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Research article

A self-adaptive viscosity-type inertial algorithm for common solutions of generalized split variational inclusion and paramonotone equilibrium problem

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Abstract: In this paper, we aimed to consider the common elements of the generalized split variational inclusion and paramonotone equilibrium problem in real Hilbert spaces. Based on the self-adaptive method, a self-adaptive viscosity-type inertial algorithm to solve the problem under consideration was introduced and the inertial technique was used to accelerate the convergence rate of the method. Under the assumption of generalized monotonicity of the related mappings, the strong convergence of the iterative algorithm was established. The results presented here improve and generalize many results in this area.

Keywords: generalized split variational inclusion; equilibrium problem; paramonotonicity; strong convergence
Mathematics Subject Classification: 90C33, 49J52

1. Introduction

Throughout the paper, unless otherwise stated, let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, and P_C and P_Q be the orthogonal projection onto C and Q , respectively. Let $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ be two maximal monotone mappings and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*.

Find

$$x^* \in C \text{such that} Ax^* \in Q, \tag{1.1}$$

which is called the split feasibility problem (SFP). It was first introduced by Censor and Elfving [1] in finite dimensional Hilbert spaces to model the inverse problem caused by medical image reconstruction. Since then, SFP has received much attention for its applications in signal processing, image reconstruction, approximate theory, control theory, biomedical engineering, communications, and geophysics. For details, the readers can refer to [1–5] and the references therein. To solve SFP, we proposed the recurrent projection algorithm, but in each iteration process, calculating the inverse of the matrix or the maximum eigenvalue of the matrix is needed. In solving the real problem, calculating the inverse of the matrix takes a lot of time and is not easy to solve. To overcome the disadvantage of finding the matrix inverse in the algorithm, in 2002, Byrne [6] presented the following CQ algorithm:

$$x_{n+1} = P_C\left(x_n - \gamma_n A^T\left(I - P_Q\right)Ax_n\right),$$

where A is the matrix operator, A^T is the transpose operator of A, $\gamma \in \left(0, \frac{2}{L}\right)$ with L the largest eigenvalue of the matrix A^TA.

In 2011, Moudafi [7] first introduced the following problem: Find $x^* \in H_1$such that

$$0 \in B\left(x^*\right) \text{and} 0 \in D\left(Ax^*\right), \tag{1.2}$$

which is called the split variational inclusion problem (for short, denoted by SVIP). It is clear that the SVIP includes the SFP as a special case. We denote the solution set of the SVIP by $SVIP(B, D) := \{x^* \in C | \, 0 \in B\left(x^*\right), 0 \in D\left(Ax^*\right)\}$. The SVIP is at the core of modeling of many inverse problems arising from phase retrieval and other real world problems, for instance, in sensor networks in computerized and data compression [8, 9]. In recent years, there has been tremendous interest in solving the SVIP, and many researchers have constructed a large number of methods to solve this problem [10–16].

In 2014, Yang and Zhao [17] defined the following: Find $x^* \in H_1$such that

$$x^* \in \cap_{i=1}^{\infty} B_i^{-1}(0) \text{andA} x^* \in \cap_{i=1}^{\infty} D_i^{-1}(0), \tag{1.3}$$

which is called the generalized split variational inclusion problem (for short, denoted by GSVIP1), where for each $i \in \mathbb{N}$, $B_i : H_1 \to 2^{H_1}$ and $D_i : H_2 \to 2^{H_2}$ are two families of maximal monotone mappings. To solve the GSVIP1, the following algorithm is introduced :

$$x_{n+1} = a_n x_n + b_n f\left(x_n\right) + \sum_{i=1}^{\infty} c_{n,i} J_{\beta_{n,i}}^{B_i}\left(I - \gamma_{n,i} A^*\left(I - J_{\beta_{n,i}}^{D_i}\right)A\right)x_n, n \geq 0,$$

where for each $i \in \mathbb{N}$, the sequences$\{a_n\}, \{b_n\}, \{c_{n,i}\} \subset (0, 1)$, $a_n + b_n + \sum_{i=1}^{\infty} c_{n,i} = 1$, $\{\beta_{n,i}\} \subset (0, \infty), \{\gamma_{n,i}\} \subset \left(0, \frac{2}{\|A\|^2+1}\right)$, f is a $k-$ contraction mapping of H_1, and the strong convergence of the above algorithm under mild assumptions has been proved.

Ogbuisi et al. [18] introduced a new inertial algorithm to solve the following problem: Find $x^* \in H_1$ such that

$$x^* \in \cap_{i=1}^{s} B_i^{-1}(0) \text{andA} x^* \in \cap_{j=1}^{t} D_j^{-1}(0), \tag{1.4}$$

which is also called the generalized split variational inclusion problem (for convenience, denoted by GSVIP2), and where for $s\ t \in \mathbb{N}$, $B_i : H_1 \to 2^{H_1}(i = 1, \cdots, s)$, and $D_j : H_2 \to 2^{H_2}(j = 1, \cdots, t)$ are two finite families of maximal monotone mappings. We denote the solution set of the GSVIP2 by $GSVIP2\left(B_i, D_j\right) := \left\{x^* \in C \,\middle|\, x^* \in \cap_{i=1}^{s} B_i^{-1}(0) \text{andA} x^* \in \cap_{j=1}^{t} D_j^{-1}(0)\right\}$. The following algorithm is introduced to solve the GSVIP2: Choose any initial value $u_0, v_1 \in H_1, \lambda > 0$. Assume u_{n-1}, u_nhave been known. Compute

$$\begin{cases} x_n = u_n + \theta_n (u_n - u_{n-1}). \\ z_n = J_{\lambda B_{i_n}} x_n. \\ y_n = A^* \left(I - J_{\lambda D_{j_n}}\right) A x_n, n \geq 1, \\ \text{where} i_n \in \left\{ i \left| \max_{1\leq i\leq s} \left\| x_n - J_{\lambda B_i} x_n \right\| \right. \right\}, j_n \in \left\{ j \left| \max_{1\leq j\leq t} \left\| A x_n - J_{\lambda D_j} A x_n \right\| \right. \right\}. \\ \text{If } \|x_n + y_n - z_n\| = 0, \text{thenstop}(x_n \text{isthedesiredsolution}); \text{otherwise, continuetocompute,} \\ u_{n+1} = (1 - \alpha_n) u_n + \alpha_n \left[x_n - \tau_n (x_n + y_n - z_n) \right], \end{cases}$$

where $\alpha_n \in (0,1), \theta_n \in [0,1], \tau_n = \gamma_n \frac{\|x_n - z_n\|^2 + \|y_n\|^2}{2\|x_n + y_n - z_n\|^2}, \gamma_n > 0$, and they show that the sequences generated by the above algorithm weakly converge to the solution of their problem.

In addition, the equilibrium problem (for short, EP) was first proposed by Nikaido and Isoda [19] in 1955, which is described as: Find $u^* \in C$ such that

$$f(u^*, v) \geq 0, \forall v \in C, \tag{1.5}$$

where H is a real Hilbert space, C is a nonempty closed convex subset of $H, f : C \times C \to R$ is a bifunction. We denote the solution set of the EP by $EP(f) := \{u^* \in C \,|\, f(u^*, v) \geq 0, \forall v \in C\}$. Noting that after the publication of the paper by Blum and Oettli [20] in 1994, the EP attracted wide attention, and many scholars published a large number of articles on the problem. The EP includes some important problems such as optimization problem, saddle point, variational inequality, and Nash equilibrium as special cases.

For solving the monotone EP, Korpelevich [21] first extended the extragradient method (double projection) of the saddle point problem to the monotone EP, and many algorithms [22–26] have been developed for solving the EP. Santos and Scheimberg [27] proposed an inexact projection subgradient method to solve the EP involving paramonotone bifunctions in finite dimensional space. It is noted that this algorithm needs only one projection per iteration, and its weak convergence was proved under mild assumptions.

