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Research article

Discussion on exact null boundary controllability of nonlinear fractional stochastic evolution equations in Hilbert spaces

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Abstract: Null boundary controllability refers to the ability to drive the state of a dynamical system to zero by applying suitable control inputs on the boundary of the domain. This research investigates the sufficient conditions for the null boundary controllability of Atangana-Baleanu (A-B) fractional stochastic differential equations involving fractional Brownian motion (fBm) within Hilbert space. We employ various tools, including fractional analysis, compact semigroup theory, fixed point theorems, and stochastic analysis, to derive the desired results. An example is included to illustrate the application of our findings.

Keywords: boundary control; fractional analysis; stochastic analysis
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1. Introduction

Research on fractional stochastic differential equations (SDEs) has received considerable interest lately due to their effectiveness in modeling complex systems affected by memory and uncertainty [1]. In contrast to conventional stochastic models, fractional SDEs use fractional derivatives, enabling them to account for anomalous diffusion and long-range dependence characteristics often seen in fields like finance, biology, and physics.

Numerous studies have explored fractional SDEs. For instance, Saravanakumar and Balasubramaniam [2] studied the non-instantaneous impulsive Hilfer fractional stochastic differential equations driven by fractional Brownian motion. Guo et al. [3] investigated the existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm. Ahmed [4] studied the Sobolev-type fractional stochastic integrodifferential equations with nonlocal conditions in Hilbert space. Makhlouf and Mchiri [5] studied the Caputo-Hadamard fractional stochastic differential equations. The averaging principle for fractional stochastic differential equations was investigated in [6–8]. Sufficient conditions for existence and uniqueness of fractional stochastic delay differential equations were discussed in [9–11].

A key component of control theory is boundary controllability, which investigates whether a system can be steered to a desired state by applying controls at the edges of its domain, for example, Li et al. [12] studied the exact boundary controllability and exact boundary synchronization for a coupled system of wave equations with coupled Robin boundary controls. Ahmed [13, 14] investigated the boundary controllability of nonlinear fractional integrodifferential systems. Baranovskii [15] explored the optimal boundary control of the Boussinesq approximation for polymeric fluids. Katz and Fridman [16] studied the boundary control of one dimension parabolic partial differential equations under point measurement. Tajani and El Alaoui [17] discussed the boundary controllability of Riemann-Liouville fractional semilinear evolution Systems. In the case of fractional SDEs, this concept is especially complex because of the interaction between fractional dynamics and stochastic effects, primarily represented by fBm. The distinctive characteristics of fBm, including its self-similarity and long-range dependence, present both challenges and opportunities for controlling these systems [18, 19].

Null controllability is the capability to drive a dynamical system from any initial state to the zero state (or equilibrium) in a finite time using suitable control inputs [20, 21]. Few authors studied the null controllability for stochastic differential systems, for example, Sathiyaraj et al. [22] investigated the null controllability results for stochastic delay systems with delayed perturbation of matrices. Wang and Ahmed [23] studied the null controllability of nonlocal Hilfer fractional stochastic differential equations. Exact null controllability of Hilfer fractional stochastic differential equations with fractional Brownian motion and Poisson jumps was discussed in [24, 25].

The A-B fractional derivative plays a crucial role in modeling physical processes characterized by non-locality and memory effects, which are prevalent in complex systems such as viscoelastic materials, anomalous diffusion, and fluid mechanics. Unlike classical derivatives, which are local operators, the A-B fractional derivative incorporates the entire history of a system using a non-singular kernel. This approach provides a more accurate representation of processes where past states significantly influence the current behavior. In the Caputo sense, the A-B fractional derivative has been effectively applied to model heat flow in heterogeneous thermal media. For more comprehensive details about the A-B fractional derivative and its applications, we direct readers to references [26–28].

Several authors have explored fractional differential equations (DEs) involving A-B fractional derivatives. For instance, Dhayal et al. [29] investigated the approximate controllability of A-B fractional stochastic differential systems with non-Gaussian processes and impulses. Kaliraj et al. [30] examined the controllability of impulsive integro-differential equations using the A-B fractional derivative. Ahmed et al. [31] studied the approximate controllability of Sobolev-type A-B fractional differential inclusions under the influence of noise and Poisson jumps. Bahaa [32] proposed an optimal control problem for variable-order fractional differential systems with time delay, involving A-B derivatives. Dineshkumar et al. [33] established the existence and approximate controllability results for Atangana-Baleanu neutral fractional stochastic hemivariational inequalities. Bedi et al. [34] studied the controllability of neutral impulsive fractional differential equations with A-B Caputo derivatives. Aimene et al. [35] investigated the controllability of semilinear impulsive A-B fractional differential equations with delay. Logeswari and Ravichandran [36] discussed the existence of fractional neutral integro-differential equations in the concept of A-B derivative. However, there have been no documented studies in existing literature concerning the null boundary controllability of A-B fractional SDEs incorporating fBm. Inspired by this gap in research, this work aims to explore the null boundary controllability of such A-B fractional SDEs with fBm in Hilbert space, structured as follows:

$$\begin{cases} {}^{ABC}D^{\mathfrak{h}}_{0+}\varkappa(\mathrm{t}) = \alpha\varkappa(\mathrm{t}) + N(\mathrm{t},\varkappa(\mathrm{t})) + W(\mathrm{t},\varkappa(\mathrm{t}))\frac{d\mathfrak{B}^H(\mathrm{t})}{dt}, \ \mathrm{t} \in \bar{J} = [0,T], \\ \gamma\varkappa(\mathrm{t}) = \bar{B}_1\psi(\mathrm{t}), \ \mathrm{t} \in \bar{J}, \\ \varkappa(0) = \varkappa_0. \end{cases} \tag{1.1}$$

The expression ${}^{ABC}D^{\mathfrak{h}}_{0+}$ represents the A-B Caputo fractional derivative of order $\mathfrak{h} \in (\frac{1}{2}, 1)$. The function $\varkappa(\cdot)$ operates in a Hilbert space denoted as $\mathcal{K}$, equipped with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The term $\mathfrak{B}^H$ signifies a fBm on another separable and real Hilbert space $\bar{Y}$, characterized by a Hurst parameter $\frac{1}{2} < H < 1$.

The control function $\psi(\cdot)$ is specified within $\mathfrak{L}_2(\bar{J},\mathfrak{U})$, where $\mathfrak{U}$ represents another separable Hilbert space. Let $\gamma : \mathfrak{D}(\gamma) \subset C(\bar{J},\mathfrak{L}_2(\Omega,\mathcal{K})) \to R(\gamma) \subset \mathcal{K}$ be a linear operator and let $\alpha : \mathfrak{D}(\alpha) \subset C(\bar{J},\mathfrak{L}_2(\Omega,\mathcal{K})) \to R(\alpha) \subset \mathcal{K}$ be a closed, densely defined linear operator. Let $\Pi : \mathcal{K} \to \mathcal{K}$ be the linear operator defined by $\mathfrak{D}(\Pi) = \{\varkappa \in \mathfrak{D}(\alpha);\ \gamma\varkappa = 0\}$, $\Pi\varkappa = \alpha\varkappa$, for $\varkappa \in \mathfrak{D}(\Pi)$, and $\bar{B}_1 : \mathfrak{U} \to \mathcal{K}$ is a linear continuous operator.

