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Research article

Development of a new statistical distribution with insights into mathematical properties and applications in industrial data in KSA

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Abstract: This study presents the development of a novel distribution through a transformation involving error functions, namely the error function inverse Weibull model, along with an overview of the fundamental characteristics of the proposed model. The hazard function of the recommended model is very flexible; it fits increasing, decreasing, and unimodal factors. For estimating the unknown parameters, we suggested two estimation methods, including the maximum likelihood estimation and Bayesian techniques. We perform a Monte Carlo simulation analysis to assess the stability of the parameter estimation procedure. The numerical results of these simulations show that the Bayesian technique under the square error loss function performs better than another method to obtain the model parameters. We thoroughly examine the significance of the proposed model and illustrate its application using three real-world data sets from the industrial sector. We compared the suitability and flexibility of the suggested distribution with several others, and the results showed that it fits the real-world data better than the competing models.

Keywords: Bayesian method; error function transformation; Hazard function; loss function; maximum likelihood estimation; simulation analysis
Mathematics Subject Classification: 60B12, 62G30

1. Introduction

Probability distributions are very important in modeling and fitting random phenomena in all areas of life. In the literature on distribution theory, there are various probability distributions for analyzing and predicting multiple kinds of data in many sectors, including life, biology, medical science, insurance, finance, engineering, and industry [1–9]. Based on existing findings, industrial data often exhibits a thick right tail, and many authors have developed several well-known right-skewed families. Afify et al. [10] defined the power-modified Kies-exponential distribution. Coşkun et al. [11] introduced the modified-Lindley distribution, and Gómez et al. [12] proposed the power piecewise exponential model. In addition, Dhungana and Kumar [13] proposed an exponentiated odd Lomax exponential distribution, while Hassan et al. [14] introduced the alpha power transformed extended exponential distribution. In the same line, Karakaya et al. [15], presented a unit-Lindley distribution, and Tung et al. [16] developed the Arcsine-X family of distributions.

To bring further flexibility to these generated distributions, various approaches of well-known models have been defined and used in several applied sciences to allow the smoothing parameter to vary across different locations in the data space. One of the new model-generating techniques is the error function (EF) transformation, which was first proposed by Fernández and De Andrade [17]. The cumulative distribution function (CDF) and the corresponding probability distribution function (PDF) of the EF transformation are as follows:

$$\Delta(y) = \operatorname{erf}\left(\frac{H(y)}{1 - H(y)}\right), \quad y \in \mathbb{R}, \tag{1.1}$$

and

$$\delta(y) = \frac{2h(y)}{\sqrt{\pi}(1 - H(y))^2} \exp\left\{-\left(\frac{H(y)}{1 - H(y)}\right)^2\right\}. \tag{1.2}$$

The EF transformation is a novel method for generalizing a given model, which transforms a distribution without adding any parameters. It is a modified version of traditional probability distributions for the relative importance or worth of data points. This strategy improves flexibility, allowing analysts to better explain real-world scenarios in which traditional random sampling fails to capture the underlying data structure. The derivation of the new attractive EF transformation to modify the existing distribution helps the fitting power of the existing distributions. The proposed method has many applications that extend to fitting, especially in industrial domains. However, recent works considering the EF technique, such as [18, 19].

The inverse Weibull (IW) distribution is widely used in reliability and lifetime modeling for mortality rates, especially when studying extreme events. Since it captures tail behavior effectively, it is effective in understanding the upper quantiles of life expectancy or survival time. The CDF of the IW distribution, denoted as $G(x)$, is defined as follows:

$$G(x) = e^{-\theta x^{-\beta}}, \qquad ; \; x, \; \theta, \beta > 0. \tag{1.3}$$

In reference to $G(x)$ as stated in Eq (1.3), the PDF $g(x)$ is formulated as:

$$g(x) = \theta\beta \, x^{-(\beta+1)} \, e^{-\theta x^{-\beta}}. \tag{1.4}$$

The IW model has undoubtedly established itself as a crucial tool for data modeling across nearly all sectors. However, despite its widespread use and advantages, the IW distribution is constrained by its inherent limitations. One of the primary constraints of the IW distribution is its capacity to represent solely monotonic forms of hazard functions, as it can only model situations where the hazard rate increases or decreases consistently over time. More papers have used the IW model for many different statistical models, such as the following : Alzeley et al. [20] discussed statistical inference under censored data for IW model, Hussam et al. [21] discussed fuzzy vs. traditional reliability models, Ahmad et al. [22] derived the new cotangent IW model, Mohamed et al. [23] discussed Bayesian and E-Bayesian estimation for an odd generalized exponential IW model. Abdelall et al. [24] introduced a new extension of the odd IW model. Al Mutairi et al. [25] obtained Bayesian and non-Bayesian inference based on a jointly type-II hybrid censoring model. Hassan et al. [26] discussed the statistical analysis of IW based on step-stress partially accelerated life testing. Alsadat et al. [27] presented novel Kumaraswamy power IW distribution with data analysis related to diverse scientific areas.

In this paper, we focus on providing a new form of the IW distribution for analyzing the datasets of different areas and highlighting specific characteristics. We extend this distribution by using the approach discussed in equation (1.1), and the resultant distribution is named the error function inverse Weibull (EF-IW) model. This heightened flexibility allows for a better fit to datasets with diverse kurtotic characteristics, enhancing the model's applicability across various scenarios. Further, the key objectives of the current study are as follows.

(1) The primary objective was extending the EF-IW distribution using the error function method, allowing for the derivation and investigation of its essential mathematical characteristics.
(2) The second main goal was to estimate the models' parameters using two different estimation methods, such as the maximum likelihood estimator (MLE) and Bayesian estimator, under different loss functions via Metropolis-Hastings (MH) algorithms. We conduct a detailed simulation study to demonstrate the behavior of derived estimators and pinpoint the most efficient estimation method.
(3) Two data sets from the industry field are utilized to illustrate the applicability and utilization of the proposed distribution.

The following is the organization of the study. Section 2 introduces the model description and the extension distribution, while Section 3 discusses various statistical properties such as moments, quantiles, and moment-generating functions. In Section 4, parameters are estimated using two different estimation methods. The performance of the EF-IW distribution using simulation is carried out and illustrated using three real industrial data sets in Sections 5 and 6, respectively. Finally, Section 7 presents the concluding remark of the paper.

2. Model construction

Here, we provide the inverse Weibull distribution as a classical distribution. Plugging Eqs (1.3) and (1.4) into Eqs (1.1) and (1.2) gives the CDF and PDF of the new EF-IW model:

$$\Xi(z) = \operatorname{erf}\left(\frac{e^{-\theta z^{-\beta}}}{1 - e^{-\theta z^{-\beta}}}\right), \; z, \; \theta, \; \beta > 0, \tag{2.1}$$

and

$$\xi(z) = \frac{2\theta\beta\, z^{-(\beta+1)}\, e^{-\theta z^{-\beta}}}{\sqrt{\pi}(1-e^{-\theta z^{-\beta}})^2}\ \exp\left\{-\left(\frac{e^{-\theta z^{-\beta}}}{1-e^{-\theta z^{-\beta}}}\right)^2\right\}, \tag{2.2}$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\, dz$. The plots of the EF-IW PDF for some parameter values given in Figure 1 reveal that this function can be decreasing, unimodal, and skewed depending on the parameter values.