In 2016, Yen et. al. [28] studied the SFP involving paramonotone equilibrium problem and convex optimization problem, which is formulated as: Find $x^* \in C$ such that

$$f(u^*, v) \geq 0, \forall v \in C \text{and} g(Au^*) \leq g(y), \forall y \in H_2, \tag{1.6}$$

where g is a properly lower semicontinuous convex function on H_2. They introduced the following algorithm:

$$\begin{cases} y_n = P_C (x_n - \alpha_n \eta_n), \\ z_n = P_C \left(y_n - \mu_n A^* \left(I - prox_{\lambda g} \right) A y_n \right), \\ x_{n+1} = a_n x_n + (1 - a_n) z_n, \end{cases}$$

for each $x_n \in C, \eta_n \in \partial_2^{\varepsilon_n} f(x_n, x_n)$ and $\alpha_n = \frac{\beta_n}{\gamma_n}$ where $\gamma_n = \max\{\delta_n, \|g_n\|\}$ and

$$\mu_n = \begin{cases} 0, if \nabla h(y_n) = 0, \\ \rho_n \frac{h(y_n)}{\|\nabla h(y_n)\|^2}, if \nabla h(y_n) \neq 0, \end{cases}$$

the selection of the sequences $\{\alpha_n\}, \{\delta_n\}, \{\beta_n\}, \{\varepsilon_n\}$ and $\{\rho_n\}$ is described in Algorithm 3.1 [28]. Moreover, they proved the strong convergence of the algorithm under mild assumptions.

The problems of finding common solutions of the set of fixed points of nonlinear mappings and the set of solutions of optimization problems with its related problems have been considered by some authors (for instance, see [29–33] and the references therein). The motivation for studying such a common solution problem lies in its potential application to mathematical models whose constraints can be expressed as fixed point problems and optimization problems. This arises in practical problems, such as signal processing, network resource allocation, and image recovery (see, for instance, [34, 35] and the references therein).

Tan, Qin and Yao [36] proposed four self-adaptive inertial algorithms with strong convergence to solve the split variational inclusion problem in real Hilbert spaces. Izuchukwu et al. [37] first proposed and studied several strongly convergent versions of the forward-reflected-backward splitting method of Malitsky and Tam for finding a zero of the sum of two monotone operators in a real Hilbert space, which required only one forward evaluation of the single-valued operator and one backward evaluation of the set-valued operator at each iteration. They also developed inertial versions of their methods with strong convergence when the set-valued operator was maximal monotone and the single-valued operator was Lipschitz continuous and monotone. Moreover, they discussed some examples from image restorations and optimal control regarding the implementations of our methods in comparisons with known related methods in the literature. Zhang and Wang [38] suggested a new inertial iterative algorithm for split null point and common fixed point problems. In [39], the authors focused on a inertial-viscosity approximation method for solving a split generalized equilibrium problem and common fixed point problem in real Hilbert spaces, their algorithm was designed such that its strong convergence did not require the norm of the bounded linear operator underlying the split equilibrium problem and under mild conditions. In [40], the authors studied the split variational inclusion and fixed point problems using Bregman weak relatively nonexpansive mappings in the $p-$uniformly convex smooth Banach spaces, they introduced an inertial shrinking projection self-adaptive iterative scheme for the problem and proved a strong convergence theorem.

Su et al. [41] constructed a multi-step inertial asynchronous sequential algorithm for common fixed point problems. Zheng et al. [42] considered a new fixed-time stability of a neural network to solve split convex feasibility problems.

Motivated and inspired by the above research work, we aim to consider the common element of the paramonotone equilibrium problem and the GSVIP2: Find $u^* \in C$ such that

$$f(u^*, u) \geq 0, \forall u \in C and 0 \in \cap_{i=1}^{s} B_i(u^*), 0 \in \cap_{j=1}^{t} D_j(Au^*), \tag{1.7}$$

where $s, t, B_i, D_j, f : C \times C \to R$ are as mentioned above. We denote the set of solutions of Problem (1.7) by

$\Gamma := \left\{ x^* \in C \,\middle|\, f(x^*, x) \geq 0, 0 \in \cap_{i=1}^{\infty} B_i(x^*), 0 \in \cap_{j=1}^{\infty} D_j(Ax^*), \forall x \in C \right\} = GSIVP2(B_i, D_j) \cap EP(f).$

It is easy to see, if $B_i = 0, D_j = 0$, then Problem (1.7) simplifies to the EP (1.5); if $f = 0$, then Problem (1.7) simplifies to the GSVIP2 (1.4); if $s = 1, t = 1$, then Problem (1.7) changes into the following problem: Find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \forall y \in C and 0 \in B(x^*), 0 \in D(Ax^*), \tag{1.8}$$

if for $s, t \in \mathbb{N}, i \in \{1, 2, \cdots, s\}, j \in \{1, 2, \cdots, t\}, B_i = N_{C_i}, D_j = N_{Q_j}$ in Problem (1.7), where N_{C_i} and N_{Q_i} are the normal cones of nonempty, closed and convex subsets $C_i \subseteq H_1$ and $Q_j \subseteq H_2$, respectively.

Then, we obtain the following multiple-sets split feasibility problem and paramonotone equilibrium problem: Find $u^* \in C$ such that

$$f(u^*, u) \geq 0, \forall u \in C and u^* \in \cap_{i=1}^{s} C_i, Au^* \in \cap_{j=1}^{t} Q_j. \tag{1.9}$$

Thus, it can be seen that Problem (1.7) considered in this paper is more general, and contains many known and new mathematical models about the common element problems, such as Problems (1.1–1.6), Problems (1.8) and (1.9) as special cases. We are committed to establishing strong convergences of a self-adaptive viscosity-type inertial algorithm for the common solutions of Problem (1.7). The advantages of the suggested iterative algorithm are that (1) the design of the algorithm is self-adaptive, the inertial term can speed up its convergence, (2) the strong convergence analysis does not require a prior estimate of the norm of bounded operator, (3) the strong convergence of the iterative algorithm is established under the weak assumption of paramonotonicity of the related mappings. Our results improve and generalize many known results in the literature [18, 27].

2. Preliminaries

In this section, we give some basic concepts, properties, and notations that will be used in the sequel. Let C be a nonempty closed and convex subset of a real Hilbert space H embed with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. For each $u, v \in H$ and $a \in R$, we have the following facts:

i)$\|u+v\|^2 = \|u\|^2 + 2\langle u, v \rangle + \|v\|^2$;

ii)$\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v \rangle$;

iii)$\|au + (1-a)v\|^2 = a\|u\|^2 + (1-a)\|v\|^2 - a(1-a)\|u-v\|^2$.

Definition 2.1 [18] (1) $F : H \to H$ is nonexpansive, if $\|Fu - Fv\| \leq \|u - v\|, \forall u, v \in H$.
(2) $F : H \to H$ is firmly nonexpansive, if $\langle Fu - Fv, u - v \rangle \geq \|Fu - Fv\|^2, \forall u, v \in H$.

Definition 2.2 [18] A mapping $F : C \to C$ is said to be demiclosed, if for any sequence $\{u_n\} \subset C$ which weakly converges to u, and the sequence $\{Fu_n\}$ strongly converges to v, then $F(u) = v$.

Lemma 2.1 [43] Let C be a nonempty, closed and convex subset of a real Hilbert space$A$, $F : C \to C$ be a nonexpansive mapping. Then, $I - F$ is demiclosed at 0.

Lemma 2.2 [44] Let B be a maximal monotone mapping on a Hilbert space H for any $r > 0$, we define the resolvent $J_r^B = (I + rB)^{-1}$, then the following hold:

(1) J_r^B is a single-valued and firmly nonexpansive mapping.