Additionally, there are nonlinear functions represented by

$$N : \bar{J}\times\mathcal{K} \to \mathcal{K} \text{ and } W : \bar{J}\times\mathcal{K} \to \mathfrak{L}^0_2(\bar{Y},\mathcal{K}).$$

2. Preliminaries

Definition 2.1. *[37] A-B Caputo fractional derivative of order* $0 < \mathfrak{h} < 1$ *is characterized by the following definition:*

$${}^{ABC}D^{\mathfrak{h}}_{0+}\mathfrak{g}(\mathrm{t}) = \frac{\varpi(\mathfrak{h})}{1-\mathfrak{h}}\int_0^{\mathrm{t}} \mathfrak{g}'(\bar{s})\mathbb{M}_{\mathfrak{h}}(-\theta(\mathrm{t}-\bar{s})^{\mathfrak{h}})d\bar{s}, \tag{2.1}$$

where the function $\theta = \frac{\mathfrak{h}}{1-\mathfrak{h}}$,

$$\mathbb{M}_{\mathfrak{h}}(\bar{G}) = \sum_{n=0}^{\infty}\frac{\bar{G}^n}{\Gamma(n\mathfrak{h}+1)}$$

denotes the Mittag-Leffler function. Additionally, the normalization function, denoted by $\varpi(\mathfrak{h})$*, is expressed as* $(1-\mathfrak{h}) + \frac{\mathfrak{h}}{\Gamma(\mathfrak{h})}$. *It is defined in such a way that* $\varpi(0) = \varpi(1) = 1$.

The expression for the fractional integral of A-B is given as

$${}^{AB}I^{\mathfrak{h}}_{0+}\mathfrak{g}(\mathrm{t}) = \frac{(1-\mathfrak{h})}{\varpi(\mathfrak{h})}\mathfrak{g}(\mathrm{t}) + \frac{\mathfrak{h}}{\varpi(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\mathfrak{g}(\bar{s})d\bar{s}. \tag{2.2}$$

$\mathrm{t} > 0$ is a fixed constant. $(\Omega,\xi,\bar{P})$ is a complete probability space equipped with a comprehensive collection of right-continuous increasing sub σ-algebras $\{\xi_{\mathrm{t}} : \mathrm{t} \in [0,T]\}$ all nested within ξ.

Here, $\mathfrak{L}(\bar{Y},\mathcal{K})$ represents the space of linear bounded operators from $\bar{Y}$ into $\mathcal{K}$. We consider an operator $\mathfrak{Q} \in \mathfrak{L}(\bar{Y},\bar{Y})$, defined by the relation $\mathfrak{Q}\tau_n = \mathfrak{b}_n\tau_n$, where the trace of $\mathfrak{Q}$, denoted by $tr\mathfrak{Q}$, is finite. Here, $\mathfrak{b}_n \geq 0$ and $\{\tau_n\}$ $(n = 1, 2, ...)$ forms a complete orthonormal basis in $\bar{Y}$. $\|\cdot\|$ constitutes the norm in $\mathfrak{L}(\bar{Y}, \mathcal{K})$, $\bar{Y}$ and $\mathcal{K}$.

We establish the fBm in $\bar{Y}$ as follows:

$$\mathfrak{B}^H(\mathrm{t}) = \mathfrak{B}^H_{\mathfrak{Q}}(\mathrm{t}) = \sum_{n=1}^{\infty} \sqrt{\mathfrak{b}_n}\tau_n\beta^H_n(\mathrm{t}).$$

The variables β^H_n represent real, independent fBms.

We introduce the space $\mathfrak{L}^0_2$, denoted as $\mathfrak{L}^0_2(\bar{Y}, \mathcal{K})$, encompassing all $\mathfrak{Q}$-Hilbert Schmidt operators $\eta : \bar{Y} \to \mathcal{K}$ if the expression $\|\eta\|^2_{\mathfrak{L}^0_2} := \sum_{n=1}^{\infty} \|\sqrt{\mathfrak{b}_n}\eta\tau_n\|^2$ is finite. Additionally, the space $\mathfrak{L}^0_2$, endowed with $\langle\vartheta, \eta\rangle_{\mathfrak{L}^0_2} = \sum_{n=1}^{\infty}\langle\vartheta\tau_n, \eta\tau_n\rangle$, forms a separable Hilbert space.

Lemma 2.2. *[38] If function $\eta : [0, T] \to \mathfrak{L}^0_2(\bar{Y}, \mathcal{K})$ meets the condition $\int_0^T \|\eta(\bar{s})\|^2_{\mathfrak{L}^0_2} < \infty$, then we can conclude that*

$$E\left\|\int_0^{\mathrm{t}} \eta(\bar{s})d\mathfrak{B}^H(\bar{s})\right\|^2 \leq 2H\mathrm{t}^{2H-1}\int_0^{\mathrm{t}} \|\eta(\bar{s})\|^2_{\mathfrak{L}^0_2}d\bar{s}.$$

Consider $C(\bar{J}, \mathfrak{L}_2(\Omega, \mathcal{K}))$, the Banach space comprising all continuous mappings from $\bar{J}$ to $\mathfrak{L}_2(\Omega, \mathcal{K})$, where each function satisfies the condition $\sup_{\mathrm{t}\in\bar{J}} E\|\varkappa(\mathrm{t})\|^2 < \infty$.

Let $\bar{C}$ denote the set $\{\varkappa : \varkappa(\cdot) \in C(\bar{J}, \mathfrak{L}_2(\Omega, \mathcal{K}))\}$, with its norm $\|\cdot\|_{\bar{C}}$ defined as

$$\|\cdot\|_{\bar{C}} = (\sup_{\mathrm{t}\in\bar{J}} E\|\varkappa(\mathrm{t})\|^2)^{\frac{1}{2}}.$$

Through this work, the operator $\Pi : \mathfrak{D}(\Pi) \subset \mathcal{K} \to \mathcal{K}$ acts as the infinitesimal generator of a family of $\mathfrak{h}$-resolvents denoted as $(\mathfrak{S}_{\mathfrak{h}}(\mathrm{t}))_{\mathrm{t}\geq 0}$ and $(\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}))_{\mathrm{t}\geq 0}$, defined on a separable Hilbert space $\mathcal{K}$.

Definition 2.3. *[39] The set of resolvent denoted $\rho(\Pi)$, consists of complex numbers ζ for which the operator $(\zeta - \Pi) : \mathfrak{D}(\Pi) \to \mathcal{K}$ is a bijective mapping. According to the closed graph theorem, the operator $\mathfrak{R}(\zeta, \Pi) = (\zeta - \Pi)^{-1}$ is bounded for $\zeta \in \rho(\Pi)$ on $\mathcal{K}$, serving as the resolvent of Π at ζ. Consequently, for all $\zeta \in \rho(\Pi)$, the equation $\Pi\mathfrak{R}(\zeta, \Pi) = \zeta\mathfrak{R}(\zeta, \Pi) - I$ holds true.*

Definition 2.4. *(See [39]) If Π is a linear and closed sectorial operator, then there exist $\hbar > 0$, $\mathfrak{I}$ real, and Λ within the interval $[\frac{\pi}{2}, \pi]$, such that (s.t.)*
(i) $\sum_{\Lambda,\mathfrak{I}} = \{\zeta \in \mathbb{C} : \zeta \neq \mathfrak{I}, |\arg(\zeta - \mathfrak{I})| < \Lambda\} \subset \rho(\Pi)$.
(ii) $\|\mathfrak{R}(\zeta, \Pi)\| \leq \frac{\hbar}{|\zeta-\mathfrak{I}|}$, $\zeta \in \sum_{\Lambda,\mathfrak{I}}$
are verified.

Let us impose the assumptions as follows:

$(H1)$ $\mathfrak{D}(\alpha) \subset \mathfrak{D}(\gamma)$ and the restriction of τ to $\mathfrak{D}(\alpha)$ is continuous concerning the graph norm of $\mathfrak{D}(\alpha)$.

$(H2)$ $\bar{B} : \mathfrak{U} \to \mathcal{K}$ is a linear operator s.t. $\forall\psi \in \mathfrak{U}$ we have $\bar{B}\psi \in \mathfrak{D}(\alpha)$, $\gamma(\bar{B}\psi) = \bar{B}_1\psi$ and $\|\bar{B}\psi\| \leq C\|\bar{B}_1\psi\|$, C is a constant.

$(H3)$ There exists a constant $M_1 > 0$ s.t. $\|\Pi\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t})\| \leq M_1$.