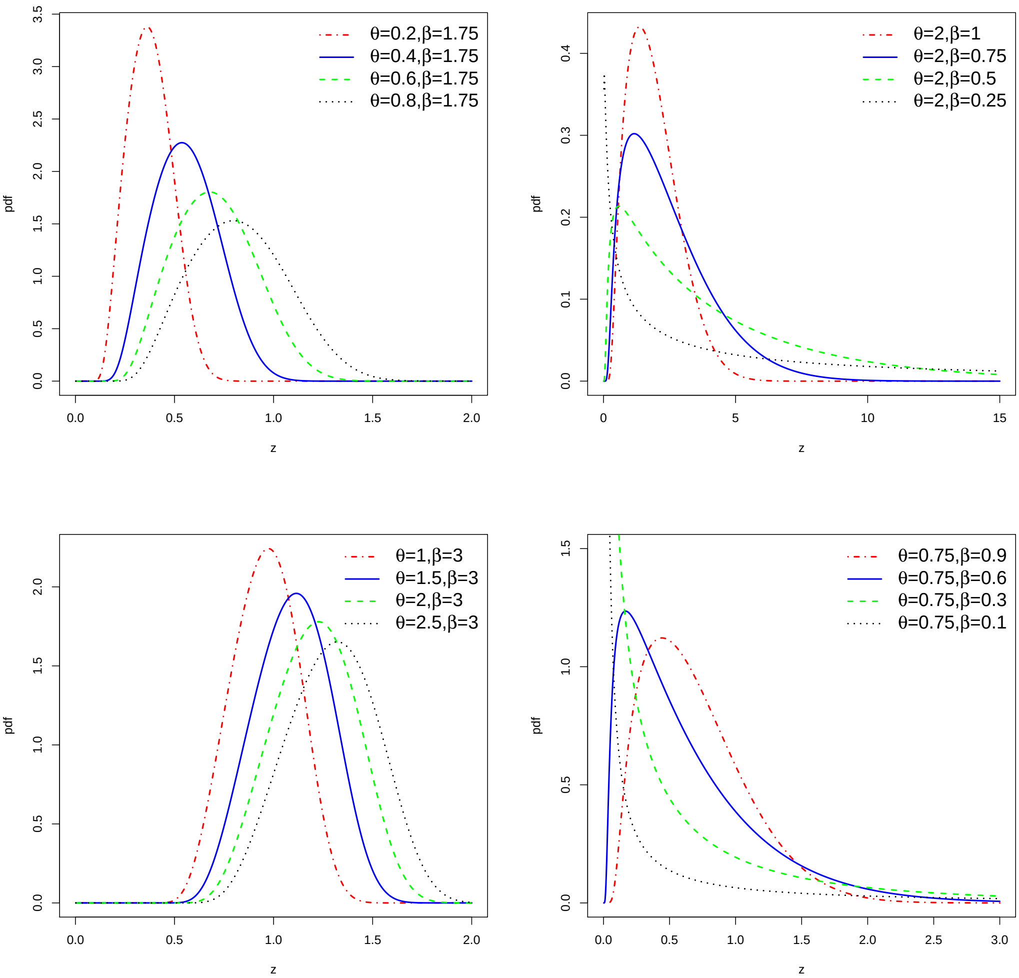

Figure 1. PDF curves for the EF-IW model based on various selected parameter values.

Suppose the random variable Z has a CDF denoted by $\Xi(z)$. Then, its survival function (SF) and hazard rate function (HRF) can then be expressed as

$$S(z) = 1 - \text{erf}\left(\frac{e^{-\theta z^{-\beta}}}{1-e^{-\theta z^{-\beta}}}\right), \tag{2.3}$$

and

$$h(z) \;=\; \frac{2\theta\beta\, z^{-(\beta+1)}\, e^{-\theta z^{-\beta}}}{\sqrt{\pi}(1-e^{-\theta z^{-\beta}})^2\,[1-\operatorname{erf}(t)]}\,\exp\{-t\}, \tag{2.4}$$

with $t=\left(\frac{e^{-\theta z^{-\beta}}}{1-e^{-\theta z^{-\beta}}}\right)^2$.

Next, the cumulative hazard rate function (CHRF) and reversed hazard rate function (RHRF) of the random variable Z can be expressed as

$$H(z) = -\log\left[1-\operatorname{erf}\left(\frac{e^{-\theta z^{-\beta}}}{1-e^{-\theta z^{-\beta}}}\right)\right], \tag{2.5}$$

and

$$R(z) = \frac{2\theta\beta\, z^{-(\beta+1)}\, e^{-\theta z^{-\beta}}}{\sqrt{\pi}(1-e^{-\theta z^{-\beta}})^2\operatorname{erf}\left(\frac{e^{-\theta z^{-\beta}}}{1-e^{-\theta z^{-\beta}}}\right)}. \tag{2.6}$$

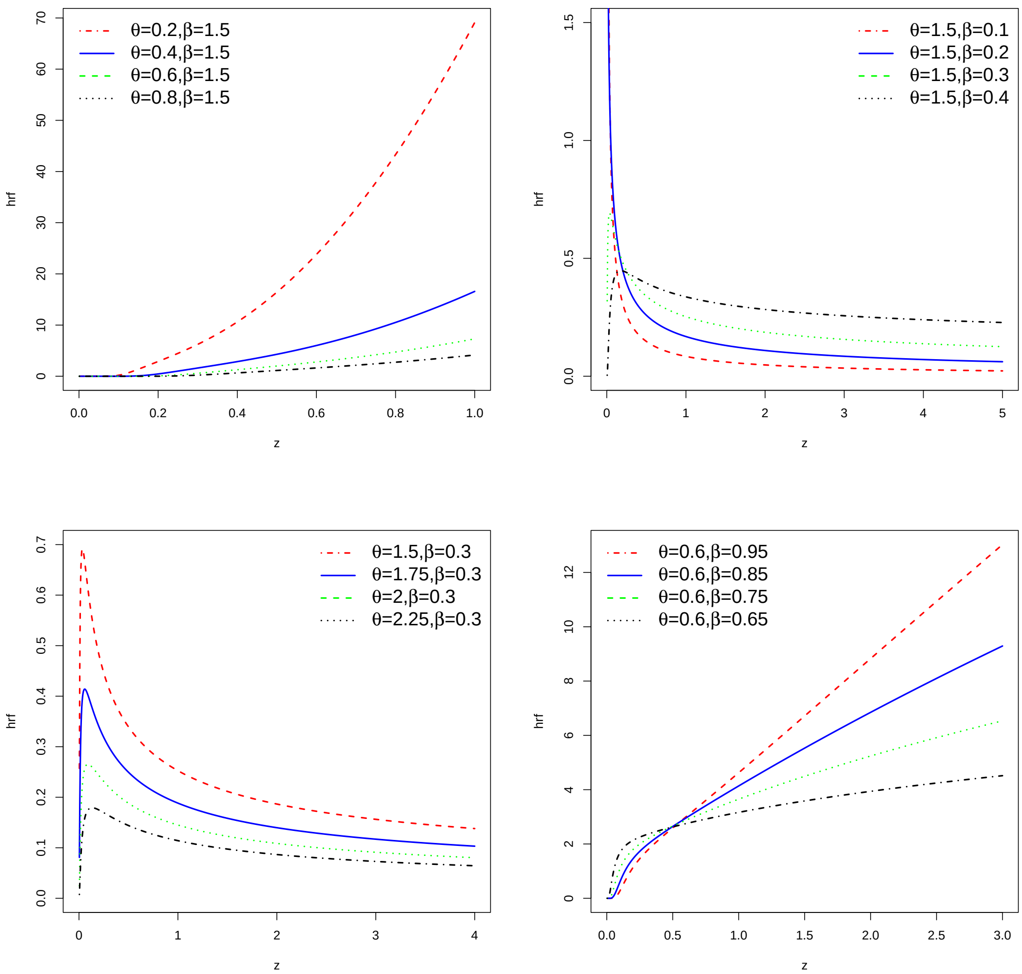

Figure 2. HRF curves for the EF-IW model based on various selected parameter values.

Figure 2 shows HRF plots of EF-IW for different sets of parameter values. It has increasing, unimodal, and decreasing shapes.

3. Statistical properties of the EF-IW distribution

3.1. The quantile function

The quantile function $\Xi^{-1}(u)$ holds significant importance in simulation studies across various disciplines due to its ability to generate random variables with desired distribution characteristics .The quantile function of the new EF-IW model can be expressed as

$$\Xi^{-1}(u) = \left[-\frac{1}{\theta}\log\left(\frac{\text{erf}^{-1}(u)}{1+\text{erf}^{-1}(u)}\right)\right]^{-1/\beta}, 0 \le u \le 1, \tag{3.1}$$

where $\text{erf}^{-1}(x) = \Phi^{-1}(x)$ is the standard normal quantile function.
Proof. By setting the Eq (2.1) equal u, we get

$$\begin{aligned}
\text{erf}\left(\frac{e^{-\theta z^{-\beta}}}{1-e^{-\theta z^{-\beta}}}\right) &= u,\\
\frac{e^{-\theta z^{-\beta}}}{1-e^{-\theta z^{-\beta}}} &= \text{erf}^{-1}(u),\\
e^{-\theta z^{-\beta}}(1+\text{erf}^{-1}(u)) &= \text{erf}^{-1}(u),\\
e^{-\theta z^{-\beta}} &= \frac{\text{erf}^{-1}(u)}{1+\text{erf}^{-1}(u)},\\
\theta z^{-\beta} &= -\log\left(\frac{\text{erf}^{-1}(u)}{1+\text{erf}^{-1}(u)}\right),\\
z &= \left[-\frac{1}{\theta}\log\left(\frac{\text{erf}^{-1}(u)}{1+\text{erf}^{-1}(u)}\right)\right]^{-1/\beta}.
\end{aligned}$$

The quantile function can be used to compute the first, second, and third quantiles by replacing u with $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

Additionally, the Bowleys skewness ($\mathcal{N}$) and Moors kurtosis ($\mathcal{M}$) of the EF-IW model are described as

$$\mathcal{N} = \frac{\Xi^{-1}(1/4) + \Xi^{-1}(3/4) - 2\Xi^{-1}(1/2)}{\Xi^{-1}(3/4) - \Xi^{-1}(1/4)},$$

and

$$\mathcal{M} = \frac{\Xi^{-1}(7/8) - \Xi^{-1}(5/8) + \Xi^{-1}(3/8) - \Xi^{-1}(1/8)}{\Xi^{-1}(6/8) - \Xi^{-1}(2/8)}.$$

