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Research article

Strong tripled fixed points under a new class of F-contractive mappings with supportive applications

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Abstract: A significant advancement in the field of fixed point theory is presented in this manuscript. The existence and uniqueness of strong tripled coincidence points for F-contractive mappings in metric spaces were investigated. An extension of this analysis to multivalued F-contractive mappings was provided, establishing the existence of tripled fixed points within this generalized setting. Existing findings in the literature were generalized and refined by these results, offering a more comprehensive understanding of fixed point phenomena. Furthermore, the practical applicability of these theoretical contributions was demonstrated through the study of solutions to various forms of nonlinear integral equations and integral-type inequalities.

Keywords: tripled fixed point; nonlinear equation; F-contraction mapping; evaluation metrics; standard metric space
Mathematics Subject Classification: 47H10, 47H09, 26A33, 54G05, 45G10

1. Introduction

The traditional silos between scientific disciplines are dissolving, fueled by transformative advancements and innovative theoretical methodologies. This interdisciplinary shift is especially pronounced in mathematics, a field undergoing a significant evolution. In an era where mathematical literacy is paramount, a deficit in understanding equates to a diminished grasp of the natural world's intricacies. Through mathematical exploration, we unlock hidden patterns and interrelationships. The ubiquitous Fibonacci sequence, for example, illuminates growth patterns in flora and reproductive strategies in fauna. Furthermore, mathematics serves as a cornerstone for modeling complex phenomena, such as epidemic spread. By employing differential or difference equations to analyze interactions within population groups, scientists glean critical insights into the dynamics of infectious disease transmission.

The intricate interplay between mathematics and physics is exemplified by functional analysis, a field that has grown in significance alongside theoretical physics. The formal framework of functional analysis provides the mathematical tools necessary for understanding quantum field theory and quantum mechanics, while these physical theories have enriched the field with new problems and inspired the development of innovative functional analytic methods.

Fixed point theory, a significant branch of functional analysis, offers a powerful and versatile toolkit with broad applicability across diverse fields. Centered on the fundamental concept of a fixed point, it has profoundly impacted areas such as topology, game theory, optimal control, artificial intelligence, logic programming, dynamical systems, differential equations, and economics, notably through the analysis of equilibrium problems and the solution of integral equations. Furthermore, the inherent ability of fixed point techniques to establish the existence and uniqueness of solutions to complex fractional differential and integral equations, which arise from the non-local nature of fractional operators, renders them indispensable in fractional calculus. By leveraging theorems like the Banach contraction principle and the Leray-Schauder alternative, researchers can effectively analyze these equations and model real-world phenomena that exhibit memory and hereditary properties, solidifying fixed point theory's role as a robust framework for both theoretical and practical applications, for examples, the existence of the solutions to Fredholm integral equations [1–3], solving factional integral systems [4–7], solving fractional differential and fractional reaction-diffusion systems [8–10], and studying the stability for mixed integral fractional delay dynamic systems and fractional pantograph differential equations [11, 12].

Building upon Bhaskar and Lakshmikantham's foundational work [13], the concept of coupled fixed points has not only expanded the theoretical landscape of fixed point theory but also spurred extensive investigations into its applications across various mathematical domains. Their subsequent contributions further refined our understanding of coupled fixed points within partially ordered metric spaces, laying the groundwork for numerous extensions and generalizations. This initial research has catalyzed a wealth of studies, exploring diverse aspects of coupled fixed points, including their existence, uniqueness, and stability, as well as their relevance in solving differential and integral equations, optimization problems, and other areas. The breadth and depth of these developments are evidenced by the comprehensive body of literature available, like coupled fixed point theorems in generalized MSs [14–17], coupled fixed point theorems in various spaces [18–21], which collectively demonstrate the continued significance and evolving nature of coupled fixed point theory.

Building upon the foundation laid by Berinde and Borcut in 2011 [22], the investigation of tripled fixed points (TFPs) within partially ordered metric spaces has become a vibrant area of research. This concept has not only enriched the theoretical framework of fixed point theory but has also opened doors to a multitude of practical applications. The initial introduction of TFPs has sparked a cascade of studies exploring various aspects, including the establishment of existence and uniqueness theorems, the development of iterative methods for finding TFPs, and the extension of these concepts to more generalized settings. The growing interest in TFPs is evidenced by the substantial body of literature, such as TCP theorems in partially ordered MSs [23, 24], and TCP theorems in various spaces [25–27], which delve into the nuanced properties and diverse applications of these points, demonstrating their ongoing relevance and the potential for further advancements in this field.

Throughout this manuscript, we assume that (χ, ϖ), $\Upsilon(\chi)$, and $CB(\chi)$ refer to a metric space (MS), a set of all nonempty subsets of χ, and a set of all nonempty closed and bounded subsets in χ, respectively. The Hausdorff metric generated by ξ is given by $\xi : CB(\chi) \times CB(\chi) \to CB(\chi)$,

$$\xi(V, W) = \max\left\{\sup_{v \in V} D(v, W), \sup_{w \in W} D(w, V)\right\},$$

where $V, W \in CB(\chi)$ and $D(v, W) = \inf\{\varpi(v, w) : w \in W\}$.

Definition 1.1. [28] The subset V of a MS (χ, ϖ) is called proximinal if for each $w \in \chi$ there is $v \in V$ such that $\varpi(w, v) = D(w, V)$.

Definition 1.2. [22, 29] Let (χ, ϖ) be a MS. A trio $(v, w, z) \in \chi^3$ is said to be a TFP of the mapping $\Im : \chi^3 \to \chi$ (where $\chi^3 = \chi \times \chi \times \chi$) if $v = \Im(v, w, z)$, $w = \Im(w, z, v)$, and $z = \Im(z, v, w)$. Moreover, if $v = w = z$, then the trio $(v, w, z) \in \chi^3$ is called a strong TFP of the mapping $\Im$, i.e., $\Im(v, v, v) = v$.

Example 1.3. Assume that $\chi = [-1, 1]$ is endowed with the distance metric $\varpi(v, w) = |v - w|$. If the mapping $\Im : \chi^3 \to \chi$ is described as:

(i) $\Im(v, w, z) = \frac{v+w+z}{3}$, then $(v, v, v) \in [-1, 1]^3$ is a strong TFP of $\Im$.
(ii) $\Im(v, w, z) = \frac{v-w-z}{3}$, then $(0, 0, 0) \in [-1, 1]^3$ is a unique strong TFP of $\Im$.
(iii) $\Im(v, w, z) = \frac{|v+w+z|}{3}$, then $(v, v, v) \in [0, 1]^3$ is a strong TFP of $\Im$.

Definition 1.4. [22] A trio $(v, v, v) \in \chi^3$ is called a strong tripled coincidence point (TCP) of the mappings $\Im : \chi^3 \to \chi$ and $\theta : \chi \to \chi$ if $\Im(v, v, v) = \theta(v)$.

F-contraction mappings, introduced by Wardowski in 2012 [30], provide a generalized framework for studying fixed point (FP) theorems. By relaxing the traditional contraction condition, F-contractions encompass a wider class of mappings while still ensuring the existence and uniqueness of fixed points. This concept has been extensively investigated in various metric spaces, resulting in significant advancements in FP theory and its applications in diverse fields such as integral equations [31–34], and functional and differential systems [35–37].

Wardowski [30] considered that F be a class of functions $F : \mathbb{R}_+ \to \mathbb{R}$ such that the following axioms are true:

(F_i) For each $v, w > 0$, if $v < w$, then $F(v) < F(w)$, that is, F is strictly increasing;
(F_{ii}) For each sequence $\{v_m\}_{m \in \mathbb{N}} \subseteq \mathbb{R}_+$, $\lim_{m\to\infty} v_m = 0$ iff $\lim_{m\to\infty} F(v_m) = -\infty$;
(F_{iii}) There is $\vartheta \in (0, 1)$ in order that $\lim_{m\to\infty} (v_m)^{\vartheta} F(v_m) = 0$.

Definition 1.5. [30] Assume that $\Im : \chi \to \chi$ is an operator defined on a MS (χ, ϖ). The mapping $\Im$ is called an $F-$contraction if there is a real number $a > 0$ such that

$$\varpi(\Im(\tau), \Im(v)) > 0 \text{ implies } a + F[\varpi(\Im(\tau), \Im(v))] \leq F(\varpi(\tau, v)),$$

for all $\tau, v \in \chi$ and $F \in F$.

According to the above definition, Wardowski [30] presented the following functions with the corresponding contractions: For each $j \in \{1,2,3,4\}$, the function F_j defined on $(0,+\infty)$ belongs to F,

$$(i)\ F_1(\varrho) = \ln(\varrho), \qquad \varpi(\Im\tau, \Im\nu) \le e^{-a}\varpi(\tau,\nu),$$

$$(ii)\ F_2(\varrho) = \ln(\varrho) + \varrho, \qquad \frac{\varpi(\Im\tau,\Im\nu)}{\varpi(\tau,\nu)} e^{\varpi(\Im\tau,\Im\nu)-\varpi(\tau,\nu)} \le e^{-a},$$

$$(iii)\ F_3(\varrho) = \frac{-1}{\sqrt{\varrho}}, \qquad \varpi(\Im\tau, \Im\nu) \le \frac{1}{\left(1+\tau\sqrt{\varpi(\tau,\nu)}\right)^2}\varpi(\tau,\nu),$$

$$(iv)\ F_4(\varrho) = \ln(\varrho^2 + \varrho), \qquad \frac{\varpi(\Im\tau,\Im\nu)(1+\varpi(\Im\tau,\Im\nu))}{\varpi(\tau,\nu)(1+\varpi(\tau,\nu))} \le e^{-a},$$

for all $\tau, \nu \in \chi$ and $\varrho > 0$ with $a > 0$ and $\Im\tau \ne \Im\nu$.

