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Research article

Bounds and complexity results of rainbow vertex-disconnection colorings

Yindi Weng*

Department of Mathematical Sciences, Zhejiang Sci-Tech University, Hangzhou 310027, China

* **Correspondence:** Email: wengyindi@zstu.edu.cn.

Abstract: A subset $Y \subseteq V(G)$ in a vertex-colored graph G is termed rainbow when vertices in Y receive distinct colors from each other. For each pair of vertices $w_1, w_2 \in V(G)$, if there exists $\mathcal{F} \subseteq V(G)$ satisfying $\mathcal{F}$ rainbow and w_1, w_2 disconnected in $G-\mathcal{F}$ for nonadjacent w_1, w_2; $\mathcal{F}+w_1$ or $\mathcal{F}+w_2$ rainbow and w_1, w_2 disconnected in $(G-w_1w_2)-\mathcal{F}$ for adjacent w_1, w_2, then G is rainbow vertex-disconnected. The smallest number needed to color G so that it is rainbow vertex-disconnected is known as the rainbow vertex-disconnection number of G, or $rvd(G)$. The RVD-Problem aims to determine whether G has a rainbow vertex-disconnection coloring with k colors given the graph G and a positive integer k. In this paper, some bounds between $rvd(G)$ and different parameters, such as diameter, independence number, and so on, are obtained. Some results of rainbow vertex-disconnection numbers of three graph products are then obtained. Last, we demonstrate that there is a polynomial time approach that approximates $rvd(G)$ of split graph G within a factor of $n^{2/3}$. We show RVD-Problem is NP-complete for induced $K_{1,t}$-free split graphs for $t \geq 4$ but polynomially solvable for $t \leq 3$.

Keywords: rainbow vertex-disconnected; graph products; complexity; approximability
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1. Introduction

In this paper, we consider simple, nontrivial connected and undirected graphs. Use $V(G)$ and $E(G)$ to respectively denote the vertex set and edge set of graph G. The notation $n = |V(G)|$ represents the *order* of G. For $v \in V(G)$, its open neighborhood is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. The degree of v is $d_G(v) = |N_G(v)|$. $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighborhood of v. The symbols $\delta(G)$ and $\Delta(G)$ represent the minimum and maximum degree of G, respectively. Let P_n be a path of order n. The k-cycle is a cycle with k vertices. Considering any two vertices u and v of G, if u is adjacent to v, for convenience, it is sometimes denoted by $u \sim v$; otherwise, $u \nsim v$.

Chartrand et al. [4] extended the concept of rainbow connection to rainbow disconnection. For a graph G with edge colored, $R \subseteq E(G)$, is an edge-cut if $G - R$ is not connected. Additionally, R is a

u_1-u_2 rainbow cut if R is rainbow and u_1, u_2 are disconnected in $G-R$. If there is a u_1-u_2 rainbow cut for any $u_1, u_2 \in G$, then G's edge coloring is a rainbow disconnection coloring. Its rainbow disconnection number, represented by $rd(G)$, means the minimum number of colors that are necessary.

Based on the perspective of vertex-cut, Bai et al. [1] presented the rainbow vertex-disconnection coloring. It is applicable to frequency distribution and cargo circulation, in which different colors of the rainbow vertex-cut is used to feedback different frequencies or interception locations.

Let w_1 and w_2 be two vertices of a vertex-colored graph G. $\mathcal{F} \subseteq V(G)$ is a w_1-w_2 vertex-cut if w_1, w_2 are in distinct components of $G - \mathcal{F}$ for $w_1 \nsim w_2$ and in distinct components of $(G - w_1w_2) - \mathcal{F}$ for $w_1 \sim w_2$. Then $\mathcal{F}$ is called rainbow if $\mathcal{F}$ has no two vertices with the same color. A w_1-w_2 rainbow vertex-cut of G, denoted by $\mathcal{F}_G(w_1, w_2)$, is a w_1-w_2 vertex-cut $\mathcal{F}$ such that if $w_1 \nsim w_2$, $\mathcal{F}$ is rainbow; if $w_1 \sim w_2$, $\mathcal{F} + w_1$ or $\mathcal{F} + w_2$ is rainbow.

G is rainbow vertex-disconnected when it contains a w_1-w_2 rainbow vertex-cut for any $w_1, w_2 \in V(G)$. The corresponding vertex-coloring c is a rainbow vertex-disconnection coloring (RVD-Coloring) of G. Its required minimum number of colors is called rainbow vertex-disconnection number, represented as $rvd(G)$. An rvd-coloring is an RVD-Coloring using $rvd(G)$ colors.

Furthermore, based on proper coloring and monochromatic coloring, proper (vertex-)disconnection coloring and monochromatic (vertex-)disconnection coloring were presented. For more details, refer to [2, 6, 9, 10].

Bai et al. [1] studied the relations between $rvd(G)$ and connected subgraph, block, connectivity, upper connectivity, and girth, respectively. Li et al. [11] obtained $\delta(G) \leq rvd(G) \leq \chi_i(G)$, where $\chi_i(G)$ is the minimum number of colors needed to make the open neighborhood of each vertex rainbow in a vertex-coloring of G. So what are the relations between other parameters and $rvd(G)$?