(2) $D\left(J_r^B\right) = H$, and$Fix\left(J_r^B\right) = B^{-1}(0)$ where $Fix\left(J_r^B\right)$ stands for the fixed point set of J_r^B.

Lemma 2.3 [18] Let $B : H \to 2^H$ be a maximal monotone mapping, then the associated resolvent J_r^B for some $r > 0$ has the following characterization:

$$\left\langle u - J_r^B(u), u - v \right\rangle \geq \left\| u - J_r^B(u) \right\|^2, \forall u \in H, v \in Fix\left(J_r^B\right).$$

Lemma 2.4 [45] Let $\{\upsilon_n\}$ and $\{\delta_n\}$ be nonnegative sequences of real numbers satisfying $\upsilon_{n+1} \leq \upsilon_n + \delta_n$ with $\sum_{n=1}^{\infty} \delta_n < +\infty$. Then, the sequence $\{\upsilon_n\}$ is convergent.

Lemma 2.5 [46] Let H be a real Hilbert space, $\{a_n\}$ be a sequence of real numbers such that $0 < a < a_n < b < 1$ for all $n \geq 1$ and $\{b_n\}, \{d_n\}$ be the sequences in H such that $\limsup_{n\to\infty} \|b_n\| \leq c$, $\limsup_{n\to\infty} \|d_n\| \leq c$, and for some $c > 0$, $\limsup_{n\to\infty} \|a_n b_n + (1 - a_n) d_n\| = c$. Then $\lim_{n\to\infty} \|b_n - d_n\| = 0$.

3. Main results

In this section, in order to prove the convergence of the algorithm, the following conditions are assumed:
(A1) Let H_1 and H_2 be two real Hilbert spaces, C be a nonempty closed convex subset of $H_1, A : H_1 \to H_2$ be a linear and bounded operator.
(A2) For $s, t \in \mathbb{N}, i \in \{1, 2, \cdots, s\}, j \in \{1, 2, \cdots, t\}, B_i : H_1 \to 2^{H_1}, D_j : H_2 \to 2^{H_2}$ are two families of maximal monotone mappings.
(A3) The bifunctions $f : C \times C \to R$ satisfies the following:
(B1) For each $u \in C, f(u, u) = 0$, and $f(u, \cdot)$ is lower semicontinuous and convex on C, $f(\cdot, u)$ is upper semicontinuous and convex on C.
(B2) $\partial_2^{\lambda} f(u, u)$is nonempty for any $\lambda > 0$ and $u \in C$, and it is bounded on any bounded subset of C where $\partial_2^{\lambda} f(u, u)$ denotes $\lambda-$subdifferential of the convex function $f(u, \cdot)$ at u, that is

$$\partial_2^{\lambda} f(u, u) := \{\eta \in H : \langle \eta, v - u \rangle + f(u, u) \le f(u, v) + \lambda, \forall v \in C\}.$$

(B3) f is pseudo-monotone on C with respect to every solution of the EP, that is $f(u, u^*) \le 0$ for $\forall u \in C, u^* \in EP(f)$. And f satisfies the following condition, which is called the paramonotonicity properly: $u^* \in EP(f), \forall v \in C, f(u^*, v) = f(v, u^*) = 0 \Rightarrow v \in EP(f)$.
(A4) $\Gamma \neq \emptyset$.

Now, we introduce a self-adaptive viscosity-type inertial algorithm to solve Problem (1.7), which is described as follows.
Algorithm 3.1. Initialization. Pick $u_0, u_1 \in H_1, r > 0$, let $\theta \in [0, 1)$, for any $n \in N$, the sequence $\{\rho_n\}, \{a_n\}, \{\beta_n\}, \{\lambda_n\}, \{\delta_n\}, \{\varepsilon_n\} \subset [0, \infty)$ satisfying the following conditions:

$$\rho_n > \rho > 0, 0 < a < a_n < b < 1, \beta_n > 0, \lambda_n > 0,$$

$$\sum_{n=1}^{\infty} \varepsilon_n < +\infty, \lim_{n\to\infty} a_n = \tfrac{1}{2}, \sum_{n=1}^{\infty} \tfrac{\beta_n}{\rho_n} = +\infty, \sum_{n=1}^{\infty} \beta_n^2 < +\infty,$$

$$\sum_{n=1}^{\infty} \tfrac{\beta_n \lambda_n}{\rho_n} < +\infty, 0 < \liminf_{n\to\infty} \delta_n \le \limsup_{n\to\infty} \delta_n < 4.$$

Step 1. Assume u_{n-1}, u_n have been known. Choose α_n such that $0 < \alpha_n \le \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\left\{\theta, \frac{\varepsilon}{\|u_n - u_{n-1}\|}\right\}, \text{if} u_n \neq u_{n-1}, \\ \theta, \text{otherwise}. \end{cases}$$

Computer

$$x_n = u_n + \alpha_n (u_n - u_{n-1}). \tag{3.1}$$

Step 2. Compute

$$y_n = J_r^{B_{i_n}} x_n, \tag{3.2}$$

$$z_n = A^* \left(I - J_r^{D_{j_n}}\right) A x_n, \forall n \ge 1, \tag{3.3}$$

where

$$i_n \in \left\{i | \max_{1 \le i \le s} \left\| x_n - J_r^{B_i} x_n \right\|\right\}, j_n \in \left\{j | \max_{1 \le j \le t} \left\| A x_n - J_r^{D_j} A x_n \right\|\right\}, \tag{3.4}$$

if

$$\|x_n + z_n - y_n\| = 0, \tag{3.5}$$

then stop; otherwise, continue and compute v_n as follows

$$v_n = P_C\left(x_n - \xi_n\left(x_n + z_n - y_n\right)\right), \tag{3.6}$$

where

$$\xi_n = \delta_n \frac{\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right)Ax_n\right\|^2}{2\|x_n - y_n + z_n\|^2}, \delta_n > 0.$$

Step 3. Take $\eta_n \in \partial_2^{\lambda_n} f\left(v_n, v_n\right)$ and define $\tau_n = \frac{\beta_n}{\gamma_n}$ with $\gamma_n = \max\left\{\rho_n, \|\eta_n\|\right\}$. Computer

$$w_n = P_C\left(v_n - \tau_n \eta_n\right). \tag{3.7}$$

Step 4. Compute

$$u_{n+1} = a_n u_n + \left(1 - a_n\right) w_n. \tag{3.8}$$

Several algorithms can be deduced from our Algorithm 3.1 for solving Problem (1.7) as follows: If $\alpha_n = 0$ in Algorithm 3.1, we have the following self-adaptive viscosity-type method:

Algorithm 3.2. Choose any initial value $u_1 \in H_1$.

Step 1. Compute

$$y_n = J_r^{B_{i_n}} u_n,$$

$$z_n = A^*\left(I - J_r^{D_{j_n}}\right)Au_n, \forall n \geq 1,$$

where

$$i_n \in \left\{i \middle| \max_{1\leq i\leq s} \left\|u_n - J_r^{B_i} u_n\right\|\right\}, j_n \in \left\{j \middle| \max_{1\leq j\leq t} \left\|Au_n - J_r^{D_j} Au_n\right\|\right\},$$

if $\|x_n + z_n - y_n\| = 0$, then stop; otherwise, continue to compute $v_n$as follows

$$v_n = P_C\left(x_n - \xi_n\left(x_n + z_n - y_n\right)\right).$$

Step 2. Take $\eta_n \in \partial_2^{\lambda_n} f\left(v_n, v_n\right)$ and define $\tau_n = \frac{\beta_n}{\gamma_n}$ with $\gamma_n = \max\left\{\rho_n, \|\eta_n\|\right\}$. Compute

$$u_{n+1} = P_C\left(v_n - \tau_n \eta_n\right),$$

where $\xi_n = \delta_n \frac{\|u_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right)Au_n\right\|^2}{2\|u_n - y_n + z_n\|^2}, \delta_n > 0$ and $\theta, r, \{\rho_n\}, \{a_n\}, \{\beta_n\}, \{\lambda_n\}, \{\varepsilon_n\}$ are updated by Algorithm 3.1.