Remark 1.6. The authors of [38] proved that, if $F(\varrho) = \frac{-1}{\sqrt[q]{\varrho}}$, where $q > 1$ and $\varrho > 0$, then $F \in F$.

Inspired by the aforementioned work, novel F-contractive θ-triplings are introduced, and the existence of TCPs and strong TCPs is established. By merging the concepts of θ-tripling and F-contractions, a comprehensive framework for analyzing these fixed-point problems is presented. Furthermore, an extension of these results to multivalued θ-tripling is provided, and their applicability to a class of nonlinear integral equations is demonstrated. In addition, integral-type results are explored.

2. Tripled coincidence points

The main results in this section are to obtain some TCPs under F-contractive-type θ-tripling in the MS. We start our task with the following definitions:

Definition 2.1. Let (χ, ϖ) be an MS and $V, W, Z \subseteq \chi$ are non-empty sets. Let $\Im : \chi^3 \to \chi$ and $\theta : \chi \to \chi$ be two mappings on χ. We say that $\Im$ is a $\theta-$tripling if

(i) $\Im(v,w,z) \in \theta(Z)$ for all $v \in V$, $w \in W$, and $z \in Z$.
(ii) $\Im(w,z,v) \in \theta(V)$ for all $v \in V$, $w \in W$, and $z \in Z$.
(iii) $\Im(z,v,w) \in \theta(W)$ for all $v \in V$, $w \in W$, and $z \in Z$.

Definition 2.2. Let (χ, ϖ) be a MS and $V, W, Z \subseteq \chi$ be non-empty sets. The mapping $\Im$ is called an $F-$contractive-type θ-tripling (FCT-θT, for short) if

(i) $\Im$ is a θ-tripling with respect to V, W, and Z,
(ii) there are a real number $a > 0$ and a function $F \in F$ in order that

$$\varpi(\Im(\tau,\kappa,\lambda), \Im(v,w,z)) > 0 \text{ implies } a + F[\varpi(\Im(\tau,\kappa,\lambda), \Im(v,w,z))] \le F(\varpi(\theta\lambda, \theta z)),$$

for all $\tau, \kappa, \lambda, v, w, z \in \chi$.

Theorem 2.3. *Let (χ, ϖ) be a complete MS and $V, W, Z \subseteq \chi$ be non-empty and closed sets. Assume that $\Im : \chi^3 \to \chi$ is an FCT-θT and the mapping $\theta : \chi \to \chi$ is continuous and sequentially convergent such that V, W, and Z are invariant under θ. Then, $V \cap W \cap Z \ne \emptyset$ and $\Im, \theta$ have a TCP in $V \cap W \cap Z$, provided that $\Im$ is continuous.*

Proof. Assume that $(v_0, w_0, z_0) \in V \times W \times Z$ such that $\begin{cases} \theta v_{m+1} = \Im(w_m, z_m, v_m), \\ \theta w_{m+1} = \Im(z_m, v_m, w_m), \\ \theta z_{m+1} = \Im(v_m, w_m, z_m). \end{cases}$

Then, $\begin{cases} \theta v_m \subset V, \\ \theta w_m \subset W, \\ \theta z_m \subset Z. \end{cases}$

Now,

$$\begin{aligned} F(\varpi(\theta v_m, \theta v_{m+1})) &= F(\varpi(\Im(w_{m-1}, z_{m-1}, v_{m-1}), \Im(w_m, z_m, , v_m))) \\ &\leq F(\varpi(\theta v_{m-1}, \theta v_m)) - a \\ &= F(\varpi(\Im(w_{m-2}, z_{m-2}, v_{m-2}), \Im(w_{m-1}, z_{m-1}, v_{m-1}))) - a \\ &\leq F(\varpi(\theta v_{m-2}, \theta v_{m-1})) - 2a \\ &\quad \dots \\ &\leq F(\varpi(\theta v_0, \theta v_1)) - ma. \end{aligned} \tag{2.1}$$

Letting $m \to \infty$ in (2.1), we have

$$\lim_{m\to\infty} F(\varpi(\theta v_m, \theta v_{m+1})) = -\infty,$$

which implies that

$$\lim_{m\to\infty} \varpi(\theta v_m, \theta v_{m+1}) = 0.$$

Assume that $\rho_m = \varpi(\theta v_m, \theta v_{m+1})$. Utilizing the condition (F_{iii}), there exists $\vartheta \in (0, 1)$ in order that $\lim_{m\to\infty} (\rho_m)^{\vartheta} F(\rho_m) = 0$. Thus, by (2.1), we get

$$F(\rho_m) \leq F(\rho_0) - ma.$$

Multiple the two sides in $(\rho_m)^{\vartheta}$, one can write

$$(\rho_m)^{\vartheta} F(\rho_m) \leq (\rho_m)^{\vartheta} F(\rho_0) - ma(\rho_m)^{\vartheta},$$

and in another form, we can write

$$(\rho_m)^{\vartheta} F(\rho_m) - (\rho_m)^{\vartheta} F(\rho_0) \leq -ma(\rho_m)^{\vartheta}.$$

Suppose that $m \to \infty$, then the above inequality gives $\lim_{m\to\infty} m(\rho_m)^{\vartheta} = 0$. Hence, there is a natural number M in order that

$$m(\rho_m)^{\vartheta} \leq 1 \text{ for all } m \geq M.$$

It follows that, for $m \geq M$,

$$\rho_m \leq \frac{1}{m^{\frac{1}{\vartheta}}}.$$

For $k > m \geq M$, we get

$$\begin{aligned} \varpi(\theta v_m, \theta v_k) &\leq \varpi(\theta v_m, \theta v_{m+1}) + \varpi(\theta v_{m+1}, \theta v_{m+2}) + \cdots + \varpi(\theta v_{k-1}, \theta v_k) \\ &= \rho_m + \rho_{m+1} + \cdots + \rho_{k-1} \end{aligned}$$

$$\leq \sum_{j=m}^{k-1}\rho_j \leq \sum_{j=m}^{\infty}\rho_j \leq \sum_{j=m}^{\infty}\frac{1}{j^{\frac{1}{\vartheta}}}.$$

As the series $\sum\limits_{j=m}^{\infty}\frac{1}{j^{\frac{1}{\vartheta}}}$ is convergence, then $\lim_{m\to\infty}\varpi(\theta v_m,\theta v_k)=0$. Thus, the sequence $\{\theta v_m\}$ is a Cauchy sequence. Since θ is continuous and sequentially convergent, and χ is complete, then we conclude that $\{\theta v_m\}$ converges to $\{\theta v\}$ (say) in V. On the other hand, if $\{v_m\}$ converges to some $v\in\chi$, then θv_m converges to $\theta v\in V$ because of the continuity of θ.

By the same procedure, one can obtain that

$$\begin{aligned}\lim_{m\to\infty} w_m &= w\in W\Rightarrow \lim_{m\to\infty}\theta w_m=\theta w\in W,\\ \lim_{m\to\infty} z_m &= z\in Z\Rightarrow \lim_{m\to\infty}\theta z_m=\theta z\in Z.\end{aligned}$$

Now,

$$\begin{aligned}F(\varpi(\theta v_m,\theta w_m)) &= F(\varpi(\Im(w_{m-1},z_{m-1},v_{m-1}),\Im(z_{m-1},v_{m-1},w_{m-1})))\\ &\leq F(\varpi(\theta(v_{m-1}),\theta(w_{m-1})))-a\\ &= F(\varpi(\Im(w_{m-2},z_{m-2},v_{m-2}),\Im(z_{m-2},v_{m-2},w_{m-2})))-a\\ &\leq F(\varpi(\theta(v_{m-2}),\theta(w_{m-2})))-2a\\ &\quad\ldots\\ &\leq F(\varpi(\theta v_0,\theta w_0))-ma.\end{aligned}$$

Passing $m\to\infty$, we have

$$\begin{aligned}\lim_{m\to\infty}F(\varpi(\theta v_m,\theta w_m)) &= 0\Rightarrow \lim_{m\to\infty}\varpi(\theta v_m,\theta w_m)=\varpi(\theta v,\theta w)=0\\ &\Rightarrow \theta v=\theta w.\end{aligned}$$

Similarly, we can prove that $\theta w=\theta z$. Hence, $\theta v=\theta w=\theta z$. Therefore, $V\cap W\cap Z\neq\emptyset$. □

Now,

$$\begin{aligned}F(\varpi(\theta v_m,\Im(v,w_{m-1},z_{m-1}))) &= F(\varpi(\Im(w_{m-1},z_{m-1},v_{m-1}),\Im(v,w_{m-1},z_{m-1})))\\ &\leq F(\varpi(\theta v_{m-1},\theta z_{m-1}))-a\\ &= F(\varpi(\Im(w_{m-2},z_{m-2},v_{m-2}),\Im(v_{m-1},w_{m-1},z_{m-1})))-a\\ &\leq F(\varpi(\theta v_{m-2},\theta z_{m-2}))-2a\\ &\quad\ldots\\ &\leq F(\varpi(\theta v_0,\theta z_0))-ma.\end{aligned}$$

Letting $m\to\infty$, and using the continuity of $\Im$, we get

$$\lim_{m\to\infty}F(\varpi(\theta v_m,\Im(v,w_{m-1},z_{m-1})))=F(\varpi(\theta v,\Im(v,w,z)))=-\infty,$$

which implies that

$$\lim_{m\to\infty}\varpi(\theta v_m,\Im(v,w_{m-1},z_{m-1}))=\varpi(\theta v,\Im(v,w,z))=0.$$

Hence, $\theta v=\Im(v,w,z)$. Analogously, we can show that $\theta w=\Im(w,z,v)$ and $\theta z=\Im(z,v,w)$. This proves that the element (v,w,z) is a TCP.