Chen et al. [5] demonstrated that, even in graph G having $\Delta(G) = 3$ or being bipartite, determining if a given vertex-colored graph G is rainbow vertex-disconnected is NP-complete. Given a positive integer k and a graph G, RVD-Problem aims to determine if G has an RVD-Coloring using k colors. The current author [14] proved RVD-Problem is NP-complete for bipartite graphs and split graphs. For every $\epsilon > 0$, it is impossible to approximate the rainbow vertex-disconnection number of any bipartite graph and split graph within a factor of $n^{\frac{1}{3}-\epsilon}$ unless $ZPP = NP$. So in this paper, we will focus on the approximate result of split graphs and the complexity of subclasses of split graphs.

The remainder of the paper is organized as follows. In Section 2, we investigate relations between different parameters and $rvd(G)$, such as diameter, edge connectivity, independence number, and so on. In Section 3, $rvd(G)$ of graph products such as Cartesian product, direct product, and lexicographic product is explored. In Section 4, the approximate result of split graph is given. We prove that for a split graph G, there exists a polynomial time algorithm that approximates $rvd(G)$ within a factor of $n^{2/3}$. We also show that RVD-Problem is NP-complete for induced $K_{1,t}$-free split graphs for $t \geq 4$ but polynomially solvable for $t \leq 3$.

2. Some bounds

In this section, we will study the relation of rainbow vertex-disconnection number with various parameters. First, we will consider the diameter.

For $w_1, w_2 \in V(G)$, we denote the distance of w_1 and w_2 in G by $d_G(w_1, w_2)$. The maximum distance, $diam(G)$, between every two vertices of a graph G is its diameter.

Theorem 2.1. *For a graph G with $diam(G) = d$, $rvd(G) \leq n - d + 2$.*

Proof. Suppose $u, v \in V(G)$ with $d_G(u, v) = d$. The shortest path of u and v is denoted by $P_{uv} = uv_1v_2 \cdots v_{d-1}v$. For convenience, let $u = v_0$ and $v = v_d$. Next, define the following vertex-coloring c of graph G. For vertices in P_{uv}, let $c(v_i) = r + 1$ where $i \equiv r \pmod 3$. We color the remaining vertices of G using different colors $4, 5, \cdots, n - d + 2$. Let w_1, w_2 be any pair of vertices of G. Then $N_G(w_1)$ is rainbow. For if w_1 is adjacent to two vertices v_p and v_q $(p < q)$ with the same color in P_{uv}, then the path $v_0v_1 \cdots v_pw_1v_qv_{q+1} \cdots v_d$ is a path between u and v that is shorter than P_{uv}, a contradiction. So if $w_1 \sim w_2$, $\mathcal{F}_G(w_1, w_2) = N_G(w_1) \setminus \{w_2\}$. If $w_1 \nsim w_2$, $\mathcal{F}_G(w_1, w_2) = N_G(w_1)$. Therefore, c is an RVD-Coloring. Specifically, $rvd(G) \leq n - d + 2$. □

The edge connectivity $\lambda(G)$ of graph G is the minimum number of edges of G whose removal results in a disconnected graph.

Theorem 2.2. *Let R be the minimum edge-cut of graph G and G_1, G_2 be connected components of $G - R$. Then $rvd(G) \leq max\{rvd(G_i)|i \in [2]\} + 2\lambda(G)$.*

Proof. Let $s = max\{rvd(G_i)|i \in [2]\}$ and c_i be an rvd-coloring on G_i, where $i \in [2]$. Use $V(R)$ to denote the endpoints of minimum edge-cut R. Based on c_i $(i \in [2])$, we recolor $V(R)$ using new colors $s + 1, s + 2, \cdots, s + |V(R)|$. Denote the vertex-coloring of G by c. Let $w, z \in V(G)$. Assume $w \in G_p$ and $z \in G_q$, where $p, q \in [2]$. If $p = q$, assuming that $\mathcal{F}_{G_p}(w, z) = S_p$ with the vertex-coloring c_p, then $\mathcal{F}_G(w, z) = S_p \cup V(R) \setminus \{w, z\}$ with the vertex-coloring c. If $p \neq q$, then $\mathcal{F}_G(w, z) = V(R) \setminus \{w, z\}$ with the vertex-coloring c. So c is an RVD-Coloring of G. Thus, $rvd(G) \leq s + |V(R)| \leq max\{rvd(G_i)|i \in [2]\} + 2\lambda(G)$. □

Lemma 2.3. *[3] Every graph with average degree at least $2k$, where k is a positive integer, has an induced subgraph with minimum degree at least $k + 1$.*

Lemma 2.4. *[1] If G is a nontrivial connected graph and H is a connected subgraph of G, then $rvd(H) \leq rvd(G)$.*

For $w_1, w_2 \in V(G)$, if $w_1 \nsim w_2$, the local connectivity $\kappa_G(w_1, w_2)$ represents the smallest number of vertices to make w_1, w_2 disconnected. If $w_1 \sim w_2$, $\kappa_G(w_1, w_2) = \kappa_{G - w_1w_2}(w_1, w_2) + 1$. The connectivity $\kappa(G)$ means the smallest number of vertices in G that, when removed, yield a trivial or disconnected graph. The upper connectivity $\kappa^+(G)$ satisfies $\kappa^+(G) = \max\{\kappa_G(w_1, w_2)|w_1, w_2 \in V(G)\}$.