If $s = 1, t = 1$ then Problem (1.7) reduces to Problem (1.8), and we consider the following Algorithm 3.3 corresponding to Algorithm 3.1 for computing the solution of Problem (1.8):

Algorithm 3.3. Choose any initial value $u_0, u_1 \in H_1$.

Step 1. Assume u_{n-1}, u_n have been known. Compute

$$x_n = u_n + \alpha_n\left(u_n - u_{n-1}\right).$$

Step 2. Compute

$$y_n = J_r^B x_n,$$

$$z_n = A^*\left(I - J_r^D\right)Ax_n, \forall n \geq 1,$$

if $\|x_n + z_n - y_n\| = 0$, then stop; otherwise, continue to compute v_n as follows

$$v_n = P_C\left(x_n - \xi_n\left(x_n + z_n - y_n\right)\right).$$

Step 3. Take $\eta_n \in \partial_2^{\lambda_n} f\left(v_n, v_n\right)$ and define $\tau_n = \frac{\beta_n}{\gamma_n}$ with $\gamma_n = \max\left\{\rho_n, \|\eta_n\|\right\}$. Compute

$$w_n = P_C\left(v_n - \tau_n \eta_n\right).$$

Step 4. Compute

$$u_{n+1} = a_n u_n + \left(1 - a_n\right) w_n,$$

where $\theta, r, \{\rho_n\}, \{a_n\}, \{\beta_n\}, \{\lambda_n\}, \{\varepsilon_n\}, \{\alpha_n\}, \{\xi_n\}$ are updated by Algorithm 3.1.

If $B_i = N_{C_i}(i = 1, 2, \cdots, s), D_j = N_{Q_j}(i = 1, 2, \cdots, t)$, then Problem (1.7) reduces to Problem (1.9), and Algorithm 3.1 reduces to the following method:

Algorithm 3.4. Choose any initial value $u_0, u_1 \in H_1$.

Step 1. Compute

$$x_n = u_n + \alpha_n\left(u_n - u_{n-1}\right).$$

Step 2. Compute

$$y_n = P_{C_{i_n}} x_n,$$

$$z_n = A^*\left(I - P_{Q_{j_n}}\right)Ax_n,$$

where

$$i_n \in \left\{i \middle| \max_{1 \leq i \leq s} \left\|x_n - P_{C_{i_n}} x_n\right\|\right\}, j_n \in \left\{j \middle| \max_{1 \leq j \leq t} \left\|Ax_n - P_{Q_{j_n}} Ax_n\right\|\right\},$$

if $\|x_n + z_n - y_n\| = 0$, then stop; otherwise, continue to compute v_n as follows Step 3. Take $\eta_n \in \partial_2^{\lambda_n} f\left(v_n, v_n\right)$ and define $\tau_n = \frac{\beta_n}{\gamma_n}$ with $\gamma_n = \max\left\{\rho_n, \|\eta_n\|\right\}$. Compute

$$w_n = P_C\left(v_n - \tau_n \eta_n\right).$$

Step 4. Compute

$$u_{n+1} = a_n u_n + \left(1 - a_n\right) w_n,$$

where $\xi_n = \delta_n \frac{\|x_n - y_n\|^2 + \left\|\left(I - P_{Q_{j_n}}\right)Ax_n\right\|^2}{2\|x_n - y_n + z_n\|^2}, \delta_n > 0$ and $\theta, \{\rho_n\}, \{a_n\}, \{\beta_n\}, \{\lambda_n\}, \{\varepsilon_n\}, \{\alpha_n\}$ are updated by Algorithm 3.1.

If $f = 0$, Problem (1.7) reduces to Problem (1.4), we consider the following inertial Algorithm 3.5 corresponding to Algorithm 3.1 for computing the solution of GSVIP2(1.4):

Algorithm 3.5. Choose any initial value $u_0, u_1 \in H_1$.

Step 1. Assume u_{n-1}, u_n have been known. Compute

$$x_n = u_n + \alpha_n\left(u_n - u_{n-1}\right).$$

Step 2. Compute

$$y_n = J_r^{B_{i_n}} x_n,$$

$$z_n = A^* \left(I - J_r^{D_{j_n}}\right) A x_n, \forall n \geq 1,$$

where

$$i_n \in \left\{ i | \max_{1\leq i\leq s} \left\| x_n - J_r^{B_i} x_n \right\| \right\}, j_n \in \left\{ j | \max_{1\leq j\leq t} \left\| A x_n - J_r^{D_j} A x_n \right\| \right\},$$

if $\|x_n + z_n - y_n\| = 0$, then stop; otherwise, continue to compute v_n as follows

$$v_n = x_n - \xi_n (x_n + z_n - y_n).$$

Step 3. Compute

$$u_{n+1} = a_n u_n + (1 - a_n) v_n,$$

where $\theta, r, \{a_n\}, \{\alpha_n\}, \{\xi_n\}$ are updated by Algorithm 3.1.

If $B_i = 0, D_j = 0$, Problem (1.7) reduces to Problem (1.5) and hence we consider the following inertial Algorithm 3.6 corresponding to Algorithm 3.1 for computing the solution of EP(1.5):
Algorithm 3.6. Initialization. Choose any initial value $u_0, u_1 \in H_1$.
Step 1. Compute

$$x_n = u_n + \alpha_n (u_n - u_{n-1}).$$

Step 2. Take $\eta_n \in \partial_2^{\lambda_n} f(v_n, v_n)$ and define $\tau_n = \frac{\beta_n}{\gamma_n}$ with $\gamma_n = \max\{\rho_n, \|\eta_n\|\}$. Compute

$$w_n = P_C (v_n - \tau_n \eta_n).$$

Step 3. Compute

$$u_{n+1} = a_n u_n + (1 - a_n) w_n,$$

where $\{\alpha_n\}, \{\rho_n\}, \{a_n\}, \{\beta_n\}, \{\lambda_n\}$ are updated by Algorithm 3.1.

In order to obtain our major results, we also need the following lemmas.
Lemma 3.1. [27] For any $n \geq 1$, the following inequalities hold:
(1)$\tau_n \|\eta_n\| \leq \beta_n$. (2)$\|w_n - v_n\| \leq \beta_n$.
Lemma 3.2. [18] The equality (3.5) holds if and only if $u_n$is a solution of $GSVIP\left(B_i, D_j\right)$.
Theorem 3.1. Suppose Assumptions (A1–A4) hold and the sequence $\{u_n\}$ generated by Algorithm 3.1 strongly converges to a solution of Problem (1.7).
Proof. We divide the proof into the following several steps.
Step 1. The sequence $\left\{\|u_n - u^*\|^2\right\}$ is convergent for all $u^* \in \Gamma$, then the sequence $\{u_n\}$ is bounded.
Indeed, for $u^* \in \Gamma$, we have

$$\langle x_n + z_n - y_n, x_n - u^* \rangle \geq \|x_n - y_n\|^2 + \left\| \left(I - J_\lambda^{D_{j_n}}\right) A x_n \right\|^2. \tag{3.9}$$

From (3.9), we get

$$
\begin{aligned}
&\|v_n - u^*\|^2 \\
&\leq \|x_n - \xi_n (x_n + z_n - y_n) - u^*\|^2 \\
&= \|x_n - u^*\|^2 - 2\xi_n \langle x_n - u^*, x_n + z_n - y_n\rangle + {\xi_n}^2\|x_n + z_n - y_n\|^2 \\
&\leq \|x_n - u^*\|^2 - 2\xi_n \left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) Ax_n\right\|^2\right) + {\xi_n}^2\|x_n + z_n - y_n\|^2 \\
&\leq \|x_n - u^*\|^2 - 2\delta_n \frac{\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) Ax_n\right\|^2}{2\|x_n - y_n + z_n\|^2} \left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) Ax_n\right\|^2\right) \\
&\quad + \delta_n^2 \frac{\left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) Ax_n\right\|^2\right)^2}{4\|x_n - y_n + z_n\|^4} \|x_n + z_n - y_n\|^2 \\
&= \|x_n - u^*\|^2 - \delta_n \left(1 - \frac{\delta_n}{4}\right) \frac{\left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) Ax_n\right\|^2\right)^2}{\|x_n - y_n + z_n\|^2}.
\end{aligned} \tag{3.10}
$$