Theorem 2.4. *With the aid of the assertions of Theorem 2.3, $\Im, \theta$ have a unique strong TCP, provided that θ is injective.*

Proof. Thanks to Theorem 2.3, $\theta v = \theta w = \theta z$. Since θ is injective, then $v = w = z$. This proves that $\theta v = \Im(v, v, v)$.

For the uniqueness, assume that ϱ is another strong TCP of $\Im$ and θ such that $\varrho \neq v$. By our contractive mapping, we have

$$F(\varpi(\theta\varrho, \theta v)) = F(\varpi(\Im(\varrho, \varrho, \varrho), \Im(v, v, v))) \leq F(\varpi(\theta\varrho, \theta v)) - a.$$

Here, a contradiction exists because $a > 0$. This illustrate that $\Im$ and θ have a unique strong TCP. □

Corollary 2.5. *Let (χ, ϖ) be a complete MS and $V, W, Z \subseteq \chi$ be non-empty and closed sets. Assume that $\theta : \chi \to \chi$ is a continuous and sequentially convergent mapping such that V, W, and Z are invariant under θ. If $\Im : \chi^3 \to \chi$ satisfies the condition*

$$\begin{aligned} \varpi(\Im(\tau, \kappa, \lambda), \Im(v, w, z)) &> 0, \text{ it implies}, \\ a + F[\varpi(\Im(\tau, \kappa, \lambda), \Im(v, w, z))] &\leq F(\max\{\varpi(\theta\tau, \theta v), \varpi(\theta\kappa, \theta w), \varpi(\theta\lambda, \theta z)\}). \end{aligned} \tag{2.2}$$

Then, $V \cap W \cap Z \neq \emptyset$ and $\Im, \theta$ have a TCP in $V \cap W \cap Z$, provided that $\Im$ is continuous.

Proof. The proof follows immediately with Theorem 2.3 if we consider

$$\begin{aligned} \max\{\varpi(\theta\tau, \theta v), \varpi(\theta\kappa, \theta w), \varpi(\theta\lambda, \theta z)\} &= \varpi(\theta\tau, \theta v), \\ \text{or } \max\{\varpi(\theta\tau, \theta v), \varpi(\theta\kappa, \theta w), \varpi(\theta\lambda, \theta z)\} &= \varpi(\theta\kappa, \theta w), \\ \text{or } \max\{\varpi(\theta\tau, \theta v), \varpi(\theta\kappa, \theta w), \varpi(\theta\lambda, \theta z)\} &= \varpi(\theta\lambda, \theta z). \end{aligned}$$

□

Definition 2.6. Let (χ, ϖ) be an MS and $V, W, Z \subseteq \chi$ be non-empty sets. We say that the mapping $\Im$ is a strict FCT-θT, if

(i) $\Im$ is a $\theta-$tripling with respect to V, W, and Z,
(ii) there are a real number $a > 0$ and a function $F \in F$ with $F(\tau + \kappa) \leq F(\tau) + F(\kappa)$ such that

$$\varpi(\Im(v, w, z), \Im(\tau, \kappa, \lambda)) > 0 \text{ implies } a + F[\varpi(\Im(v, w, z), \Im(\tau, \kappa, \lambda))] \leq F(\varpi(\theta z, \theta\lambda)),$$

for all $\tau, z \in V$, $\kappa, w \in W$, $\lambda, v \in Z$.

Theorem 2.7. *Let (χ, ϖ) be a complete MS and $V, W, Z \subseteq \chi$ be non-empty and closed sets. Assume that $\Im : \chi^3 \to \chi$ is a strict FCT-θT and $\theta : \chi \to \chi$ is a continuous and sequentially convergent mapping such that V, W, and Z are invariant under θ. Then, $V \cap W \cap Z \neq \emptyset$ and $\Im, \theta$ have a TCP in $V \cap W \cap Z$, provided that $\Im$ is continuous.*

Proof. Assume that $(v_0, w_0, z_0) \in V \times W \times Z$ such that $\begin{cases} \theta v_{m+1} = \Im(w_m, z_m, v_m), \\ \theta w_{m+1} = \Im(z_m, v_m, w_m), \\ \theta z_{m+1} = \Im(v_m, w_m, z_m). \end{cases}$

Then $\begin{cases} \theta v_m \subset V, \\ \theta w_m \subset W, \\ \theta z_m \subset Z. \end{cases}$

Now,

$$\begin{aligned}
F\left(\varpi\left(\theta v_m, \theta v_{m+1}\right)\right) &\leq F\left(\varpi\left(\theta v_m, \theta w_m\right)+\varpi\left(\theta w_m, \theta v_{m+1}\right)\right) \\
&\leq F\left(\varpi\left(\theta v_m, \theta w_m\right)\right)+F\left(\varpi\left(\theta w_m, \theta v_{m+1}\right)\right) \\
&= F\left(\varpi\left(\Im\left(w_{m-1}, z_{m-1}, v_{m-1}\right), \Im\left(z_{m-1}, v_{m-1}, w_{m-1}\right)\right)\right) \\
&\quad +F\left(\varpi\left(\Im\left(z_{m-1}, v_{m-1}, w_{m-1}\right), \Im\left(w_m, z_m, v_m\right)\right)\right) \\
&\leq F\left(\varpi\left(\theta v_{m-1}, \theta w_{m-1}\right)\right)+F\left(\varpi\left(\theta w_{m-1}, \theta v_m\right)\right)-2a \\
&= F\left(\varpi\left(\Im\left(w_{m-2}, z_{m-2}, v_{m-2}\right), \Im\left(z_{m-2}, v_{m-2}, w_{m-2}\right)\right)\right) \\
&\quad +F\left(\varpi\left(\Im\left(z_{m-2}, v_{m-2}, w_{m-2}\right), \Im\left(w_{m-1}, z_{m-1}, v_{m-1}\right)\right)\right)-2a \\
&\leq F\left(\varpi\left(\theta v_{m-2}, \theta w_{m-2}\right)\right)+F\left(\varpi\left(\theta w_{m-2}, \theta v_{m-1}\right)\right)-4a \\
&\quad \cdots \\
&\leq F\left(\varpi\left(\theta v_0, \theta w_0\right)\right)+F\left(\varpi\left(\theta w_0, \theta v_1\right)\right)-2ma.
\end{aligned}$$

Letting $m \rightarrow \infty$ in the above inequality, we get

$$\lim_{m \rightarrow \infty} F\left(\varpi\left(\theta v_m, \theta v_{m+1}\right)\right)=-\infty,$$

which yields

$$\lim_{m \rightarrow \infty} \varpi\left(\theta v_m, \theta v_{m+1}\right)=0.$$

Utilizing the triangle inequality, for $l>m$, one can write

$$\varpi\left(\theta v_m, \theta v_l\right) \leq \varpi\left(\theta v_m, \theta v_{m+1}\right)+\varpi\left(\theta v_{m+1}, \theta v_{m+2}\right)+\cdots+\varpi\left(\theta v_{l-1}, \theta v_l\right) \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Thus, $\{\theta v_m\}$ is a Cauchy sequence. Similar to the proof of Theorem 2.3, we conclude that

$$\begin{aligned}
\lim_{m \rightarrow \infty} v_m &= v \in V \Rightarrow \lim_{m \rightarrow \infty} \theta v_m=\theta v \in V, \\
\lim_{m \rightarrow \infty} w_m &= w \in W \Rightarrow \lim_{m \rightarrow \infty} \theta w_m=\theta w \in W, \\
\lim_{m \rightarrow \infty} z_m &= z \in Z \Rightarrow \lim_{m \rightarrow \infty} \theta z_m=\theta z \in Z.
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.3. Hence, $V \cap W \cap Z \neq \emptyset$ and the trio (v, w, z) is a TCP. □

Theorem 2.8. *With the aid of the assumptions of Theorem 2.7, $\Im, \theta$ have a unique strong TCP, provided that θ is injective.*

Proof. The proof follows immediately from Theorem 2.4. □

Remark 2.9. In Theorems 2.4 and 2.8, if we considered $\theta(v)=v$, then $\Im, \theta$ have a unique strong TFP.

Example 2.10. Assume that $\chi=[-2,2]$ equipped with a metric $\varpi=|v-w|$. Assume that $V=[0,2]$, $W=[0,1]$, and $Z=[-2,0]$. Define the mappings $\Im: \chi^3 \rightarrow \chi$ and $\theta: \chi \rightarrow \chi$ by

$$\Im(v, w, z)= \begin{cases}\frac{-|z|}{6}, & \text{if } (v, w, z) \in V \times W \times Z, \\ \frac{v}{6}, & \text{if } (w, z, v) \in W \times Z \times V, \\ \frac{w}{6}, & \text{if } (z, v, w) \in Z \times V \times W,\end{cases}$$

and $\theta(v) = \frac{v}{3}$ for $v \in \chi$. Clearly, θ is an injective mapping. Furthermore, suppose that $F(\varrho) = \ln(\varrho)$, $\varrho > 0$. Then,

$$\begin{aligned} F\left[\varpi\left(\Im\left(\tau,\kappa,\lambda\right),\Im\left(v,w,z\right)\right)\right] &= F\left[\varpi\left(\frac{-|\lambda|}{6},\frac{-|z|}{6}\right)\right] = \ln\left|\frac{|\lambda|}{6}-\frac{|z|}{6}\right| \\ &\leq \ln\left|\frac{\lambda}{3}-\frac{z}{3}\right| - \ln(2) = F\left(\varpi\left(\theta\lambda,\theta z\right)\right) - \ln(2). \end{aligned}$$

Therefore, all requirements of Theorem 2.4 are fulfilled with $a = \ln(2) > 0$. Hence, $\Im, \theta$ have a unique strong TCP. The unique strong TCP is $(0,0,0) \in V \cap W \cap Z$.