Lemma 2.5. *[11] For a graph G with $\Delta(G) = \Delta$, $\delta(G) \leq \kappa^+(G) \leq rvd(G) \leq \chi_i(G) \leq \Delta(\Delta - 1) + 1$.*

Theorem 2.6. *For a graph G with order n and size m, $rvd(G) \geq \lfloor \frac{m}{n} \rfloor + 1$.*

Proof. Suppose that the average degree of graph G is $\bar{d}$. So $\bar{d} = \frac{2m}{n} \geq 2\lfloor \frac{m}{n} \rfloor$. By Lemma 2.3, there is an induced subgraph H with $\delta(H) \geq \lfloor \frac{m}{n} \rfloor + 1$. Therefore, $rvd(G) \geq rvd(H) \geq \delta(H) \geq \lfloor \frac{m}{n} \rfloor + 1$ by Lemmas 2.4 and 2.5. □

For $S \subseteq V(G)$, if it contains no two adjacent vertices, S is referred to as an independent set of G. Furthermore, when there is no independent set containing more vertices than S, S is maximum. The independence number of G, indicated by $\alpha(G)$, is the number of vertices in a maximum independent set of G.

Lemma 2.7. *[1] Let G be a nontrivial connected graph. Then* $rvd(G) = 1$ *if and only if G is a tree.*

Theorem 2.8. *For a graph G of order n,* $rvd(G) \geq \lceil \frac{n}{2\alpha(G)} \rceil$*, and the bound is sharp.*

Proof. Suppose that $rvd(G) = k$. Denote the color classes of graph G by $V_1, V_2, \cdots, V_k$. Then each V_i ($i \in [k]$) induces a tree or a forest by Lemma 2.7. Assume that $T_1, T_2, \cdots, T_\ell$ are the connected components of V_i. Since T_j ($j \in [\ell]$) is bipartite, we have $\alpha(T_j) \geq \lceil \frac{|T_j|}{2} \rceil$ ($j \in [\ell]$). We obtain

$$\alpha(V_i) \geq \sum_{j\in[\ell]} \lceil \frac{|T_j|}{2} \rceil \geq \lceil \frac{|V_i|}{2} \rceil.$$

Since there exists V_s with $|V_s| \geq \lceil \frac{n}{k} \rceil$ ($s \in [k]$), we have $\alpha(G) \geq max_{i\in[k]}\alpha(V_i) \geq \alpha(V_s) \geq \lceil \frac{n}{2k} \rceil$. Let $G = P_n$. When n is odd, we have $\alpha(P_n) = \frac{n+1}{2}$ and $rvd(P_n) = 1$. When n is even, we have $\alpha(P_n) = \frac{n}{2}$ and $rvd(P_n) = 1$. So the bound is tight. □

A vertex-coloring of G is proper if any two adjacent vertices receive different colors. The chromatic number of G is the minimum number of colors such that G has a proper coloring, denoted by $\chi(G)$. G is k-chromatic if $\chi(G) = k$. If $\chi(H) < \chi(G)$ for every proper subgraph H of G, G is critical. A graph is k-critical if it is critical and $\chi(G) = k$.

Lemma 2.9. *[7] A k-chromatic graph contains a k-critical subgraph.*

Lemma 2.10. *[7] Let G be a connected* $(k + 1)$*-critical graph. Then* $\delta(G) \geq k$*.*

Lemma 2.11. *[11] Let G be a connected graph of order n with minimum degree* δ*. If* $\delta \geq \frac{n+2}{2}$*, then* $rvd(G) = n$*.*

Theorem 2.12. *For a graph G of order n, if* $\chi(G) \geq \frac{n+4}{2}$*, then* $rvd(G) = \chi_i(G) = n$*.*

Proof. Assume that $\chi(G) = k$. There exists a k-critical subgraph H by Lemma 2.9. Then $\delta(G) \geq \delta(H) \geq k - 1 \geq \frac{n+4}{2} - 1 = \frac{n+2}{2}$ by Lemma 2.10. According to Lemma 2.11 and 2.5, $rvd(G) = \chi_i(G) = n$. □

Next, we get Theorem 2.15 to show the gap of $rvd(G)$ and $\chi_i(G)$ arbitrarily large.

A block of a graph G is a maximal 2-connected subgraph of G.

Lemma 2.13. *[1] Let G be a nontrivial connected graph, and let B be a block of G such that* $rvd(B)$ *is maximum among all blocks of G. Then* $rvd(G) = rvd(B)$*.*

Lemma 2.14. *[1] Let* $G = K_{n_1,n_2,\ldots,n_k}$ *be a complete k-partite graph of order n, where* $k \geq 2$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ *and* $n_k \geq 2$*. Then*

$$\mathrm{rvd}(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} n, & if\ k \geq 4\ or\ k = 3,\ n_3 \geq n_2 \geq n_1 \geq 2, \\ n - n_{k-1}, & if\ k = 3,\ n_1 = 1\ or\ k = 2,\ n_2 \geq n_1 \geq 2, \\ 1, & if\ k = 2\ and\ n_1 = 1. \end{cases}$$

Theorem 2.15. *For any positive integers a and b with* $a \leq b$*, there exists a connected graph such that* $rvd(G) = a$ *and* $\chi_i(G) = b$*.*

Proof. If $a = b = 1$, $rvd(K_2) = \chi_i(K_2) = 1$. Consider that $b > a = 1$ or $b \geq a \geq 2$. Assume that $K_{2,a}$ is a complete bipartite graph with bipartition (V_1, V_2), where $V_1 = \{x, y\}$ and $V_2 = \{v_1, v_2, \cdots, v_a\}$. Add $b - a$ pendant edges to x. The new graph from $K_{2,a}$ is denoted by G, where the set of new $b - a$ vertices is $V_3 = \{v_{a+1}, v_{a+2}, \cdots, v_b\}$. Then $rvd(G) = rvd(K_{2,a}) = a$ by Lemmas 2.13 and 2.14. Since $\Delta(G) = b$, we have $\chi_i(G) \geq b$. Now give a vertex-coloring c on G. Assume $c(x) = 1$, $c(y) = 2$, and $c(v_i) = i$ for $i \in [b]$. Then for any vertex v of G, $N_G(v)$ is rainbow. So c is an injective coloring of G and $\chi_i(G) \leq b$. □

A graph is minimally k-connected if it is k-connected, but omitting any of the edges, the resulting graph is no longer k-connected.

