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Research article

Novel iterative criteria for oscillatory behavior in nonlinear neutral differential equations

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Abstract: The purpose of this study was to investigate the oscillation criteria for nonlinear second-order neutral differential equations with deviating arguments, with a particular emphasis on their non-canonical forms. The primary goal was to expand the current theoretical framework by introducing new relations that improved the monotonicity of positive solutions. To attain this purpose, an iterative technique was used to deduce new oscillation criteria, which helped to enhance present understanding in this field. The study process was based on a thorough review of previous literature, followed by the creation of new oscillation criteria with both theoretical and applied significance. The obtained results were validated by three illustrative instances, demonstrating the importance and influence of these criteria in the study of neutral differential equations, particularly in the study of neutral differential equations, especially in nonlinear contexts.

Keywords: oscillatory behavior; differential equations; deviating arguments; noncanonical form; nonlinear equations, second order
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this paper, we investigate the oscillatory properties of nonlinear second-order neutral differential equations (NDEs) of the form:

$$(\kappa(s)\left(\omega'(s)\right)^{\alpha})' + \int_a^b \mathrm{h}(s,\ell)\mathrm{y}^{\beta}(\sigma(s,\ell))\mathrm{d}\ell = 0, \tag{1.1}$$

where $\omega(\mathrm{s}) = \mathrm{y}(\mathrm{s}) + \mathrm{u}(\mathrm{s})\,\mathrm{y}(\tau(\mathrm{s}))$. The following hypotheses are assumed throughout this study:

(**Hyp.**1) $0 < \alpha \leq 1$, $\alpha \geq \beta$ are ratios of odd positive integers;

(**Hyp.**2) $\mathrm{h} \in C([\mathrm{s}_0, \infty) \times (a, b), \mathbb{R})$ and $\mathrm{h}(\mathrm{s}, \ell) \geq 0$;

(**Hyp.**3) $\mathrm{u} \in C([\mathrm{s}_0, \infty), (0, \infty))$, $0 \leq \mathrm{u}(\mathrm{s}) < 1$, $\tau \in C^1([\mathrm{s}_0, \infty), \mathbb{R})$, $\sigma \in C^1([\mathrm{s}_0, \infty) \times (a, b), \mathbb{R})$, $\tau(\mathrm{s}) \leq \mathrm{s}$, $\sigma(\mathrm{s}, \ell) \leq \mathrm{s}$, σ has nonnegative partial derivatives with respect to s and nondecreasing with respect to ℓ, $\lim_{\mathrm{s}\to\infty} \tau(\mathrm{s}) = \infty$, and $\lim_{\mathrm{s}\to\infty} \sigma(\mathrm{s}, \ell) = \infty$ for $\ell \in [a, b]$;

(**Hyp.**4) $\kappa \in C([\mathrm{s}_0, \infty), \mathbb{R}^+)$ satisfies the noncanonical case. That is

$$\xi(\mathrm{s}_0) := \int_{\mathrm{s}_0}^{\infty} \frac{1}{\kappa^{1/\alpha}(\varrho)} \mathrm{d}\varrho < \infty, \tag{1.2}$$

where

$$\xi(\mathrm{s}) := \int_{\mathrm{s}}^{\infty} \frac{1}{\kappa^{1/\alpha}(\varrho)} \mathrm{d}\varrho;$$

(**Hyp.**5) $\mathrm{u}(\mathrm{s}) < \xi(\mathrm{s})/\xi(\tau(\mathrm{s}))$.

Below we provide some basic definitions [1]:

(i) A function $\mathrm{y}(\mathrm{s}) \in C([\mathrm{s}_\mathrm{y}, \infty), \mathbb{R})$, $\mathrm{s}_\mathrm{y} \geqslant \mathrm{s}_0$, is said to be a solution of (1.1) which has the property $\kappa(\mathrm{s})(\omega'(\mathrm{s}))^\alpha \in C^1[\mathrm{s}_\mathrm{y}, \infty)$, and it satisfies (1.1) for all $\mathrm{s} \in [\mathrm{s}_\mathrm{y}, \infty)$. We consider only those solutions y(s) of (1.1) that are defined on a half-line $[\mathrm{s}_\mathrm{y}, \infty)$ and satisfy the condition

$$\sup\{|\mathrm{y}(\mathrm{s})| : \mathrm{s} \geqslant \mathrm{S}\} > 0, \text{ for all } \mathrm{S} \geq \mathrm{s}_\mathrm{y}.$$

(ii) A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

(iii) The Eq (1.1) is said to be oscillatory if all its solutions are oscillatory.

The study of differential equations (DEs) is a cornerstone of mathematical analysis, particularly in understanding dynamic systems that arise in various scientific and engineering applications. Among these, (NDEs) play a critical role in modeling phenomena where the derivative of the unknown function depends not only on the function itself but also on its delayed or advanced argument. In recent decades, there has been a growing interest in the qualitative analysis of such equations, particularly regarding their oscillatory behavior. This interest stems from the fact that oscillatory solutions often represent stable or periodic phenomena in real-world systems; see [2–4].

Oscillation theorems are pivotal in the analysis of DEs, as they provide critical insight into the nature of solutions, particularly in identifying whether these solutions exhibit oscillatory behavior over time. These theorems are essential tools for mathematicians and scientists alike, helping to predict and understand the dynamics of various physical, biological, and engineering systems. Historically, oscillation criteria have been developed and refined to handle a wide array of DEs, from simple linear forms to more intricate nonlinear systems. In recent years, there has been significant progress in extending these classical theorems to accommodate the growing complexity of DEs, including those with non-standard, non-canonical forms. These advancements reflect the continuous evolution of mathematical methods and the increasing sophistication of the systems being studied, making oscillation theorems more relevant and applicable than ever before in addressing contemporary challenges across multiple disciplines; see [5–9].

Second-order non-linear neutral differential equations (NDEs) with deviating arguments constitute a specialized class of DEs that have garnered significant attention due to their broad applications in physics, engineering, and biological systems. These equations are distinguished by terms involving delays or advanced arguments, adding layers of complexity to their analysis and necessitating advanced mathematical techniques for understanding their behavior. While previous studies have extensively examined the existence and stability of solutions, less attention has been given to their oscillatory behavior. The oscillation of solutions to such equations, however, remains an active area of research, motivated by the need to establish precise conditions under which solutions oscillate or converge. In particular, the interplay between nonlinear terms and deviating arguments presents unique challenges that require refined criteria and novel analytical approaches, see [10–12].

In recent years, the study of the oscillatory and exponential behavior of DEs with delays and neutral terms across different orders has seen increasing interest, as illustrated by the work of Han et al. [13], Baculíková [14], Džurina et al. [15], Jadlovská et al. [16], Bazighifan et al. [17], Moaaz et al. [18], and Aldiaiji et al [19, 20]. This broad interest has led to major advances in the understanding of complex periodic solutions ranging from simple harmonic motion to chaotic oscillations and has enabled accurate analyses of critical properties such as amplitude, frequency, and stability. Here is a comprehensive review of the foundational studies that have contributed significantly to this field:
Baculíková [21] investigated the second-order delay differential equations (DDEs) oscillatory characteristics:

$$(\kappa(\mathrm{s})\mathrm{y}'(\mathrm{s}))' + \mathrm{h}(\mathrm{s})\mathrm{y}(\sigma(\mathrm{s})) = 0, \tag{1.3}$$

under the case (1.2). However, both Sun and Meng [22], and Kusano et al. [23] noted that NDEs had the following characteristics:

$$\left(\kappa(\mathrm{s})\left|\mathrm{y}'(\mathrm{s})\right|^{\alpha-1}\mathrm{y}'(\mathrm{s})\right)' + \mathrm{h}(\mathrm{s})\left|\mathrm{y}(\sigma(\mathrm{s}))\right|^{\alpha-1}\mathrm{y}(\sigma(\mathrm{s})) = 0, \tag{1.4}$$

and the linear form that corresponds to them

$$(\kappa(\mathrm{s})\mathrm{y}'(\mathrm{s}))' + \mathrm{h}(\mathrm{s})\mathrm{y}(\mathrm{s}) = 0. \tag{1.5}$$