3.2. Mixture representation of the EF-IW

In this part, we provide a series representation of the EF-IW CDF and PDF by employing the erf series, see Fernández and De Andrade [17] and Ajongba et al. [18],

$$\text{erf}(t) = \frac{2}{\sqrt(\pi)}\sum_{l=0}^{\infty}\frac{(-1)^l t^{2l+1}}{l!(2l+1)},$$

and by applying the expansion

$$\frac{t}{1-t}=\sum_{j=0}^{\infty} t^{j},\ |t|<1,$$

the corresponding CDF of the EF-IW distribution can be rewritten as:

$$\Xi(z)=\frac{2}{\sqrt{(\pi)}}\sum_{l=0}^{\infty}\frac{(-1)^{l}}{l!(2l+1)}\left[\sum_{j=0}^{\infty} e^{-\theta z^{-\beta}}\right]^{2l+1}.$$

Now, consider the series expansion

$$\left[\sum_{j=0}^{\infty} a_j t^{j}\right]^{k}=\sum_{n=0}^{\infty}\mathcal{D}_{k,n}\, t^{n},$$

where $\mathcal{D}_{k,0}=a_0^{k}$ and $\mathcal{D}_{k,n}=\dfrac{1}{n\,a_0}\sum_{s=1}^{n}(sk-n+s)\,a_s\,\mathcal{D}_{k,n-s},\ \ n\geq 1.$

Consequently, the EF-IW CDF takes the expression

$$\Xi(z)=\frac{2}{\sqrt{\pi}}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{l}\mathcal{D}_{2l+1,n}}{l!(2l+1)}\,e^{-\theta(n+2l+1)z^{-\beta}}=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}C_{l,n}\,e^{-\theta(n+2l+1)z^{-\beta}},$$

with $C_{l,n}=\dfrac{2(-1)^{l}\mathcal{D}_{2l+1,n}}{\sqrt{\pi}l!(2l+1)}$, $\mathcal{D}_{2l+1,n}=\dfrac{1}{n}\sum_{s=1}^{n}[2s(l+1)-n]\,\mathcal{D}_{2l+1,n-s}$ and $\mathcal{D}_{2l+1,0}=1.$

Similarly, the density of the recommended EF-IW model becomes

$$\xi(z)=\theta\beta\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{H}_{l,n}\,z^{-\beta-1}\,e^{-\theta(n+2l)z^{-\beta}},$$

with $\mathcal{H}_{l,n}=C_{l,n}(n+2l+1)$.

3.3. Moments and related measures

One of the efficient statistical criteria that can calculate symmetry, spread-ness, and asymmetry is the ordinary moment. The r-th moment of the EF-IW distribution, whose PDF is given in Eq (2.2), can be determined as follows:

$$\mu_r'=\theta\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{H}_{l,n}\,\frac{\Gamma\left(1-\frac{r}{\beta}\right)}{[\theta(2l+n)]^{1-\frac{r}{\beta}}},\tag{3.2}$$

where $\Gamma(.)$ represents the gamma function.

Thus, for $r=1$ and $r=2$, the mean (μ_1') and second moment (μ_2') of the EF-IW distribution are defined, respectively, as

$$\mu_1'=\theta\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{H}_{l,n}\,\frac{\Gamma\left(1-\frac{1}{\beta}\right)}{[\theta(2l+n)]^{1-\frac{1}{\beta}}},$$

and

$$\mu_2' = \theta \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{H}_{l,n} \frac{\Gamma\left(1 - \frac{2}{\beta}\right)}{[\theta(2l+n)]^{1-\frac{2}{\beta}}}.$$

The variance (Var_z) with a corresponding coefficient of variation (CV) for the EF-IW model are obtained to be

$$\mathrm{Var}_Z = \mu_2' - \mu_1'^2,$$

and

$$\mathrm{CV} = \frac{\mathrm{Var}_Z}{\mu_1'}.$$

Table 1 defined various proposed mathematical characteristics of the suggested EF-IW. In addition, Figure 3 shows the 3D plots of these statistical properties.

Table 1. Statistical properties of EF-IW with different values of parameters.

	β	μ_1	Var_Z	CV	$\mathcal{N}$	$\mathcal{M}$
θ=0.4	0.3	0.0874	0.0234	1.7508	4.3227	30.106
	0.6	0.2287	0.0351	0.8190	1.5817	3.3285
	0.9	0.3488	0.0369	0.5506	0.8949	0.7335
	1.2	0.4412	0.034	0.4179	0.5665	0.0143
θ=0.6	0.3	0.3375	0.3492	1.7508	4.3227	30.106
	0.6	0.4495	0.1355	0.8190	1.5817	3.3285
	0.9	0.5473	0.0908	0.5506	0.8949	0.7335
	1.2	0.6186	0.0668	0.4179	0.5665	0.0143
θ=0.8	0.3	0.8806	2.3770	1.7508	4.3227	30.106
	0.6	0.7260	0.3535	0.8190	1.5817	3.3285
	0.9	0.7534	0.1721	0.5506	0.8949	0.7335
	1.2	0.7862	0.1080	0.4179	0.5665	0.0143
θ=1.2	0.3	3.4022	35.479	1.7508	4.3227	30.106
	0.6	1.4270	1.3659	0.8190	1.5817	3.3285
	0.9	1.1822	0.4238	0.5506	0.8949	0.7335
	1.2	1.1022	0.2122	0.4179	0.5665	0.0143
θ=1.5	0.3	7.1581	157.05	1.7508	4.3227	30.106
	0.6	2.0699	2.8737	0.8190	1.5817	3.3285
	0.9	1.5149	0.6958	0.5506	0.8949	0.7335
	1.2	1.3274	0.3078	0.4179	0.5665	0.0143

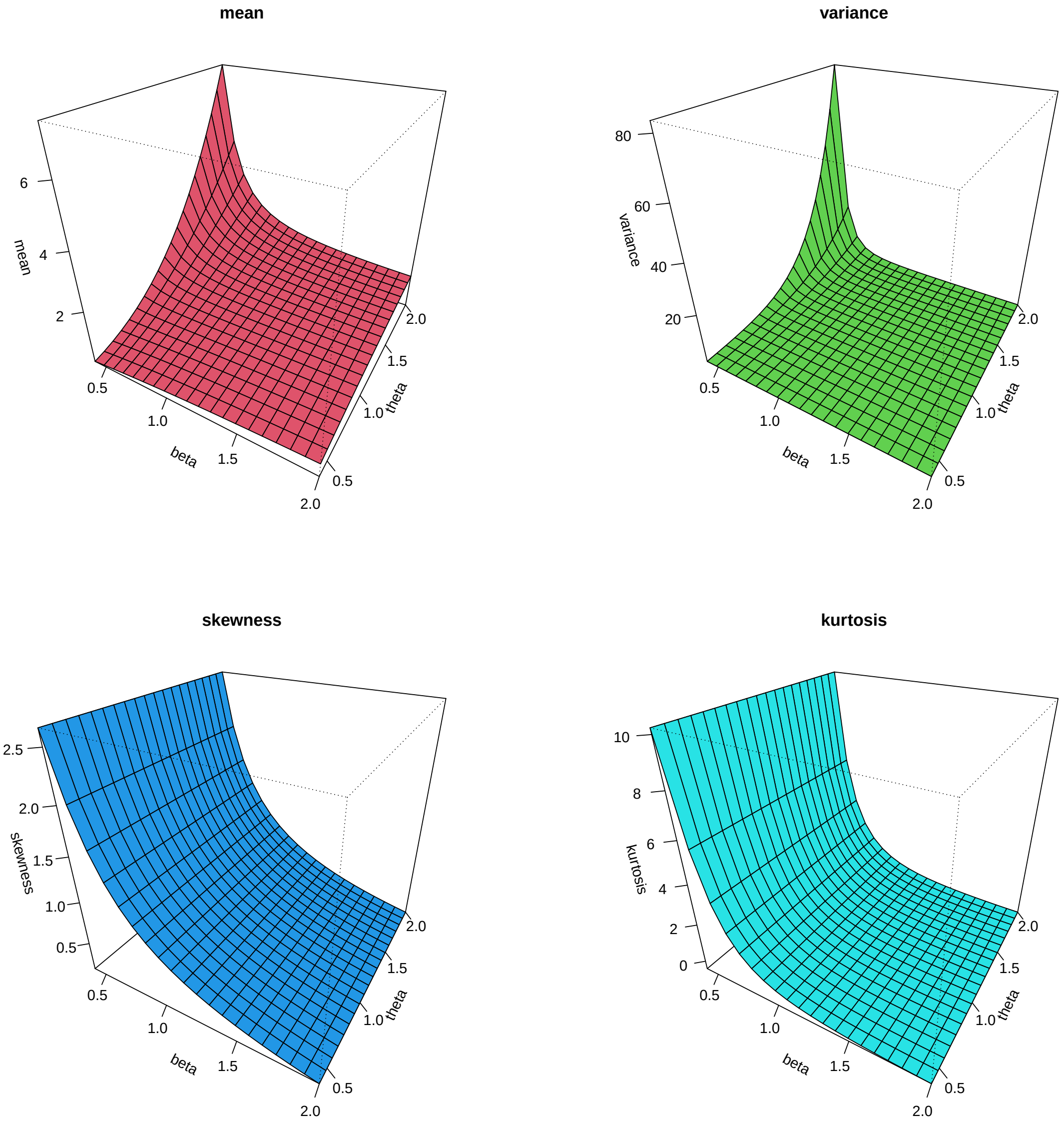

Figure 3. Plots for μ_1, Var_Z, ID, $\mathcal{N}$, and $\mathcal{M}$ for distinct parameter choice.