3. Multivalued F-contractive mappings and TCPs

In this section, we obtain some TCPs for multivalued F-contractive-type θ-tripling (FCT-θT, for short).

Definition 3.1. Let (χ, ϖ) be an MS, $V, W, Z \subseteq \chi$ be non-empty sets, and $\Theta = V \cup W \cup Z$. Suppose that $\theta : \Theta \to \Theta$ is a given mapping. We say that the mapping $\Im : \Theta^3 \to \Upsilon(\Theta)$ is a multivalued FCT-θT, if

(i) $\Im\left(V \times W \times Z\right) \subset \theta\left(Z\right)$, $\Im\left(W \times Z \times V\right) \subset \theta\left(V\right)$, and $\Im\left(Z \times V \times W\right) \subset \theta\left(W\right)$,
(ii) there are a real number $a > 0$ and a function $F \in F$ with $F(\tau + \kappa) \leq F(\tau) + F(\kappa)$ such that

$$\begin{aligned} \xi\left(\Im\left(v,w,z\right),\Im\left(\tau,\kappa,\lambda\right)\right) &> 0 \text{ implies }, \\ a + F\left[\xi\left(\Im\left(v,w,z\right),\Im\left(\tau,\kappa,\lambda\right)\right)\right] &\leq F\left(\varpi\left(\theta z,\theta\lambda\right)\right). \end{aligned}$$

for all $v, w, z, \tau, \kappa, \lambda \in \Theta$.

Theorem 3.2. *Let (χ, ϖ) be a complete MS and $V, W, Z \subseteq \chi$ be non-empty, closed, and bounded sets. Assume that $\Im : \Theta^3 \to \Upsilon_{Prox}(\Theta)$ is a multivalued FCT-θT and $\theta : \Theta \to \Theta$ is a continuous and sequentially convergent mapping such that V, W, and Z are invariant under θ. Then, $V \cap W \cap Z \neq \emptyset$ and $\Im, \theta$ have a TCP in $V \cap W \cap Z$, whenever $\Im$ is continuous.*

Proof. Assume that $v_0 \in V$, $w_0 \in W$, and $z_0 \in Z$. Then, V, W, and Z are invariant under θ. Further, $\Im(v_0, w_0, z_0) \in \Upsilon_{\text{Prox}}(\Theta)$, $\Im(w_0, z_0, v_0) \in \Upsilon_{\text{Prox}}(\Theta)$, and $\Im(z_0, v_0, w_0) \in \Upsilon_{\text{Prox}}(\Theta)$. Thus, there exist $v_1 \in V$, $w_1 \in W$, and $z_1 \in Z$ such that $\theta v_1 = \Im(w_0, z_0, v_0)$, $\theta w_1 = \Im(z_0, v_0, w_0)$, and $\theta z_1 = \Im(v_0, w_0, z_0)$. Also, we can write

$$\begin{aligned} \varpi\left(\theta v_0,\theta v_1\right) &= D\left(\theta v_0,\Im\left(w_0,z_0,v_0\right)\right), \\ \varpi\left(\theta w_0,\theta w_1\right) &= D\left(\theta w_0,\Im\left(z_0,v_0,w_0\right)\right), \\ \varpi\left(\theta z_0,\theta z_1\right) &= D\left(\theta z_0,\Im\left(v_0,w_0,z_0\right)\right). \end{aligned}$$

Since v_1, w_1, and z_1 exist, then $\Im(w_1, z_1, v_1)$, $\Im(z_1, v_1, w_1)$, and $\Im(v_1, w_1, z_1)$ exist in $\Upsilon_{\text{Prox}}(\Theta)$.

Again, there are $v_2 \in V$, $w_2 \in W$, and $z_2 \in Z$ such that $\theta v_2 = \Im(w_1, z_1, v_1)$, $\theta w_2 = \Im(z_1, v_1, w_1)$, and $\theta z_2 = \Im(v_1, w_2, z_2)$. Moreover, we have

$$\varpi\left(\theta v_1,\theta v_2\right) = D\left(\theta v_1,\Im\left(w_1,z_1,v_1\right)\right),$$

$$\begin{aligned} \varpi(\theta w_1, \theta w_2) &= D(\theta w_1, \Im(z_1, v_1, w_1)), \\ \varpi(\theta z_1, \theta z_2) &= D(\theta z_1, \Im(v_1, w_1, z_1)). \end{aligned}$$

Repeating the above process, we have sequences $\begin{cases} \theta v_m \subset V, \\ \theta w_m \subset W, \\ \theta z_m \subset Z, \end{cases}$ such that $\begin{cases} \theta v_{m+1} \in \Im(w_m, z_m, v_m), \\ \theta w_{m+1} \in \Im(z_m, v_m, w_m), \\ \theta z_{m+1} \in \Im(v_m, w_m, z_m), \end{cases}$

and

$$\begin{cases} \varpi(\theta v_m, \theta v_{m+1}) = D(\theta v_m, \Im(w_m, z_m, v_m)), \\ \varpi(\theta w_m, \theta w_{m+1}) = D(\theta w_m, \Im(z_m, v_m, w_m)), \\ \varpi(\theta z_m, \theta z_{m+1}) = D(\theta z_m, \Im(v_m, w_m, z_m)). \end{cases}$$

Now, assume that $\theta v_m \notin \Im(w_m, z_m, v_m)$. Then, $D(\theta v_m, \Im(w_m, z_m, v_m)) > 0$ and

$$\begin{aligned} F(\varpi(\theta v_m, \theta v_{m+1})) &= F(D(\theta v_m, \Im(w_m, z_m, v_m))) \\ &\leq F(\xi(\Im(w_{m-1}, z_{m-1}, v_{m-1}), \Im(w_m, z_m, v_m))) \\ &\leq F(\varpi(\theta v_{m-1}, \theta v_m)) - a \\ &= F(D(\Im(w_{m-2}, z_{m-2}, v_{m-2}), \Im(w_{m-1}, z_{m-1}, v_{m-1}))) - a \\ &\leq F(\varpi(\theta v_{m-2}, \theta v_{m-1})) - 2a \\ &\quad \dots \\ &\leq F(\varpi(\theta v_0, \theta v_1)) - ma. \end{aligned} \tag{3.1}$$

Taking $m \to \infty$ in (3.1), we have

$$\lim_{m\to\infty} F(\varpi(\theta v_m, \theta v_{m+1})) = -\infty,$$

which implies that

$$\lim_{m\to\infty} \varpi(\theta v_m, \theta v_{m+1}) = 0.$$

Assume that $\rho_m = \varpi(\theta v_m, \theta v_{m+1})$. Using the condition ($F_{iii}$), there exists $\vartheta \in (0,1)$ such that $\lim_{m\to\infty} (\rho_m)^\vartheta F(\rho_m) = 0$. Thus, by (3.1), we get

$$F(\rho_m) \leq F(\rho_0) - ma,$$

which yields

$$(\rho_m)^\vartheta F(\rho_m) \leq (\rho_m)^\vartheta F(\rho_0) - ma(\rho_m)^\vartheta,$$

and in another form, we can write

$$(\rho_m)^\vartheta F(\rho_m) - (\rho_m)^\vartheta F(\rho_0) \leq -ma(\rho_m)^\vartheta.$$

Assume that $m \to \infty$, then the above inequality gives $\lim_{m\to\infty} m(\rho_m)^\vartheta = 0$. Hence, there is a natural number M in order that

$$m(\rho_m)^\vartheta \leq 1 \text{ for all } m \geq M.$$

It follows that, for $m \geq M$,

$$\rho_m \leq \frac{1}{m^{\frac{1}{\vartheta}}}.$$

For $k > m \geq M$, we get

$$\begin{aligned}\varpi(\theta v_m, \theta v_k) &\leq \varpi(\theta v_m, \theta v_{m+1}) + \varpi(\theta v_{m+1}, \theta v_{m+2}) + \cdots + \varpi(\theta v_{k-1}, \theta v_k) \\ &= \rho_m + \rho_{m+1} + \cdots + \rho_{k-1} \\ &\leq \sum_{j=m}^{k-1} \rho_j \leq \sum_{j=m}^{\infty} \rho_j \leq \sum_{j=m}^{\infty} \frac{1}{j^{\frac{1}{\vartheta}}}.\end{aligned}$$

As the series $\sum\limits_{j=m}^{\infty} \frac{1}{j^{\frac{1}{\vartheta}}}$ is convergence, then $\lim_{m\to\infty} \varpi(\theta v_m, \theta v_k) = 0$. Thus, the sequence $\{\theta v_m\}$ is a Cauchy sequence. Since θ is continuous and sequentially convergent, and χ is complete, then we conclude that $\{\theta v_m\}$ converges to $\{\theta v\}$ (say) in V. On the other hand, if $\{v_m\}$ converges to some $v \in \chi$, then θv_m converges to $\theta v \in V$ due to the continuity of θ.

