\frac{N_n}{n}=0,\quad \text{q.s.}\mathbb{V}, \tag{4.12}$$

and

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{M_n}{n}>\mathrm{A}\bar{\mu}+\epsilon\right)=0,\quad\forall\epsilon>0. \tag{4.13}$$

To prove(4.12), we need to prove $\lim_{\mathrm{t}\to\infty}\sum_{i=\mathrm{t}}^{\infty}\alpha_i\varepsilon_{\mathrm{t}-i}=0,\text{q.s.}\mathbb{V}$.

For any $\epsilon>0$, using the Chebyshev inequality in Lemma 2.2, and the countable sub-additivity of $\hat{\mathbb{E}}$, we obtain the following:

$$
\begin{aligned}
\sum_{\mathrm{t}=1}^{\infty} \mathbb{V}\left(\left|\sum_{i=\mathrm{t}}^{\infty} \alpha_i \varepsilon_{\mathrm{t}-i}\right| > \epsilon\right) &= \sum_{\mathrm{t}=1}^{\infty} \frac{\hat{\mathbb{E}}\left[\left|\sum_{i=\mathrm{t}}^{\infty} \alpha_i \varepsilon_{\mathrm{t}-i}\right|\right]}{\epsilon} \\
&\le \frac{1}{\epsilon} \sum_{\mathrm{t}=1}^{\infty} \sum_{i=\mathrm{t}}^{\infty} |\alpha_i| \hat{\mathbb{E}} |\varepsilon_{\mathrm{t}-i}| \\
&\le \frac{1}{\epsilon} \sup_{i \in \mathbb{Z}} \hat{\mathbb{E}} |\varepsilon_i| \sum_{\mathrm{t}=1}^{\infty} \sum_{i=\mathrm{t}}^{\infty} |\alpha_i| \\
&= \frac{1}{\epsilon} \sup_{i \in \mathbb{Z}} \hat{\mathbb{E}} |\varepsilon_i| \sum_{i=1}^{\infty} i |\alpha_i| < \infty.
\end{aligned}
$$

By Lemma 2.1, it follows that

$$
\mathbb{V}\left(\limsup_{\mathrm{t}\to\infty} \left|\sum_{i=\mathrm{t}}^{\infty} \alpha_i \varepsilon_{\mathrm{t}-i}\right| > \epsilon\right) = 0.
$$

Therefore, by the arbitrariness of ϵ , it follows that

$$
\lim_{\mathrm{t}\to\infty} \sum_{i=\mathrm{t}}^{\infty} \alpha_i \varepsilon_{\mathrm{t}-i} = 0, \quad \text{q.s.}\mathbb{V}.
$$

Thus, (4.12) holds. Let $a_{n-i} = \sum_{\mathrm{r}=0}^{n-i} \alpha_{\mathrm{r}}$ and $\mathrm{T} = \sup_{k\ge 0} |a_k| < \infty$; we prove Eq (4.13) in two steps.

Step 1: If for any $i \ge 1$ we have $|\varepsilon_i - \bar{\mu}| \le \frac{2ci}{\ln(1+i)}, c > 0$, then we can directly utilize the conclusion of Lemma 2.10; for any $\mathrm{t} \ge 1$, we have the following:

$$
\sup_{n\ge 1} \hat{\mathbb{E}}\left[\exp\left(\frac{\mathrm{t}\ln(1+n)}{n} \sum_{i=1}^{n} a_{n-i} \left(\varepsilon_i - \bar{\mu}\right)\right)\right] < \infty.
$$

Since $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_{n-k}}{n} = \mathrm{A}$, then $\mathbb{V}\left(\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_{n-i}(\varepsilon_i - \bar{\mu})}{n} > \epsilon\right) = 0$ is equivalent to (4.13). Choosing a suitable t, such that $\mathrm{t} > \frac{1}{\epsilon}$, using the Chebyshev inequality in Lemma 2.2, we have the following:

$$\mathbb{V}\left(\frac{\sum_{i=1}^{n} a_{n-i}(\varepsilon_i-\bar{\mu})}{n}\geq\epsilon\right)=\mathbb{V}\left(\frac{\mathrm{t}\ln(1+n)\sum_{i=1}^{n} a_{n-i}(\varepsilon_i-\bar{\mu})}{n}\geq\epsilon \mathrm{t}\ln(1+n)\right)$$
$$\leq\frac{1}{(1+n)^{\epsilon\mathrm{t}}}\sup_{n\geq1}\hat{\mathbb{E}}\left[\exp\left(\frac{\mathrm{t}\ln(1+n)}{n}\sum_{i=1}^{n} a_{n-i}(\varepsilon_i-\bar{\mu})\right)\right].$$