Lemma 2.16. *[13] Let G be a minimally k-connected graph of order n. If $n \geq 3k - 2$, then $|E(G)| \leq k(n - k)$. Furthermore, if $n \geq 3k - 1$, equality holds if and only if $G = K_{k,n-k}$.*

Lemma 2.17. *[1] For integers k and n with $1 \leq k \leq n$, the minimum size of a connected graph G of order $n \geq 4$ with $rvd(G) = k$ is*

$$|E(G)|_{\min} = \begin{cases} n + k - 2, & 1 \leq k \leq n - 1, \\ 2n - 4 + \lceil \frac{n}{2} \rceil, & k = n. \end{cases}$$

Theorem 2.18. *Let G be a minimally 2-connected graph with order n. If $n \geq 4$, $rvd(G) \leq n - 2$. Furthermore, $rvd(G) = n - 2$ if and only if $G = K_{2,n-2}$.*

Proof. By Lemma 2.16, we have $|E(G)| \leq 2n - 4$. By Lemma 2.17, $rvd(G) \leq n - 2$. If $n = 4$, G is $K_{2,2}$, and $rvd(G) = 2$. If $n \geq 5$, by Lemma 2.16, $rvd(G) = n - 2$ if and only if $G = K_{2,n-2}$. □

A graph G is outerplanar if it has a planar embedding $\widetilde{G}$ in which all vertices lie on the boundary of its outer face.

Lemma 2.19. *[3] Every simple 2-connected outerplanar graph other than K_2 has a vertex of degree two.*

Theorem 2.20. *For an outerplanar graph G, $rvd(G) \leq 4$.*

Proof. By Lemma 2.7, consider that G is not a tree. If G is a triangle, then $rvd(G) = 2$. So consider graph G with order $n \geq 4$. If there is a cut vertex in G, assuming that the 2-connected subgraph of G with maximum rainbow vertex-disconnection number is G', we obtain $rvd(G) = rvd(G')$ by Lemma 2.13. So suppose that G is 2-connected. If there is an interior face with length more than 3, then we add some edges to make each interior face a 3-cycle. The resulting graph is denoted by H.

Next, we prove there exists an RVD-Coloring c_H of graph H: $V(H) \to [4]$, satisfying the vertex set of each 4-cycle is rainbow. For simplicity, we say that each 4-cycle is rainbow.

For $n = 4$, we have $H = K_4 - e$ and it suffices to color each vertex using different color. Assume that for graphs of order n, the assertion is true. Then consider graph H with order $n + 1$. By Lemma 2.19, select a vertex v with $d_H(v) = 2$. Let $N_H(v) = \{v_1, v_2\}$. Since H is an outerplanar graph, H is $K_{2,3}$ minor-free. So let $N_H(v_1) \cap N_H(v_2) = \{v, v_3\}$. Since $H - v$ is 2-connected, there is an RVD-Coloring c_{H-v}: $V(H - v) \to [4]$ such that each 4-cycle is rainbow. By coloring the vertex v using the color different from $c_{H-v}(v_1), c_{H-v}(v_2), c_{H-v}(v_3)$, we obtain a vertex-coloring c_H of graph H. Obviously, the 4-cycle $vv_1v_3v_2$ is rainbow.

Let $w, z \in V(H)$. If $w = v$ or $z = v$, then $\mathcal{F}_H(w,z) = \{v_1, v_2\} \setminus \{z\}$ or $\{v_1, v_2\} \setminus \{w\}$. If $\{w, z\} = \{v_1, v_2\}$, then $\mathcal{F}_H(w,z) = \{v, v_3\}$. Except for them, $\mathcal{F}_H(w,z) = \mathcal{F}_{H-v}(w,z)$. So c_H is an RVD-Coloring of H such that each 4-cycle is rainbow. Thus, $rvd(G) \leq 4$ by Lemma 2.4. □

3. Results of graph products

Next, we study rainbow vertex-disconnection numbers of three kinds of graph products. First, we give the definitions of three kinds of graph products as follows.

Give two internal disjoint graphs G, H. The graph with vertex set $V(G) \times V(H)$ is Cartesian product $G \Box H$, where $(u, x) \sim (v, y)$ in $G \Box H$ if and only if either $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $x = y$.

The direct product $G \times H$ is the graph with vertex set $V(G) \times V(H)$, where $(u, x) \sim (v, y)$ if and only if both $uv \in E(G)$ and $xy \in E(H)$.

The lexicographic product $G \circ H$ is the graph with vertex set $V(G) \times V(H)$, where $(u, x) \sim (v, y)$ if and only if $uv \in E(G)$, or $u = v$ and $xy \in E(H)$.