By the same procedure, one can obtain that

$$\begin{aligned}\lim_{m\to\infty} w_m &= w \in W \Rightarrow \lim_{m\to\infty} \theta w_m = \theta w \in W, \\ \lim_{m\to\infty} z_m &= z \in Z \Rightarrow \lim_{m\to\infty} \theta z_m = \theta z \in Z.\end{aligned}$$

Now,

$$\begin{aligned}F(D(\theta v_m, \Im(v, w_{m-1}, z_{m-1}))) &= F(\xi(\Im(w_{m-1}, z_{m-1}, v_{m-1}), \Im(v, w_{m-1}, z_{m-1}))) \\ &\leq F(\varpi(\theta v_{m-1}, \theta z_{m-1})) - a \\ &\leq F(\varpi(\theta v_{m-1}, \theta v_m) + \varpi(\theta v_m, \theta z_{m-1})) - a \\ &\leq F(\varpi(\theta v_{m-1}, \theta v_m)) + F(\varpi(\theta v_m, \theta z_{m-1})) - a \\ &= F(D(\theta v_{m-1}, \Im(w_{m-1}, z_{m-1}, v_{m-1}))) + F(\varpi(\theta v_m, \theta z_{m-1})) - a \\ &\leq F(\xi(\Im(w_{m-2}, z_{m-2}, v_{m-2}), \Im(w_{m-1}, z_{m-1}, v_{m-1}))) \\ &\quad + F(\varpi(\theta v_m, \theta z_{m-1})) - a \\ &\leq F(\varpi(\theta v_{m-2}, \theta v_{m-1})) + F(\varpi(\theta v_m, \theta z_{m-1})) - 2a \\ &\quad \ldots \\ &\leq F(\varpi(\theta v_0, \theta v_1)) + F(\varpi(\theta v_m, \theta z_{m-1})) - ma.\end{aligned}$$

Letting $m \to \infty$, and using the continuity of $\Im$, we get

$$\lim_{m\to\infty} F(D(\theta v_m, \Im(v, w_{m-1}, z_{m-1}))) = F(D(\theta v, \Im(v, w, z))) = -\infty,$$

which implies that

$$\lim_{m\to\infty} D(\theta v_m, \Im(v, w_{m-1}, z_{m-1})) = D(\theta v, \Im(v, w, z)) = 0.$$

Hence, $\theta v = \Im(v, w, z)$. Similarly, we can show that $\theta w = \Im(w, z, v)$ and $\theta z = \Im(z, v, w)$. This proves that the trio (v, w, z) is a TCP of θ and $\Im$. □

4. Supportive applications

This section is important as it highlights the practical applications of our research. By showing how our techniques can solve nonlinear integral systems, a topic of significant interest, we emphasize the broader impact of FP theory.

4.1. Some integral-type results

Assume that $\mho$ is a family of functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the axioms below:

(i) ϕ is a positive Lebesgue integrable mapping on each compact subset of $\mathbb{R}_+$,
(ii) for all $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(r)dr > 0$.

Corollary 4.1. *Replacing the contractive condition of Theorem 2.3 by the formula*

$$\int_0^{\varpi(\Im(\tau,\kappa,\lambda),\Im(v,w,z))} \phi(r)dr > 0 \text{ implies } \int_0^{a+F[\varpi(\Im(\tau,\kappa,\lambda),\Im(v,w,z))]} \phi(r)dr \le \int_0^{F(\varpi(\theta\lambda,\theta z))} \phi(r)dr, \tag{4.1}$$

where $\phi \in \mho$. If the rest of the requirements of Theorem 2.3 hold, then there exists a TCP of the mapping $\Im$ and θ.

Proof. Consider the function $\Lambda(\beta) = \int_0^{\beta} \phi(r)dr$ such that $\Lambda(r_1) \le \Lambda(r_2)$ implies that $r_1 \le r_2$ for each $r_1, r_1 \in \mathbb{R}_+$. Then, (4.1) can be expressed as

$$\Lambda(a + F[\varpi(\Im(\tau,\kappa,\lambda), \Im(v,w,z))]) \le \Lambda(F(\varpi(\theta\lambda,\theta z))),$$

which yields

$$a + F[\varpi(\Im(\tau,\kappa,\lambda), \Im(v,w,z))] \le F(\varpi(\theta\lambda,\theta z)),$$

provided that $\varpi(\Im(\tau,\kappa,\lambda), \Im(v,w,z)) > 0$. □

Corollary 4.2. *Replacing the contractive condition of Theorem 3.2 by the formula*

$$\int_0^{\xi(\Im(v,w,z),\Im(\tau,\kappa,\lambda))} \phi(r)dr > 0 \text{ implies } \int_0^{a+F[\xi(\Im(v,w,z),\Im(\tau,\kappa,\lambda))]} \phi(r)dr \le \int_0^{F(\varpi(\theta z,\theta\lambda))} \phi(r)dr, \tag{4.2}$$

where $\phi \in \mho$. If the rest of the hypotheses of Theorem 3.2 are satisfied, then there exists a TCP of the mapping $\Im$ and θ.

Proof. The proof is similar to Corollary 4.1. □

Remark 4.3. If we take the mapping θ as an injective mapping in Corollaries 4.1 and 4.2, we have a strong TCP of $\Im$ and θ.

Motivated by [39], assume that $\varphi \in \mathbb{N}$ is a fixed number and $\{\phi_k\}_{k\in[1,\varphi]}$ is a family of φ functions contained on $\mho$. For each $r \ge 0$, we define

$$\begin{aligned}
\aleph_1(r) &= \int_0^r \phi_1(r)dr,\\
\aleph_2(r) &= \int_0^{\aleph_1(r)} \phi_2(r)dr = \int_0^{\int_0^r \phi_1(r)dr} \phi_2(r)dr,\\
\aleph_3(r) &= \int_0^{\aleph_2(r)} \phi_3(r)dr = \int_0^{\int_0^{\int_0^r \phi_1(r)dr} \phi_2(r)dr} \phi_3(r)dr,\\
&\cdots\\
\aleph_\varphi(r) &= \int_0^{\aleph_{(\varphi-1)}(r)} \phi_\varphi(r)dr.
\end{aligned}$$

We have the following result:

Corollary 4.4. *Exchange the contractive condition of Theorem 2.3 by the hypotheses below*

$$\aleph_{\varphi}\left(a+F\left[\varpi\left(\Im\left(\tau,\kappa,\lambda\right),\Im\left(v,w,z\right)\right)\right]\right)\leq\aleph_{\varphi}\left(F\left(\varpi\left(\theta\lambda,\theta z\right)\right)\right). \tag{4.3}$$

If the rest of the requirements of Theorem 2.3 are satisfied, then there exists a TCP of the mapping $\Im$ and θ.

Proof. Specify the function $\aleph_{\varphi}(r)$ such that $\aleph_{\varphi}(r_1)\leq\aleph_{\varphi}(r_2)$ implies that $r_1\leq r_2$ for each $r_1,r_1\in\mathbb{R}_+$. Then, (4.3) can be written as

$$a+F\left[\varpi\left(\Im\left(\tau,\kappa,\lambda\right),\Im\left(v,w,z\right)\right)\right]\leq\left(F\left(\varpi\left(\theta\lambda,\theta z\right)\right)\right),$$

provided that $\varpi\left(\Im\left(\tau,\kappa,\lambda\right),\Im\left(v,w,z\right)\right)>0$. The proof can be completed by Theorem 2.3. □

Corollary 4.5. *Exchange the contractive condition of Theorem 3.2 by the following assumption:*

$$\aleph_{\varphi}\left(a+F\left[\xi\left(\Im\left(v,w,z\right),\Im\left(\tau,\kappa,\lambda\right)\right)\right]\right)\leq\aleph_{\varphi}\left(F\left(\varpi\left(\theta z,\theta\lambda\right)\right)\right).$$

If the rest of the axioms of Theorem 3.2 are true, then, there exists a TCP of the mapping $\Im$ and θ.

Proof. The proof is similar to Corollary 4.4. □

Remark 4.6. If θ is an injective mapping in Corollaries 4.4 and 4.5, we have a strong TCP of $\Im$ and θ.

4.2. Solving a system of nonlinear integral equations

Assume that $\chi=C\left([0,l],\mathbb{R}\right)$ is the set of all continuous and sequential convergence functions described on $[0,l]$. Define a metric distance $\varpi:\chi\times\chi\to\mathbb{R}$ by $\varpi\left(\tau,\kappa\right)=\sup_{s\in[0,l]}|\tau(s)-\kappa(s)|$ for all $\tau,\kappa\in\chi$. Clearly, the pair (χ,ϖ) is a complete MS.

Suppose we have the following system:

$$\begin{cases}\tau\left(s\right)=\int_0^l\supset\left(s,r\right)\Xi\left(r,\tau\left(r\right),\kappa(r),\lambda(r)\right)dr, & r\in[0,l],\\ \kappa\left(s\right)=\int_0^l\supset\left(s,r\right)\Xi\left(r,\kappa(r),\lambda(r),\tau\left(r\right)\right)dr, & r\in[0,l],\\ \lambda\left(s\right)=\int_0^l\supset\left(s,r\right)\Xi\left(r,\lambda(r),\tau\left(r\right),\kappa(r)\right)dr, & r\in[0,l],\end{cases} \tag{4.4}$$

where $l\in(0,\infty)$ is a real number, $\supset:[0,l]\times[0,l]\to\mathbb{R}$, and $\Xi:[0,l]\times\mathbb{R}^3\to\mathbb{R}$.