(H_4) $(\mathfrak{S}_{\mathfrak{h}})(\mathrm{t})_{\mathrm{t}\geq 0}$ and $(\mathfrak{Q}_{\mathfrak{h}})(\mathrm{t})_{\mathrm{t}\geq 0}$ are compact.

(H_5) The fractional linear system described by Eq (3.1) is exactly null controllable over $\bar{J}$.

(H_6) $N : \bar{J} \times \mathcal{K} \to \mathcal{K}$ meets the following:

(i) N is continuous. Suppose $N \in \bar{C}$ $\forall$ $\mathcal{K} \in \bar{C}$, which guarantees ${}^{ABC}D^{\mathfrak{h}}_{0+}\mathcal{K} \in \bar{C}$ exists.

(ii) $\forall \mathfrak{q} \in \mathfrak{N},\ \mathfrak{q} > 0$, there exists a positive function $N_{\mathfrak{q}}(\cdot) : \bar{J} \to \mathfrak{R}^+$ s.t.

$$\sup_{\|\varkappa\|^2 \le \mathfrak{q}} E\|N(\mathfrak{t}, \varkappa)\|^2 \le N_{\mathfrak{q}}(\mathfrak{t}),$$

$s \to (\mathfrak{t} - \bar{s})^{\mathfrak{h}-1} N_{\mathfrak{q}}(\bar{s}) \in \mathfrak{L}^1([0, \mathfrak{t}], \mathfrak{R}^+)$, and

$$\liminf_{\mathfrak{q}\to\infty} \frac{\int_0^{\mathfrak{t}} (\mathfrak{t} - \bar{s})^{\mathfrak{h}-1} N_{\mathfrak{q}}(\bar{s}) d\bar{s}}{q} = \delta < \infty,\ \mathfrak{t} \in \bar{J},\ \delta > 0.$$

(H_7) $W : \bar{J} \times \mathcal{K} \to \mathfrak{L}_2^0(\mathfrak{K}, \mathcal{K})$ fulfills the following:
(i) $W : J \times \mathcal{K} \to \mathfrak{L}_2^0(\mathfrak{K}, \mathcal{K})$ is a continuous function.
(ii) $\forall \mathfrak{q} > 0;\ \mathfrak{q} \in \mathfrak{N}$, there exists a positive function $\mathfrak{g}_{\mathfrak{q}}(\cdot) : \bar{J} \to \mathfrak{R}^+$ s.t.

$$\sup_{\|\varkappa\|^2 \le q} E\|W(\mathfrak{t}, x)\|^2_{\mathfrak{L}_2^0} \le \mathfrak{g}_{\mathfrak{q}}(\mathfrak{t}),$$

$s \to (\mathfrak{t} - \bar{s})^{\mathfrak{h}-1} \mathfrak{g}_{\mathfrak{q}}(\bar{s}) \in \mathfrak{L}^1([0, \mathfrak{t}], \mathfrak{R}^+)$, and $\exists\, \delta > 0$ s. t.

$$\liminf_{\mathfrak{q}\to\infty} \frac{\int_0^{\mathfrak{t}} (\mathfrak{t} - \bar{s})^{\mathfrak{h}-1} \mathfrak{g}_{\mathfrak{q}}(\bar{s}) d\bar{s}}{\mathfrak{q}} = \delta < \infty,\ \mathfrak{t} \in \bar{J},\ \delta > 0.$$

Let $\varkappa(\mathfrak{t})$ be the solution of (1.1). Then, let $\bar{X}(\mathfrak{t}) = \varkappa(\mathfrak{t}) - \bar{B}\psi(\mathfrak{t})$, $\bar{X}(\mathfrak{t}) \in \mathfrak{D}(\Pi)$. Thus, Eq (1.1) can be represented using Π and $\bar{B}$ as

$$\begin{cases} {}^{ABC}D_{0+}^{\mathfrak{h}} \bar{X}(\mathfrak{t}) = \Pi \bar{X}(\mathfrak{t}) + \alpha \bar{B}\psi(\mathfrak{t}) - \bar{B}\ {}^{ABC}D_{0+}^{\mathfrak{h}} \psi(\mathfrak{t}) + N(\mathfrak{t}, \varkappa(\mathfrak{t})) + W(\mathfrak{t}, \varkappa(\mathfrak{t})) \frac{d\mathfrak{B}^H(\mathfrak{t})}{dt},\ \mathfrak{t} \in \bar{J}, \\ \bar{X}(0) = \varkappa(0) - \bar{B}\psi(0). \end{cases} \tag{2.3}$$

Applying ${}^{AB}I_{0+}^{\mathfrak{h}}$ to both sides of (2.3), then, we obtain

$$\begin{aligned} \varkappa(\mathfrak{t}) - \bar{B}\psi(\mathfrak{t}) &= \varkappa_0 - \bar{B}\psi(0) + {}^{AB}I_{0+}^{\mathfrak{h}} \Pi \varkappa(\mathfrak{t}) - {}^{AB}I_{0+}^{\mathfrak{h}} \Pi \bar{B}\psi(\mathfrak{t}) + {}^{AB}I_{0+}^{\mathfrak{h}} \alpha \bar{B}\psi(\mathfrak{t}) \\ &\quad - \bar{B}\psi(\mathfrak{t}) + \bar{B}\psi(0) + {}^{AB}I_{0+}^{\mathfrak{h}} N(\mathfrak{t}, \varkappa(\mathfrak{t})) + {}^{AB}I_{0+}^{\mathfrak{h}} W(\mathfrak{t}, \varkappa(\mathfrak{t})) \frac{d\mathfrak{B}^H(\mathfrak{t})}{d\mathfrak{t}}. \end{aligned}$$

Hence,

$$\begin{aligned} \varkappa(\mathfrak{t}) &= \varkappa_0 + \frac{1-\mathfrak{h}}{\varpi(\mathfrak{h})} \Pi \varkappa(\mathfrak{t}) + \frac{\mathfrak{h}}{\varpi(\mathfrak{h})\Gamma(\mathfrak{h})} \int_0^{\mathfrak{t}} (\mathfrak{t} - \bar{s})^{\mathfrak{h}-1} \Pi \varkappa(\bar{s}) d\bar{s} \\ &\quad + \frac{1-\mathfrak{h}}{\varpi(\mathfrak{h})} (\alpha - \Pi) \bar{B}\psi(\mathfrak{t}) + \frac{\mathfrak{h}}{\varpi(\mathfrak{h})\Gamma(\mathfrak{h})} \int_0^{\mathfrak{t}} (\mathfrak{t} - \bar{s})^{\mathfrak{h}-1} (\alpha - \Pi) \bar{B}\psi(\bar{s}) d\bar{s} \\ &\quad + \frac{1-\mathfrak{h}}{\varpi(\mathfrak{h})} N(\mathfrak{t}, \varkappa(\mathfrak{t})) + \frac{\mathfrak{h}}{\varpi(\mathfrak{h})\Gamma(\mathfrak{h})} \int_0^{\mathfrak{t}} (\mathfrak{t} - \bar{s})^{\mathfrak{h}-1} N(\bar{s}, \varkappa(\bar{s})) d\bar{s} \\ &\quad + \frac{1-\mathfrak{h}}{\varpi(\mathfrak{h})} W(\mathfrak{t}, \varkappa(\mathfrak{t})) \frac{d\mathfrak{B}^H(\mathfrak{t})}{d\mathfrak{t}} + \frac{\mathfrak{h}}{\varpi(\mathfrak{h})\Gamma(\mathfrak{h})} \int_0^{\mathfrak{t}} (\mathfrak{t} - \bar{s})^{\mathfrak{h}-1} W(\bar{s}, \varkappa(\bar{s}) d\mathfrak{B}^H(\bar{s}). \end{aligned} \tag{2.4}$$