The moment generating function (MGF), $M(t)$ of the KMIW model is derived as

$$M(t) \quad = \quad \theta \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\mathcal{H}_{l,n}\, t^r}{r!} \frac{\Gamma\left(1 - \frac{r}{\beta}\right)}{[\theta(2l+n)]^{1-\frac{r}{\beta}}}.$$

3.4. Order statistics of KMIW

The PDF of the r^{th}-order statistics for a sample of size m taken from the EF-IW model is expressed as follows:

$$k_{(r)}(z) = \frac{m!\xi(z)}{(r-1)!(m-r)!} [\Xi(z)]^{r-1} [1 - \Xi(z)]^{m-r}$$

$$= \frac{m!}{(r-1)!(m-r)!}\,\theta\beta \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{H}_{l,n}\, z^{-\beta-1}\, e^{-\theta(n+2l)z^{-\beta}} \left[\sum_{l=0}^{\infty}\sum_{n=0}^{\infty} C_{l,n}\, e^{-\theta(n+2l+1)z^{-\beta}}\right]^{r-1}$$
$$\times \left[1 - \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} C_{l,n}\, e^{-\theta(n+2l+1)z^{-\beta}}\right]^{m-r}.$$

In a special case, the PDF of the minimum 1^{th} and maximum m^{th} order statistics of the EF-IW distribution can be given below as

$$k_{(1)}(z) = m\theta\beta \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{H}_{l,n}\, z^{-\beta-1}\, e^{-\theta(n+2l)z^{-\beta}} \left[1 - \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} C_{l,n}\, e^{-\theta(n+2l+1)z^{-\beta}}\right]^{1-r},$$

and

$$k_{(m)}(z) = m\theta\beta \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{H}_{l,n}\, z^{-\beta-1}\, e^{-\theta(n+2l)z^{-\beta}} \left[\sum_{l=0}^{\infty}\sum_{n=0}^{\infty} C_{l,n}\, e^{-\theta(n+2l+1)z^{-\beta}}\right]^{m-1}.$$

The corresponding CDF of the EF-IW model can be written as

$$K_{(r)}(z) = \sum_{k=0}^{m} \Xi^{k}(z)[1-\Xi(z)]^{m-k}$$
$$= \sum_{k=0}^{m} \left[\sum_{l=0}^{\infty}\sum_{n=0}^{\infty} C_{l,n}\, e^{-\theta(n+2l+1)z^{-\beta}}\right]^{k} \left[1 - \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} C_{l,n}\, e^{-\theta(n+2l+1)z^{-\beta}}\right]^{m-k}.$$

4. Parameter estimation

In this part of the study, we estimate the models' parameters $\boldsymbol{\eta} = (\beta, \theta)$ using two different estimation methods. For this purpose, the maximum likelihood and Bayesian estimators are the estimation methods used.

4.1. Maximum likelihood estimation

Assuming $\{z_1, z_2, \ldots, z_m\}$ are the observed values of a random sample $\{Z_1, Z_2, \ldots, Z_m\}$ from the EF-IW distribution with vector of parameters $\boldsymbol{\eta} = (\beta, \theta)$, the log-likelihood function can be obtained to be

$$\mathcal{LL}(z) = \sum_{i=1}^{m} \log \xi(z)$$
$$= \sum_{i=1}^{m} \log\left(\frac{2\theta\beta\, z^{-(\beta+1)}\, e^{-\theta z^{-\beta}}}{\sqrt{\pi}(1-e^{-\theta z^{-\beta}})^2}\, \exp\left\{-\left(\frac{e^{-\theta z^{-\beta}}}{1-e^{-\theta z^{-\beta}}}\right)^2\right\}\right)$$
$$\propto m\log\theta + m\log\beta - 2\sum_{i=1}^{m}\log(1-e^{-\theta z_i^{-\beta}}) - \theta\sum_{i=1}^{m} z_i^{-\beta} - (\beta+1)\sum_{i=1}^{m}\log z_i - \sum_{i=1}^{m}\left(\frac{e^{-\theta z_i^{-\beta}}}{1-e^{-\theta z_i^{-\beta}}}\right)^2. \quad (4.1)$$

With the vector of the parameters $\boldsymbol{\eta} = (\beta, \theta)$, the corresponding partial derivatives of Eq (4.1) are obtained as:

$$\frac{\partial \mathcal{LL}(z;\boldsymbol{\vartheta})}{\partial \theta} = \frac{m}{\theta} - 2\sum_{i=1}^{m} \frac{z_i^{-\beta} e^{-\theta z_i^{-\beta}}}{1-e^{-\theta z_i^{-\beta}}} - \sum_{i=1}^{m} z_i^{-\beta} + 2\left[\sum_{i=1}^{m} \frac{z_i^{-\beta}\, e^{-3\theta z_i^{-\beta}}}{(1-e^{-\theta z_i^{-\beta}})^3} + \sum_{i=1}^{m} \frac{z_i^{-\beta}\, e^{-2\theta z_i^{-\beta}}}{(1-e^{-\theta z_i^{-\beta}})^2}\right], \tag{4.2}$$

and

$$\frac{\partial \mathcal{LL}(z;\boldsymbol{\vartheta})}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m} \log z_i^{-\beta} - \theta \sum_{i=1}^{m} z_i^{-\beta} \log z_i + 2\theta\left[\sum_{i=1}^{m} \frac{z_i^{-\beta} \log z_i\, e^{-3\theta z_i^{-\beta}}}{(1-e^{-\theta z_i^{-\beta}})^3} + \sum_{i=1}^{m} \frac{z_i^{-\beta} \log z_i\, e^{-2\theta z_i^{-\beta}}}{(1-e^{-\theta z_i^{-\beta}})^2}\right]. \tag{4.3}$$

The parameter estimates for the parameters $\boldsymbol{\eta} = (\beta, \theta)$ can be obtained by solving the above non-linear equations with respect to the parameters. It might be difficult to obtain a precise solution to the derived equations, and thus one option to optimize them is to use techniques like the Newton-Raphson algorithm. We used the R software's optimize function in this case.

4.2. Bayesian estimation

We proceed based on the information available on the unknown parameters obtained from the opinions of the researchers. The interpretation of the informative prior is rarely precise enough to determine a single prior distribution. However, there are laws calibrated according to the distribution of observations, called the conjugate prior or the gamma prior. For more details, see Xu [28] and Zhuang [29]. Assuming that the unknown parameters β and θ are random variables that have a Gamma distribution with PDF expressed as

$$\pi_1(\theta) = \frac{b_1^{a_1}}{\Gamma(a_1)}\, \theta^{a_1-1}\, e^{-b_1\theta}, \quad \theta, a_1, b_1 > 0,$$

and

$$\pi_1(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2 s)}\, \beta^{a_2-1}\, e^{-b_2\beta}, \quad \beta, a_2, b_2 > 0.$$

Henceforth, the joint prior PDF of $\boldsymbol{\eta} = (\beta, \theta)$ can be derived as

$$\pi(\boldsymbol{\vartheta}) \propto \theta^{a_1-1}\, \beta^{a_2-1}\, e^{-b_1\theta - b_2\beta}.$$

Next, the joint posterior PDF of $\boldsymbol{\eta} = (\beta, \theta)$ is

$$\begin{aligned} \pi^*(\boldsymbol{\vartheta} \mid z) &= \mathcal{L}(\boldsymbol{\vartheta}))\pi(\boldsymbol{\vartheta}) \mid z) \\ &\propto \theta^{m+a_1-1}\, \beta^{m+a_2-1}\, e^{b_1\theta - b_2\beta} \prod_{i=1}^{m} = \frac{z_i^{-(\beta+1)}\, e^{-\theta z_i^{-\beta}}}{(1-e^{-\theta z_i^{-\beta}})^2}\, \exp\left\{-\left(\frac{e^{-\theta z_i^{-\beta}}}{1-e^{-\theta z_i^{-\beta}}}\right)^2\right\}. \end{aligned}$$

The Bayes estimates of the parametric function $\boldsymbol{\eta} = (\beta, \theta)$ under the assumption of the square error loss function ($\mathrm{B_{SE}}$) is the posterior mean of $\boldsymbol{\eta}$. The $\mathrm{B_{SE}}$ is

$$\hat{f}_{SE} = \int_{\boldsymbol{\eta}} f\, \pi^*(\boldsymbol{\vartheta} \mid z)\, d\eta. \tag{4.4}$$

Now, the Bayes estimator under linear exponential loss function (B_{LI}), can be written $f = e^{\delta(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})} - \delta(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}})$. The B_{LI} is

$$\hat{f}_{LI} = -\frac{1}{\delta}\log\left(\int_{\boldsymbol{\eta}} e^{-\delta f}\, \pi^*(\boldsymbol{\eta} \mid z)\, d\boldsymbol{\eta}\right). \tag{4.5}$$

In the end, the Bayes estimator under general entropy loss function (B_{GE}), defined as $f = \left(\frac{\hat{\boldsymbol{\eta}}}{\boldsymbol{\eta}}\right)^{\delta} - \delta\log\left(\frac{\hat{\boldsymbol{\eta}}}{\boldsymbol{\eta}}\right) - 1$, is

$$\hat{f}_{GE} = \left(\int_{\boldsymbol{\eta}} f^{-\delta}\, \pi^*(\boldsymbol{\eta} \mid z)\, d\boldsymbol{\eta}\right)^{-1/\delta}, \tag{4.6}$$

with $\delta \neq 0$. It is difficult to obtain analytical expressions of Eqs (4.4)–(4.6). To solve this issue, we have considered the Metropolis Hasting (MH) algorithm for this purpose.