Sufficient criteria have been established by Agarwal et al. [24] to guarantee the oscillatory behavior of second-order DEs with a neutral term:

$$\left(\kappa(\mathrm{s})\left(\mathrm{y}(\mathrm{s}) + \mathrm{u}(\mathrm{s})\mathrm{y}^{\alpha}(\tau(\mathrm{s}))\right)'\right)' + \mathrm{h}(\mathrm{s})\mathrm{y}(\sigma(\mathrm{s})) = 0, \tag{1.6}$$

under the conditions:

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\kappa(\varrho)}\mathrm{d}\varrho = \infty,$$

and

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\kappa(\varrho)}\mathrm{d}\varrho < \infty.$$

Han et al. [25] reviewed oscillations in second-order linear NDEs (1.6) where $\alpha = 1$, and introduced criteria under the condition $0 \leq u(\mathrm{s}) \leq u_0 < \infty$. This analysis was expanded upon by Grace and Lalli [26] to the equation

$$\left(\kappa(\mathrm{s})\left(\mathrm{y}(\mathrm{s}) + \mathrm{u}(\mathrm{s})\mathrm{y}(\mathrm{s}-\tau)\right)'\right)' + \mathrm{h}(\mathrm{s})f(\mathrm{y}(\mathrm{s}-\sigma)) = 0, \tag{1.7}$$

where $f(\mathrm{y})/\mathrm{y} \geq k > 0$ and $\int_{s_0}^{\infty} 1/\kappa(\ell)\, d\ell = \infty$.

Bohner et al. [27] also investigated the oscillations of the second-order quasi-linear NDEs

$$(\kappa(\mathrm{s})[\omega'(\mathrm{s})]^{\alpha})' + \mathrm{h}(\mathrm{s})\,\mathrm{y}^{\alpha}(\sigma(\mathrm{s})) = 0, \tag{1.8}$$

under the condition (1.2).

In similar studies, Zhang et al. [28] considered a particular type of second-order NDEs

$$\left(\kappa(\mathrm{s})|\omega'(\mathrm{s})|^{\alpha-1}\omega'(\mathrm{s})\right)' + \mathrm{h}(\mathrm{s})\left|\mathrm{y}(\sigma(\mathrm{s}))\right|^{\alpha-1}\mathrm{y}(\sigma(\mathrm{s})) = 0, \tag{1.9}$$

where $\omega(\mathrm{s}) = \mathrm{y}(\mathrm{s}) + \sum_{i=1}^{m} u_i(\mathrm{s})\,\mathrm{y}(\tau_i(\mathrm{s}))$, which helps simplify the analysis of these equations.

In the same context, Sun [29] established oscillation criteria for second-order nonlinear NDEs

$$\left(\kappa(\mathrm{s})|\omega'(\mathrm{s})|^{\gamma-1}\omega'(\mathrm{s})\right)' + \mathrm{h}(\mathrm{s})\,f(\mathrm{s},\mathrm{y}(\sigma(\mathrm{s}))) = 0, \tag{1.10}$$

they relied on a new variational principle to extract these criteria.

Finally, Moaaz et al. [30] presented a study on the oscillation properties of NDEs

$$(\kappa(\mathrm{s})(\omega'(\mathrm{s}))^{\alpha})' + \sum_{i=1}^{n} \mathrm{h}_i(\mathrm{s})\mathrm{y}^{\alpha}(\sigma_i(\mathrm{s})) = 0. \tag{1.11}$$

They proposed new properties characterized by a recursive nature, and extracted oscillation conditions that guarantee the oscillation of all solutions. Alemam et al. [31] also made an in-depth study of the oscillatory properties of the second-order NDEs:

$$\left(\kappa(\mathrm{s})\left[(\mathrm{y}(\mathrm{s}) + \mathrm{u}(\mathrm{s})\,\mathrm{y}^{\gamma}(\tau(\mathrm{s})))'\right]^{\alpha}\right)' + \sum_{i=1}^{n} \mathrm{h}_i(\mathrm{s})\mathrm{y}^{\beta}(\sigma_i(\mathrm{s})) = 0, \tag{1.12}$$

by using the Riccati transformation method to establish oscillation criteria.

While much of the previous research has concentrated on the oscillatory properties of linear and quasi-linear second-order NDEs, resulting in significant advancements in the understanding of their behavior, the oscillatory characteristics of nonlinear second-order NDEs have not received the same level of attention, leaving a notable gap in the literature. This study aims to address this gap by extending the investigation of oscillatory behavior to encompass nonlinear second-order equations. Building on the work of [30], which explored the oscillatory properties of quasi-linear second-order equations, this paper adapts and extends the approach to include nonlinear terms. Through this extension, new oscillation criteria are introduced, tailored to the distinctive features of nonlinear equations, thereby offering a more comprehensive and nuanced understanding of their oscillatory dynamics.

2. Preliminary results

Let us define

$$\gamma := \begin{cases} 1, & \text{if } \alpha = \beta, \\ \gamma_1, & \text{if } \alpha > \beta; \end{cases}$$

and

$$\widehat{\mathrm{h}}(\mathrm{s}) := \int_a^b \mathrm{h}(\mathrm{s},\ell)\left(1 - \frac{\xi(\tau(\sigma(\mathrm{s},\ell)))}{\xi(\sigma(\mathrm{s},\ell))}\mathrm{u}(\sigma(\mathrm{s},\ell))\right)^{\beta} d\ell, \tag{2.1}$$

for $\mathrm{s} \in [\mathrm{s}_0, \infty)$.

Lemma 2.1. *[32] Assume that* $\mathrm{y(s)}$ *is an eventually positive solution of (1.1), then the corresponding function* $\omega(\mathrm{s})$ *satisfies one of two cases eventually:*

$$\begin{aligned}(\mathrm{C}_1) &: \omega(\mathrm{s}) > 0,\ \omega'(\mathrm{s}) > 0, (\kappa(\mathrm{s})(\omega'(\mathrm{s}))^{\alpha})' < 0,\\ (\mathrm{C}_2) &: \omega(\mathrm{s}) > 0,\ \omega'(\mathrm{s}) < 0, (\kappa(\mathrm{s})(\omega'(\mathrm{s}))^{\alpha})' < 0,\end{aligned}$$

for $\mathrm{s} \geqslant \mathrm{s}_1 \geqslant \mathrm{s}_0$.

The subsequent considerations aim to demonstrate that the class (C_2) is fundamental.

Lemma 2.2. *If*

$$\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{\kappa(\nu)}\int_{\mathrm{s}_0}^{\nu}\widehat{\mathrm{h}}(\varrho)\,d\varrho\right)^{1/\alpha} d\nu = \infty, \tag{2.2}$$

then, the positive solution $\mathrm{y(s)}$ *of (1.1) satisfies* (C_2) *in Lemma 2.1 and, moreover*

$(\mathrm{A}_{1,1})$ $\kappa^{1/\alpha}(\mathrm{s})\omega'(\mathrm{s})\xi(\mathrm{s}) + \omega(\mathrm{s}) \geqslant 0$;
$(\mathrm{A}_{1,2})$ $\omega(\mathrm{s})/\xi(\mathrm{s})$ *is increasing*;
$(\mathrm{A}_{1,3})$ $\omega^{\beta/\alpha-1}(\mathrm{s}) \geq \gamma$;
$(\mathrm{A}_{1,4})$ $(\kappa(\mathrm{s})(\omega'(\mathrm{s}))^{\alpha})' \leq -\omega^{\beta}(\sigma(\mathrm{s},b))\widehat{\mathrm{h}}(\mathrm{s})$;
$(\mathrm{A}_{1,5})$ $\lim_{\mathrm{s}\to\infty}\omega(\mathrm{s}) = 0$.