Definition 2.5. *We define $\varkappa \in \bar{C}$ as a mild solution to (2.4) if it meets the condition:*

$$\begin{aligned}\varkappa(\mathrm{t}) &= F\mathfrak{S}_{\mathfrak{h}}(\mathrm{t})\varkappa_0 + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\Pi\varkappa(\bar{s})d\bar{s}\\ &+\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\alpha-\Pi)(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\bar{B}\psi(\bar{s})d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\\ &+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})N(\bar{s},\varkappa(\bar{s}))d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}\Pi\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})\varkappa(\bar{s})d\bar{s}\\ &+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}(\alpha-\Pi)\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})\bar{B}\psi(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s}),\end{aligned}$$

where $F = \vartheta^*(\vartheta^* I - \Pi)^{-1}$ and $\wp = -\delta^*\Pi(\vartheta^* I - \Pi)^{-1}$, with $\vartheta^* = \frac{V(\mathfrak{h})}{1-\mathfrak{h}}$, $\delta^* = \frac{\mathfrak{h}}{1-\mathfrak{h}}$,

$$\begin{aligned}\mathfrak{S}_{\mathfrak{h}}(\mathrm{t}) &= \mathbb{M}_{\mathfrak{h}}(-\wp\mathrm{t}^{\mathfrak{h}}) = \frac{1}{2\pi i}\int_{\Upsilon} e^{\bar{s}t}\bar{s}^{\mathfrak{h}-1}(\bar{s}^{\mathfrak{h}}I-\wp)^{-1}d\bar{s},\\ \mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}) &= \mathrm{t}^{\mathfrak{h}-1}\mathbb{M}_{\mathfrak{h},\mathfrak{h}}(-\wp\mathrm{t}^{\mathfrak{h}}) = \frac{1}{2\pi i}\int_{\Upsilon} e^{\bar{s}t}(\bar{s}^{\mathfrak{h}}I-\wp)^{-1}d\bar{s},\end{aligned}$$

and the path Υ is lying on $\sum_{\Lambda,\mathfrak{I}}$.

3. Null controllability investigation

Here, we examine the null controllability for (1.1).

If $\Pi \in \Pi^{\varepsilon}(\varrho_0, \varsigma_0)$, then for $C_1 > 0$ and $C_2 > 0$, the following holds:

$$\|\mathfrak{S}_{\mathfrak{h}}(\mathrm{t})\| \le C_1 e^{\mathfrak{I}\mathrm{t}} \text{ and } \|\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t})\| \le C_2 e^{\mathfrak{I}\mathrm{t}}(1+\mathrm{t}^{\mathfrak{h}-1}), \text{ for every } \mathrm{t}>0,\ \mathfrak{I}>\mathfrak{I}_0.$$

Let $C_3 = \sup_{\mathrm{t}\ge 0}\|\mathfrak{S}_{\mathfrak{h}}(\mathrm{t})\|$, $C_4 = \sup_{\mathrm{t}\ge 0} C_2 e^{\mathfrak{I}\mathrm{t}}(1+\mathrm{t}^{\mathfrak{h}-1})$. So we get $\|\mathfrak{S}_{\mathfrak{h}}(\mathrm{t})\| \le C_3$, $\|\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t})\| \le C_4\,\mathrm{t}^{\mathfrak{h}-1}$ [33].

To examine the null boundary controllability of Eq (1.1), we analyze the fractional stochastic linear system

$$\begin{cases} {}^{ABC}D^{\mathfrak{h}}_{0+}\lambda(\mathrm{t}) = \alpha\lambda(\mathrm{t}) + N(\mathrm{t}) + W(\mathrm{t})\frac{d\mathfrak{B}^H(\mathrm{t})}{dt},\ \mathrm{t}\in\bar{J}=[0,T],\\ \gamma\lambda(\mathrm{t}) = \bar{B}_1\psi(\mathrm{t}),\ \mathrm{t}\in\bar{J},\\ \lambda(0)=\lambda_0, \end{cases} \tag{3.1}$$

associated with the system (1.1).

Consider

$$\begin{aligned}\mathfrak{L}_0^T\psi =& \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^T (T-\bar{s})^{\mathfrak{h}-1}(\alpha-\Pi)\mathfrak{B}\psi(\bar{s})d\bar{s}\\ &+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^T \mathfrak{Q}_{\mathfrak{h}}(T-\bar{s})(\alpha-\Pi)\mathfrak{B}\psi(\bar{s})d\bar{s}\ :\mathfrak{L}_2(\bar{J},\mathfrak{U})\to\mathcal{K},\end{aligned}$$

where $\mathfrak{L}_0^T\psi$ possesses a bounded inverse operator denoted as $(\mathfrak{L}_0)^{-1}$, operating within the space $\mathfrak{L}_2(\bar{J},\mathfrak{U})/\ker(\mathfrak{L}_0^T)$, and

$$\mathfrak{N}_0^T(\lambda, N, W) = F\mathfrak{S}_{\mathfrak{h}}(T)\lambda + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^T (T-\bar{s})^{\mathfrak{h}-1}N(\bar{s})d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^T (T-\bar{s})^{\mathfrak{h}-1}W(\bar{s})d\mathfrak{B}^{\mu}(\bar{s}) + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^T \mathfrak{Q}_{\mathfrak{h}}(T-\bar{s})N(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^T \mathfrak{Q}_{\mathfrak{h}}(T-\bar{s})W(\bar{s})d\mathfrak{B}^H(\bar{s}) : \mathcal{K}\times\mathfrak{L}_2(\bar{J},\mathcal{K}) \to \mathcal{K}.$$

Definition 3.1. *[40] The system described by Eq (3.1) is termed exact null controllable over $\bar{J}$ if $Im\mathfrak{L}_0^T \supset Im\mathfrak{N}_0^T$ or there exists $\kappa > 0$ s.t. $\|(\mathfrak{L}_0^T)^*\lambda\|^2 \geq \kappa\|(\mathfrak{N}_0^T)^*\lambda\|^2$ for $\forall\lambda \in \mathcal{K}$.*

Lemma 3.2. *[41] Assume that (3.1) exhibits exactly null boundary controllability over the interval $\bar{J}$. Consequently, the operator $(\mathfrak{L}_0)^{-1}\mathfrak{N}_0^T \times \mathfrak{L}_2(\bar{J},\mathcal{K}) \to \mathfrak{L}_2(\bar{J},\psi)$ is bounded, and the control*

$$\psi(\mathrm{t}) = -(\mathfrak{L}_0)^{-1}\Big[F\mathfrak{S}_{\mathfrak{h}}(T)\lambda_0 + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^T (T-\bar{s})^{\mathfrak{h}-1}N(\bar{s})d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^T (T-\bar{s})^{\mathfrak{h}-1}W(\bar{s})d\mathfrak{B}^H(\bar{s}) + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^T \mathfrak{Q}_{\mathfrak{h}}(T-\bar{s})N(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^T \mathfrak{Q}_{\mathfrak{h}}(T-\bar{s})W(\bar{s})d\mathfrak{B}^H(\bar{s})\Big](\mathrm{t})$$

drives the system described by Eq (3.1) from an initial state λ_0 to the zero state. Here, $\mathfrak{L}_0$ represents the restriction of $\mathfrak{L}_0^T$ to $[\ker \mathfrak{L}_0^T]^{\perp}$, while N belongs to $\mathfrak{L}_2(\bar{J},\mathcal{K})$ and W belongs to $\mathfrak{L}_2^0(\bar{J},\mathfrak{L}(\lambda,\mathcal{K}))$.