Form $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 4$ that $\left(1 - \frac{\delta_n}{4}\right) > 0$ and

$$
\|v_n - u^*\|^2 \leq \|x_n - u^*\|^2. \tag{3.11}
$$

From the definition of x_n that

$$
\begin{aligned}
&\|x_n - u^*\|^2 \\
&= \|u_n + \alpha_n (u_n - u_{n-1}) - u^*\|^2 \\
&= \|u_n - u^*\|^2 + 2\alpha_n \langle u_n - u^*, u_n - u_{n-1}\rangle + \alpha_n^2\|u_n - u_{n-1}\|^2 \\
&= \|u_n - u^*\|^2 + \alpha_n \left(\|u_n - u^*\|^2 + \|u_n - u_{n-1}\|^2 - \|u_{n-1} - u^*\|^2\right) + \alpha_n^2\|u_n - u_{n-1}\|^2 \\
&= \|u_n - u^*\|^2 + \alpha_n \left(\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2\right) + \alpha_n (1 + \alpha_n) \|u_n - u_{n-1}\|^2 \\
&\leq \|u_n - u^*\|^2 + \alpha_n \left(\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2\right) + 2\alpha_n\|u_n - u_{n-1}\|^2 \\
&\leq \|u_n - u^*\|^2 + \alpha_n (\|u_n - u^*\| + \|u_{n-1} - u^*\|) \|u_n - u_{n-1}\| + 2\alpha_n\|u_n - u_{n-1}\|^2 \\
&= \|u_n - u^*\|^2 + \alpha_n (\|u_n - u^*\| + \|u_{n-1} - u^*\| + 2 \|u_n - u_{n-1}\|) \|u_n - u_{n-1}\| \\
&= \|u_n - u^*\|^2 + \alpha_n c_1 \|u_n - u_{n-1}\|,
\end{aligned} \tag{3.12}
$$

where $c_1 = \|u_n - u^*\| + \|u_{n-1} - u^*\| + 2 \|u_n - u_{n-1}\|$. By (3.11) and (3.12), we have

$$
\|v_n - u^*\|^2 \leq \|u_n - u^*\|^2 + \alpha_n c_1 \|u_n - u_{n-1}\|. \tag{3.13}
$$

Noting that

$$
\|w_n - u^*\|^2 = \|w_n - v_n + v_n - u^*\|^2 \leq \|v_n - u^*\|^2 + 2 \langle v_n - w_n, u^* - w_n\rangle. \tag{3.14}
$$

By the definition of w_n and the projection property, we have $\langle w_n - v_n + \tau_n\eta_n, u^* - w_n\rangle \geq 0$. so $\langle \tau_n\eta_n, u^* - w_n\rangle \geq \langle v_n - w_n, u^* - w_n\rangle$. From (3.14) that

$$
\begin{aligned}
\|w_n - u^*\|^2 &\leq \|v_n - u^*\|^2 + 2 \langle \tau_n\eta_n, u^* - w_n\rangle \\
&= \|v_n - u^*\|^2 + 2 \langle \tau_n\eta_n, u^* - v_n\rangle + 2 \langle \tau_n\eta_n, v_n - w_n\rangle
\end{aligned}. \tag{3.15}
$$

It follows from $\eta_n \in \partial_2^{\lambda_n} f(v_n, v_n)$ that $f(v_n, u^*) - f(v_n, v_n) \geq \langle \eta_n, u^* - v_n \rangle - \lambda_n$. Then

$$f(v_n, u^*) + \lambda_n \geq \langle \eta_n, u^* - v_n \rangle. \tag{3.16}$$

Moreover, by Lemma 3.1, we get

$$\langle \tau_n \eta_n, v_n - w_n \rangle \leq \tau_n \|\eta_n\| \|v_n - w_n\| \leq \beta_n^2. \tag{3.17}$$

By (3.15)–(3.17), we have

$$\|w_n - u^*\|^2 \leq \|v_n - u^*\|^2 + 2\tau_n f(v_n, u^*) + 2\tau_n \lambda_n + 2\beta_n^2. \tag{3.18}$$

Combining (3.11), we get

$$\|w_n - u^*\|^2 \leq \|x_n - u^*\|^2 + 2\tau_n f(v_n, u^*) + 2\tau_n \lambda_n + 2\beta_n^2. \tag{3.19}$$

Since $u^* \in \Gamma$, then $u^* \in EP(f)$. And f is pseudomonotone on C with respect to every solution of $EP(f)$, we have $f(v_n, u^*) \leq 0$. By the definition of u_{n+1},we obtain

$$\|u_{n+1} - u^*\|^2 = \|a_n u_n + (1 - a_n) w_n - u^*\|^2 \leq a_n \|u_n - u^*\|^2 + (1 - a_n) \|w_n - u^*\|^2. \tag{3.20}$$

By the definition of y_n and (3.12) , we get

$$\|y_n - u^*\|^2 = \left\| J_r^{B_{i_n}} x_n - u^* \right\|^2 \leq \|x_n - u^*\|^2 \leq \|u_n - u^*\|^2 + \alpha_n c_1 \|u_n - u_{n-1}\|.$$

From (3.12), (3.19) and (3.20) that

$$\begin{aligned}
&\|u_{n+1} - u^*\|^2 \\
&\leq a_n \|u_n - u^*\|^2 + (1 - a_n) \left(\|x_n - u^*\|^2 + 2\tau_n f(v_n, u^*) + 2\tau_n \lambda_n + 2\beta_n^2 \right) \\
&\leq a_n \|u_n - u^*\|^2 + (1 - a_n) \left(\|u_n - u^*\|^2 + \alpha_n c_1 \|u_n - u_{n-1}\| + 2\tau_n f(v_n, u^*) + 2\tau_n \lambda_n + 2\beta_n^2 \right) \\
&= \|u_n - u^*\|^2 + (1 - a_n) \left[\alpha_n c_1 \|u_n - u_{n-1}\| + 2\tau_n f(v_n, u^*) + 2\tau_n \lambda_n + 2\beta_n^2 \right] \qquad (3.21) \\
&\leq \|u_n - u^*\|^2 + (1 - a_n) \alpha_n c_1 \|u_n - u_{n-1}\| + 2(1 - a_n) \tau_n \lambda_n + 2(1 - a_n) \beta_n^2 \\
&= \|u_n - u^*\|^2 + (1 - a_n) \left[\alpha_n c_1 \|u_n - u_{n-1}\| + 2\tau_n f(v_n, u^*) + 2\tau_n \lambda_n + 2\beta_n^2 \right], \qquad (3.22)
\end{aligned}$$

where $\Lambda_n = 2(1 - a_n) \left(\tau_n \lambda_n + \beta_n^2 \right)$. since $\tau_n = \frac{\beta_n}{\gamma_n}$ $\gamma_n = \max\{\rho_n, \|\eta_n\|\}$, then

$$\sum_{n=1}^{\infty} \tau_n \lambda_n = \sum_{n=1}^{\infty} \frac{\beta_n}{\gamma_n} \lambda_n \leq \sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n} \lambda_n < +\infty.$$

Noting $\sum_{n=1}^{\infty} \beta_n^2 < +\infty, 0 < a < a_n < b < 1$, we have $\sum_{n=1}^{\infty} \Lambda_n < 2(1 - a) \sum_{n=1}^{\infty} \left(\tau_n \lambda_n + \beta_n^2 \right) < +\infty$. By (3.1), we have $\alpha_n \|u_n - u_{n-1}\| \leq \bar{\alpha}_n \|u_n - u_{n-1}\| \leq \varepsilon_n$, noting $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, so

$$\sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\| < +\infty.$$

From Lemma 2.4 and (3.22), we can see that $\left\{\|u_n - u^*\|^2\right\}$ is convergent for all $u^* \in \Gamma$. Hence $\{u_n\}$ is bounded, consequently, so are the sequences $\{x_n\}, \{y_n\}, \{v_n\}$ and $\{w_n\}$.