Proof. Suppose on the contrary that y is a positive solution to (1.1) that meets case (C_1) in Lemma 2.1 for $\mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0$. Then there exists a constant $c_0 > 0$ such that $\omega(\mathrm{s}) \geq c_0$ and $\omega(\sigma(\mathrm{s},\ell)) \geq c_0$ eventually. Using the definition of ω, we deduce that

$$\mathrm{y(s)} = \omega(\mathrm{s}) - \mathrm{u}(\mathrm{s})\,\mathrm{y}(\tau(\mathrm{s})) \geqslant \omega(\mathrm{s}) - \mathrm{u}(\mathrm{s})\,\omega(\tau(\mathrm{s})) \geqslant (1-\mathrm{u}(\mathrm{s}))\,\omega(\mathrm{s}).$$

Then (1.1) becomes

$$\begin{aligned}(\kappa(\mathrm{s})(\omega'(\mathrm{s}))^{\alpha})' &= -\int_a^b \mathrm{h}(\mathrm{s},\ell)\mathrm{y}^{\beta}(\sigma(\mathrm{s},\ell))d\ell\\ &\leq -\int_a^b \mathrm{h}(\mathrm{s},\ell)(1-\mathrm{u}(\sigma(\mathrm{s},\ell)))^{\beta}\,\omega^{\beta}(\sigma(\mathrm{s},\ell))\,d\ell.\end{aligned} \tag{2.3}$$

Since $\xi'(\mathrm{s}) < 0$ and $\tau(\mathrm{s}) \leq \mathrm{s}$, we get

$$\frac{\xi(\tau(\sigma(\mathrm{s},\ell)))}{\xi(\sigma(\mathrm{s},\ell))} \geq 1,$$

and then

$$1 - \mathrm{u}(\sigma(\mathrm{s},\ell)) \geq 1 - \frac{\xi(\tau(\sigma(\mathrm{s},\ell)))}{\xi(\sigma(\mathrm{s},\ell))}\mathrm{u}(\sigma(\mathrm{s},\ell)). \tag{2.4}$$

By combining (2.3) and (2.4) and integrating the resulting inequality from s_1 to ∞, we conclude that

$$\begin{aligned}\kappa(s_1)\left(\omega'(s_1)\right)^{\alpha} &\geq \int_{s_1}^{\infty}\int_{a}^{b} h(s,\ell)\left(1-\frac{\xi\left(\tau\left(\sigma\left(s,\ell\right)\right)\right)}{\xi\left(\sigma\left(s,\ell\right)\right)}u\left(\sigma\left(s,\ell\right)\right)\right)^{\beta}\omega^{\beta}\left(\sigma(s,\ell)\right)d\ell d\varrho \\ &\geq c_0^{\beta}\int_{s_1}^{\infty}\int_{a}^{b} h(s,\ell)\left(1-\frac{\xi\left(\tau\left(\sigma\left(s,\ell\right)\right)\right)}{\xi\left(\sigma\left(s,\ell\right)\right)}u\left(\sigma\left(s,\ell\right)\right)\right)^{\beta} d\ell d\varrho \\ &\geq c_0^{\beta}\int_{s_1}^{\infty}\widehat{h}\left(\varrho\right)d\varrho, \end{aligned} \tag{2.5}$$

It follows from (2.2) and (hyp.5) that $\int_{s_1}^{s}\widehat{h}\left(\varrho\right)d\varrho$ must be unbounded. Furthermore, since $\xi'\left(s\right)<0$, it's clear that

$$\int_{s_1}^{s}\widehat{h}\left(\varrho\right)d\varrho \to \infty \text{ as } s\to\infty, \tag{2.6}$$

which with (2.5) gives a contradiction.

$(A_{1,1})$ Based on case (C_2) of Lemma 2.1, it follows that $\omega\left(s\right)$ is positive and decreases for every $s \geq s_1 \geq s_0$. By the definition of $\omega\left(s\right)$, we obtain $\omega\left(s\right)\geq y\left(s\right)$ and

$$y\left(s\right)\geq\omega\left(s\right)-u\left(s\right)\omega\left(\tau\left(s\right)\right),\ s\geq s_1\geq s_0. \tag{2.7}$$

Since $\kappa\left(s\right)\left(\omega'\left(s\right)\right)^{\alpha}$ is decreasing, we get

$$\kappa^{1/\alpha}\left(s\right)\omega'\left(s\right)\geq\kappa^{1/\alpha}\left(l\right)\omega'\left(l\right)\ \text{ for } l\geq s.$$

Dividing the resulting inequality by $\kappa^{1/\alpha}\left(l\right)$ and then integrating from s to ∞, we get

$$\kappa^{1/\alpha}(s)\omega'(s)\xi(s)+\omega(s)\geqslant 0. \tag{2.8}$$

$(A_{1,2})$ From (2.8), we obtain

$$\left(\frac{\omega(s)}{\xi(s)}\right)' = \frac{\kappa^{1/\alpha}(s)\omega'(s)\xi(s)+\omega(s)}{\kappa^{1/\alpha}(s)\xi^{2}(s)}\geqslant 0.$$

$(A_{1,3})$ In the case where $\alpha=\beta$, it is easy to see that $\omega^{\beta/\alpha-1}\left(s\right)=1$. Now, let $\alpha>\beta$. Since $\omega'(s)<0$, there exists a constant $l>0$, such that

$$\omega(s)\leq l,$$

and consequently,

$$\omega^{\beta/\alpha-1}(s)\geq l^{\beta/\alpha-1}=\gamma_1.$$

$(A_{1,4})$ Since $\omega(s)/\xi(s)$ is increasing, we get

$$\omega(\tau\left(s\right))\leq\frac{\xi(\tau\left(s\right))}{\xi(s)}\omega(s).$$

In view of the definition of ω, we get

$$y(s)=\omega\left(s\right)-u\left(s\right)y\left(\tau\left(s\right)\right)\geqslant\omega\left(s\right)-u\left(s\right)\omega\left(\tau\left(s\right)\right)\geqslant\omega\left(s\right)\left(1-u\left(s\right)\frac{\xi\left(\tau\left(s\right)\right)}{\xi\left(s\right)}\right).$$

Thus, (1.1) becomes

$$\begin{aligned}(\kappa(s)(\omega'(s))^{\alpha})' &= -\int_a^b h(s,\ell)y^{\beta}(\sigma(s,\ell))d\ell \\ &\leq -\int_a^b h(s,\ell)\left(1-u(\sigma(s,\ell))\frac{\xi(\tau(\sigma(s,\ell)))}{\xi(\sigma(s,\ell))}\right)^{\beta}\omega^{\beta}(\sigma(s,\ell))\,d\ell \\ &\leq -\omega^{\beta}(\sigma(s,b))\widehat{h}(s),\end{aligned}$$

that is,

$$(\kappa(s)(\omega'(s))^{\alpha})' \leq -\omega^{\beta}(\sigma(s,b))\widehat{h}(s). \tag{2.9}$$

($A_{1,5}$) Since $\omega(s) > 0$, and $\omega'(s) < 0$, then $\lim_{s\to\infty}\omega(s) = c_1 \geq 0$. We assert that $c_1 = 0$. If not, $\omega(s) \geq c_1 > 0$ for $s \geq s_2 \geq s_1$. Integrating (1.1) from s_1 to s yields

$$\kappa(s)(\omega'(s))^{\alpha} \leq \kappa(s_1)(\omega'(s_1))^{\alpha} - \int_{s1}^{s}\omega^{\beta}(\sigma(\varrho,b))\widehat{h}(\varrho)\,d\varrho \leq -c_1^{\beta}\int_{s_1}^{s}\widetilde{h}(\varrho)\,d\varrho,$$

and so

$$\omega'(s) \leq -\frac{c_1^{\beta/\alpha}}{\kappa^{1/\alpha}(s)}\left(\int_{s_1}^{s}\widetilde{h}(\varrho)\,d\varrho\right)^{1/\alpha}.$$

Integrating this inequality from s_1 to ∞, we find

$$\omega(s_1) \geq c_1^{\beta/\alpha}\int_{s_1}^{\infty}\left(\frac{1}{\kappa(\nu)}\int_{s1}^{\nu}\widetilde{h}(\varrho)\,d\varrho\right)^{1/\alpha}d\nu \to \infty \text{ as } s\to\infty,$$

which contradicts (2.2). Therefore, $c_1 = 0$.

As a result, the lemma has been completely proven. □

3. Main results

In this section, we will discuss new monotonic properties for the solutions of (1.1).