By Lemma 2.10 and the convergence of infinite series $\sum_{n=1}^{\infty}\frac{1}{(1+n)^{\epsilon\mathrm{t}}}$, we obtain the following:

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{\sum_{i=1}^{n} a_{n-i}(\varepsilon_i-\bar{\mu})}{n}\geq\epsilon\right)\leq\sum_{n=1}^{\infty}\frac{1}{(1+n)^{\epsilon\mathrm{t}}}\sup_{n\geq1}\hat{\mathbb{E}}\left[\exp\left(\frac{\mathrm{t}\ln(1+n)}{n}\sum_{i=1}^{n} a_{n-i}(\varepsilon_i-\bar{\mu})\right)\right]<\infty.$$

By Lemma 2.1, it follows that

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{\sum_{i=1}^{n} a_{n-i}(\varepsilon_i-\bar{\mu})}{n}>\epsilon\right)=0.$$

Therefore, (4.13) is proven.

Step 2: Assume that $\{\varepsilon_i, i\in\mathbb{Z}\}$ only satisfies the conditions of Theorem 3.2. Let $g(x)\in C_{l,Lip}(\mathbb{R})$; for any x, we have $0\leq g(x)\leq 1$, $g(x)=1$ when $|x|\leq\mu$, and $g(x)=0$ when $|x|>1$. Then we have the following:

$$\begin{aligned} I(|x|\leq\mu)\leq g(x)\leq I(|x|\leq 1),\\ I(|x|>1)\leq 1-g(x)\leq I(|x|>\mu). \end{aligned}\tag{4.14}$$

Let $\tilde{\varepsilon}_i=-\hat{\mathbb{E}}\left[(\varepsilon_i-\bar{\mu})\,g\left(\frac{\mu(\varepsilon_i-\bar{\mu})\ln(1+i)}{i}\right)\right]+(\varepsilon_i-\bar{\mu})\,g\left(\frac{\mu(\varepsilon_i-\bar{\mu})\ln(1+i)}{i}\right)+\bar{\mu}$; for any $i\geq 1$, we have $\hat{\mathbb{E}}(\tilde{\varepsilon}_i)=\bar{\mu}$ and $|\tilde{\varepsilon}_i-\bar{\mu}|\leq\frac{2ci}{\ln(1+i)}$. Then, $\{\tilde{\varepsilon}_i, i\geq 1\}$ satisfies the conditions of Lemma 2.10. Let $\widetilde{M}_n=\sum_{i=1}^{n}a_{n-i}\tilde{\varepsilon}_i$; similar to the proof of step 1,we obtain the following:

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{\widetilde{M}_n}{n}>\mathrm{A}\bar{\mu}+\epsilon\right)=0,\quad\forall\epsilon>0.\tag{4.15}$$

By the definition of $\tilde{\varepsilon}_i$, we have the following:

$$\varepsilon_i=\tilde{\varepsilon}_i+\hat{\mathbb{E}}\left[(\varepsilon_i-\bar{\mu})\,g\left(\frac{\mu(\varepsilon_i-\bar{\mu})\ln(1+i)}{i}\right)\right]+(\varepsilon_i-\bar{\mu})\left[1-g\left(\frac{\mu(\varepsilon_i-\bar{\mu})\ln(1+i)}{i}\right)\right].$$

Since $\mathrm{T} = \sup_{k\geq 0} |a_k| < \infty$, then we have the following:

$$\begin{aligned}\frac{M_n}{n} \leq& \frac{\widetilde{M}_n}{n} + \frac{\mathrm{T}}{n}\sum_{i=1}^{n} \hat{\mathbb{E}}\left[(\varepsilon_i - \bar{\mu})\, g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right] \\ &+ \frac{\mathrm{T}}{n}\sum_{i=1}^{n} (\varepsilon_i - \bar{\mu})\left[1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right].\end{aligned} \tag{4.16}$$