Definition 3.3. *The system defined by Eq (1.1) is deemed exactly null boundary controllable over $\bar{J}$ if there exists a stochastic control $\psi \in \mathfrak{L}_2(\bar{J},\mathfrak{U})$ s.t. the solution $\varkappa(\mathrm{t})$ of (1.1) meets the condition $\varkappa(T) = 0$.*

Theorem 3.4. *Let $(H_1)-(H_7)$ hold, then (1.1) is exactly null boundary controllable over $\bar{J}$ s.t.*

$$\Big\{\frac{32\delta T^{\mathfrak{h}} + 16\delta H T^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\Big[\frac{\|\wp\|^2\|F\|^2(1-\mathfrak{h})^2}{V^2(\mathfrak{h})\Gamma^2(\mathfrak{h})} + \frac{\mathfrak{h}^2\|F\|^4C_4^2}{V^2(\mathfrak{h})}\Big] + 16\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\|\Pi\|^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1} + 16\Big[\frac{\mathfrak{h}\|F\|^2M_1}{V(\mathfrak{h})}\Big]^2T\Big\}\Big\{1 + 16\|\bar{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\Big(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{(\|\alpha\|^2+\|\Pi\|^2)T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1} + \Big[\frac{\mathfrak{h}\|F\|^2}{V(\mathfrak{h})}\Big]^2\Big(\frac{\|\alpha\|^2C_4^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1} + M_1^2T\Big)\Big)\Big\} < 1. \tag{3.2}$$

Proof. For any function $\varkappa(\cdot)$, the operator Φ on $\bar{C}$ is defined in the following manner:

$$(\Phi\varkappa)(\mathrm{t}) = F\mathfrak{S}_{\mathfrak{h}}(\mathrm{t})\varkappa_0 + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}} (\mathrm{t}-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}} (\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\Pi\varkappa(\bar{s})d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}} (\alpha-\Pi)(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\bar{B}\psi(\bar{s})d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}} (\mathrm{t}-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s}) + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}} \mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})N(\bar{s},\varkappa(\bar{s}))d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}} \Pi\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})\varkappa(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}} (\alpha-\Pi)\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})\bar{B}\psi(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}} \mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s}), \tag{3.3}$$

where

$$\begin{aligned}\psi(\mathfrak{t}) &= -(\mathfrak{L}_0)^{-1}\Big[F\mathfrak{S}_{\mathfrak{h}}(T)\varkappa_0 + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^T (T-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))d\bar{s}\\ &+\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^T (T-\bar{s})^{\mathfrak{h}-1}\Pi\varkappa(\bar{s})d\bar{s}\\ &+\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^T (T-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s}) + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^T \mathfrak{Q}_{\mathfrak{h}}(T-\bar{s})N(\bar{s},\varkappa(\bar{s}))d\bar{s}\\ &+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^T \Pi\mathfrak{Q}_{\mathfrak{h}}(T-\bar{s})\varkappa(\bar{s})d\bar{s}\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^T \mathfrak{Q}_{\mathfrak{h}}(T-\bar{s})W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\Big].\end{aligned}$$

We will demonstrate that Φ, mapping from $\bar{C}$ to itself, possesses a fixed point. For all integer $\mathfrak{q} > 0$, put $\mathfrak{B}_{\mathfrak{q}} = \{\iota \in \bar{C}, \|\iota\|^2_{\bar{C}} \leq \mathfrak{q}\}$. We assume that there exists $\mathfrak{q} > 0$ s.t. $\Phi(\mathfrak{B}_{\mathfrak{q}}) \subseteq \mathfrak{B}_{\mathfrak{q}}$. If it is not true, then, $\forall \mathfrak{q} > 0$, there exists a function $\varkappa_{\mathfrak{q}}(\cdot) \in \mathfrak{B}_{\mathfrak{q}}$, s.t. $\Phi(\varkappa_{\mathfrak{q}}) \notin \mathfrak{B}_{\mathfrak{q}}$. Specifically, $\exists\ \mathfrak{t} = \mathfrak{t}(\mathfrak{q}) \in \bar{J}$, where $\mathfrak{t}(\mathfrak{q})$ depends on $\mathfrak{q}$, s.t. $\|\Phi(\varkappa_{\mathfrak{q}}))(\mathfrak{t})\|^2_{\bar{C}} > \mathfrak{q}$.

From (H_6) in conjunction with the Hölder inequality, we derive

$$\begin{aligned}&\sup_{\mathfrak{t}\in\bar{J}} E\Big\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathfrak{t}} \mathfrak{Q}_{\mathfrak{h}}(\mathfrak{t}-\bar{s})N(\bar{s},\varkappa(\bar{s}))d\bar{s}\Big\|^2\\ \leq\ & \Big\{\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2 + \Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\Big\}E\Big[\int_0^{\mathfrak{t}}\|(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))\|d\bar{s}\Big]^2\\ \leq\ & \Big\{\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2 + \Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\Big\}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}d\bar{s}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}E\|N(\bar{s},\varkappa(\bar{s}))\|^2d\bar{s}\\ \leq\ & \frac{T^{\mathfrak{h}}}{\mathfrak{h}}\Big\{\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2 + \Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\Big\}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}N_{\mathfrak{q}}(\bar{s})d\bar{s}. \qquad (3.4)\end{aligned}$$

Also, from Burkholder-Gungy's inequality and Lemma 2.2 along with $(H7)$, it yields

$$\begin{aligned}&\sup_{\mathfrak{t}\in\bar{J}} E\Big\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s}) + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathfrak{t}} \mathfrak{Q}_{\mathfrak{h}}(\mathfrak{t}-\bar{s})W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\Big\|^2\\ \leq\ & 2HT^{2H-1}\Big\{\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2 + \Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\Big\}E\Big[\int_0^{\mathfrak{t}}\|(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))\|_{\mathfrak{L}_2^0}d\bar{s}\Big]^2\\ \leq\ & 2HT^{2H-1}\Big\{\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2 + \Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\Big\}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}d\bar{s}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}E\|W(\bar{s},\varkappa(\bar{s}))\|^2_{\mathfrak{L}_2^0}d\bar{s}\\ \leq\ & \frac{2HT^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\Big\{\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2 + \Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\Big\}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}\mathfrak{g}_q(\bar{s})d\bar{s}. \qquad (3.5)\end{aligned}$$

From (H_3), we derive

$$\begin{aligned}&\sup_{\mathfrak{t}\in\bar{J}} E\Big\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathfrak{t}}(\mathfrak{t}-\bar{s})^{\mathfrak{h}-1}\Pi\varkappa(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathfrak{t}} \Pi\mathfrak{Q}_{\mathfrak{h}}(\mathfrak{t}-\bar{s})\varkappa(\bar{s})d\bar{s}\Big\|^2\\ \leq\ & \Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\mathfrak{q}\|\Pi\|^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1} + \Big[\frac{\mathfrak{h}\|F\|^2M_1}{V(\mathfrak{h})}\Big]^2\mathfrak{q}T. \qquad (3.6)\end{aligned}$$

However, from (3.4)–(3.6), we obtain

$$\begin{aligned}
&\sup_{\mathrm{t}\in\bar{J}} E\left\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\alpha-\Pi)(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\bar{B}\psi(\bar{s})d\bar{s}+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}(\alpha-\Pi)\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})\bar{B}\psi(\bar{s})d\bar{s}\right\|^2\\
\le\ & 16\|\bar{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\bigg(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{(\|\alpha\|^2+\|\Pi\|^2)T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+\Big[\frac{\mathfrak{h}\|F\|^2}{V(\mathfrak{h})}\Big]^2\Big(\frac{\|\alpha\|^2C_4^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+M_1^2T\Big)\bigg)\\
&\times\bigg\{\|F\|^2C_3^2E\|\varkappa_0\|^2+\frac{T^{\mathfrak{h}}}{\mathfrak{h}}\bigg(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2+\Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\bigg)\int_0^T(T-\bar{s})^{\mathfrak{h}-1}N_{\mathfrak{q}}(\bar{s})d\bar{s}\\
&+\frac{2HT^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\bigg(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2+\Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\bigg)\int_0^T(T-\bar{s})^{\mathfrak{h}-1}\mathfrak{g}_{\mathfrak{q}}(\bar{s})d\bar{s}\\
&+\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\mathfrak{q}\|\Pi\|^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+\Big[\frac{\mathfrak{h}\|F\|^2M_1}{V(\mathfrak{h})}\Big]^2\mathfrak{q}T\bigg\}.
\end{aligned}\tag{3.7}$$