Lemma 3.1. *Let* y(s) *be a positive solution of (1.1), and assume that (2.2) holds. If* $\delta_0 \in (0,1)$ *with*

$$\frac{1}{\alpha}\kappa^{1/\alpha}(s)\widehat{h}(s)\xi^{\alpha+1}(s) \geqslant \delta_0^{\alpha},\ \ \rho_0 = \gamma\delta_0, \tag{3.1}$$

then
($A_{2,1}$) $\omega(s)/\xi^{\rho_0}(s)$ *is decreasing;*
($A_{2,2}$) $\lim_{s\to\infty}\omega(s)/\xi^{\rho_0}(s) = 0$;
($A_{2,3}$) $\omega(s)/\xi^{1-\rho_0}(s)$ *is increasing.*

Proof. For the purposes of this discussion, let y(s) be an eventually positive solution of (1.1). From (3.1), it follows that:

$$\int_{s_0}^{\infty}\left(\frac{1}{\kappa(\nu)}\int_{s_1}^{\nu}\widehat{h}(\varrho)\,d\varrho\right)^{1/\alpha}d\nu \ \geq\ \alpha^{1/\alpha}\delta_0\int_{s_0}^{\infty}\left(\frac{1}{\kappa(\nu)}\int_{s_1}^{\nu}\frac{1}{\kappa^{1/\alpha}(\varrho)\xi^{\alpha+1}(\varrho)}d\varrho\right)^{1/\alpha}d\nu$$

$$= \alpha^{1/\alpha}\delta_0 \int_{s_0}^{\infty} \frac{1}{\kappa^{1/\alpha}(\nu)} \left(\int_{s_1}^{\nu} \frac{1}{\kappa^{1/\alpha}(\varrho)\,\xi^{\alpha+1}(\varrho)} d\varrho \right)^{1/\alpha} d\nu$$
$$= \delta_0 \int_{s_0}^{\infty} \frac{1}{\kappa^{1/\alpha}(\nu)} \left(\xi^{-\alpha}(\nu) - \xi^{-\alpha}(s_1)\right)^{1/\alpha} d\nu.$$

From $(A_{1,5})$, we know that $\lim_{s\to\infty} \omega(s) = 0$. Then, there exists $s_1 \geq s_0$ such that $\xi^{-\alpha}(s) - \xi^{-\alpha}(s_1) \geq \epsilon\xi^{-\alpha}(s)$ where $\epsilon \in (0, 1)$. Thus, we have

$$\int_{s_0}^{\infty} \left(\frac{1}{\kappa(\nu)} \int_{s_1}^{\nu} \widehat{h}(\varrho)\, d\varrho \right)^{1/\alpha} d\nu \geq \epsilon^{1/\alpha}\delta_0 \int_{s_0}^{\infty} \frac{1}{\kappa^{1/\alpha}(\nu)\xi(\nu)} d\nu$$
$$= \epsilon^{1/\alpha}\delta_0 \lim_{s\to\infty} \ln \frac{\xi(s_0)}{\xi(s)} \to \infty.$$

Hence, from Lemma 2.2, we have that $(A_{1,1})$–$(A_{1,4})$ hold.

$(A_{2,1})$ Integrating $(A_{1,4})$ from s_1 to s, we obtain

$$-\kappa(s)\left(\omega'(s)\right)^{\alpha} \geq -\kappa(s_1)\left(\omega'(s_1)\right)^{\alpha} + \int_{s_1}^{s} \omega^{\beta}\left(\sigma(\varrho, b)\right) \widehat{h}(\varrho)\, d\varrho$$
$$\geq -\kappa(s_1)\left(\omega'(s_1)\right)^{\alpha} + \omega^{\beta}\left(\sigma(s, b)\right) \int_{s_1}^{s} \widehat{h}(\varrho)\, d\varrho.$$

By using (3.1), we get

$$-\kappa(s)\left(\omega'(s)\right)^{\alpha} \geqslant -\kappa(s_1)\left(\omega'(s_1)\right)^{\alpha} + \omega^{\beta}(s) \int_{s_1}^{s} \frac{\alpha\delta_0^{\alpha}}{\kappa^{1/\alpha}(\varrho)\,\xi^{\alpha+1}(\varrho)} d\varrho$$
$$= -\kappa(s_1)\left(\omega'(s_1)\right)^{\alpha} + \delta_0^{\alpha}\frac{\omega^{\beta}(s)}{\xi^{\alpha}(s)} - \delta_0^{\alpha}\frac{\omega^{\beta}(s)}{\xi^{\alpha}(s_1)}. \tag{3.2}$$

Since $\omega(s) \to 0$ as $t \to \infty$, as stated in $(A_{1,5})$, we have

$$-\kappa(s_1)\left(\omega'(s_1)\right)^{\alpha} - \delta_0^{\alpha}\frac{\omega^{\beta}(s)}{\xi^{\alpha}(s_1)} \geq 0, \ s \geq s_2,$$

and so, (3.2) becomes

$$-\kappa^{1/\alpha}(s)\omega'(s) \geqslant \delta_0 \frac{\omega^{\beta/\alpha}(s)}{\xi(s)},$$

and so,

$$\kappa^{1/\alpha}(s)\xi(s)\omega'(s) + \delta_0\omega^{\beta/\alpha}(s) \leq 0. \tag{3.3}$$

Furthermore, from $(A_{1,3})$, we see that

$$\kappa^{1/\alpha}(s)\xi(s)\omega'(s) + \gamma\delta_0\omega(s) \leq \kappa^{1/\alpha}(s)\xi(s)\omega'(s) + \delta_0\omega^{\beta/\alpha}(s) \leq 0.$$

This results in

$$\kappa^{1/\alpha}(s)\xi(s)\omega'(s) + \rho_0\omega(s) \leq 0. \tag{3.4}$$

Consequently,

$$\left(\frac{\omega(s)}{\xi^{\rho_0}(s)}\right)' = \frac{\kappa^{1/\alpha}(s)\xi(s)\omega'(s) + \rho_0\omega(s)}{\kappa^{1/\alpha}(s)\xi^{1+\rho_0}(s)} \le 0.$$

$(A_{2,2})$ Since $\omega(s)/\xi^{\rho_0}(s)$ is positive and decreasing, $\lim_{s\to\infty}\omega(s)/\xi^{\rho_0}(s) = c_1 \geqslant 0$. We assert that $c_2 = 0$. If not, eventually $\omega(s)/\xi^{\rho_0}(s) \geqslant c_2 > 0$. We now present the function

$$w(s) = \left(\kappa^{1/\alpha}(s)\omega'(s)\xi(s) + \omega(s)\right)\xi^{-\rho_0}(s).$$

We observe that $w(s) > 0$ in context of $(A_{1,1})$ in Lemma 2.2, and

$$\begin{aligned}
w'(s) &= \left(\kappa^{1/\alpha}(s)\omega'(s)\right)'\xi^{1-\rho_0}(s) - (1-\rho_0)\,\omega'(s)\xi^{-\rho_0}(s) + \omega'(s)\xi^{-\rho_0}(s) + \rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} \\
&= \frac{1}{\alpha}\left(\kappa(s)\left(\omega'(s)\right)^{\alpha}\right)'\left(\kappa^{1/\alpha}(s)\omega'(s)\right)^{1-\alpha}\xi^{1-\rho_0}(s) + \rho_0\omega'(s)\xi^{-\rho_0}(s) + \rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} \\
&= -\frac{1}{\alpha}\left(\kappa^{1/\alpha}(s)\omega'(s)\right)^{1-\alpha}\xi^{1-\rho_0}(s)\int_a^b h(s,\ell)y^{\beta}(\sigma(s,\ell))d\ell + \rho_0\omega'(s)\xi^{-\rho_0}(s) + \rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} \\
&\le -\frac{1}{\alpha}\left(\kappa^{1/\alpha}(s)\omega'(s)\right)^{1-\alpha}\xi^{1-\rho_0}(s)\omega^{\beta}(\sigma(s,b))\widehat{h}(s) + \rho_0\omega'(s)\xi^{-\rho_0} + \rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)}.
\end{aligned}$$

By using $(A_{1,3})$, (3.1), (3.3) and (3.4), we find

$$\begin{aligned}
w'(s) &\le -\left(\frac{\delta_0\omega^{\beta/\alpha}(s)}{\xi(s)}\right)^{1-\alpha}\xi^{1-\rho_0}(s)\frac{\delta_0^{\alpha}}{\kappa^{1/\alpha}(s)\xi^{\alpha+1}(s)}\omega^{\beta}(s) + \rho_0\omega'(s)\xi^{-\rho_0} + \rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} \\
&\le -\delta_0\omega^{\beta/\alpha}(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} + \rho_0\omega'(s)\xi^{-\rho_0}(s) + \rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} \\
&\le -\gamma\delta_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} + \rho_0\omega'(s)\xi^{-\rho_0}(s) + \rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} \\
&\le -\rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} + \rho_0\omega'(s)\xi^{-\rho_0}(s) + \rho_0\omega(s)\frac{\xi^{-1-\rho_0}(s)}{\kappa^{1/\alpha}(s)} \\
&\le \rho_0\omega'(s)\xi^{-\rho_0}(s) \le -\rho_0\xi^{-\rho_0}(s)\frac{\rho_0\omega(s)}{\kappa^{1/\alpha}(s)\xi(s)} \le -\frac{\rho_0^2}{\kappa^{1/\alpha}(s)\xi(s)}\frac{\omega(s)}{\xi^{\rho_0}(s)}.
\end{aligned}$$

Using the fact that $\omega(s)/\xi^{\rho_0}(s) \geqslant c_2$, we get

$$w'(s) \leqslant -\frac{\rho_0^2 c_2}{\kappa^{1/\alpha}(s)\xi(s)} < 0.$$

When we integrate the previous inequality from s_1 to s, we get

$$w(s_1) \ge \rho_0^2 c_2 \ln\frac{\xi(s_1)}{\xi(s)} \to \infty \text{ as } s \to \infty,$$

which is a contradiction. Thus, $c_2 = 0$.