5. Simulation investigation

In this section, a detailed simulation study is carried out to examine the behavior of two derived estimators using the R software to evaluate the efficiency of the recommended estimators. The results are presented for various sample sizes $m = \{30, 60, 80, 100\}$ from the proposed EF-IW distribution and several parameter values of $\boldsymbol{\eta} = (\beta, \theta)$ (Set 1: (0.5, 0.75), Set 2: (0.8, 1.25), and Set 3: (1.2, 1.5)) to provide more accurate and comprehensive results. The Monte Carlo simulations are repeated 1000 times, and the estimates are assessed based on the mean estimate (AEs) and mean squared errors (MSEs). The empirical results are illustrated in Tables (2)–(4), and in this simulation, we choose $\delta = 1.5$ to compute the B_{LI} and B_{GE}. To check that the iterative non-linear method converges to the MLEs, we have applied the Newton Raphson technique with some other initial estimates, and it converges to the same set of estimates, which ensures that the estimates obtained via the suggested Newton Raphson method converges to the MLEs. The following conclusions are drawn from these tables.

(1) All estimation approaches produce estimates that converge toward the true parameter values as the sample size increases, which confirm that they are consistent and asymptotically unbiased.
(2) In most cases, the value of MSEs decreases as the value of m increases.
(3) As m increases, the Bayes estimates tends to perform efficiently based on MSE as an optimal criterion. On the contrary, B_{SE} is more appropriate than B_{LI} and B_{GE}.
(4) Figure (4) ensures the same conclusion.

Table 2. Numerical values of EF-IW model simulation for Set 1.

m		MLE		B_{SE}		B_{LI}		B_{GE}	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
30	θ	0.4919	0.0039	0.4502	0.0035	0.4505	0.0037	0.4496	0.0039
	β	0.7802	0.0115	0.7496	0.0008	0.7497	0.0101	0.7494	0.0103
60	θ	0.4928	0.0018	0.5093	0.0010	0.5093	0.0013	0.5085	0.0015
	β	0.7681	0.0051	0.7395	0.0007	0.7396	0.0009	0.7393	0.0101
80	θ	0.5007	0.0011	0.4888	0.0006	0.4888	0.0008	0.4886	0.0010
	β	0.7586	0.0034	0.7882	0.0005	0.7883	0.0008	0.7880	0.0009
100	θ	0.4943	0.0010	0.5091	0.0004	0.5093	0.0005	0.5087	0.0008
	β	0.7591	0.0024	0.7577	0.0004	0.7581	0.0006	0.7573	0.0008

Table 3. Numerical values of EF-IW model simulation for Set 2.

m		MLE		B_{SE}		B_{LI}		B_{GE}	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
30	θ	0.7998	0.0055	0.8643	0.0039	0.8645	0.0041	0.4816	0.0043
	β	1.2981	0.0309	1.3373	0.0121	1.3384	0.0123	1.3364	0.0125
60	θ	0.8001	0.0024	0.8358	0.0015	0.8359	0.0017	0.8357	0.0019
	β	1.2646	0.0148	1.1742	0.0075	1.1746	0.0078	1.1738	0.0079
80	θ	0.7993	0.0023	0.7856	0.0014	0.7859	0.0016	0.7852	0.0018
	β	1.2641	0.0075	1.2241	0.0017	1.2244	0.0020	1.2239	0.0021
100	θ	0.7948	0.0014	0.8046	0.0007	0.8048	0.0009	0.8044	0.0011
	β	1.2673	0.0072	1.2320	0.0011	1.2322	0.0013	1.2319	0.0015

Table 4. Numerical values of EF-IW model simulation for Set 3.

m		MLE		B_{SE}		B_{LI}		B_{GE}	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
30	θ	1.2240	0.0188	1.2206	0.0051	1.2217	0.0053	1.2196	0.0055
	β	1.5799	0.0559	1.6200	0.0245	1.6225	0.0248	1.6184	0.0249
60	θ	1.2049	0.0065	1.2310	0.0032	1.2316	0.0034	1.2306	0.0036
	β	1.5556	0.0215	1.5530	0.0058	1.5538	0.0061	1.5526	0.0062
80	θ	1.2035	0.0040	1.2276	0.0023	1.2280	0.0025	1.2272	0.0028
	β	1.5524	0.0159	1.4679	0.0032	1.4685	0.0034	1.4676	0.0035
100	θ	1.2139	0.0041	1.2160	0.0019	1.2165	0.0022	1.2157	0.0024
	β	1.5158	0.0113	1.4818	0.0022	1.4822	0.0023	1.4815	0.0025

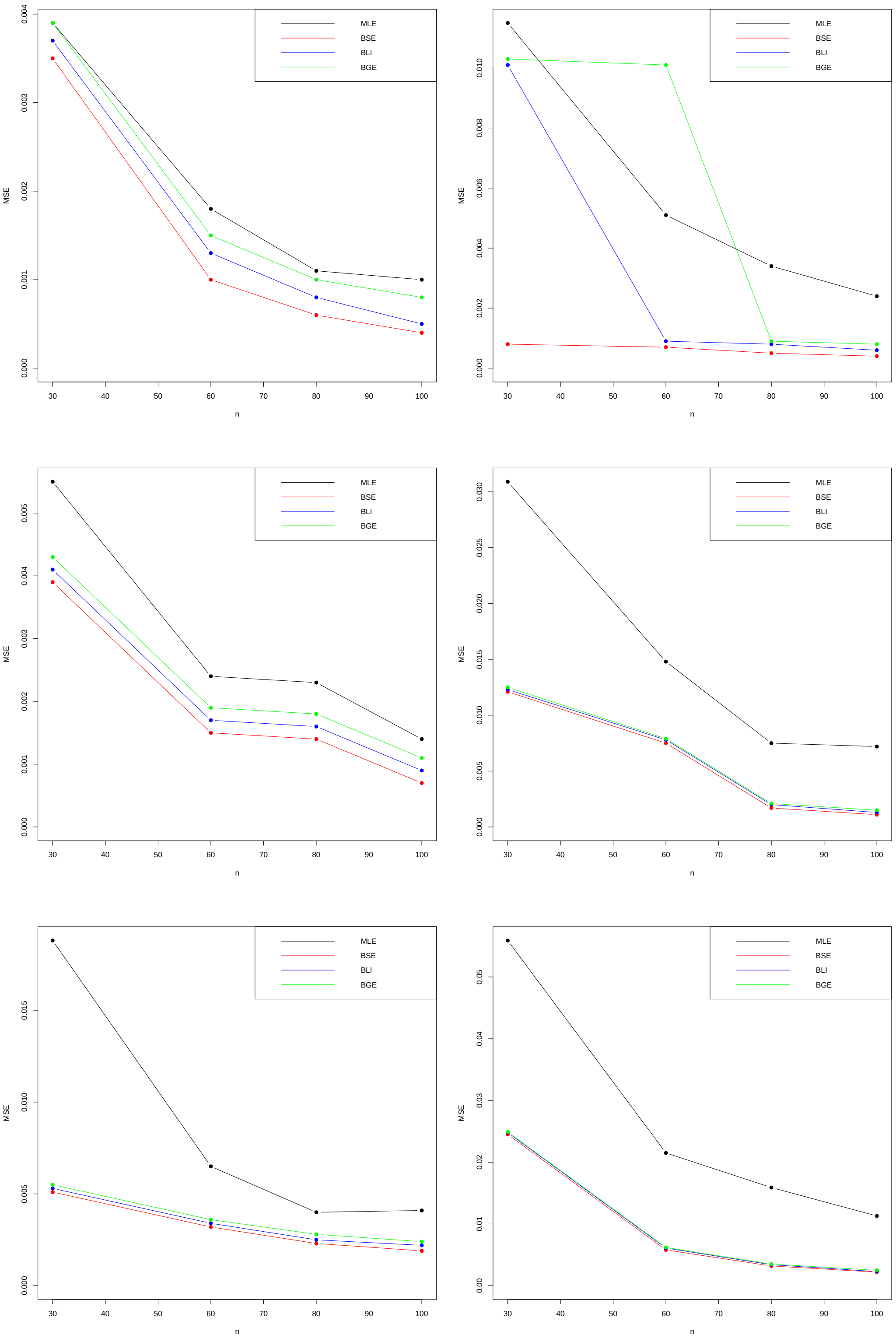

Figure 4. MSE plots based on all proposed estimators using various selected parameter values.