The stacked book graph is defined as $B_{m,n} \cong S_m \Box P_n$, where S_m is a star with order $m + 1$, $m \geq 2$, and $n \geq 2$.

Theorem 3.1. $rvd(B_{m,n}) = m + 1$.

Proof. Let $V(S_m) = \{s_i | i \in [m + 1]\}$, where $d_{S_m}(s_1) = m$. Let $V(P_n) = \{p_j | j \in [n]\}$. For convenience, denote vertex (s_i, p_j) in $B_{m,n}$ by v_{ij}. Since $\kappa_{B_{m,n}}(v_{11}, v_{12}) = m + 1$, $rvd(B_{m,n}) \geq m + 1$. Give a vertex-coloring c of $B_{m,n}$ using $m+1$ colors as follows. Let $c(v_{1j}) = m$ for $j \in [n]$. Let $c(v_{ij}) = \lceil \frac{j}{2} \rceil + i - 1 \pmod m$ for $i \geq 2$ and $i \in [m+1]$ and $j \in [n]$. Except for v_{1j}, the open neighborhood of each vertex of $B_{m,n}$ is rainbow. Consider any two vertices v_{1j_1} and v_{1j_2}, where $j_1 < j_2$. If $j_1 = 1$, then $N_{B_{m,n}}(v_{1j_1})$ is rainbow; otherwise, $N_{B_{m,n}}(v_{1j_1}) \setminus \{v_{1j_1-1}\}$ is rainbow. Thus, there exists $\mathcal{F}_{B_{m,n}}(v_{1j_1}, v_{1j_2})$. So $rvd(B_{m,n}) \leq m + 1$. □

Lemma 3.2. *[1] Let G be a nontrivial connected graph, and let u and v be two vertices of G having at least two common neighbors. Then u and v receive different colors in any rvd-coloring of G.*

Lemma 3.3. *[1] Let G be a nontrivial connected graph of order n. Then rvd(G) = n if and only if any two vertices of G have at least two common neighbors.*

Theorem 3.4. *For nontrivial connected graphs G and H,* $max\{\kappa^+(G) + \Delta(H), \kappa^+(H) + \Delta(G)\} \leq rvd(G \Box H) \leq |V(G)| \cdot |V(H)|$. *Moreover, the upper and lower bounds are sharp.*

Proof. The upper bound is obtained from $rvd(G \Box H) \leq |G \Box H| = |V(G)| \cdot |V(H)|$. According to Lemma 3.3, we have $rvd(K_p \Box K_q) = pq$, where $p \geq 4$ and $q \geq 4$. So the upper bound is sharp. Select row i where each vertex has the maximum degree in the corresponding graph G. In the same row i, we select two vertices $x_{i,\ell}$ and $x_{i,s}$ satisfying $\kappa_H(x_{i,\ell}, x_{i,s}) = \kappa^+(H)$. Assume that $N_G(x_{i,\ell}) = \{x_{j_1,\ell}, x_{j_2,\ell}, \cdots, x_{j_\Delta,\ell}\}$. Consider the number of internally disjoint paths between $x_{i,\ell}$ and $x_{i,s}$. Since H is connected, there is a path between $x_{j_t,\ell}$ and $x_{j_t,s}$ for $t \in \{1, 2, \cdots, \Delta\}$, say P_{j_t}, only through the vertices in row j_t. So paths $x_{i,\ell}P_{j_1}x_{i,s}, x_{i,\ell}P_{j_2}x_{i,s}, \cdots, x_{i,\ell}P_{j_\Delta}x_{i,s}$ are $\Delta(G)$ internally disjoint paths between $x_{i,\ell}$ and $x_{i,s}$. There exist at least $\kappa^+(H)$ internally disjoint paths in row i. Thus, $\kappa_{G \Box H}(x_{i,\ell}, x_{i,s}) \geq \Delta(G) + \kappa^+(H)$. According to Lemma 2.5, $rvd(G \Box H) \geq \Delta(G) + \kappa^+(H)$. By symmetry, the lower bound is obtained. For a stacked book graph $B_{m,n}$, since $\Delta(S_m) = m$, $\kappa^+(P_n) = 1$, and $rvd(B_{m,n}) = m + 1$ by Theorem 3.1, the lower bound is sharp. □

Theorem 3.5. *For graphs G, H with $\Delta(G) \geq 2$ and $\Delta(H) \geq 2$, max*$\{\Delta(G), \Delta(H)\} \leq rvd(G \times H) \leq |V(H)||V(G)|$.

Proof. Denote each vertex of $G \times H$ by (u, v), where $u \in V(G)$ and $v \in V(H)$. Let $u_0 \in V(G)$ with $d_G(u_0) = \Delta(G)$ and $N_G(u_0) = \{u_1, u_2, \cdots, u_{\Delta(G)}\}$. Let $v_0 \in V(H)$ with $d_H(v_0) = \Delta(H)$ and $N_H(v_0) = \{v_1, v_2, \cdots, v_{\Delta(H)}\}$. Then $(u_1, v_0), (u_2, v_0), (u_3, v_0), \cdots, (u_{\Delta(G)}, v_0)$ are the common neighbors of (u_0, v_1) and (u_0, v_2). So $rvd(G\times H) \geq \Delta(G)$. Since $(u_0, v_1), (u_0, v_2), \cdots, (u_0, v_{\Delta(H)})$ are the neighbors that (u_0, v_1) and (u_0, v_2) have in common, we have $rvd(G \times H) \geq \Delta(H)$. Since $rvd(P_3 \times K_{1,n-1}) = \Delta(K_{1,n-1})$, the lower bound is sharp. Obviously, $rvd(G \times H) \leq |V(H)||V(G)|$. Since $rvd(K_p \times K_q) = pq$, where $p \geq 4$ and $q \geq 4$, the upper bound is sharp. □