Step 2. For any $u^* \in \Gamma$, $\limsup_{n\to\infty} f(v_n, u^*) = 0$. Indeed, from (3.21), we have

$$-2(1-a_n)\tau_n f(v_n, u^*) \leq \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2 + (1-a_n)\alpha_n c_1 \|u_n - u_{n-1}\| + \Lambda_n. \tag{3.23}$$

Consequently, $\sum_{n=1}^{\infty} -2(1-a_n)\tau_n f(v_n, u^*) < +\infty$. It follows from Assumption (B2) and the boundedness of $\{u_n\}$, we get that $\|\eta_n\|$ is bounded. Thus, for each $n \geq 1$, there is a constant $L > \rho$ such that $\|\eta_n\| \leq L$ then $\frac{\gamma_n}{\rho_n} = \max\left\{1, \frac{\|\eta_n\|}{\rho_n}\right\} \leq \frac{L}{\rho}$, so $\tau_n = \frac{\beta_n}{\gamma_n} > \frac{\rho}{L}\frac{\beta_n}{\rho_n}$. Since $u^* \in \Gamma$, it follows from the pseudomonotonicity of f that $-f(v_n, u^*) \geq 0$ and combine $0 < a < a_n < b < 1$, we have $\sum_{n=1}^{\infty}(1-b)\frac{\beta_n}{\rho_n}\left[-f(v_n, u^*)\right] < +\infty$. Since $\sum_{n=1}^{\infty}\frac{\beta_n}{\rho_n} = +\infty$, then $\limsup_{n\to\infty} f(v_n, u^*) = 0$.

Step 3. For any $u^* \in \Gamma$, let $\{v_{n_k}\}$ be a subsequence of $\{v_n\}$, such that $\limsup_{n\to\infty} f(v_n, u^*) = \lim_{j\to\infty} f(v_{n_k}, u^*)$, and v^* be a weak cluster point of $\{v_{n_k}\}$, then $v^* \in EP(f)$. Indeed, if $v_{n_k} \rightharpoonup v^* \ (k \to \infty)$, since $f(\cdot, u^*)$ is upper semi-continuous and by Step2, we have $f(v^*, u^*) \geq \limsup_{j\to\infty} f(v_{n_k}, u^*) = 0$. Since $u^* \in \Gamma$, and f is pseudomonotone, we have $f(v^*, u^*) \leq 0$, and so $f(v^*, u^*) = 0$. Again, by the pseudomonotonicity of f, $f(u^*, v^*) \leq 0$, and $f(u^*, v^*) \geq 0$, we obtain $f(u^*, v^*) = 0$. Then, $f(v^*, u^*) = f(u^*, v^*) = 0$. Thus, by the pseudomonotonicity of f we have $v^* \in EP(f)$.

Step 4. Every weak cluster point $\bar{u}$ of the sequence $\{u_n\}$ all belongs to $GSVIP\left(B_i, D_j\right)$. Let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$ such that $u_{n_k} \rightharpoonup \bar{u}$. Easy to know that $\sum_{n=1}^{\infty}\|x_n - u_n\| = \sum_{n=1}^{\infty}\alpha_n\|u_n - u_{n-1}\| < \infty$. Implying that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.24}$$

Therefore $x_{n_k} \rightharpoonup \bar{u}$, where $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. It follows from (3.10), (3.12), (3.18) and (3.20) that

$$\begin{aligned}
&\|u_{n+1} - u^*\|^2 \\
&= \|a_n u_n + (1-a_n) w_n - u^*\|^2 \\
&\leq a_n\|u_n - u^*\|^2 + (1-a_n)\|w_n - u^*\|^2 \\
&\leq a_n\|u_n - u^*\|^2 + (1-a_n)\left(\|v_n - u^*\|^2 + 2\tau_n f(v_n, u^*) + 2\tau_n\lambda_n + 2\beta_n^2\right) \\
&\leq a_n\|u_n - u^*\|^2 + (1-a_n)\|v_n - u^*\|^2 + \Lambda_n \\
&\leq a_n\|u_n - u^*\|^2 + (1-a_n)\left(\|x_n - u^*\|^2 \right.\\
&\qquad \left. -\delta_n\left(1 - \frac{\delta_n}{4}\right)\frac{\left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right)Ax_n\right\|^2\right)^2}{\|x_n - y_n + z_n\|^2}\right) + \Lambda_n \\
&\leq a_n\|u_n - u^*\|^2 + (1-a_n)\left(\|u_n - u^*\|^2 + \alpha_n c_1\|u_n - u_{n-1}\| \right.\\
&\qquad \left. -\delta_n\left(1 - \frac{\delta_n}{4}\right)\frac{\left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right)Ax_n\right\|^2\right)^2}{\|x_n - y_n + z_n\|^2}\right) + \Lambda_n
\end{aligned}$$

$$
\begin{aligned}
&= \|u_n - u^*\|^2 + (1-a_n)\,\alpha_n c_1 \|u_n - u_{n-1}\| \\
&\quad - (1-a_n)\,\delta_n \left(1 - \frac{\delta_n}{4}\right) \frac{\left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) A x_n\right\|^2\right)^2}{\|x_n - y_n + z_n\|^2} + \Lambda_n.
\end{aligned} \tag{3.25}
$$

Implying that

$$
\begin{aligned}
&(1-a_n)\,\delta_n \left(1 - \frac{\delta_n}{4}\right) \frac{\left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) A x_n\right\|^2\right)^2}{\|w_n - u_n + v_n\|^2} \\
&\leq \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2 + (1-a_n)\,\alpha_n c_1 \|u_n - u_{n-1}\| + \Lambda_n.
\end{aligned} \tag{3.26}
$$

Observe that

$$
\begin{aligned}
&(1-b) \sum_{n=1}^{\infty} \delta_n \left(1 - \tfrac{\delta_n}{4}\right) \frac{\left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) A x_n\right\|^2\right)^2}{\|w_n - u_n + v_n\|^2} \\
&\leq \|u_0 - u^*\|^2 + (1-a)\, c_1 \sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\| + \sum_{n=1}^{\infty} \Lambda_n < \infty.
\end{aligned}
$$

Thus

$$
\lim_{n\to\infty} \delta_n \left(1 - \tfrac{\delta_n}{4}\right) \frac{\left(\|x_n - y_n\|^2 + \left\|\left(I - J_\lambda^{D_{j_n}}\right) A x_n\right\|^2\right)^2}{\|x_n - y_n + z_n\|^2} = 0.
$$