Note that

$$\hat{\mathbb{E}}\left[(\varepsilon_i - \bar{\mu})\, g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right] \leq \hat{\mathbb{E}}\left[|\varepsilon_i - \bar{\mu}|\left(1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right)\right]. \tag{4.17}$$

Substituting (4.17) into (4.16), we only need to prove

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \hat{\mathbb{E}}\left[|\varepsilon_i - \bar{\mu}|\left(1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right)\right] = 0, \tag{4.18}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |\varepsilon_i - \bar{\mu}|\left[1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right] = 0, \quad \text{q.s.}\mathbb{V}. \tag{4.19}$$

By (4.14), we have the following:

$$|\varepsilon_i - \bar{\mu}|\left[1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right] \leq |\varepsilon_i - \bar{\mu}|\, I\left(|\varepsilon_i - \bar{\mu}| > \frac{i}{\ln(1+i)}\right) \leq \frac{|\varepsilon_i - \bar{\mu}|\,\phi(|\varepsilon_i - \bar{\mu}|)}{\phi\left(\frac{i}{\ln(1+i)}\right)}.$$

Then, combining $\sup_{i\in\mathbb{Z}} \hat{\mathbb{E}}\left[|\varepsilon_i - \bar{\mu}|\,\phi(|\varepsilon_i - \bar{\mu}|)\right] < \infty$ and Lemma 2.8, we obtain the following:

$$\begin{aligned}&\sum_{i=1}^{\infty} \frac{1}{i}\hat{\mathbb{E}}\left[|\varepsilon_i - \bar{\mu}|\left(1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right)\right] \\ \leq& \sup_{i\in\mathbb{Z}} \hat{\mathbb{E}}\left[|\varepsilon_i - \bar{\mu}|\phi(|\varepsilon_i - \bar{\mu}|)\right]\sum_{i=1}^{\infty} \frac{1}{i\phi\left(\frac{i}{\ln(1+i)}\right)} \\ <& \infty.\end{aligned}$$

By Lemma 2.3, (4.18) holds.

Since $\hat{\mathbb{E}}$ is countably sub-additive, we have the following:

$$\hat{\mathbb{E}}\left[\sum_{i=1}^{\infty} \frac{1}{i}|\varepsilon_i - \bar{\mu}|\left(1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right)\right] \leq \sum_{i=1}^{\infty} \frac{1}{i}\hat{\mathbb{E}}\left[|\varepsilon_i - \bar{\mu}|\left(1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right)\right] < \infty.$$

From Lemma 2.7, we obtain the following:

$$\sum_{i=1}^{\infty} \frac{1}{i}|\varepsilon_i - \bar{\mu}|\left(1 - g\left(\frac{\mu(\varepsilon_i - \bar{\mu})\ln(1+i)}{i}\right)\right) < \infty, \quad \text{q.s.}\mathbb{V}.$$

By Lemma 2.3, (4.19) holds. Combining (4.14), (4.18), and (4.19), it follows that (4.13) holds .

Similarly, for $\{-\varepsilon_i, i \in \mathbb{Z}\}$, and $\hat{\mathbb{E}}(-\varepsilon_i) = -\bar{\mu}$, we obtain the following:

$$\mathbb{V}\left(\liminf_{n\to\infty} \frac{\mathrm{T}_n}{n} < \mathrm{A}\underline{\mu}\right) = 0.$$

Using the sub-additivity of $\mathbb{V}$, the proof of Theorem 3.2 is completed.

5. Conclusions

In the framework of sublinear expectations, we established the Marcinkiewicz-Zygmund type weak law of large numbers, and the strong law of large numbers for WND random variables using the Chebyshev inequality, the C_r inequality, and so on. Theorem 3.1 extends the result of Fu [24] from the Kolmogorov type weak law of large numbers to the Marcinkiewicz-Zygmund type weak law of large numbers. Theorem 3.2 extends the result of Zhang [16] from IID random variables to WND random variables under a more general moment condition. In the future, we will try to develop broader results for other sequences of dependent random variables under sublinear expectations.

Author contributions

Yuyan Wei: conceptualization, formal analysis, investigation, methodology, writing-original draft, writing-review and editing; Xili Tan: funding acquisition, project administration, supervision; Peiyu Sun: formal analysis, writing-review and editing; Shuang Guo: writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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