6. Practical application of the EF-IW model

In this section, we utilized two data sets from the industrial field to show the EF-IW model introduced in Section 2. We demonstrate the flexibility of this new distribution by analyzing two real-world datasets drawn from industrial areas in the Kingdom of Saudi Arabia (KSA).

6.1. First application

The data set represents the quarterly evolution of the number of foreign licenses in the construction sector in KSA. It was obtained from `https://datasaudi.sa/en/sector/construction#real-sector-indicators`. The values of the data set are summarized in Table (5).

Table 5. The quarterly evolution of the number of foreign licenses data set.

8	6	8	16	23	20	28	40
43	50	54	32	52	29	33	42
41	52	56	79	155	84	95	111
136	161	204	241				

6.2. Second application

The second application introduced the scale efficiency of the construction industry in KSA between 2013 and 2022. The suggested data set was considered by Yu et al. [30], and the values are presented in Table (6).

Table 6. The scale efficiency of the construction industry data set.

Zone	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022
Mecca	9.39	9.71	9.83	9.96	9.97	9.95	9.98	9.97	10.005	9.96
Eastern	8.92	9.23	9.43	9.56	9.58	9.71	9.78	9.72	9.82	9.87
Al-Madinah	7.46	7.47	7.81	8.52	8.62	8.61	8.73	8.43	8.74	8.77
Jizan	6.66	6.69	6.84	7.64	7.71	7.75	7.75	7.68	7.82	7.82
Al-Qassim	6.6	6.62	6.67	7.47	7.51	7.53	7.67	7.6	7.73	7.73
Tabuk	5.31	5.46	5.66	6.41	6.54	6.52	6.54	6.43	6.67	6.6
Ha'il	4.23	4.27	4.29	5.31	5.47	5.47	5.59	5.14	5.62	5.72

6.3. Third application

This recommended data set is about the efficiency of the pure technical construction industry between 2013 and 2022 in the KSA. The proposed data was considered by Yu et al. [30], and its records can be reported in Table 7.

Table 7. The efficiency of the pure technical construction industry data set.

Zone	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022
Eastern	2.55	3.90	4.59	6.37	7.11	7.38	7.55	7.17	7.89	8.54
Al Madinah	3.26	3.46	3.47	4.99	6.38	6.42	6.81	6.16	6.77	7.21
Asir	3.41	3.81	3.98	4.65	5.47	5.74	5.92	6.17	6.13	6.53
Jizan	3.42	3.39	3.62	4.46	5.37	5.71	5.56	5.49	5.64	5.80
Al-Qassim	3.43	3.45	3.37	4.11	4.46	4.81	5.10	5.07	5.24	5.45
Tabuk	2.99	2.78	2.96	3.96	4.48	4.96	4.82	4.75	4.89	5.13
Ha'il	2.89	2.59	2.73	3.59	4.19	4.59	4.52	4.50	4.70	4.75
Al Jawf	2.29	2.75	2.48	3.35	4.22	4.42	4.55	4.44	4.63	4.71
Najran	2.83	2.92	2.62	3.33	4.02	4.38	4.47	4.44	4.61	4.8
Northern Borders	1.51	1.51	1.6	2.79	3.95	4.04	3.99	4.08	4.4	4.48

Table (8) presents a statistical summary of the three data sets. Furthermore, Figure 5 shows several significant plots (scaled total time on test (TTT), quantile-quantile (Q-Q), and box plots) derived from the three industrial datasets. These plots help analyze the historical performance of the industrial sectors.

Table 8. Numerical values of descriptive statistics based on the three data sets.

Data	Q_1	Q_2	μ_1'	Q_3	CV	$\mathcal{N}$	$\mathcal{M}$
1	28.75	46.50	67.82	27.60	54.57	1.31	0.83
2	6.55	7.72	7.67	9.35	0.35	-0.18	-0.96
3	3.445	4.475	4.521	5.39	0.35	0.3589	-0.0846

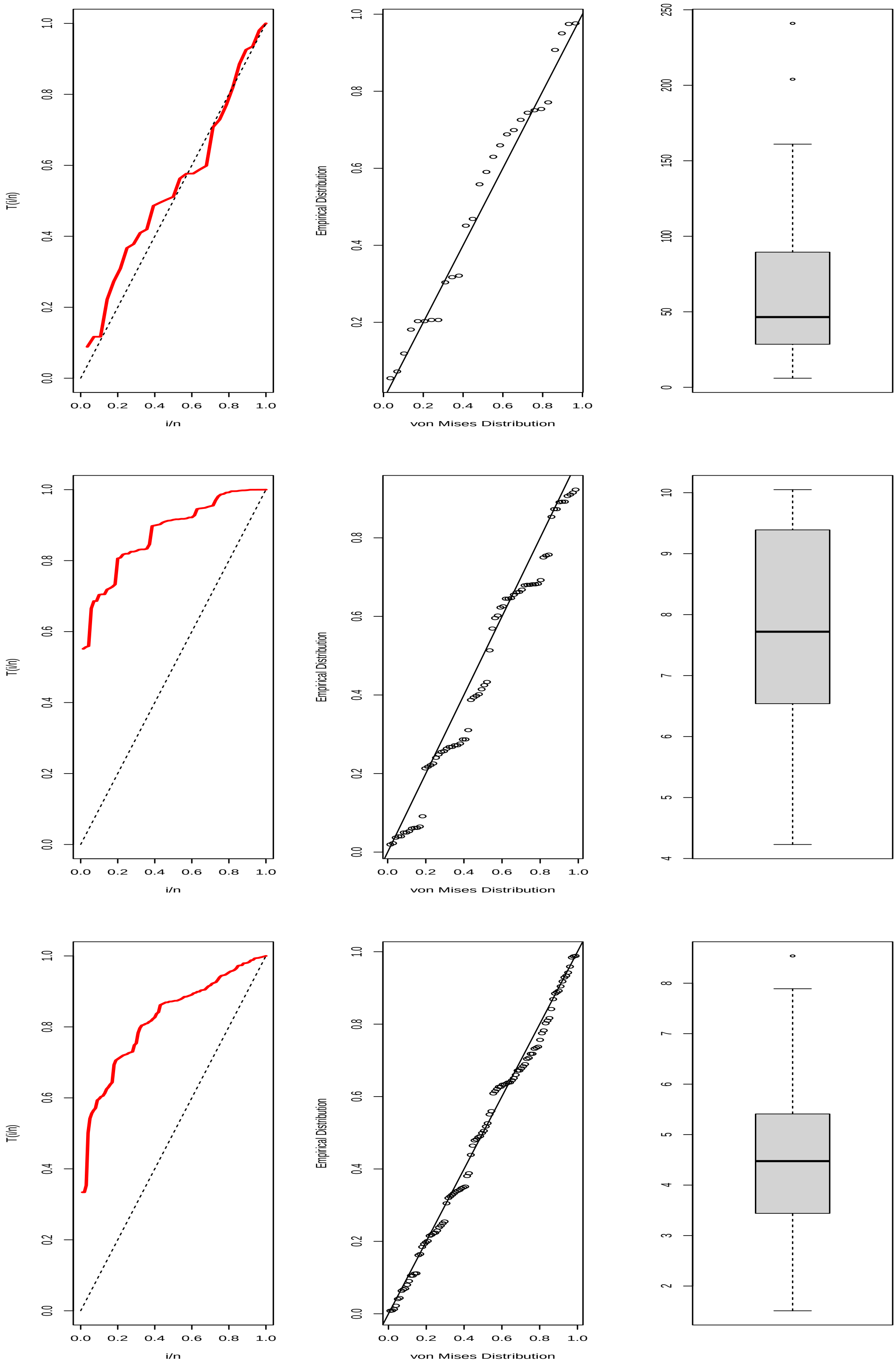

Figure 5. Various non-parametric plots of the suggested data sets.

Additionally, we would like to select the more appropriate fitting model for the two proposed data sets. We consider several renowned competitive probability distributions to compare with the results of EF-IW, including the inverse Weibull (IW), error function Weibull (EF-W), error function exponential (EF-E), power Burr X (PBX), and generalized exponential (GE) models.