$$\begin{aligned}
\mathfrak{q}\ \le\ & \|\Phi(\varkappa_{\mathfrak{q}})(\mathrm{t})\|_{\bar{C}}^2=\sup_{\mathrm{t}\in\bar{J}}E\|\Phi(\varkappa_{\mathfrak{q}})(\mathrm{t})\|^2\\
\le\ & 16\sup_{\mathrm{t}\in\bar{J}}E\|F\mathfrak{S}_{\mathfrak{h}}(\mathrm{t})\varkappa_0\|^2+16\sup_{\mathrm{t}\in\bar{J}}E\bigg\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))d\bar{s}\\
&+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})N(\bar{s},\varkappa(\bar{s}))d\bar{s}\bigg\|^2\\
&+16\sup_{\mathrm{t}\in\bar{J}}E\bigg\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\alpha-\Pi)(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\bar{B}\psi(\bar{s})d\bar{s}+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}(\alpha-\Pi)\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})\bar{B}\psi(\bar{s})d\bar{s}\bigg\|^2\\
&+16\sup_{\mathrm{t}\in\bar{J}}E\bigg\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\bigg\|^2\\
&+16\sup_{\mathrm{t}\in\bar{J}}E\bigg\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\Pi\varkappa(\bar{s})d\bar{s}+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}}\Pi\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}-\bar{s})\varkappa(\bar{s})d\bar{s}\bigg\|^2\\
\le\ & 16\|F\|^2C_3^2E\|\varkappa_0\|^2+\frac{16T^{\mathfrak{h}}}{\mathfrak{h}}\bigg(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2+\Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\bigg)\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}N_{\mathfrak{q}}(\bar{s})d\bar{s}\\
&+\frac{32HT^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\bigg(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2+\Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\bigg)\int_0^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\mathfrak{g}_q(\bar{s})d\bar{s}\\
&+16\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\mathfrak{q}\|\Pi\|^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+16\Big[\frac{\mathfrak{h}\|F\|^2M_1}{V(\mathfrak{h})}\Big]^2\mathfrak{q}T\\
&+256\|\bar{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\bigg(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{(\|\alpha\|^2+\|\Pi\|^2)T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+\Big[\frac{\mathfrak{h}\|F\|^2}{V(\mathfrak{h})}\Big]^2\Big(\frac{\|\alpha\|^2C_4^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+M_1^2T\Big)\bigg)\\
&\times\bigg\{\|F\|^2C_3^2E\|\varkappa_0\|^2+\frac{T^{\mathfrak{h}}}{\mathfrak{h}}\bigg(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2+\Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\bigg)\int_0^T(T-\bar{s})^{\mathfrak{h}-1}N_{\mathfrak{q}}(\bar{s})d\bar{s}\\
&+\frac{2HT^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\bigg(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2+\Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\bigg)\int_0^T(T-\bar{s})^{\mathfrak{h}-1}\mathfrak{g}_{\mathfrak{q}}(\bar{s})d\bar{s}\\
&+\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\mathfrak{q}\|\Pi\|^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+\Big[\frac{\mathfrak{h}\|F\|^2M_1}{V(\mathfrak{h})}\Big]^2\mathfrak{q}T\bigg\}\\
&+16\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\mathfrak{q}\|\Pi\|^2T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+16\Big[\frac{\mathfrak{h}\|F\|^2M_1}{V(\mathfrak{h})}\Big]^2\mathfrak{q}T\bigg\}.
\end{aligned}\tag{3.8}$$

By dividing both sides of (3.8) by $\mathfrak{q}$ and letting $\mathfrak{q} \to +\infty$, we obtain

$$\Big\{\frac{32\delta T^{\mathfrak{h}} + 16\delta H T^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\Big[\frac{\|\wp\|^2\|F\|^2(1-\mathfrak{h})^2}{V^2(\mathfrak{h})\Gamma^2(\mathfrak{h})} + \frac{\mathfrak{h}^2\|F\|^4 C_4^2}{V^2(\mathfrak{h})}\Big] + 16\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\|\Pi\|^2 T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}$$
$$+16\Big[\frac{\mathfrak{h}\|F\|^2 M_1}{V(\mathfrak{h})}\Big]^2 T\Big\}\Big\{1 + 16\|\bar{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\Big(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{(\|\alpha\|^2+\|\Pi\|^2)T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}$$
$$+\Big[\frac{\mathfrak{h}\|F\|^2}{V(\mathfrak{h})}\Big]^2\Big(\frac{\|\alpha\|^2 C_4^2 T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1} + M_1^2 T\Big)\Big)\Big\} \geq 1.$$

This contradicts (3.2). Therefore, $\Phi(\mathfrak{B}_\mathfrak{q}) \subseteq \mathfrak{B}_\mathfrak{q}$, for $\mathfrak{q} > 0$.

Indeed, Φ maps $\mathfrak{B}_\mathfrak{q}$ into a compact subset of $\mathfrak{B}_\mathfrak{q}$. To establish this, we begin by demonstrating that $\mathfrak{B}_\mathfrak{q}(\mathrm{t}) = \{(\Phi\varkappa)(\mathrm{t}) : \varkappa \in \mathfrak{B}_\mathfrak{q}\}$ is precompact in $\mathcal{K}$, $\forall\ \mathrm{t} \in \bar{J}$. This is trivial for $\mathrm{t} = 0$, because $\mathfrak{B}_\mathfrak{q}(0) = \{\varkappa_0\}$. Now, consider a fixed t, where $0 < \mathrm{t} \leq T$. For $0 < \epsilon < \mathrm{t}$, take

$$\begin{aligned}
(\Phi^\epsilon\varkappa)(\mathrm{t}) &= F\mathfrak{S}_\mathfrak{h}(\mathrm{t})\varkappa_0 + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}-\epsilon}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}-\epsilon}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\Pi\varkappa(\bar{s})d\bar{s} \\
&+\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}-\epsilon}(\alpha-\Pi)(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\bar{B}\psi(\bar{s})d\bar{s} + \frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}-\epsilon}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s}) \\
&+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}-\epsilon}\mathfrak{Q}_\mathfrak{h}(\mathrm{t}-\bar{s})N(\bar{s},\varkappa(\bar{s}))d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}-\epsilon}\Pi\mathfrak{Q}_\mathfrak{h}(\mathrm{t}-\bar{s})\varkappa(\bar{s})d\bar{s} \\
&+\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}-\epsilon}(\alpha-\Pi)\mathfrak{Q}_\mathfrak{h}(\mathrm{t}-\bar{s})\bar{B}\psi(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}-\epsilon}\mathfrak{Q}_\mathfrak{h}(\mathrm{t}-\bar{s})W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s}).
\end{aligned}$$

From (H_4), the set $\mathfrak{V}^\epsilon(\mathrm{t}) = \{(\Phi^\epsilon\varkappa)(\mathrm{t}) : \varkappa \in \mathfrak{B}_\mathfrak{q}\}$ is a precompact set in $\mathcal{K}$ for all ϵ, where $0 < \epsilon < \mathrm{t}$.