Corollary 3.6. *For nontrivial connected graphs G and H, $rvd(G\times H) = 1$ if and only if $G\times H = K_2\times T$ or $T \times K_2$, where T is a tree.*

Proof. Assume that $rvd(G\times H) = 1$. Then $\Delta(G) = 1$ or $\Delta(H) = 1$ by Theorem 3.5. Suppose that $\Delta(G) = 1$ by symmetry. Then G is K_2. Let $V(G) = \{x, y\}$. Suppose that H contains a cycle $C_l = v_1v_2 \cdots v_lv_1$. If l is even, there exists a cycle $(x, v_1)(y, v_2)(x, v_3)(y, v_4) \cdots (y, v_l)(x, v_1)$ in $G\times H$. If l is odd, there exists a cycle $(x, v_1)(y, v_2)(x, v_3) \cdots (x, v_l)(y, v_1)(x, v_2)(y, v_3) \cdots (y, v_l)(x, v_1)$ in $G \times H$. So $rvd(G \times H) \geq 2$. It is a contradiction. Thus, H is a tree.

For the converse, we prove $rvd(K_2 \times T) = 1$ by contradiction. Suppose that there is a tree T_0 satisfying $rvd(K_2 \times T_0) \geq 2$. Then there is a cycle C in $K_2 \times T_0$. By symmetry, assume that $C = (x, v_{i_1})(y, v_{i_2})(x, v_{i_3}) \cdots (y, v_{i_t})(x, v_{i_1})$. If the cycle C passes $(y, v_{i_k}) = (y, v_{i_1})$, then there exists a cycle $v_{i_1}v_{i_2} \cdots v_{i_{k-1}}v_{i_1}$ in T_0; otherwise, there exists a cycle $v_{i_1}v_{i_2} \cdots v_{i_t}v_{i_1}$ in T_0. It is a contradiction. □

The subsequent corollary is obtained by Corollary 3.6 and Lemma 2.14. For $K_2 \times C_n$, where C_n is a cycle with n vertices, the lower bound is sharp. The upper bound is sharp for $K_2 \times K_p$, where $p \geq 4$.

Corollary 3.7. *For a 2-connected graph G: $2 \leq rvd(K_2 \times G) \leq |V(G)|$.*

Any pair of vertices at a distance ≤ 2 receives different colors in a 2-*distance coloring* of a graph G. The smallest number of colors required in a 2-distance coloring of G is 2-*distance chromatic number*, represented by $\chi^2(G)$. For any graph G, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$, where $\Delta = \Delta(G)$. As shown in [12], the upper bound is sharp for Moore graphs of type $(\Delta, 2)$, which are graphs where all vertices have degree Δ, are at distance at most two from each other, and the total number of vertices is $\Delta^2 + 1$.

Theorem 3.8. *For nontrivial connected graphs G, H with $\Delta(G) = \Delta$, $rvd(G \circ H) \leq \chi^2(G)|V(H)| \leq (\Delta^2 + 1)|V(H)|$.*

Proof. Let $V(G) = \{u_0, u_1, \cdots, u_p\}$ and $V(H) = \{v_0, v_1, \cdots, v_q\}$, where u_0 is the vertex with the maximum degree of G. Let $S_i = \{(u_i, v_j)|j = 0, 1, 2, \cdots, q\}$. Since $(u_{i'}, v_0)$ and $(u_{i'}, v_1)$ are two common neighbors for any pair of vertices in S_i, where u_i is adjacent to $u_{i'}$ in G, S_i is rainbow under any rvd-coloring by Lemma 3.2.

Assume that $N_G(u_0) = \{u_1, u_2, \cdots, u_\Delta\}$. For $(u_{i_1}, v_{j_1}) \in S_{i_1}$ and $(u_{i_2}, v_{j_2}) \in S_{i_2}$, where $i_1, i_2 \in [\Delta]$, they have at least two common neighbors (u_0, v_0) and (u_0, v_1). So $\bigcup_{i=1}^{\Delta} S_i$ is rainbow. For $(u_0, v_t) \in S_0$ and $(u_i, v_j) \in S_i$, where $i \in [\Delta]$, they have at least two common neighbors $(u_0, v_{t'})$ and $(u_i, v_{j'})$, where

$v_t \sim v_{t'}$ and $v_j \sim v_{j'}$ in H. So $\bigcup_{i=0}^{\Delta} S_i$ is rainbow. Thus, $rvd(G) \geq (\Delta+1)|V(H)|$. Similarly, $S_i \bigcup (\bigcup_{t=p_1}^{p_d} S_t)$ is rainbow for any $u_i \in V(G)$, where $N_G(u_i) = \{u_{p_1}, u_{p_2}, \cdots, u_{p_d}\}$.

For a 2-distance coloring c' of G: $V(G) \mapsto [\chi^2(G)]$, we expand each color k to a color set $c_k = \{(k-1)|V(H)|+1, (k-1)|V(H)|+2, \cdots, k|V(H)|\}$. If $c'(u_i) = k$, then color S_i using c_k. The resulting coloring of $G \circ H$ is an RVD-Coloring of $G \circ H$. So $\chi^2(G)|V(H)| \geq rvd(G \circ H)$.