Akaike information criterion ($\mathcal{A}$), Bayesian information criterion ($\mathcal{B}$), Hannan-Quin information criterion ($\mathcal{C}$), correction Akaike information criterion ($\mathcal{D}$), Kolmogorov-Smirnov ($\mathcal{KS}$) statistics with its associated $\mathcal{P}$-values are considered when comparing the model and recommending the best model. By calculating and comparing the proposed measures, we gain a clear understanding of the relative performance of each model. Models with the lowest values for these statistics will be considered for the best fit of the given data set. This approach reflects the strengths of the new distribution in terms of its suitability for different data structures and ensures that the model selection process takes into account both the complexity of the model and the goodness of fit across multiple aspects of the data distribution. Table (9) summarizes the final estimates of the unknown parameters with their corresponding log-likelihood ($\mathcal{LL}$). Consequently, the recommended EF-IW model emerges as the most favorable distribution for modeling the three data sets. Henceforth, the empirical v.s. the fitted (PDF and CDF) plots for the proposed model with its competitors are generated and reported in Figures (6)–(8) using the two data sets. These visual plots demonstrate that the EF-IW distribution works well with the three data sets.

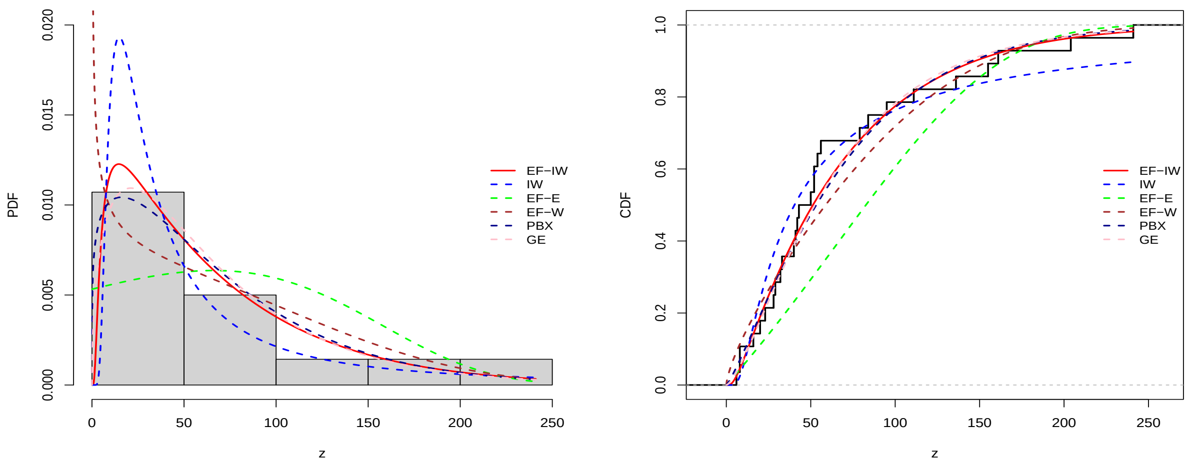

Figure 6. Fitted density and CDF for the fitting models to data set 1.

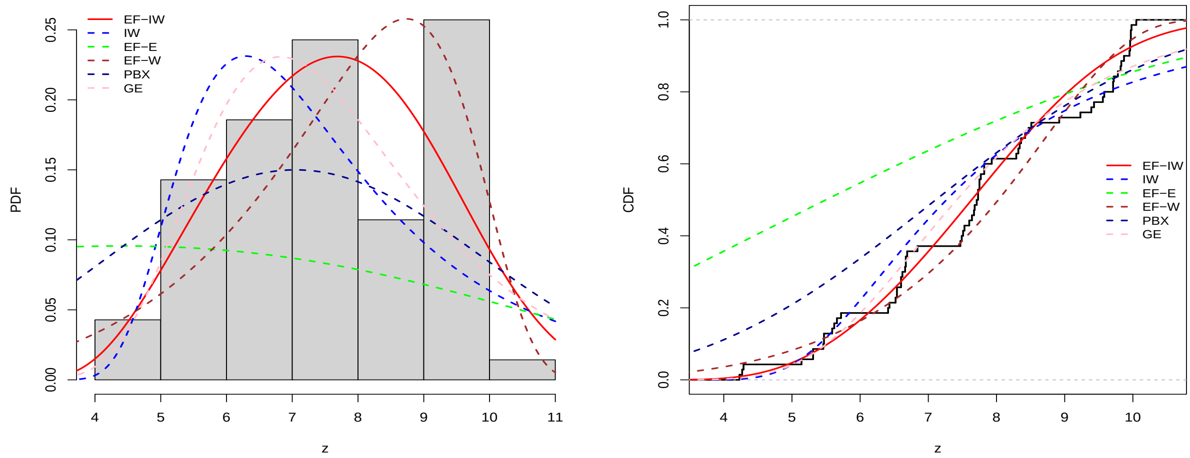

Figure 7. Fitted density and CDF for the fitting models to data set 2.

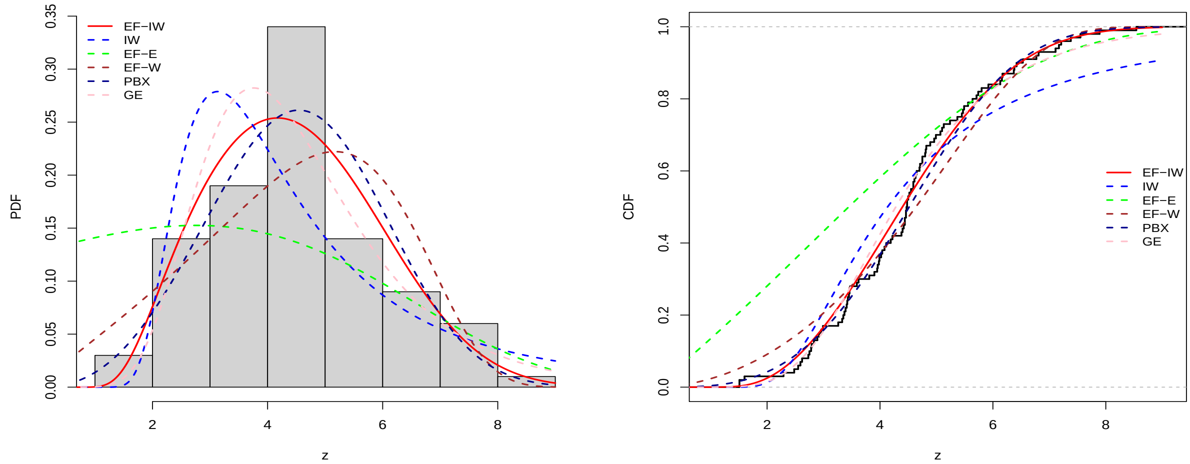

Figure 8. Fitted density and CDF for the fitting models to data set 3.

Table 9. Parameter estimations with various statistic comparison measures for the three considered data sets.

Data	Model	$\hat{\theta}$	$\hat{\beta}$	$\mathcal{KS}$	$\mathcal{P}$-value	$\mathcal{LL}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
	EF-IW	10.609	0.5686	0.1425	0.6197	-144.601	293.202	295.866	294.016	293.682
	IW	8.5601	0.6934	0.2135	0.1557	-152.624	309.269	311.934	300.958	300.624
1	EF-W	224.821	0.7022	0.1961	0.2316	-147.318	298.637	301.301	299.451	299.117
	EF-E		0.0047	0.3475	0.0023	-149.955	301.911	303.243	302.318	302.064
	PBX	0.0771	0.5983	0.1575	0.4901	-145.356	294.713	297.378	295.603	295.269
	GE	0.0188	1.4887	0.1495	0.5582	-145.041	294.082	296.746	294.897	294.562
	EF-IW	169.011	2.4633	0.1114	0.3500	-132.473	268.947	273.444	270.734	269.126
	IW	2168.36	4.0579	0.1697	0.0354	-146.885	297.771	302.268	299.557	297.950
2	EF-W	10.415	3.6140	0.1468	0.0978	-132.736	269.472	273.969	271.259	269.651
	EF-E		0.0709	0.4251	< 0.0001	-179.247	360.494	362.743	361.387	360.553
	PBX	0.0407	1.5378	0.2102	0.0041	-146.950	297.936	302.433	299.723	298.115
	GE	0.6226	69.883	0.1399	0.1290	-139.298	282.596	287.093	284.382	282.775
	EF-IW	11.193	1.5481	0.0756	0.6168	-177.432	358.864	364.074	360.973	358.988
	IW	25.075	2.5284	0.1395	0.0408	-198.317	400.634	405.845	402.743	400.758
3	EF-W	1.9167	7.5766	0.1301	0.0676	-185.737	375.475	380.686	377.584	375.599
	EF-E		0.1131	0.3139	< 0.0001	-210.347	422.694	425.300	423.749	422.735
	PBX	0.0639	1.7021	0.0909	0.3791	-177.673	359.347	364.557	361.455	359.470
	GE	0.7435	16.358	0.1060	0.2107	-180.239	364.478	69.688	366.587	364.602

Finally, the estimates of the model parameters using the Bayesian technique under several loss functions of the EF-IW distribution by applying the three data sets are computed and reported in Table (10). Also, Figures (9)–(11) show the histogram and trace plots of MH results.

Table 10. Bayesian estimates under various loss functions for the EF-IW model using the three data sets.