Furthermore, for any $\varkappa \in \mathfrak{B}_\mathfrak{q}$, we have

$$\begin{aligned}
&\|(\Phi\varkappa)(\mathrm{t}) - (\Phi^\epsilon\varkappa)(\mathrm{t})\|_{\bar{C}}^2 \\
=\ & \sup_{\mathrm{t}\in\bar{J}} E\|(\Phi\varkappa)(\mathrm{t}) - (\Phi^\epsilon\varkappa)(\mathrm{t})\|^2 \\
\leq\ & 16\sup_{\mathrm{t}\in\bar{J}} E\Big\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}\mathfrak{Q}_\mathfrak{h}(\mathrm{t}-\bar{s})N(\bar{s},\varkappa(\bar{s}))d\bar{s}\Big\|^2 \\
&+16\sup_{\mathrm{t}\in\bar{J}} E\Big\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}(\alpha-\Pi)(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\bar{B}\psi(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}(\alpha-\Pi)\mathfrak{Q}_\mathfrak{h}(\mathrm{t}-\bar{s})\bar{B}\psi(\bar{s})d\bar{s}\Big\|^2 \\
&+16\sup_{\mathrm{t}\in\bar{J}} E\Big\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s}) + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}\mathfrak{Q}_\mathfrak{h}(\mathrm{t}-\bar{s})W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\Big\|^2 \\
&+16\sup_{\mathrm{t}\in\bar{J}} E\Big\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\Pi\varkappa(\bar{s})d\bar{s} + \frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}\Pi\mathfrak{Q}_\mathfrak{h}(\mathrm{t}-\bar{s})\varkappa(\bar{s})d\bar{s}\Big\|^2 \\
\leq\ & \frac{16\epsilon^{\mathfrak{h}}}{\mathfrak{h}}\Big(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2 + \Big[\frac{\mathfrak{h}\|F\|^2 C_4}{V(\mathfrak{h})}\Big]^2\Big)\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}N_\mathfrak{q}(\bar{s})d\bar{s} \\
&+\frac{32H\epsilon^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\Big(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2 + \Big[\frac{\mathfrak{h}\|F\|^2 C_4}{V(\mathfrak{h})}\Big]^2\Big)\int_{\mathrm{t}-\epsilon}^{\mathrm{t}}(\mathrm{t}-\bar{s})^{\mathfrak{h}-1}\mathfrak{g}_\mathfrak{q}(\bar{s})d\bar{s} \\
&+16\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\mathfrak{q}\|\Pi\|^2\epsilon^{2\mathfrak{h}-1}}{2\mathfrak{h}-1} + \Big[\frac{\mathfrak{h}\|F\|^2 M_1}{V(\mathfrak{h})}\Big]^2\mathfrak{q}\epsilon
\end{aligned}$$

$$+16\|\bar{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\Big(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{(\|\alpha\|^2+\|\Pi\|^2)\epsilon^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+\Big[\frac{\mathfrak{h}\|F\|^2}{V(\mathfrak{h})}\Big]^2\Big(\frac{\|\alpha\|^2C_4^2\epsilon^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+M_1^2\epsilon\Big)\Big)$$
$$\times\Big\{\|F\|^2C_3^2E\|\varkappa_0\|^2+\frac{\epsilon^{\mathfrak{h}}}{\mathfrak{h}}\Big(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2+\Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\Big)\int_{T-\epsilon}^{T}(T-\bar{s})^{\mathfrak{h}-1}N_{\mathfrak{q}}(\bar{s})d\bar{s}$$
$$+\frac{2H\epsilon^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\Big(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2+\Big[\frac{\mathfrak{h}\|F\|^2C_4}{V(\mathfrak{h})}\Big]^2\Big)\int_{T-\epsilon}^{T}(T-\bar{s})^{\mathfrak{h}-1}\mathfrak{g}_{\mathfrak{q}}(\bar{s})d\bar{s}$$
$$+\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\mathfrak{q}\|\Pi\|^2\epsilon^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}+\Big[\frac{\mathfrak{h}\|F\|^2M_1}{V(\mathfrak{h})}\Big]^2\mathfrak{q}\epsilon\Big\}.$$

We observe that $\forall\, \varkappa \in \mathfrak{B}_{\mathfrak{q}}$, $\|(\Phi\varkappa)(\mathrm{t})-(\Phi^{\epsilon}\varkappa)(\mathrm{t})\|_{\bar{C}}^2 \to 0$ as ϵ approaches 0^+. Thus, there exists precompact sets arbitrarily close to the set $\mathfrak{B}_{\mathfrak{q}}(\mathrm{t})$, indicating that $\mathfrak{B}_{\mathfrak{q}}(\mathrm{t})$ itself is precompact in $\mathcal{K}$.

Next, we demonstrate that $\{\Phi\varkappa : \varkappa \in \mathfrak{B}_{\mathfrak{q}}\}$ is an equicontinuous family of functions. Let $\varkappa \in \mathfrak{B}_{\mathfrak{q}}$ and $\mathrm{t}_1, \mathrm{t}_2 \in \bar{J}$ such that $0 < \mathrm{t}_1 < \mathrm{t}_2$, then

$$\begin{aligned}
&\|(\Phi x)(\mathrm{t}_2)-(\Phi x)(\mathrm{t}_1)\|_{\bar{C}}^2\\
\leq\ & 16\|F\mathfrak{S}_{\mathfrak{h}}(\mathrm{t}_2)\varkappa_0-F\mathfrak{S}_{\mathfrak{h}}(\mathrm{t}_1)x_0\|_{\bar{C}}^2+16\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}_1}[(\mathrm{t}_2-\bar{s})^{\mathfrak{h}-1}-(\mathrm{t}_1-\bar{s})^{\mathfrak{h}-1}]N(\bar{s},\varkappa(\bar{s}))d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_{\mathrm{t}_1}^{\mathrm{t}_2}(\mathrm{t}_2-\bar{s})^{\mathfrak{h}-1}N(\bar{s},\varkappa(\bar{s}))d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}_1}[(\mathrm{t}_2-\bar{s})^{\mathfrak{h}-1}-(\mathrm{t}_1-\bar{s})^{\mathfrak{h}-1}]\Pi\varkappa(\bar{s})d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_{\mathrm{t}_1}^{\mathrm{t}_2}(\mathrm{t}_2-\bar{s})^{\mathfrak{h}-1}\Pi\varkappa(\bar{s})d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}_1}(\alpha-\Pi)[(\mathrm{t}_2-\bar{s})^{\mathfrak{h}-1}-(\mathrm{t}_1-\bar{s})^{\mathfrak{h}-1}]\bar{B}\psi(\bar{s})d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_{\mathrm{t}_1}^{\mathrm{t}_2}(\mathrm{t}_2-\bar{s})^{\mathfrak{h}-1}\bar{B}\psi(\bar{s})d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_0^{\mathrm{t}_1}[(\mathrm{t}_2-\bar{s})^{\mathfrak{h}-1}-(\mathrm{t}_1-\bar{s})^{\mathfrak{h}-1}]W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\|_{\bar{C}}^2\\
&+16\|\frac{\wp F(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\int_{\mathrm{t}_1}^{\mathrm{t}_2}(\mathrm{t}_2-\bar{s})^{\mathfrak{h}-1}W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\|_{\bar{C}}^2\\
&+16\|\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}_1}[\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}_2-\bar{s})-\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}_1-\bar{s})]N(\bar{s},\varkappa(\bar{s}))d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{\mathrm{t}_2}^{\mathrm{t}_1}\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}_2-\bar{s})N(\bar{s},\varkappa(\bar{s}))d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}_1}[\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}_2-\bar{s})-\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}_1-\bar{s})]\Pi\varkappa(\bar{s})d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{\mathrm{t}_2}^{\mathrm{t}_1}\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}_2-\bar{s})\Pi\varkappa(\bar{s})d\bar{s}\|_{\bar{C}}^2\\
&+16\|\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_0^{\mathrm{t}_1}(\alpha-\Pi)[\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}_2-\bar{s})-\mathfrak{Q}_{\mathfrak{h}}(\mathrm{t}_1-\bar{s})]\bar{B}\psi(\bar{s})d\bar{s}\|_{\bar{C}}^2
\end{aligned}$$

$$+16\|\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{t_2}^{t_1}(\alpha-\Pi)\mathfrak{Q}_{\mathfrak{h}}(t_2-\bar{s})\bar{B}\psi(\bar{s})d\bar{s}\|_{\bar{C}}^2$$
$$+16\|\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{0}^{t_1}[\mathfrak{Q}_{\mathfrak{h}}(t_2-\bar{s})-\mathfrak{Q}_{\mathfrak{h}}(t_1-\bar{s})]W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\|_{\bar{C}}^2$$
$$+16\|\frac{\mathfrak{h}F^2}{V(\mathfrak{h})}\int_{t_1}^{t_2}\mathfrak{Q}_{\mathfrak{h}}(t_2-\bar{s})W(\bar{s},\varkappa(\bar{s}))d\mathfrak{B}^H(\bar{s})\|_{\bar{C}}^2.$$

Based on the earlier observation, we note that $\|(\Phi\varkappa)(t_2)-(\Phi\varkappa)(t_1)\|_{\bar{C}}^2 \to 0$ independently of $\varkappa \in \mathfrak{B}_{\mathfrak{q}}$ as t_2 tends to t_1. The compactness of $\mathfrak{S}\mathfrak{h}(t)$ and $\mathfrak{Q}\mathfrak{h}(t)$ for $t > 0$ ensures that continuity is maintained in the uniform operator topology.