□

4. Complexity results

In this section, the complexity of computing rvd number of split graphs is studied. If a split graph G is partitioned into a clique C and an independent set I, then define (C, I) as a *split partition* of G. If $|C| = 1$, G is a tree and $rvd(G) = 1$. If $|C| = 2$, $rvd(G) = t+1$, where t is the number of vertices in I that have degree two. So in the rest of this section, we consider $|C| \geq 3$.

Lemma 4.1. *[1] For an integer $n \geq 2$,*

$$\mathrm{rvd}(K_n) = \begin{cases} n-1, & if\ n = 2, 3, \\ n, & if\ n \geq 4. \end{cases}$$

Lemma 4.2. *Let G be a graph from K_n ($n \geq 3$) by adding a vertex v with degree ≥ 3 to K_n. Then $rvd(G) = n+1$.*

Proof. When $n = 3$, G is K_4, and $rvd(G) = 4$ by Lemma 4.1. When $n \geq 4$, assume that $x_1, x_2, x_3 \in N_G(v)$. Since there exist x_1-x_2, x_1-x_3, x_2-x_3 rainbow vertex-cuts under any rvd-coloring of G, v's color differs from K_n. So $rvd(G) = n+1$. □

Theorem 4.3. *For a split graph G, there exists a polynomial time algorithm that approximates $rvd(G)$ within a factor of $n^{\frac{2}{3}}$.*

Proof. Let G be a connected graph with a clique X and an independent set Y. Consider that G is 2-connected by Lemma 2.13. So $\delta(G) \geq 2$. Let $p = |X|-1$ for $|X| = 3$ and $p = |X|$ for $|X| \geq 4$. Construct a new graph H using Y. If $u, v \in Y$ and u, v have at least two common neighbors in X, then $u \sim v$; otherwise, $u \nsim v$.

Now we claim $p + \chi(H) - 1 \leq rvd(G) \leq |X| + \chi(H)$. First, provide a vertex-coloring c on G. Color X rainbow using $|X|$ colors. Color Y using new $\chi(H)$ colors according to the proper coloring of H. Let $w, z \in V(G)$. If $w, z \in X$, then $\mathcal{F}_G(w, z) = X \cup M_Y(w, z) \setminus \{w, z\}$, where $M_Y(w, z) = N_Y(w) \cap N_Y(z)$. If $w \in Y$ or $z \in Y$, $\mathcal{F}_G(w, z) = X \setminus \{w, z\}$. So $rvd(G) \leq |X| + \chi(H)$. We prove the lower bound by contradiction. Assume that $rvd(G) \leq p + \chi(H) - 2$. Denote an RVD-Coloring of G by c_0 using $p + \chi(H) - 2$ colors. Since X is a complete graph, $rvd(X) = p$. Then Y has at most $\chi(H) - 2$ colors different from X, denoted by $[\chi(H) - 2]$. Let $S = \{v | v \in Y \text{ and } c_0(v) \in c_0(X)\}$. For $v \in S$, by Lemma 4.2, we have $d_G(v) = 2$. Now recolor S to get a vertex-coloring c' on G. Assume that $N_G(v) = \{u_1, u_2\}$. If $M_Y(u_1, u_2) = \{v\}$, recolor v such that $c'(v) \in [\chi(H) - 2]$; otherwise, recolor v using a new color. So Y uses at most $\chi(H) - 1$ colors in c' and has no color appearing in X. Color H as Y. Then we obtain a proper coloring of H using at most $\chi(H) - 1$ colors, which is a contradiction. So $rvd(G) \geq p + \chi(H) - 1$.

Consider color classes $V_1, V_2, \cdots, V_{\chi(H)}$ of H. Since $\delta(G) \geq 2$ and any two vertices in V_i ($i \in [\chi(H)]$) have at most one common neighbor in X, we have $\binom{|X|}{2} \geq |V_i|$ for any $i \in [\chi(H)]$. Thus, $\binom{|X|}{2} \geq \frac{|V(H)|}{\chi(H)}$. We obtain $\chi(H) \geq \frac{2|V(H)|}{|X|^2}$. Any graph with t edges can be colored properly with at most $\frac{1}{2} + \sqrt{2t + \frac{1}{4}}$ colors in chapter 5 [8]. We can obtain an RVD-Coloring c_0 of G using at most $|X| + 1 + \sqrt{2|E(H)|}$ colors. So we obtain the following inequalities for $n \geq 22$:

$$\frac{|X| + 1 + \sqrt{2|E(H)|}}{rvd(G)} \leq \frac{|X| + 1 + \sqrt{2|E(H)|}}{p - 1 + \frac{2|V(H)|}{|X|^2}} \leq \frac{n+1}{|X| + \frac{2n}{|X|^2} - \frac{8}{3}} \leq n^{\frac{2}{3}}.$$

Hence, c_0 is an $n^{\frac{2}{3}}$-approximation of $rvd(G)$.

□

Next, we consider RVD-Problem, which aims to determine whether G has an RVD-Coloring with k colors given the graph G and a positive integer k.

We have proved Lemma 4.4 in [14]. The main idea of Lemma 4.4 is to construct a split graph H from any graph G satisfying $rvd(H) \leq k + 3|E(G)|$ if and only if $\chi(G) \leq k$. A graph is *induced $K_{1,t}$-free* if it does not contain $K_{1,t}$ as an induced subgraph. In fact, H is an induced $K_{1,t}$-free split graph for $t \geq 4$. So we easily obtain Theorem 4.5.