$(A_{2,3})$ Finally, we have

$$\left(\kappa^{1/\alpha}(s)\omega'(s)\xi(s) + \omega(s)\right)' = \left(\kappa^{1/\alpha}(s)\omega'(s)\right)'\xi(s) - \omega'(s) + \omega'(s)$$

$$
\begin{aligned}
&= \left(\kappa^{1/\alpha}(\mathrm{s})\omega'(\mathrm{s})\right)'\xi(\mathrm{s}) \\
&= \frac{1}{\alpha}\left(\kappa(\mathrm{s})\left(\omega'(\mathrm{s})\right)^{\alpha}\right)'\left(\kappa^{1/\alpha}(\mathrm{s})\,\omega'(\mathrm{s})\right)^{1-\alpha}\xi(\mathrm{s}) \\
&\le -\frac{1}{\alpha}\widehat{\mathrm{h}}(\mathrm{s})\,\omega^{\beta}(\mathrm{s})\left(\kappa^{1/\alpha}(\mathrm{s})\,\omega'(\mathrm{s})\right)^{1-\alpha}\xi(\mathrm{s}) \\
&\le -\delta_0^{\alpha}\frac{1}{\kappa^{1/\alpha}(\mathrm{s})\,\xi^{1+\alpha}(\mathrm{s})}\omega^{\beta}(\mathrm{s})\left(-\delta_0\frac{\omega^{\beta/\alpha}(\mathrm{s})}{\xi(\mathrm{s})}\right)^{1-\alpha}\xi(\mathrm{s}) \\
&\le -\delta_0^{\alpha}\frac{1}{\kappa^{1/\alpha}(\mathrm{s})\,\xi^{\alpha}(\mathrm{s})}\omega^{\beta}(\mathrm{s})\left(\delta_0\frac{\omega^{\beta/\alpha}(\mathrm{s})}{\xi(\mathrm{s})}\right)^{1-\alpha} \\
&\le \frac{-\delta_0}{\kappa^{1/\alpha}(\mathrm{s})\,\xi(\mathrm{s})}\omega^{\beta/\alpha}(\mathrm{s}) \\
&\le \frac{-\gamma\delta_0}{\kappa^{1/\alpha}(\mathrm{s})\,\xi(\mathrm{s})}\omega(\mathrm{s}) \le \frac{-\rho_0}{\kappa^{1/\alpha}(\mathrm{s})\,\xi(\mathrm{s})}\omega(\mathrm{s}).
\end{aligned}
$$

When we integrate the previous inequality from s to ∞, we get

$$
\kappa^{1/\alpha}(\mathrm{s})\omega'(\mathrm{s})\,\xi(\mathrm{s}) + \omega(\mathrm{s}) \ge \rho_0\int_{\mathrm{s}}^{\infty}\frac{1}{\kappa^{1/\alpha}(\varrho)}\frac{\omega(\varrho)}{\xi(\varrho)}\mathrm{d}\varrho \ge \rho_0\frac{\omega(\mathrm{s})}{\xi(\mathrm{s})}\int_{\mathrm{s}}^{\infty}\frac{1}{\kappa^{1/\alpha}(\varrho)}\mathrm{d}\varrho \ge \rho_0\omega(\mathrm{s}).
$$

Thus

$$
\kappa^{1/\alpha}(\mathrm{s})\omega'(\mathrm{s})\,\xi(\mathrm{s}) + (1-\rho_0)\,\omega(\mathrm{s}) \ge 0,
$$

and hence

$$
\left(\frac{\omega(\mathrm{s})}{\xi^{1-\rho_0}(\mathrm{s})}\right)' = \frac{\kappa^{1/\alpha}(\mathrm{s})\xi(\mathrm{s})\omega'(\mathrm{s}) + (1-\rho_0)\omega(\mathrm{s})}{\kappa^{1/\alpha}(\mathrm{s})\xi^{2-\rho_0}(\mathrm{s})} \geqslant 0.
$$

Hence, the proof is complete. □

Theorem 3.1. *Assume that (2.2) and (3.1) hold. If*

$$
\rho_0 > \frac{1}{2}, \tag{3.5}
$$

then, (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that y is an eventually positive solution of (1.1). Referring to the proof of Lemma 3.1, we obtain

$$
\kappa^{1/\alpha}(\mathrm{s})\omega'(\mathrm{s})\,\xi(\mathrm{s}) + \rho_0\omega(\mathrm{s}) \le 0, \tag{3.6}
$$

and

$$
\kappa^{1/\alpha}(\mathrm{s})\omega'(\mathrm{s})\,\xi(\mathrm{s}) + (1-\rho_0)\,\omega(\mathrm{s}) \ge 0. \tag{3.7}
$$

By combining (3.6) and (3.7), we find

$$
\begin{aligned}
0 &\le \kappa^{1/\alpha}(\mathrm{s})\omega'(\mathrm{s})\,\xi(\mathrm{s}) + (1-\rho_0)\,\omega(\mathrm{s}) \\
&= \kappa^{1/\alpha}(\mathrm{s})\omega'(\mathrm{s})\,\xi(\mathrm{s}) + \rho_0\omega(\mathrm{s}) + (1-2\rho_0)\,\omega(\mathrm{s}) \\
&\le (1-2\rho_0)\,\omega(\mathrm{s}).
\end{aligned}
$$

Since $\omega(\mathrm{s}) > 0$, it must hold that $1 - 2\rho_0 \geq 0$, which implies that

$$\rho_0 \leq 1/2,$$

which leads to a contradiction. This completes the proof. □

When $\rho_0 \leq \frac{1}{2}$, it is possible to refine the results stated in Lemma 3.1. Since $\xi(\mathrm{s})$ is a decreasing function, there exists a constant $\lambda \geq 1$ such that

$$\frac{\xi(\sigma(\mathrm{s}, b))}{\xi(\mathrm{s})} \geq \lambda. \tag{3.8}$$

We introduce the constant $\rho_1 > \rho_0$ as follows

$$\rho_1 = \rho_0 \sqrt[\alpha]{\frac{\lambda^{\beta\rho_0}}{1 - \frac{\beta}{\alpha}\rho_0}}. \tag{3.9}$$

Lemma 3.2. *Assume (2.2) and (3.1) hold. If* $\mathrm{y}(\mathrm{s})$ *is a positive solution of (1.1), then*
$(A_{3,1})$ $\omega(\mathrm{s})/\xi^{\rho_1}(\mathrm{s})$ *is decreasing;*
$(A_{3,2})$ $\lim_{\mathrm{s}\to\infty} \omega(\mathrm{s})/\xi^{\rho_1}(\mathrm{s}) = 0$;
$(A_{3,3})$ $\omega(\mathrm{s})/\xi^{1-\rho_1}(\mathrm{s})$ *is increasing.*

Proof. Assume that $\mathrm{y}(\mathrm{s})$ is an eventually positive solution of (1.1) satisfying condition (C_2) in Lemma 2.1 for $\mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0$. From Lemma 2.2, we have that $(A_{1,1})$–$(A_{1,5})$ hold. Additionally, Lemma 3.1 implies that conditions $(A_{2,1})$-$(A_{2,3})$ are satisfied.