Data	Par	Bayes		
		B_{SE}	B_{LI}	B_{GE}
1	θ	10.468	10.470	10.468
	β	0.5765	0.5766	0.5764
2	θ	168.989	168.990	168.989
	β	2.457	2.457	2.457
3	θ	11.692	11.689	11.694
	β	1.5635	1.5637	1.5634

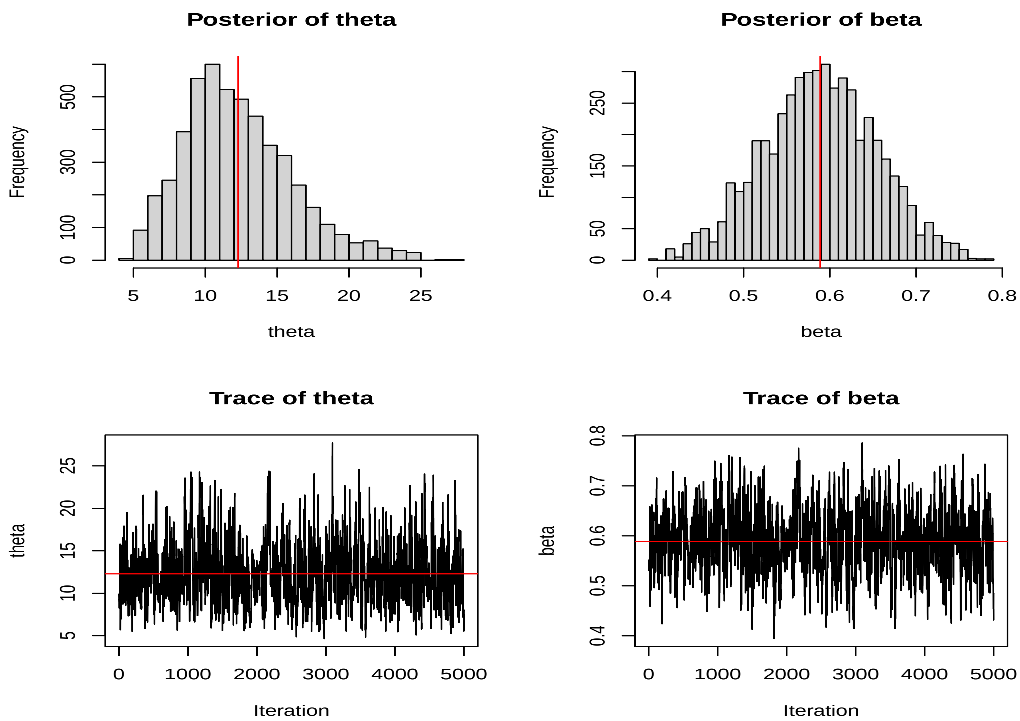

Figure 9. Histogram and trace plots applying MH technique for the first data set.

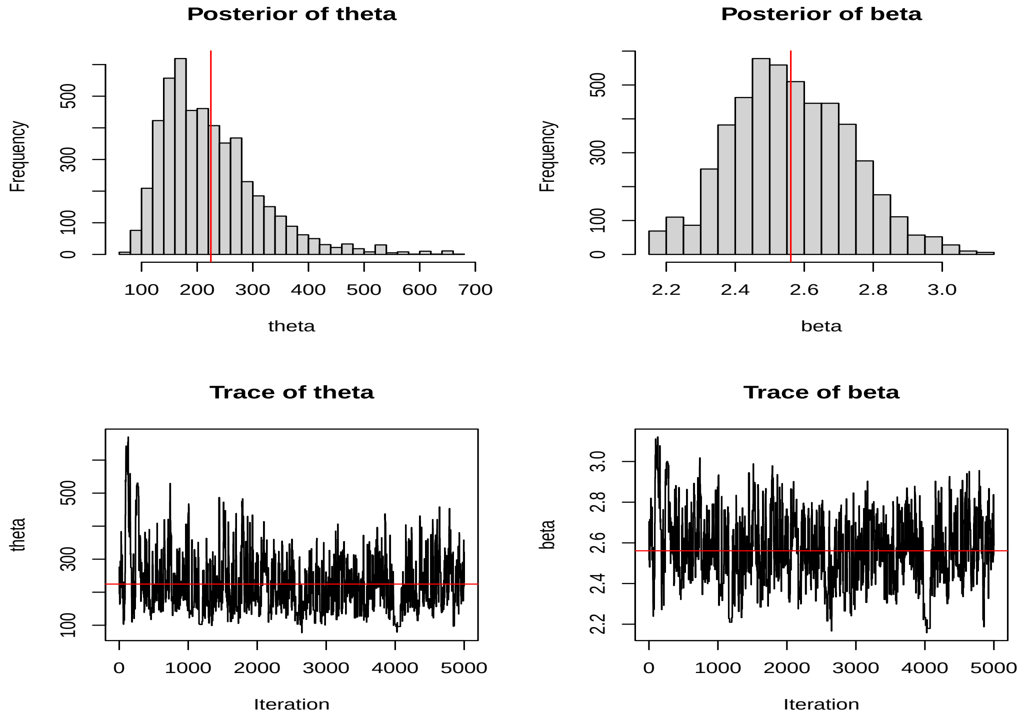

Figure 10. Histogram and trace plots applying MH technique for the second data set.

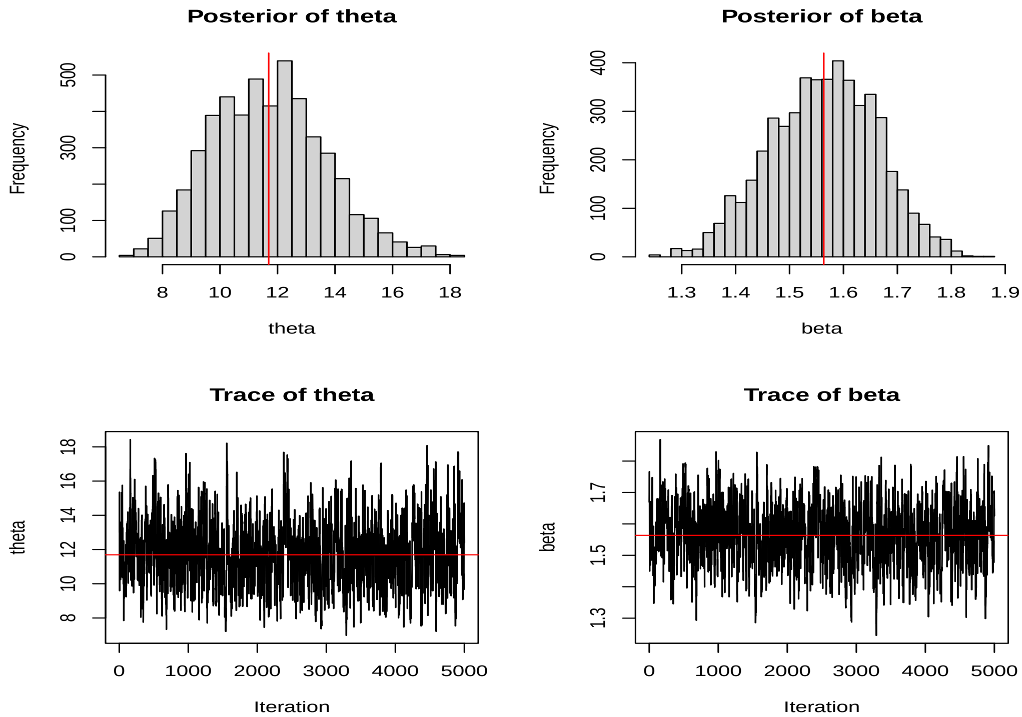

Figure 11. Histogram and trace plots applying MH technique for the third data set.

7. Conclusions

This study introduces a new probability distribution, and its mathematical properties are thoroughly explored. The new model is named the error function inverse Weibull distribution. The model parameters are estimated using two different estimation methods, and extensive simulation studies are conducted to identify the most efficient estimation technique. To demonstrate the versatility and practical usefulness of the EF-IW distribution, the new distribution is applied to three datasets, demonstrating its ability to adapt to varied data properties. The findings of these applications show that the EF-IW distribution surpasses considered competitive probability distributions previously studied in the literature, giving more accurate and efficient outcomes in terms of fit and prediction. These findings show the novel distribution's potential as a robust tool for modeling data across several domains, providing a promising alternative to established models.

8. Future work

Future work on the EF-IW distribution may include expanding modifications, estimation, and applications. Some potential directions include the following

(1) New extended forms of the EF-IW distribution can be proposed, such as truncation, zero-inflation, and Neutrosophic extension for imprecise datasets.
(2) The progressive censoring type may also be used to obtain the model parameter estimations.
(3) Future studies should focus on the utilization of the EF-IW distribution to handle ranked set sampling data, which is frequently seen in survival and reliability analysis studies. Enhancing the distribution applicability and usefulness will require developing parameter estimation approaches for censored and uncensored data with a cure fraction.

Author contributions

All authors contributed equally to this paper. Badr Aloraini and Abdulaziz S. Alghamdi did the writing and mathematics, Mohammad Zaid Alaskar and Maryam Ibrahim Habadi did the revising, editing, and validating.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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