Before we present our main results, we need the following hypotheses:

(H_1) The function $\Xi:[0,l]\times\mathbb{R}^3\to\mathbb{R}$ is continuous.

(H_2) There are closed subsets V, W, and Z such that for $\tau,\kappa,\lambda,v,w,z\in V\cup W\cup Z$, we have

$$\left|\Xi\left(r,\tau\left(r\right),\kappa(r),\lambda(r)\right)-\Xi\left(r,v\left(r\right),w(r),z(r)\right)\right|\leq\frac{1}{a}\left|\lambda(r)-z(r)\right|,\ \text{where } a>0.$$

(H_3) $\sup_{s\in[0,l]}\supset\left(s,r\right)\leq1$.

Theorem 4.7. *Under the hypotheses (H_1)–(H_3), the nonlinear problem (4.4) has a solution on χ.*

Proof. The mechanism of the FP technique is summarized in equating a given operator with the problem under study and searching for a unique FP for this operator that is considered a unique solution to the problem presented. So, we define the mapping $\Im : \chi^3 \to \chi$ by

$$\Im(\tau, \kappa, \lambda)(s) = \int_0^l \backepsilon(s, r)\, \Xi(r, \tau(r), \kappa(r), \lambda(r))\, dr, \;\; r \in [0, l], \;\; \tau, \kappa, \lambda \in \chi,$$

and $\theta(\tau)(s) = \tau(s)$. Then, for each $\lambda(r) \in V$, $\kappa(r) \in W$, and $\tau(r) \in Z$, the problem (4.4) yields

$$\begin{aligned} \Im(\tau, \kappa, \lambda)(s) &= \int_0^l \backepsilon(s, r)\, \Xi(r, \tau(r), \kappa(r), \lambda(r))\, dr \\ &= \tau(s) \\ &= \theta(\tau)(s) \in \theta(Z), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \Im(\kappa, \lambda, \tau)(s) &= \int_0^l \backepsilon(s, r)\, \Xi(r, \kappa(r), \lambda(r), \tau(r))\, dr \\ &= \kappa(s) \\ &= \theta(\kappa)(s) \in \theta(W), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \Im(\lambda, \tau, \kappa)(s) &= \int_0^l \backepsilon(s, r)\, \Xi(r, \lambda(r), \tau(r), \kappa(r))\, dr \\ &= \lambda(s) \\ &= \theta(\lambda)(s) \in \theta(V). \end{aligned} \tag{4.7}$$

It follows from (4.5) and (4.6) that the mapping $\Im$ is a θ-tripling with respect to V, W, and Z.

Next, we show that the mapping $\Im$ is an FCT-θT. Assume that $\tau, \kappa, \lambda, v, w, z \in V \cup W \cup Z$. Then

$$\begin{aligned} & \left|\Im(\tau, \kappa, \lambda)(s) - \Im(v, w, z)(s)\right| \\ = & \left|\int_0^l \backepsilon(s, r)\, \Xi(r, \tau(r), \kappa(r), \lambda(r))\, dr - \int_0^l \backepsilon(s, r)\, \Xi(r, v(r), w(r), z(r))\, dr\right| \\ = & \left|\int_0^l \backepsilon(s, r)\, [\Xi(r, \tau(r), \kappa(r), \lambda(r)) - \Xi(r, v(r), w(r), z(r))]\, dr\right| \\ \leq & \int_0^l \backepsilon(s, r)\, |\Xi(r, \tau(r), \kappa(r), \lambda(r)) - \Xi(r, v(r), w(r), z(r))|\, dr \\ \leq & \int_0^l \frac{1}{a} |\lambda(r) - z(r)|\, \backepsilon(s, r)\, dr \\ \leq & \int_0^l \frac{1}{a} \sup_{q \in [0, l]} |\theta(\lambda)(q) - \theta(z)(q)|\, \backepsilon(s, r)\, dr \\ \leq & \frac{1}{a} \varpi(\theta(\lambda), \theta(z)) \int_0^l \backepsilon(s, r)\, dr \\ \leq & \frac{1}{a} \varpi(\theta(\lambda), \theta(z)). \end{aligned}$$

This leads to

$$\sup_{s\in[0,l]}\left|\Im(\tau,\kappa,\lambda)(s)-\Im(v,w,z)(s)\right|\leq\frac{1}{a}\varpi(\theta(\lambda),\theta(z)),$$

that is,

$$\varpi(\Im(\tau,\kappa,\lambda),\Im(v,w,z))\leq\frac{1}{a}\varpi(\theta(\lambda),\theta(z)).$$

Taking the natural logarithm on both sides, we have

$$\begin{aligned}\ln(\varpi(\Im(\tau,\kappa,\lambda),\Im(v,w,z))) &\leq \ln\left(\frac{1}{a}\varpi(\theta(\lambda),\theta(z))\right)\\ &= \ln(\varpi(\theta(\lambda),\theta(z)))-\ln(a).\end{aligned}$$

Thus, $\Im$ is an FCT-θT with $F(\varrho)=\ln(\varrho)$, $\varrho>0$. Consequently, all requirements of Theorem 2.3 are fulfilled. Hence $\Im$ and θ have a TCP $(v,w,z)\in V\cup W\cup Z$, which is a solution to the problem (4.5). □

5. Solving another form of integral equations

Let $\chi=C([0,l],\mathbb{R})$ be defined in the above part and (χ,ϖ) is a complete MS under the distance $\varpi(\tau,\kappa)=\max_{s\in[0,l]}|\tau(s)-\kappa(s)|$ for all $\tau,\kappa\in\chi$. Consider the following system:

$$\begin{cases}\widetilde{\tau}(s)=\ell(s)+\int_0^l\widetilde{\eth}(s,r)\left[\Xi_1(r,\widetilde{\tau}(r))+\Xi_2(r,\widetilde{\kappa}(r))+\Xi_3\left(r,\widetilde{\lambda}(r)\right)\right]dr,\\ \widetilde{\kappa}(s)=\ell(s)+\int_0^l\widetilde{\eth}(s,r)\left[\Xi_1(r,\widetilde{\kappa}(r))+\Xi_2\left(r,\widetilde{\lambda}(r)\right)+\Xi_3(r,\widetilde{\tau}(r))\right]dr,\\ \widetilde{\lambda}(s)=\ell(s)+\int_0^l\widetilde{\eth}(s,r)\left[\Xi_1\left(r,\widetilde{\lambda}(r)\right)+\Xi_2(r,\widetilde{\tau}(r))+\Xi_3(r,\widetilde{\kappa}(r))\right]dr,\end{cases}\tag{5.1}$$

for all $r\in[0,l]$. Assume that the following assertions hold:

(A_1) The functions $\ell:[0,l]\to\mathbb{R}$, $\widetilde{\eth}:[0,l]\times\mathbb{R}\to\mathbb{R}$, and $\Xi_j:[0,l]\times\mathbb{R}\to\mathbb{R}$ $(j=1,2,3)$ are continuous.

(A_2) There exist closed subsets V, W, and Z and there exists a positive constant η such that for $\widetilde{\tau},\widetilde{\kappa},\widetilde{\lambda}\in V\cup W\cup Z$, we get

$$\begin{aligned}|\Xi_1(r,\widetilde{\tau}(r))-\Xi_1(r,\widetilde{\kappa}(r))| &\leq \eta\left|\widetilde{\tau}-\widetilde{\kappa}\right|,\\ \left|\Xi_2(r,\widetilde{\kappa}(r))-\Xi_2\left(r,\widetilde{\lambda}(r)\right)\right| &\leq \eta\left|\widetilde{\lambda}-\widetilde{\kappa}\right|,\\ \left|\Xi_3\left(r,\widetilde{\lambda}(r)\right)-\Xi_3(r,\widetilde{\tau}(r))\right| &\leq \eta\left|\widetilde{\tau}-\widetilde{\lambda}\right|.\end{aligned}$$

(A_3)

$$\eta\max_{s\in[0,l]}\int_0^l\widetilde{\eth}(s,r)\leq\frac{1}{3a},\ a>0.$$

Our main theorem in this part is as follows:

Theorem 5.1. *Via the assertions (A_1)–(A_3), the considered problem (5.1) has a solution on χ.*