$(A_{3,1})$ Integrating $(A_{1,4})$ from s_1 to s, we get

$$-\kappa(\mathrm{s})\left(\omega'(\mathrm{s})\right)^{\alpha} \geqslant -\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} + \int_{\mathrm{s}_1}^{\mathrm{s}} \omega^{\beta}\left(\sigma\left(\varrho, b\right)\right)\widehat{\mathrm{h}}\left(\varrho\right)\mathrm{d}\varrho.$$

By using the fact $\omega(\mathrm{s})/\xi^{\rho_0}(\mathrm{s})$ is decreasing, we have

$$\begin{aligned} -\kappa(\mathrm{s})\left(\omega'(\mathrm{s})\right)^{\alpha} &\geqslant -\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} + \int_{\mathrm{s}_1}^{\mathrm{s}} \left(\frac{\omega(\varrho, b)}{\xi^{\rho_0}\left(\varrho, b\right)}\right)^{\beta} \xi^{\beta\rho_0}(\sigma\left(\varrho, b\right))\widehat{\mathrm{h}}\left(\varrho\right)\mathrm{d}\varrho \\ &\geqslant -\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} + \left(\frac{\omega(\mathrm{s}, b)}{\xi^{\rho_0}\left(\mathrm{s}, b\right)}\right)^{\beta} \int_{\mathrm{s}_1}^{\mathrm{s}} \xi^{\beta\rho_0}(\sigma\left(\varrho, b\right))\widehat{\mathrm{h}}\left(\varrho\right)\mathrm{d}\varrho. \end{aligned}$$

By using (3.1) and (3.8), we get

$$\begin{aligned} -\kappa(\mathrm{s})\left(\omega'(\mathrm{s})\right)^{\alpha} &\geqslant -\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} + \left(\frac{\omega(\mathrm{s})}{\xi^{\rho_0}\left(\mathrm{s}\right)}\right)^{\beta} \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\alpha\delta_0^{\alpha}}{\kappa^{1/\alpha}(\varrho)\xi^{\alpha+1}(\varrho)} \xi^{\beta\rho_0}(\sigma\left(\left(\varrho, b\right)\right))\mathrm{d}\varrho \\ &\geq -\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} + \left(\frac{\omega(\mathrm{s})}{\xi^{\rho_0}\left(\mathrm{s}\right)}\right)^{\beta} \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\alpha\delta_0^{\alpha}\lambda^{\beta\rho_0}}{\kappa^{1/\alpha}(\varrho)\xi^{\alpha+1}(\varrho)} \xi^{\beta\rho_0}(\varrho)\mathrm{d}\varrho \\ &\geqslant -\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} + \alpha\delta_0^{\alpha}\lambda^{\beta\rho_0}\left(\frac{\omega(\mathrm{s})}{\xi^{\rho_0}(\mathrm{s})}\right)^{\beta} \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\xi^{-1-\alpha+\beta\rho_0}(\varrho)}{\kappa^{1/\alpha}(\varrho)}\mathrm{d}\varrho \\ &\geqslant -\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} + \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1 - \frac{\beta}{\alpha}\rho_0)}\left(\frac{\omega(\mathrm{s})}{\xi^{\rho_0}(\mathrm{s})}\right)^{\beta}\left[\xi^{\beta\rho_0-\alpha}(\mathrm{s}) - \xi^{\beta\rho_0-\alpha}(\mathrm{s}_1)\right] \end{aligned}$$

$$\geqslant \quad -\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} - \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\xi^{\beta\rho_0-\alpha}(\mathrm{s}_1)\left(\frac{\omega(\mathrm{s})}{\xi^{\rho_0}(\mathrm{s})}\right)^{\beta} + \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\frac{\omega^{\beta}(\mathrm{s})}{\xi^{\alpha}(\mathrm{s})}.$$

Since $\frac{\omega(\mathrm{s})}{\xi^{\rho_0}(\mathrm{s})} \to 0$ as $t \to \infty$, as stated in $(A_{2,2})$, we have

$$-\kappa(\mathrm{s}_1)\left(\omega'(\mathrm{s}_1)\right)^{\alpha} - \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\xi^{\beta\rho_0-\alpha}(\mathrm{s}_1)\left(\frac{\omega(\mathrm{s})}{\xi^{\rho_0}(\mathrm{s})}\right)^{\beta} \geq 0,$$

and hence

$$-\kappa(\mathrm{s})\left(\omega'(\mathrm{s})\right)^{\alpha} \geqslant \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\frac{\omega^{\beta}(\mathrm{s})}{\xi^{\alpha}(\mathrm{s})}.$$

This implies that

$$\begin{aligned}
\omega'(\mathrm{s}) &\geq \delta_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\xi(\mathrm{s})\,\kappa^{1/\alpha}(\mathrm{s})}\omega^{\beta/\alpha}(\mathrm{s}) \\
&\geq \gamma\delta_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\xi(\mathrm{s})\,\kappa^{1/\alpha}(\mathrm{s})}\omega(\mathrm{s}) \\
&= \rho_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\xi(\mathrm{s})\,\kappa^{1/\alpha}(\mathrm{s})}\omega(\mathrm{s}) \\
&= \rho_1\frac{1}{\xi(\mathrm{s})\,\kappa^{1/\alpha}(\mathrm{s})}\omega(\mathrm{s}),
\end{aligned}$$

which is equivalent to

$$\kappa^{1/\alpha}(\mathrm{s})\xi(\mathrm{s})\omega'(\mathrm{s}) + \rho_1\omega(\mathrm{s}) \leq 0. \tag{3.10}$$

Consequently,

$$\left(\frac{\omega(\mathrm{s})}{\xi^{\rho_1}(\mathrm{s})}\right)' = \frac{\kappa^{1/\alpha}(\mathrm{s})\xi(\mathrm{s})\omega'(\mathrm{s}) + \rho_1\omega(\mathrm{s})}{\kappa^{1/\alpha}(\mathrm{s})\xi^{1+\rho_1}(\mathrm{s})} \leqslant 0.$$

So $\omega(\mathrm{s})/\xi^{\rho_1}(\mathrm{s})$ is decreasing.

The same procedures as in the Lemma 3.1 proof can be used to verify that conditions $(A_{3,2})$ and $(A_{3,3})$ are satisfied. □

If $\rho_1 < 1/2$, we can repeat the previous process and deduce that $\delta_2 > \delta_1$ as follows

$$\rho_2 = \rho_0\sqrt[\alpha]{\frac{\lambda^{\beta\rho_1}}{1-\frac{\beta}{\alpha}\rho_1}}.$$

More generally, if $\rho_i < 1/2$ for $i = 1, 2, ..., n-1$, it is possible to describe

$$\rho_n = \rho_0\sqrt[\alpha]{\frac{\lambda^{\beta\rho_{n-1}}}{1-\frac{\beta}{\alpha}\rho_{n-1}}}. \tag{3.11}$$

Additionally, by taking the identical actions as in the Lemma 3.2 proof, we may verify the following:
($A_{n,1}$) $\omega(s)/\xi^{\rho_n}(s)$ is decreasing;
($A_{n,2}$) $\lim_{s\to\infty}\omega(s)/\xi^{\rho_n}(s) = 0$;
($A_{n,3}$) $\omega(s)/\xi^{1-\rho_n}(s)$ is increasing.

Theorem 3.2. *Let (2.2) and (3.1) hold. If there exists a $n \in \mathbb{N}$ such that*

$$\rho_n > \frac{1}{2}, \tag{3.12}$$

then (1.1) is oscillatory.

Theorem 3.3. *Let (2.2) and (3.1) hold. If there exists $n \in \mathbb{N}$ such that*

$$\liminf_{s\to\infty} \int_{\sigma(s,b)}^{s} \frac{\xi(\varrho)\widehat{h}(\varrho)}{\xi^{1-\alpha}(\sigma(\varrho,b))} d\varrho > \frac{\alpha\gamma^{-\alpha}\rho_n^{\alpha-1}(1-\rho_n)}{e}, \tag{3.13}$$

then (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that y(s) is an eventually positive solution of (1.1). Condition (2.2) guarantees that y(s) satisfies (C_2). From Lemma 2.2, we have that ($A_{1,1}$)–($A_{1,4}$) hold. We generate the sequence $\{\rho_n\}$ using (3.11).