Since $\{x_n + z_n - y_n\}$ is bounded, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim_{n\to\infty} \left\|\left(I - J_\lambda^{D_{j_n}}\right) A x_n\right\| = 0$. Thus $\lim_{n\to\infty} \left\|\left(I - J_r^{B_{i_n}}\right) x_n\right\| = 0$ and $\lim_{n\to\infty} \left\|\left(I - J_r^{D_{j_n}}\right) A x_n\right\| = 0$. Note that $J_r^{B_{i_n}}$ and $J_r^{D_{j_n}}$ are nonexpansive, then $\left(I - J_r^{B_{i_n}}\right)$ and $\left(I - J_r^{D_{j_n}}\right)$ are demiclosed at 0. Thus, it follows from $x_{n_k} \rightharpoonup \bar{u}$ that $\left(I - J_r^{B_{i_n}}\right)\bar{u} = 0$, and $\left(I - J_r^{D_{j_n}}\right) A\bar{u} = 0$ due to the linearity of A. That is, $\bar{u} \in \cap_{i=1}^{s} B_i^{-1}(0)$ and $A\bar{u} \in \cap_{j}^{t} D_j^{-1}(0)$. So $\bar{u} \in GSVIP\left(B_i, D_j\right)$. Noting that by Step 1, we can assume that $\lim_{n\to\infty} \|u_n - \bar{u}\| = c < +\infty$. From (3.11) and Lemma 3.1(2), we have

$$
\begin{aligned}
\|w_n - \bar{u}\| &\leq \|w_n - v_n\| + \|v_n - \bar{u}\| \\
&\leq \beta_n + \|x_n - \bar{u}\| \\
&= \|u_n + \alpha_n (u_n - u_{n-1}) - \bar{u}\| + \beta_n \\
&\leq \|u_n - \bar{u}\| + |\alpha_n| \|u_n - u_{n-1}\| + \beta_n.
\end{aligned}
$$

This means that

$$
\limsup_{n\to\infty} \|w_n - \bar{u}\| \leq \limsup_{n\to\infty} \left(\|u_n - \bar{u}\| + \alpha_n \|u_n - u_{n-1}\| + \beta_n\right) = c.
$$

Since $\lim_{n\to\infty} \|a_n (u_n - \bar{u}) + (1-a_n)(w_n - \bar{u})\| = \lim_{n\to\infty} \|u_{n+1} - \bar{u}\| = c$. By Lemma 2.5, we have

$$
\lim_{n\to\infty} \|w_n - u_n\| = 0. \tag{3.27}
$$

From Lemma 3.1 (2) and $\sum_{n=1}^{\infty} \beta_n^2 < +\infty$ that $\lim_{n\to\infty} \|v_n - w_n\| = 0$, then $\lim_{n\to\infty} \|v_n - u_n\| = 0$. Noting the fact that $\bar{u}$ is a weak cluster point of the sequence $\{u_n\}$, we are easy to see that $\bar{u}$ is also a weak cluster point of the sequence$\{v_n\}$, thus $\bar{u} \in EP(f)$, then $\bar{u} \in \Gamma$.

Step 5. Finally, we show that the sequence $\{u_n\}$ converges strongly to $\bar{u} \in \Gamma$. Indeed, combining (3.27) and the fact that $\bar{u}$ is a weak cluster point of the sequence $\{u_n\}$, we can see that $\bar{u}$ is also a weak cluster point of the sequence$\{w_n\}$. Suppose $w_{n_k} \rightharpoonup \bar{u}$, we get

$$\begin{aligned}\left\|u_{n_k+1} - P_\Gamma(u_{n_k+1})\right\|^2 &\le \left\|u_{n_k+1} - P_\Gamma(u_{n_k})\right\|^2 \\ &= \left\|a_{n_k}u_{n_k} + (1-a_{n_k})\,u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 \\ &\le a_{n_k}\left\|u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 + (1-a_{n_k})\left\|w_{n_k} - P_\Gamma(u_{n_k})\right\|^2.\end{aligned} \tag{3.28}$$

Observe that

$$\begin{aligned}\left\|w_{n_k} - P_\Gamma(u_{n_k})\right\|^2 &= \left\|w_{n_k} - u_{n_k} + u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 \\ &= \left\|w_{n_k} - u_{n_k}\right\|^2 - \left\|u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 - 2\left\langle w_{n_k} - P_\Gamma(u_{n_k}), P_\Gamma(u_{n_k}) - u_{n_k}\right\rangle.\end{aligned} \tag{3.29}$$

By (3.28) and (3.29), we have

$$\begin{aligned}&\left\|u_{n_k+1} - P_\Gamma(u_{n_k+1})\right\|^2 \\ &\le a_{n_k}\left\|u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 + (1-a_{n_k})\Big(\left\|w_{n_k} - u_{n_k}\right\|^2 - \left\|u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 \\ &\quad -2\left\langle w_{n_k} - P_\Gamma(u_{n_k}), P_\Gamma(u_{n_k}) - u_{n_k}\right\rangle) \\ &= (2a_{n_k}-1)\left\|u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 + (1-a_{n_k})\left\|w_{n_k} - u_{n_k}\right\|^2 \\ &\quad - 2(1-a_{n_k})\left\langle w_{n_k} - P_\Gamma(u_{n_k}), P_\Gamma(u_{n_k}) - u_{n_k}\right\rangle \\ &= (2a_{n_k}-1)\left\|u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 + (1-a_{n_k})\left\|w_{n_k} - u_{n_k}\right\|^2 \\ &\quad - 2(1-a_{n_k})\left\langle w_{n_k} - \bar{u}, P_\Gamma(u_{n_k}) - u_{n_k}\right\rangle \\ &\quad - 2(1-a_{n_k})\left\langle \bar{u} - P_\Gamma(u_{n_k}), P_\Gamma(u_{n_k}) - u_{n_k}\right\rangle.\end{aligned} \tag{3.30}$$

Since $\bar{u} \in \Gamma$, we have $\left\langle \bar{u} - P_\Gamma(u_{n_k}), P_\Gamma(u_{n_k}) - u_{n_k}\right\rangle \ge 0$, Also, observe that the sequence $\{u_{n_k}\}$ is bounded, then so is $\{u_{n_k} - P_\Gamma(u_{n_k})\}$. It follows from $\lim_{n\to\infty}\left\|w_{n_k} - u_{n_k}\right\| = 0, \lim_{n\to\infty} a_{n_k} = \frac{1}{2}$ and (3.30), we have

$$\lim_{n\to\infty}\left\|u_{n_k+1} - P_\Gamma(u_{n_k+1})\right\| = 0. \tag{3.31}$$

Next, we show that $\{P_\Gamma(u_{n_k})\}$ is a Cauchy sequence. Indeed, for any $m > k$, we obtain

$$\begin{aligned}&\left\|P_\Gamma(u_{n_m}) - P_\Gamma(u_{n_k})\right\|^2 = \left\|P_\Gamma(u_{n_m}) - u_{n_m} + u_{n_m} - P_\Gamma(u_{n_k})\right\|^2 \\ &= 4\left\|\left(\frac{1}{2}(P_\Gamma(u_{n_m}) - u_{n_m}) + \frac{1}{2}(u_{n_m} - P_\Gamma(u_{n_k}))\right)\right\|^2 \\ &= 2\left\|P_\Gamma(u_{n_m}) - u_{n_m}\right\|^2 + 2\left\|u_{n_m} - P_\Gamma(u_{n_k})\right\|^2 \\ &\quad - 4\left\|u_{n_m} - \frac{1}{2}(P_\Gamma(u_{n_m}) + P_\Gamma(u_{n_k}))\right\|^2 \\ &\le 2\left\|P_\Gamma(u_{n_m}) - u_{n_m}\right\|^2 + 2\left\|u_{n_m} - P_\Gamma(u_{n_k})\right\|^2 - 4\left\|u_{n_m} - P_\Gamma(u_{n_m})\right\|^2 \\ &= 2\left\|P_\Gamma(u_{n_k}) - u_{n_m}\right\|^2 - 2\left\|u_{n_m} - P_\Gamma(u_{n_m})\right\|^2.\end{aligned} \tag{3.32}$$