Proof. Describe the mapping $\Im:\chi^3\to\chi$ by

$$\Im\left(\widetilde{\tau},\widetilde{\kappa},\widetilde{\lambda}\right)(s)=\ell(s)+\int_0^l\widetilde{\eth}(s,r)\left[\Xi_1(r,\widetilde{\tau}(r))+\Xi_2(r,\widetilde{\kappa}(r))+\Xi_3\left(r,\widetilde{\lambda}(r)\right)\right]dr,$$

and $\theta(\widetilde{\tau})(s) = \widetilde{\tau}(s)$. Then, for each $\widetilde{\lambda}(r) \in V, \widetilde{\kappa}(r) \in W$, and $\widetilde{\tau}(r) \in Z$, the problem (5.1) implies that

$$\begin{aligned}\Im\left(\widetilde{\tau},\widetilde{\kappa},\widetilde{\lambda}\right)(s) &= \ell(s) + \int_0^l \widetilde{\eth}(s,r)\left[\Xi_1(r,\widetilde{\tau}(r)) + \Xi_2(r,\widetilde{\kappa}(r)) + \Xi_3\left(r,\widetilde{\lambda}(r)\right)\right]dr \\ &= \widetilde{\tau}(s) \\ &= \theta(\widetilde{\tau})(s) \in \theta(Z),\end{aligned}$$

$$\begin{aligned}\Im\left(\widetilde{\kappa},\widetilde{\lambda},\widetilde{\tau}\right)(s) &= \ell(s) + \int_0^l \widetilde{\eth}(s,r)\left[\Xi_1(r,\widetilde{\kappa}(r)) + \Xi_2\left(r,\widetilde{\lambda}(r)\right) + \Xi_3(r,\widetilde{\tau}(r))\right]dr \\ &= \widetilde{\kappa}(s) \\ &= \theta(\widetilde{\kappa})(s) \in \theta(W),\end{aligned}$$

and

$$\begin{aligned}\Im\left(\widetilde{\lambda},\widetilde{\tau},\widetilde{\kappa}\right)(s) &= \ell(s) + \int_0^l \widetilde{\eth}(s,r)\left[\Xi_1\left(r,\widetilde{\lambda}(r)\right) + \Xi_2(r,\widetilde{\tau}(r)) + \Xi_3(r,\widetilde{\kappa}(r))\right]dr \\ &= \widetilde{\lambda}(s) \\ &= \theta(\widetilde{\lambda})(s) \in \theta(V).\end{aligned}$$

From the above three inequalities, we have that the mapping $\Im$ is a $\theta-$tripling with respect to V, W, and Z. □

Now, for $\widetilde{\tau},\widetilde{\kappa},\widetilde{\lambda},\widetilde{v},\widetilde{w},\widetilde{z} \in V \cup W \cup Z$, we have

$$\begin{aligned}&\left|\Im\left(\widetilde{\tau},\widetilde{\kappa},\widetilde{\lambda}\right)(s) - \Im\left(\widetilde{v},\widetilde{w},\widetilde{z}\right)(s)\right| \\ =\ &\left|\int_0^l \widetilde{\eth}(s,r)\left(\left[\Xi_1(r,\widetilde{\tau}(r)) - \Xi_1(r,\widetilde{v}(r))\right] + \left[\Xi_2(r,\widetilde{\kappa}(r)) - \Xi_2(r,\widetilde{w}(r))\right]\right.\right. \\ &\left.\left. - \left[\Xi_3\left(r,\widetilde{\lambda}(r)\right) - \Xi_3\left(r,\widetilde{\lambda}(r)\right)\right]\right)dr\right| \\ \leq\ &\int_0^l \widetilde{\eth}(s,r)\left(\left|\Xi_1(r,\widetilde{\tau}(r)) - \Xi_1(r,\widetilde{v}(r))\right| + \left|\Xi_2(r,\widetilde{\kappa}(r)) - \Xi_2(r,\widetilde{w}(r))\right| +\right. \\ &\left. + \left|\Xi_3\left(r,\widetilde{\lambda}(r)\right) - \Xi_3(r,\widetilde{z}(r))\right|\right)dr \\ \leq\ &\int_0^l \eta\widetilde{\eth}(s,r)\left(\left|\widetilde{\tau}(r) - \widetilde{v}(r)\right| + \left|\widetilde{\kappa}(r) - \widetilde{w}(r)\right| + \left|\widetilde{\lambda}(r) - \widetilde{z}(r)\right|\right)dr \\ \leq\ &\frac{3}{3a}\max_{r\in[0,l]}\left\{\left|\widetilde{\tau}(r) - \widetilde{v}(r)\right|, \left|\widetilde{\kappa}(r) - \widetilde{w}(r)\right|, \left|\widetilde{\lambda}(r) - \widetilde{z}(r)\right|\right\}\ (\text{since } d+e+f \leq 3\max\{d,e,f\}) \\ =\ &\frac{1}{a}\max_{r\in[0,l]}\left\{\left|\theta(\widetilde{\tau})(r) - \theta(\widetilde{v})(r)\right|, \left|\theta(\widetilde{\kappa})(r) - \theta(\widetilde{w})(r)\right|, \left|\theta\left(\widetilde{\lambda}\right)(r) - \theta(\widetilde{z})(r)\right|\right\} \\ \leq\ &\frac{1}{a}\max\left\{\varpi(\theta(\widetilde{\tau}),\theta(\widetilde{v})), \varpi(\theta(\widetilde{\kappa}),\theta(\widetilde{w})), \varpi\left(\theta(\widetilde{\lambda}),\theta(\widetilde{z})\right)\right\},\end{aligned}$$

which implies that

$$\max_{s\in[0,l]}\left|\Im\left(\widetilde{\tau},\widetilde{\kappa},\widetilde{\lambda}\right)(s) - \Im\left(\widetilde{v},\widetilde{w},\widetilde{z}\right)(s)\right| \leq \frac{1}{a}\max\left\{\varpi(\theta(\widetilde{\tau}),\theta(\widetilde{v})), \varpi(\theta(\widetilde{\kappa}),\theta(\widetilde{w})), \varpi\left(\theta(\widetilde{\lambda}),\theta(\widetilde{z})\right)\right\},$$

that is,

$$\varpi\left(\Im\left(\widetilde{\tau},\widetilde{\kappa},\widetilde{\lambda}\right),\Im\left(\widetilde{v},\widetilde{w},\widetilde{z}\right)\right)\leq\frac{1}{a}\max\left\{\varpi\left(\theta(\widetilde{\tau}),\theta(\widetilde{v})\right),\varpi\left(\theta(\widetilde{\kappa}),\theta(\widetilde{w})\right),\varpi\left(\theta(\widetilde{\lambda}),\theta(\widetilde{z})\right)\right\}.$$

Taking the natural logarithm on both sides, we have

$$\begin{aligned}&\ln\left(\Im\left(\widetilde{\tau},\widetilde{\kappa},\widetilde{\lambda}\right),\Im\left(\widetilde{v},\widetilde{w},\widetilde{z}\right)\right)\\ \leq\ &\ln\left(\frac{1}{a}\max\left\{\varpi\left(\theta(\widetilde{\tau}),\theta(\widetilde{v})\right),\varpi\left(\theta(\widetilde{\kappa}),\theta(\widetilde{w})\right),\varpi\left(\theta(\widetilde{\lambda}),\theta(\widetilde{z})\right)\right\}\right)\\ =\ &\ln\left(\max\left\{\varpi\left(\theta(\widetilde{\tau}),\theta(\widetilde{v})\right),\varpi\left(\theta(\widetilde{\kappa}),\theta(\widetilde{w})\right),\varpi\left(\theta(\widetilde{\lambda}),\theta(\widetilde{z})\right)\right\}\right)-\ln(a).\end{aligned}$$

Hence, the condition (2.2) of Corollary 2.5 is fulfilled with $F(\varrho)=\ln(\varrho),\ \varrho>0$. Consequently, all assumptions of Corollary 2.5 are satisfied. Then, $\Im$ and θ have a TCP $(v,w,z)\in V\cup W\cup Z$, which is a solution to the problem (5.1).

Remark 5.2. If we consider $\chi=C([m,n],\mathbb{R})$ is a complete MS equipped with the same distance defined in the above part, Corollary 2.5 can be applied to solve the following problem:

$$\begin{aligned}\widehat{\tau}(s)\ =\ &\widehat{\ell}(s)+\int_m^n\left(\eth_1(s,r)+\eth_1(s,r)+\eth_2(s,r)\right)\\ &\times\left(\Omega_1\left(r,\widehat{\tau}(r)\right)+\Omega_2\left(r,\widehat{\kappa}(r)\right)+\Omega_3\left(r,\widehat{\lambda}(r)\right)\right)dr,\end{aligned}$$

for all $s\in[m,n]$, under the following conditions:

(C_1) The functions $\widehat{\ell}:[m,n]\to\mathbb{R}$, $\eth_j:[m,n]\times[m,n]\to\mathbb{R}$, and $\Omega_j:[m,n]\times\mathbb{R}\to\mathbb{R}$ $(j=1,2,3)$ are continuous.

(C_2) There exist closed subsets V, W, and Z and there exist constants $\eta_1,\eta_2,\eta_3>0$ such that for $\widehat{\tau},\widehat{\kappa},\widehat{\lambda}\in V\cup W\cup Z$, we have

$$\begin{aligned}\left|\Omega_1\left(r,\widetilde{\tau}(r)\right)-\Omega_1\left(r,\widetilde{\kappa}(r)\right)\right|&\leq\eta_1\left|\widetilde{\tau}-\widetilde{\kappa}\right|,\\ \left|\Omega_2\left(r,\widetilde{\kappa}(r)\right)-\Omega_2\left(r,\widetilde{\lambda}(r)\right)\right|&\leq\eta_2\left|\widetilde{\lambda}-\widetilde{\kappa}\right|,\\ \left|\Omega_3\left(r,\widetilde{\lambda}(r)\right)-\Omega_3\left(r,\widetilde{\tau}(r)\right)\right|&\leq\eta_3\left|\widetilde{\tau}-\widetilde{\lambda}\right|.\end{aligned}$$

(C_3) We suppose that

$$\max\{\eta_1,\eta_2,\eta_3\}\left(\max_{s\in[m,n]}\int_m^n\left(\eth_1(s,r)+\eth_1(s,r)+\eth_2(s,r)\right)\right)\leq\frac{1}{3a}.$$

6. Conclusions

Results confirming the existence of a TCP have been obtained, along with the definition of the F-contractive-type θ-coupling. For the previously mentioned mapping, we have established both the existence and uniqueness of a strong TCP. These findings make a significant contribution to the field of fixed point theory, broadening the scope of existing results and providing a new framework for analyzing various mathematical problems. Furthermore, we have demonstrated the practical applicability of our theoretical results by showing their relevance to the existence of solutions for certain types of nonlinear integral equations and other integral-type problems.