We now define the function:

$$\Psi(s) = \kappa^{1/\alpha}(s)\omega'(s)\xi(s) + \omega(s). \tag{3.14}$$

Based on ($A_{1,1}$) in Lemma 2.2, we can conclude that $\Psi(s) \geq 0$. Furthermore, from ($A_{n,1}$), we can derive

$$\kappa^{1/\alpha}(s)\omega'(s)\xi(s) + \rho_n\omega(s) \leq 0.$$

Next, based on the definition of $\Psi(s)$, we get

$$\begin{aligned} \Psi(s) &= \kappa^{1/\alpha}(s)\omega'(s)\xi(s) + \rho_n\omega(s) - \rho_n\omega(s) + \omega(s) \\ &\leq (1-\rho_n)\omega(s). \end{aligned} \tag{3.15}$$

From (3.14), ($A_{1,4}$) and (3.4), we deduce that

$$\begin{aligned} \Psi'(s) &= \left(\kappa^{1/\alpha}(s)\omega'(s)\right)'\xi(s) - \omega'(s) + \omega'(s) \\ &= \left(\kappa^{1/\alpha}(s)\omega'(s)\right)'\xi(s) \\ &\leq \frac{1}{\alpha}\left(\kappa(s)(\omega'(s))^{\alpha}\right)'\left(\kappa^{1/\alpha}(s)\omega'(s)\right)^{1-\alpha}\xi(s) \\ &\leq -\frac{1}{\alpha}\widehat{h}(s)\omega^{\beta}(\sigma(s,b))\left(\kappa^{1/\alpha}(s)\omega'(s)\right)^{1-\alpha}\xi(s) \\ &\leq -\frac{1}{\alpha}\widehat{h}(s)\omega^{\beta}(\sigma(s,b))\left(\rho_n\frac{\omega(s)}{\xi(s)}\right)^{1-\alpha}\xi(s) \\ &\leq -\frac{1}{\alpha}\rho_n^{1-\alpha}\xi(s)\widehat{h}(s)\omega^{\beta}(\sigma(s,b))\left(\frac{\omega(s)}{\xi(s)}\right)^{1-\alpha}. \end{aligned} \tag{3.16}$$

We observe that $\omega(\mathrm{s})/\xi(\mathrm{s})$ is increasing from $(A_{1,2})$ in Lemma 2.2, then

$$\frac{\omega(\sigma(\mathrm{s},b))}{\xi(\sigma(\mathrm{s},b))}\leq\frac{\omega(\mathrm{s})}{\xi(\mathrm{s})}.$$

Considering that $0<\alpha\leq 1$, then

$$\left(\frac{\omega(\sigma(\mathrm{s},b))}{\xi(\sigma(\mathrm{s},b))}\right)^{1-\alpha}\leq\left(\frac{\omega(\mathrm{s})}{\xi(\mathrm{s})}\right)^{1-\alpha}.$$

Hence, (3.16) yields

$$\begin{aligned}\Psi'(\mathrm{s}) &\leq -\frac{1}{\alpha}\rho_n^{1-\alpha}\widehat{\mathrm{h}}(\mathrm{s})\,\xi(\mathrm{s})\,\omega^{\beta}(\sigma(\mathrm{s},b))\left(\frac{\omega(\sigma(\mathrm{s},b))}{\xi(\sigma(\mathrm{s},b))}\right)^{1-\alpha}\\ &\leq -\frac{1}{\alpha}\rho_n^{1-\alpha}\widehat{\mathrm{h}}(\mathrm{s})\,\frac{\xi(\mathrm{s})}{\xi^{1-\alpha}(\sigma(\mathrm{s},b))}\omega^{\beta-\alpha}(\sigma(\mathrm{s},b))\,\omega(\sigma(\mathrm{s},b)).\end{aligned}$$

From $(A_{1,3})$ we know that $\omega^{\beta-\alpha}(\sigma(\mathrm{s},b))\geq\gamma^{\alpha}$. Therefore the above inequality leads to

$$\Psi'(\mathrm{s})\leq-\frac{\gamma^{\alpha}}{\alpha}\rho_n^{1-\alpha}\widehat{\mathrm{h}}(\mathrm{s})\,\frac{\xi(\mathrm{s})}{\xi^{1-\alpha}(\sigma(\mathrm{s},b))}\omega(\sigma(\mathrm{s},b)).$$

By using (3.15) we see that $w(\mathrm{s})$ is a positive solution of

$$\Psi'(\mathrm{s})+\frac{\gamma^{\alpha}}{\alpha}\frac{\rho_n^{1-\alpha}}{(1-\rho_n)}\frac{\xi(\mathrm{s})\,\widehat{\mathrm{h}}(\mathrm{s})}{\xi^{1-\alpha}(\sigma(\mathrm{s},b))}\Psi(\sigma(\mathrm{s},b))\leq 0. \tag{3.17}$$

This results in a contradiction, as Theorem 2.1.1 in [33] ensures that condition (3.13) implies (3.17) has no positive solution. This contradiction concludes the proof of the theorem. □

4. Examples

We provide examples to demonstrate the significance of the obtained results.

Example 4.1. *Consider*

$$\left(\mathrm{s}^{2\alpha}\left(\left(\mathrm{y}(\mathrm{s})+\mathrm{u}_0\mathrm{y}(\tau_0\mathrm{s})\right)'\right)^{\alpha}\right)'+\int_a^b \mathrm{h}_0\mathrm{s}^{\alpha-1}\mathrm{y}^{\beta}(\sigma_0\mathrm{s}\ell)\,\mathrm{d}\ell=0,\ \ \mathrm{s}\geq 1, \tag{4.1}$$

where $\alpha\geq\beta$, $0\leq\mathrm{u}_0<1$, τ_0, $\sigma_0\in(0,1)$, $\sigma_0\ell\leq 1$, *and* $\mathrm{h}_0>0$. *When comparing (1.1) and (4.1), we can see that* $\kappa(\mathrm{s})=\mathrm{s}^{2\alpha}$, $\mathrm{h}(\mathrm{s},\ell)=\mathrm{h}_0\mathrm{s}^{\alpha-1}$, $\mathrm{u}(\mathrm{s})=\mathrm{u}_0$, $\sigma(\mathrm{s},\ell)=\sigma_0\mathrm{s}\ell$, *and* $\tau(\mathrm{s})=\tau_0\mathrm{s}$. *It is easy to find that*

$$\xi(\mathrm{s})=\frac{1}{\mathrm{s}},\ \frac{\xi(\tau(\sigma(\mathrm{s},\ell)))}{\xi(\sigma(\mathrm{s},\ell))}=\frac{1}{\tau_0},$$

and

$$\widehat{\mathrm{h}}(\mathrm{s})=(b-a)\,\mathrm{h}_0\mathrm{s}^{\alpha-1}\left(1-\frac{1}{\tau_0}\mathrm{u}_0\right)^{\beta}.$$

For (3.1), we set

$$\delta_0 = \sqrt[\alpha]{\frac{h_0 (b-a) \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta}}{\alpha}}.$$

Applying (3.8), we obtain $\lambda = \frac{1}{\sigma_0}$. *Now, we define the sequence* $\{\rho_n\}_{n=1}^{m}$ *as*

$$\rho_n = \rho_0 \sqrt[\alpha]{\frac{1}{1 - \frac{\beta}{\alpha}\rho_{n-1}} \left(\frac{1}{\sigma_0}\right)^{\beta\rho_{n-1}}},$$

with

$$\rho_0 = \gamma \sqrt[\alpha]{\frac{h_0 (b-a) \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta}}{\alpha}}.$$