Therefore, $\Phi(\mathfrak{B}_{\mathfrak{q}})$ exhibits both boundedness and equicontinuity. According to Arzela-Ascoli theorem, $\Phi(\mathfrak{B}_{\mathfrak{q}})$ is precompact in $\mathcal{K}$. Therefore, the operator Φ is completely continuous on $\mathcal{K}$. By Schauder's fixed point theorem, Φ possesses a fixed point in $\mathfrak{B}_{\mathfrak{q}}$. Any fixed point of Φ serves as a mild solution to (1.1) over $\bar{J}$. Consequently, (1.1) has exact null controllability on $\bar{J}$. □

4. Illustration

To validate the obtained results, we examine the A-B fractional stochastic PDE with fBm and control on the boundary as follows:

$$\begin{cases} {}^{ABC}D_{0+}^{\frac{3}{5}}\varkappa(t,\mathfrak{f}) = \frac{\partial^2}{\partial\mathfrak{f}^2}\varkappa(t,\mathfrak{f})+\psi(t,\mathfrak{f})+N(t,\varkappa(t,\mathfrak{f}))+W(t,\varkappa(t,\mathfrak{f}))\frac{d\mathfrak{B}^H(t)}{dt}, \quad t\in\bar{J},\ \mathfrak{f}\in\Xi, \\ \varkappa(t,\mathfrak{f})=\psi(t,\mathfrak{f}), \quad t\in\bar{J},\ \mathfrak{f}\in\Delta, \\ \varkappa(0,\mathfrak{f})=\varkappa_0(\mathfrak{f}),\ \mathfrak{f}\in\Xi, \end{cases} \tag{4.1}$$

where ${}^{ABC}D_{0+}^{\frac{3}{5}}$ is the A-B derivative, of order $\frac{3}{5}$, Ξ is a bounded open set in $\Re$ that has Δ as sufficiently smooth boundary, while $\mathfrak{B}^H$ is a fBm. Let $\varkappa(t)(\mathfrak{f})=\varkappa(t,\mathfrak{f})$, $N(t,\varkappa(t))(\mathfrak{f})=N(t,\varkappa(t,\mathfrak{f}))$ and $W(t,\varkappa(t))(\mathfrak{f}) = W(t,\varkappa(t,\mathfrak{f}))$.

Here, consider $\mathfrak{U}=\mathfrak{L}^2(\Delta)$, $\mathcal{K}=\bar{Y}=\mathfrak{L}^2(\Xi)$, $\bar{B}_1=I$, where I is the identity operator, and $\Pi:\mathfrak{D}(\Pi)\subset\mathcal{K}\to\mathcal{K}$ is given by $\Pi=\frac{\partial^2}{\partial\mathfrak{f}^2}$ with $\mathfrak{D}(\Pi)=\{\varkappa\in\mathcal{K};\varkappa,\frac{\partial\varkappa}{\partial\mathfrak{f}}$ are absolutely continuous, $\frac{\partial^2\varkappa}{\partial\mathfrak{f}^2}\in\mathfrak{L}^2(\Xi)\}$.

We define the operator $\mho:\mathfrak{D}(\mho)\subset\mathfrak{L}^2(\Xi)\to\mathfrak{L}^2(\Xi)$ is given by $\mho\varkappa=\Pi\varkappa$. Then, $\mho$ can be written as

$$\mho\varkappa=\sum_{n=1}^{\infty}(-n)^2(\varkappa,\varkappa_n)\varkappa_n, \quad \varkappa\in\mathfrak{D}(\mho).$$

In this context, $\varkappa_n(\mathfrak{f})=(\sin(n\mathfrak{f}))\sqrt{\frac{2}{\pi}}$, $n\in\mathbb{N}$ denotes the orthogonal set of eigenvectors of $\mho$.

For $\varkappa\in\mathcal{K}$, we have

$$\mathfrak{S}(t)x=\sum_{n=1}^{\infty}e^{\frac{-n^2t}{1+n^2}}(\varkappa,\varkappa_n)\varkappa_n, \quad \varkappa\in\varkappa.$$

$\mho$ generates a compact semigroup $\mathfrak{S}(t), t>0$ on $\mathcal{K}$ with $\|\mathfrak{S}(t)\|\le 1$.

Now, Eq (4.1) can be expressed in the abstract form of (1.1).

Set $\mathfrak{h} = \frac{3}{5}$, $H = 1$, $\|\wp\| = 1$, $\|F\| = 1$, $V(\mathfrak{h}) = 1$, $\Gamma(\mathfrak{h}) = 1$, $\delta = 0.01$, $T = 1$, $C_4 = 1$, $M_1 = 1$, $\|\bar{B}\| = 0.5$, $\|\mathfrak{L}_0^{-1}\| = 1$, $\|\alpha\| = 0.1$, $\|\Pi\| = 0.1$. Then, all the conditions of Theorem 3.4 have been satisfied, along with

$$\begin{aligned}&\Big\{\frac{32\delta T^{\mathfrak{h}} + 16\delta H T^{2H+\mathfrak{h}-1}}{\mathfrak{h}}\Big[\frac{\|\wp\|^2\|F\|^2(1-\mathfrak{h})^2}{V^2(\mathfrak{h})\Gamma^2(\mathfrak{h})} + \frac{\mathfrak{h}^2\|F\|^4 C_4^2}{V^2(\mathfrak{h})}\Big] + 16\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{\|\Pi\|^2 T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}\\&+16\Big[\frac{\mathfrak{h}\|F\|^2 M_1}{V(\mathfrak{h})}\Big]^2 T\Big\}\Big\{1 + 16\|\bar{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\Big(\Big[\frac{\|\wp\|\|F\|(1-\mathfrak{h})}{V(\mathfrak{h})\Gamma(\mathfrak{h})}\Big]^2\frac{(\|\alpha\|^2 + \|\Pi\|^2)T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1}\\&+\Big[\frac{\mathfrak{h}\|F\|^2}{V(\mathfrak{h})}\Big]^2\Big(\frac{\|\alpha\|^2 C_4^2 T^{2\mathfrak{h}-1}}{2\mathfrak{h}-1} + M_1^2 T\Big)\Big)\Big\} < 1.\end{aligned}$$

Therefore, (4.1) achieves exactly null boundary controllability over $\bar{J}$.

5. Conclusions

This paper introduced a novel control model incorporating A-B fractional derivative and fractional Brownian motion. This study investigated the sufficient conditions for null boundary controllability of A-B fractional SDEs that involve fBm in a Hilbert space. Techniques such as fractional analysis, compact semigroup theory, fixed point theorems, and stochastic analysis were commonly employed to establish controllability results. An example is included to demonstrate the theoretical results.

Author contributions

Noorah Mshary: Formal analysis, Writing–review & editing; Hamdy M. Ahmed: Validation, Methodology. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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