During our research, we encountered the following challenges:

- Tripled best proximity point: We were unable to achieve a tripled best proximity point within the context of TFP and TCP. This limitation highlights an opportunity for future exploration, especially in the areas of cyclic mappings, cyclic F-contractive-type mappings, and their applications.
- Wardowski's function: While the authors in Remark 1.6 proposed an alternative form of Wardowski's function, the associated conditions, characteristics, and the potential for unique fixed points and TFP remain unexplored.

7. Abbreviations

MS$\Longrightarrow$metric space.
TFP$\Longrightarrow$tripled fixed point.
TCP$\Longrightarrow$tripled coincidence point.
FCT-θT$\Longrightarrow$F-contractive-type θ-tripling.
MFCT-θT$\Longrightarrow$multivalued F-contractive-type θ-tripling.

Author contributions

H. A. Hammad: Writing-original draft, Conceptualization, Investigation, Methodology; D. A. Kattan: Writing-review-editing, Formal analysis, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflicts of interest

The authors declare that they have no conflicts of interest.

References

1. E. I. Fredholm, Sur une classe d'équations fonctionnelles, *Acta Math.*, **27** (1903), 365–390. https://doi.org/10.1007/BF02421317
2. M. D. Rus, A note on the existence of positive solution of Fredholm integral equations, *Fix. Point Theory*, **5** (2004), 369–377.
3. M. I. Berenguer, M. V. F. Munoz, A. I. G. Guillem, M. R. Galan, Numerical treatment of fixed point applied to the nonlinear Fredholm integral equation, *Fixed Point Theory Appl.*, **2009** (2009), 735638. https://doi.org/10.1155/2009/735638
4. K. Zhao, J. Liu, X. Lv, A unified approach to solvability and stability of multipoint BVPs for Langevin and Sturm-Liouville equations with CH-fractional derivatives and impulses via coincidence theory, *Fractal Fract.*, **8** (2024), 111. https://doi.org/10.3390/fractalfract8020111

5. K. Zhao, Study on the stability and its simulation algorithm of a nonlinear impulsive ABC-fractional coupled system with a Laplacian operator via F-contractive mapping, *Adv. Cont. Discr. Mod.*, **2024** (2024), 5. https://doi.org/10.1186/s13662-024-03801-y

6. I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

7. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

8. H. A. Hammad, M. De la Sen, Involvement of the fixed point technique for solving a fractional differential system, *AIMS Math.*, **7** (2022), 7093–7105. https://doi.org/10.3934/math.2022395

9. Y. Kao, Y. Cao, Y. Chen, Projective synchronization for uncertain fractional reaction-diffusion systems via adaptive sliding mode control based on finite-time scheme, *IEEE Trans. Neural Netw. Learn. Syst.*, **35** (2023), 1–9. https://doi.org/10.1109/TNNLS.2023.3288849

10. Y. Kao, Y. Li, J. H. Park, X. Chen, Mittag-Leffler synchronization of delayed fractional memristor neural networks via adaptive control, *IEEE Trans. Neural Netw. Learn. Syst.*, **32** (2021), 2279–2284. https://doi.org/10.1109/TNNLS.2020.2995718

11. H. A. Hammad, M. De la Sen, Stability and controllability study for mixed integral fractional delay dynamic systems endowed with impulsive effects on time scales, *Fractal Frac.*, **7** (2023), 92. https://doi.org/10.3390/fractalfract7010092

12. H. A. Hammad, R. A. Rashwan, A. Nafea, M. E. Samei, S. Noeiaghdam, Stability analysis for a tripled system of fractional pantograph differential equations with nonlocal conditions, *J. Vibration Control*, **30** (2024), 632–647. https://doi.org/10.1177/10775463221149232

13. T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal. TMA*, **65** (2006), 1379–1393. https://doi.org/10.1016/j.na.2005.10.017

14. M. Abbas, M. A. Khan, S. Radenović, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, *Appl. Math. Comput.*, **217** (2010), 195–202. https://doi.org/10.1016/j.amc.2010.05.042

15. H. Aydi, M. Postolache, W. Shatanawi, Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces, *Comput. Math. Appl.*, **63** (2012), 298–309. https://doi.org/10.1016/j.camwa.2011.11.022

16. V. Berinde, Coupled fixed point theorems for contractive mixed monotone mappings in partially ordered metric spaces, *Nonlinear Anal.*, **75** (2012), 3218–3228. https://doi.org/10.1016/j.na.2011.12.021

17. B. S. Choudhury, P. Maity, Coupled fixed point results in generalized metric spaces, *Math. Comput. Model.*, **54** (2011), 73–79. https://doi.org/10.1016/j.mcm.2011.01.036

18. H. A. Hammad, F. M. Bota, L. Guran, Wardowski's contraction and fixed point technique for solving systems of functional and integral equations, *J. Func. Spaces*, **2021** (2021), 7017046. https://doi.org/10.1155/2021/7017046

19. N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, *Nonlinear Anal.*, **74** (2011), 983–992. https://doi.org/10.1016/j.na.2010.09.055

20. B. Samet, C. Vetro, Coupled fixed point, F-invariant set and fixed point of N-order, *Ann. Funct. Anal.*, **1** (2010), 46–56. https://doi.org/10.15352/afa/1399900586
21. B. Zlatanov, Coupled best proximity points for cyclic contractive maps and their applications, *Fix. Point Theory*, **22** (2021), 431–452. https://doi.org/10.24193/fpt-ro.2021.1.29
22. V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, *Nonlinear Anal.*, **74** (2011), 4889–4897. https://doi.org/10.1016/j.na.2011.03.032
23. M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, *Appl. Math. Comput.* **218** (2012), 7339–7346. https://doi.org/10.1016/j.amc.2011.11.049
24. B. S. Choudhury, E. Karapınar, A. Kundu, Tripled coincidence point theorems for nonlinear contractions in partially ordered metric spaces, *Int. J. Math. Math. Sci.*, **2012** (2012), 329298. https://doi.org/ 10.1155/2012/329298
25. Z. Mustafa, J. R. Roshan, V. Parvaneh, Existence of a tripled coincidence point in ordered *Gb*-metric spaces and applications to a system of integral equations, *J. Inequal. Appl.*, **2013** (2013), 453. https://doi.org/10.1186/1029-242X-2013-453
26. H. Aydi, E. Karapinar, M. Postolache, Tripled coincidence point theorems for weak ϕ-contractions in partially ordered metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 44. https://doi.org/10.1186/1687-1812-2012-44
27. A. llchev, V. Ivanova, H. Kulina, P. Yaneva, B. Zlatanov, Investigation of equilibrium in oligopoly markets with the help of tripled fixed points in Banach spaces, *Econometrics*, **12** (2024), 18. https://doi.org/10.3390/econometrics12020018
28. G. Godefroy, V. Indumathi, Strong proximinality and polyhedral spaces, *Rev. Mat. Complut.*, **14** (2001), 105–125.
29. R. A. Rashwan, H. A. Hammad, A. Nafea, A new contribution in fuzzy cone metric spaces by strong fixed point techniques with supportive application, *J. Intell. Fuzzy Sys.*, **42** (2022), 3923–3943. https://doi.org/10.3233/JIFS-212188
30. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 94. https://doi.org/10.1186/1687-1812-2012-94
31. M. U. Ali, T. Kamran, Multivalued F-contractions and related fixed point theorems with an application, *Filomat*, **30** (2016), 3779–3793. https://doi.org/10.2298/FIL1614779A
32. T. Hamaizia, Fixed point theorems for generalized (ψ, ϕ, F)-contraction type mappings in $b-$metric spaces with applications, *Open J. Math. Anal.*, **5** (2021), 35–41. https://doi.org/10.30538/psrp-oma2021.0080
33. A. Asif, M. Nazam, M. Arshad, S. O. Kim, F-metric, F-contraction and common fixed-point theorems with applications, *Mathematics*, **7** (2019), 586. https://doi.org/10.3390/math7070586
34. S. Luambano, S. Kumar, G. Kakiko, Fixed point theorem for F-contraction mappings, in partial metric spaces, *Lobachevskii J. Math.*, **40** (2019), 183–188. https://doi.org/10.1134/S1995080219020094
35. M. Nazam, C. Park, M. Arshad, Fixed point problems for generalized contractions with applications, *Adv. Differ. Equ.*, **2021** (2021), 247. https://doi.org/10.1186/s13662-021-03405-w

36. C. Mongkolkeha, D. Gopal, Some common fixed point theorems for generalized F-contraction involving w-distance with some applications to differential equations, *Mathematics*, **7** (2019), 32. https://doi.org/10.3390/math7010032

37. K. Sawangsup, W. Sintunavarat, Y. J. Cho, Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces, *J. Fixed Point Theory Appl.*, **22** (2020), 10. https://doi.org/10.1007/s11784-019-0737-4

38. H. A. Hammad, M. De la Sen, A coupled fixed point technique for solving coupled systems of functional and nonlinear integral equations, *Mathematics* **7** (2019), 634. https://doi.org/10.3390/math7070634

39. H. K. Nashine, B. Samet, Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces, *Nonlinear Anal.*, **74** (2011), 2201–2209. https://doi.org/10.1016/j.na.2010.11.024