Set $u^* = P_\Gamma(u_{n_k})$ in (3.22), we have

$$\begin{aligned}\left\|u_{n_m} - P_\Gamma(u_{n_k})\right\|^2 &\le \left\|u_{n_m-1} - P_\Gamma(u_{n_k})\right\|^2 + (1-a_{n_m-1})\,\alpha_{n_m-1}c_1\left\|u_{n_m-1} - u_{n_m-2}\right\| + \Lambda_{n_m-1} \\ &\;\;\vdots \\ &\le \left\|u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 + \sum_{i=n_k}^{n_m-1}(1-a_i)\alpha_i c_1\left\|u_i - u_{i-1}\right\| + \sum_{i=n_k}^{n_m-1}\Lambda_i.\end{aligned} \tag{3.33}$$

From (3.32) and (3.33) that

$$\left\|P_\Gamma(u_{n_m}) - P_\Gamma(u_{n_k})\right\|^2 \le 2\left\|u_{n_k} - P_\Gamma(u_{n_k})\right\|^2 + 2\sum_{i=n_k}^{n_m-1}(1-a_i)\alpha_i c_1\|u_i - u_{i-1}\| + 2\sum_{i=n_k}^{n_m-1}\Lambda_i - 2\left\|u_{n_m} - P_\Gamma(u_{n_m})\right\|^2.$$

It follows from (3.31) and the fact that $\lim_{k\to\infty}\sum_{i=n_k}^{n_m-1}\Lambda_i = 0$ and $\lim_{k\to\infty}\sum_{i=n_k}^{n_m-1}(1-a_i)\alpha_i c_1\|u_i - u_{i-1}\| = 0$, we obtain that $\{P_\Gamma(u_{n_k})\}$ is a Cauchy sequence. Hence $\{P_\Gamma(u_{n_k})\}$ strongly converges to some $u \in \Gamma$. Noting $\lim_{k\to\infty}\left\|u_{n_k+1} - P_\Gamma(u_{n_k+1})\right\| = 0$, we know that $\{u_{n_k}\}$ also strongly converges to $u \in \Gamma$. Thus $\lim_{n\to\infty} u_n = \bar{u}$, which completing the proof.

As the consequences of Theorem 3.1 with suitable choices of $B_i, D_j (i \in \{1,2,\cdots,s\}, j \in \{1,2,\cdots,t\})$ and f, we derive several interesting corollaries as follows.

Corollary 3.1. Suppose Assumptions (A1–A4) hold and let $s = 1, t = 1$ in (A2), then, the sequence $\{x_n\}$ generated by Algorithm 3.3 strongly converges to a solution of Problem (1.8).

Corollary 3.2. Suppose Assumptions (A1–A4) hold, then the sequence $\{x_n\}$ generated by Algorithm 3.4 strongly converges to a solution of Problem (1.9).

Corollary 3.3. Suppose Assumptions (A1–A4) hold, then the sequence $\{x_n\}$ generated by Algorithm 3.5 strongly converges to a solution of Problem (1.5).

Corollary 3.4. Suppose Assumptions (A1–A4) hold, then the sequence $\{x_n\}$ generated by Algorithm 3.6 strongly converges to a solution of Problem (1.4).

Remark 3.1. (i) Suppose $\alpha_n = 0$ in Algorithm 3.1, then Algorithm 3.1 reduces to self-adaptive viscosity-type Algorithm 3.2 for solving Problem (1.7).

(ii) Suppose $s = 1, t = 1$, then Algorithm 3.1 reduces to Algorithm 3.3 for solving Problem (1.8).

(iii) Suppose $B_i = N_{C_i}(i = 1,2,\cdots,s), D_j = N_{Q_j}(i = 1,2,\cdots,t)$ in Problem (1.7), then Algorithm 3.1 reduces to Algorithm 3.4 for solving Problem (1.9).

(iv) Suppose $f = 0$ in Problem (1.7), then Problem (1.7) reduces to Problem (1.4) studied by Ogbuisi et al. in [18] and Algorithm 3.1 reduces to Algorithm 3.5 for solving Problem (1.4). So our Algorithm 3.1 and Theorem 3.1 generalize the corresponding results in [18].

(v) Suppose $B_i = 0, D_j = 0$ in Problem (1.7), then Problem (1.7) reduces to Problem (1.5) studied by Santos et al. in [27] and Algorithm 3.1 reduces to Algorithm 3.6 for solving Problem (1.5). So our Algorithm 3.1 and Theorem 3.1 generalize the corresponding results in [27].

At last, we give two examples to illustrate the validity of our considered common solution Problem (1.7). In two examples, we take $s = 1, t = 1$ in Problem (1.7).

Example 3.1. Let $H_1 = H_2 = R^2, C = \left\{u \in R^2 \,|-10e_1 \le u \le 10e_1\right\}, e_1 = (1,1)$. We define the operators $B : H_1 \to 2^{H_1}, D : H_2 \to 2^{H_2}, A : H_1 \to H_2$ by

$$B\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, D\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, A\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix},$$

respectively. Define mapping $f(u,v) = u_1^5(v_1 - u_1) + u_2^3(v_2 - u_2), \forall u, v \in C$. Let us observe that A1–A4 hold and A is bounded linear mapping. In addition, $u^* = (0,0)$ is the unique solution of $SVIP(B,D)$. Furthermore, $EP(f)$has a unique solution $u^* = (0,0)$. Since $f(v,u^*) = -v_1^6 - v_2^4 \le 0$ for all $y \in C$ and $f(u^*,\bar{u}) = 0 = f(\bar{u},u^*) = -\bar{u}_1^6 - \bar{u}_2^4$, which implies $\bar{u} = (0,0) \in EP(f)$. Hence,

$\Gamma = SVIP(B, D) \cap EP(f) = \{(0, 0)\}$.

Example 3.2. Let $H_1 = R^2, H_2 = R^3, C = \left\{u \in R^2_+ \,|u_1 + u_2 = 1\right\} \subset H_1$. We define $B_1 : R^2 \to R^2, B_2 : R^3 \to R^3$ be

$$B_1 \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_2 \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix},$$

respectively. Let $A = \begin{bmatrix} 2 & -4 \\ -4 & 2 \\ 2 & -4 \end{bmatrix}$. We consider the equilibrium problem with the bifunction $f(u, v) = 2|v_1| - |u_1| + 2v_2^2 - u_2^2, \forall v \in C$. Suppose A1-A4 hold, the optimal point of $EP(f)$ is $u^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the partial subdifferential of f is given by

$$\partial_2 f(u, u) = \begin{cases} (2, 4u_2) \, \mathrm{if} \mathrm{u}_1 > 0, \\ ([-2, 2], 4u_2) \, \mathrm{if} \mathrm{u}_1 = 0, \\ (-2, 4u_2) \, \mathrm{if} \mathrm{u}_1 < 0. \end{cases}$$

Furthermore, we aim to find $u^* = \left(u_1^*, u_2^*\right) \in R^2$ such that $B_1(u^*) = (0, 0), B_2(Au^*) = (0, 0, 0)$. Then, we can easy to see that $x^* = \left(\frac{1}{2}, \frac{1}{2}\right) \in SVIP(B_1, B_2)$. Hence, $\Gamma = SVIP(B_1, B_2) \cap EP(f) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$.

4. Conclusions

In this paper, a new inertial form algorithm is introduced to approximate the common solutions of the generalized split variational inclusion problem and paramonotone equilibrium problem in real Hilbert spaces. The design of the algorithm is self-adaptive, the inertial term can speed up its convergence, and the strong convergence analysis does not require a prior estimate of the norm of bounded operators. Under the assumption of generalized monotonicity of the correlation mappings, we prove the strong convergence of our iterative algorithms. The results presented here improve and generalize many known results in [18, 27].

It should be noted that the way of choosing the inertial parameter α_k in our algorithm 3.1 is known as the on-line rule. As part of our future project, following the method in [47], we consider a strong convergence of our proposed algorithm under some conditions on the iterative parameter without on-line rule assumption.

Author contributions

Yali Zhao: Supervision, Conceptualization, Writing-review & editing; Qixin Dong: Writing-review & editing, Project administration; Xiaoqing Huang: Writing-original draft, Formal analysis. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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