Then, condition (3.5) reduces to

$$h_0 > \frac{\alpha}{(b-a) 2^{\alpha} \gamma^{\alpha} \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta}}, \tag{4.2}$$

and condition (3.13) becomes

$$\begin{aligned}
\liminf_{s\to\infty} \int_{\sigma(s,b)}^{s} \frac{\xi(\varrho)\widehat{h}(\varrho)}{\xi^{1-\alpha}(\sigma(\varrho,b))} d\varrho &= \liminf_{s\to\infty} \int_{\sigma_0 b s}^{s} \frac{\xi(\varrho)\widehat{h}(\varrho)}{\xi^{1-\alpha}(\sigma_0 b \varrho)} d\varrho \\
&= \liminf_{s\to\infty} \int_{\sigma_0 b s}^{s} \frac{1}{\varrho} \varrho^{1-\alpha} (\sigma_0 b)^{\alpha-1} (b-a) h_0 \varrho^{\alpha-1} \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta} d\varrho \\
&= \liminf_{s\to\infty} \int_{\sigma_0 b s}^{s} (\sigma_0 b)^{\alpha-1} (b-a) h_0 \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta} \frac{1}{\varrho} d\varrho \\
&= (b-a)(\sigma_0 b)^{\alpha-1} h_0 \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta} \liminf_{s\to\infty} \int_{\sigma_0 b s}^{s} \frac{1}{\varrho} d\varrho \\
&= (b-a)(\sigma_0 b)^{\alpha-1} h_0 \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta} \liminf_{s\to\infty} \ln \frac{1}{\sigma_0 b} \\
&= (b-a)(\sigma_0 b)^{\alpha-1} h_0 \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta} \ln \frac{1}{\sigma_0 b},
\end{aligned}$$

which leads to

$$h_0 > \frac{\alpha \gamma^{-\alpha} \rho_n^{\alpha-1} (1-\rho_n)}{(b-a)(\sigma_0 b)^{\alpha-1} \left(1 - \frac{1}{\tau_0} u_0\right)^{\beta} \ln \frac{1}{\sigma_0 b}} \frac{1}{e}. \tag{4.3}$$

Theorems 3.1 and 3.3 show that the solution of (4.1) is oscillatory if either (4.2) or (4.3) holds.

Example 4.2. *Consider the NDE*

$$\left(s^{2/3} \left(\left(y(s) + \frac{1}{4} y\left(\frac{1}{2} s\right)\right)'\right)^{1/3}\right)' + \int_{1/2}^{1} h_0 s^{-2/3} y^{1/5} \left(\frac{\ell}{3} s\right) d\ell = 0. \tag{4.4}$$

Clearly:

$a=1/2$, $b=1$, $\alpha=1/3$, $\beta=1/5$, $\kappa(\mathrm{s})=\mathrm{s}^{2/3}$, $\mathrm{h}(\mathrm{s},\ell)=\mathrm{h}_0\mathrm{s}^{-2/3}$, $\mathrm{u}(\mathrm{s})=1/4$, $\sigma(\mathrm{s},\ell)=\frac{\ell}{3}\mathrm{s}$ and $\tau(\mathrm{s})=\frac{1}{2}\mathrm{s}$. It is easy to find that

$$\xi(\mathrm{s})=\frac{1}{\mathrm{s}},\ \frac{\xi(\tau(\sigma(\mathrm{s},\ell)))}{\xi(\sigma(\mathrm{s},\ell))}=2,$$

and

$$\widehat{\mathrm{h}}(\mathrm{s})=\frac{1}{2^{6/5}}\mathrm{h}_0\mathrm{s}^{-2/3}.$$

For (3.1), we set

$$\delta_0=2.2267\mathrm{h}_0^3.$$

Using (3.8), we have $\lambda=3$. Here, we define the sequence $\{\rho_n\}_{n=1}^m$ as

$$\rho_n=\rho_0\frac{1}{\left(1-\frac{3}{5}\rho_{n-1}\right)^3}3^{\frac{3\rho_{n-1}}{5}},$$

with

$$\rho_0=2.2267\mathrm{h}_0^3\gamma,\ \gamma>0.$$

Then, condition (3.12) reduces to

$$\mathrm{h}_0>\frac{0.60781}{\sqrt[3]{\gamma}},\tag{4.5}$$

and condition (3.13) becomes

$$\begin{aligned}\liminf_{\mathrm{s}\to\infty}\int_{\sigma(\mathrm{s},b)}^{\mathrm{s}}\frac{\xi(\varrho)\widehat{\mathrm{h}}(\varrho)}{\xi^{1-\alpha}(\sigma(\varrho,b))}\mathrm{d}\varrho &= \liminf_{\mathrm{s}\to\infty}\int_{\frac{\mathrm{s}}{3}}^{\mathrm{s}}\frac{1}{\varrho}\frac{\varrho^{2/3}}{3^{2/3}}\frac{1}{2^{6/5}}\mathrm{h}\varrho^{-2/3}\mathrm{d}\varrho\\ &= \frac{1}{2^{6/5}}\frac{1}{3^{2/3}}\mathrm{h}_0\liminf_{\mathrm{s}\to\infty}\int_{\frac{\mathrm{s}}{3}}^{\mathrm{s}}\frac{1}{\varrho}\mathrm{d}\varrho\\ &= \frac{1}{2^{6/5}}\frac{1}{3^{2/3}}\ln(3)\,\mathrm{h}_0=0.22989\mathrm{h}_0,\end{aligned}$$

which leads to

$$\mathrm{h}_0>\frac{\gamma^{-1/3}\rho_n^{-2/3}(1-\rho_n)}{0.7\mathrm{e}},\ \ \gamma>0.\tag{4.6}$$

Theorems 3.1 and 3.3 show that the solution of (4.4) is oscillatory if either (4.5) or (4.6) holds.

Example 4.3. *Consider*

$$\left(\mathrm{s}^2\left(\mathrm{y}(\mathrm{s})+\frac{1}{16}\mathrm{y}\left(\frac{1}{2}\mathrm{s}\right)\right)'\right)'+\int_0^1\mathrm{h}_0\mathrm{y}^{1/3}\left(\frac{\ell}{4}\mathrm{s}\right)\mathrm{d}\ell=0,\tag{4.7}$$

Clearly:

$\alpha=1$, $\beta=1/3$, $\kappa(\mathrm{s})=\mathrm{s}^2$, $\mathrm{h}(\mathrm{s},\ell)=\mathrm{h}_0$, $\mathrm{u}(\mathrm{s})=1/16$, $\sigma(\mathrm{s},\ell)=\frac{\ell}{4}\mathrm{s}$ and $\tau(\mathrm{s})=\frac{1}{2}\mathrm{s}$. It can be easily verified that

$$\xi(\mathrm{s})=\frac{1}{\mathrm{s}},\ \frac{\xi(\tau(\sigma(\mathrm{s},\ell)))}{\xi(\sigma(\mathrm{s},\ell))}=2,$$

and

$$\widehat{\mathrm{h}}(\mathrm{s})=0.956\,47\mathrm{h}_0.$$

For (3.1), we set

$$\delta_0 = 0.956\,47\mathrm{h}_0.$$

From (3.8), we obtain $\lambda = 4$. *The sequence* $\{\rho_n\}_{n=1}^{m}$ *is then defined as*

$$\rho_n = \frac{\rho_0}{1 - \frac{1}{3}\rho_{n-1}} 2^{\frac{2\rho_{n-1}}{3}},$$

with

$$\rho_0 = 0.956\,47\mathrm{h}_0\gamma, \ \gamma > 0.$$

On the other hand, if we choose $\mathrm{h}_0 = 0.8$ *and* $\gamma = 0.7$, *then* $\rho_0 = 0.535\,62$ *and condition (3.5) is satisfied, which implise that (4.7) is oscillatory.*

For $\mathrm{h}_0 = 0.7$ *and* $\gamma = 0.5$, *we compute*

$$\rho_0 = 0.334\,76, \ \rho_1 = 0.43985, \ \rho_2 = 0.480\,69, \ \rho_3 = 0.49778, \ \rho_4 = 0.50516,$$

and (3.12) holds for $n = 4$, *which implies that for* $\mathrm{h}_0 = 0.7$ *and* $\gamma = 0.5$ *(4.7) is oscillatory.*

5. Conclusions

This research has established sufficient conditions to ensure the oscillatory behavior of all solutions within a certain class of second-order nonlinear NDEs. By focusing on the noncanonical forms of these equations, we have revealed new monotonic properties of positive solutions and proposed novel oscillation criteria, which expand the scope of current research in the field of second-order quasilinear NDEs. The contributions made in this study are an important step toward building a more comprehensive theoretical framework for understanding the oscillatory nature of these systems and paving the way for future research. Applying these analytical methods to higher-order nonlinear NDEs represents a promising path that may reveal more complex dynamics and novel oscillatory behaviors, greatly enhancing the understanding of this complex field and deepening theoretical and experimental studies in it.

Author contributions

Fahd Masood: Methodology, investigation, Writing—original draft preparation, Writing—review and editing; Salma Aljawi: Methodology, investigation, Writing—original draft preparation, Writing—review and editing; Omar Bazighifan: Methodology, investigation, Writing—review and editing, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

There are no competing interests.

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