Lemma 4.4. *[14] RVD-Problem is NP-complete for split graphs.*

Theorem 4.5. *RVD-Problem is NP-complete for induced $K_{1,t}$-free split graphs for $t \geq 4$.*

Furthermore, we consider induced $K_{1,t}$-free split graphs with $t \leq 3$.

Lemma 4.6. *If G is an induced $K_{1,2}$-free connected graph, then G is a complete graph.*

Proof. Assume that G is not complete. Then there exist two vertices, u and v, such that $uv \notin E(G)$. Since G is connected, there is a path from u to v, denoted by $P_{uv} = uw_1w_2 \cdots w_kv$. Since G is induced $K_{1,2}$-free, we have $uw_2 \in E(G)$ and get a shorter path $uw_2 \cdots w_kv$. Similarly, we have $uw_3, uw_4 \cdots uw_k, uv \in E(G)$, which is a contradiction. □

According to Lemmas 4.1 and 4.6, we get the following result.

Theorem 4.7. *For an induced $K_{1,2}$-free split graph G with order n,*

$$\text{rvd}(G) = \begin{cases} n-1, & if\ n = 2, 3, \\ n, & if\ n \geq 4. \end{cases}$$

.

Theorem 4.8. *For an induced $K_{1,3}$-free split graph G, $rvd(G)$ can be determined in polynomial time.*

Proof. Let G be an induced $K_{1,3}$-free split graph with split partition (C, I). Since G is induced $K_{1,3}$-free, each vertex in C has at most two neighbors in I (**Rule1**). For $u, v \in I$, if $N_G(u) \cap N_G(v) \neq \emptyset$, then $N_G(u) \cup N_G(v) = C$ (**Rule2**). Otherwise, there exists a vertex $v_0 \in C$ satisfying $v_0 \nsim u$ and $v_0 \nsim v$. Let $v' \in N_G(u) \cap N_G(v)$. The vertices v', u, v, v_0 form a $K_{1,3}$, which is a contradiction.

For $|C| = 3$, if there is no vertex in I with degree≥ 2, then $rvd(G) = 2$; if there is a vertex with degree two but no vertex with degree three in I, then $rvd(G) = 3$; if there is a vertex with degree three in I, then C is rainbow under any rvd-coloring of G. So we only need to consider the last case for $|C| = 3$ and $|C| \geq 4$.

Let $S = \{(u, v) \| |N_G(u) \cap N_G(v)| \geq 2 \text{ and } d_G(u) \geq 3, d_G(v) \geq 3, \text{ and } u, v \in I\}$. If $(u, v) \in S$ and $(x, y) \in S$, let $v_s \in N_G(u) \cap N_G(v)$; then $v_s \sim x$ or $v_s \sim y$ by Rule 2. We obtain $d_I(v_s) \geq 3$, which is a contradiction to Rule 1. So S has no two disjoint pairs of vertices. Here are two cases.

Case 1. Assume that $(u, v), (u, x), (v, x) \in S$.

By Rule 2, each vertex in C has two edges to $\{u, v, x\}$. So by Rule 1, $|I| = 3$ and $I = \{u, v, x\}$. Thus, $S = \{(u, v), (u, x), (v, x)\}$.

Case 2. $|S| = 1$ or every pair of vertices in S contains the same vertex.

Assume that the same vertex is u. Then let $S = \{(u, u_1), (u, u_2) \cdots\}$.

Next, we give a vertex-coloring c of the induced $K_{1,3}$-free split graph G with split partition (C, I) as follows.

Algorithm 1 rvd-coloring of G

Input: An induced $K_{1,3}$-free split graph G with split partition (C, I)
Output: An rvd-coloring c of G

Color C rainbow using colors $[|C|]$.
Set $Q_1 = Q_2 = Q_3 = \emptyset$.
for each $v \in I$ **do**
  if $d_G(v) \leq 2$ **then**
    color v with the same color as one of its neighbors
    continue;
  if $d_G(v) \geq 3$ and $|Q_1| = |Q_2| = 1$ and v and every vertex in $Q_1 \cup Q_2$ have at least two common neighbors **then**
    color v using color $|C| + 3$ and $Q_3 = Q_3 \cup \{v\}$.
    break;
  if $d_G(v) \geq 3$ and there exists $u \in Q_1$ such that u and v have at least two common neighbors **then**
    color v using color $|C| + 2$ and $Q_2 = Q_2 \cup \{v\}$.
  else
    color v using color $|C| + 1$ and $Q_1 = Q_1 \cup \{v\}$.

We now assert that c is G's rvd-coloring. By Lemmas 4.2 and 3.2, the number of colors used in vertex-coloring c is the lower bound of $rvd(G)$. Let $q, q' \in V(G)$. For $q, q' \in C$, since G is induced $K_{1,3}$-free, there is at most one common neighbor with degree two in I. Let $N_I(q, q')$ be the common neighbors of q and q' in I. Then $\mathcal{F}_G(q, q') = C \cup N_I(q, q') \setminus \{q, q'\}$. For $q, q' \in I$, $\mathcal{F}_G(q, q') = C$. For $q \in C$ and $q' \in I$, $\mathcal{F}_G(q, q') = C \setminus \{q\}$.

□

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No conflict of interest exists in the submission of this manuscript.

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