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*Research article*

# Geometric applications for meromorphic functions involving $\mathfrak{q}$-linear operators

**Ekram E. Ali**[1,2,*], **Rabha M. El-Ashwah**[3] **and Abeer M. Albalahi**[1]

[1] Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 81451, Saudi Arabia

[2] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said 42521, Egypt

[3] Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt

* **Correspondence:** Email: e.ahmad@uoh.edu.sa.

**Abstract:** This paper investigated the $\mathfrak{q}$-analogue of the multiplier-Ruscheweyh operator acting on meromorphic analytic functions, denoted by $\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)$. By applying tools from $\mathfrak{q}$-calculus together with the principle of subordination, we developed several analytical results that deepened the understanding of geometric function theory (GFT) in the setting of meromorphic functions. The study focused on constructing new subclasses of meromorphic univalent functions associated with the operator $\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)$, characterized by $\mathfrak{q}$-starlikeness, $\mathfrak{q}$-convexity, and related geometric classes. Various inclusion relationships, differential inequalities, and integral preservation properties were examined to establish the structural behavior of these families of functions. The findings generalized and unified several existing results in the literature concerning different operators and extended their applications to broader contexts within meromorphic function theory with $\mathfrak{q}$-calculus operator.

**Keywords:** analytic function; meromorphic function; $\mathfrak{q}$-difference operator; $\mathfrak{q}$-analogue multiplier Catas operator; $\mathfrak{q}$-analogue of Ruscheweyh operator
**Mathematics Subject Classification:** 26A33, 30C45, 30C50, 30C80

## 1. Introduction

Numerous fields of mathematics and science including complex analysis, hypergeometric series, particle physics, and most importantly geometric function theory (GFT), have found extensive uses for the idea of $\mathfrak{q}$-calculus operators. Ismail and his associates' introduction of the idea of $\mathfrak{q}$-starlike functions in 1990 [1] was a significant turning point in this direction and signaled the start of a new field of study in GFT.

Jackson [2,3] established the foundation for $\mathfrak{q}$-analogues of classical operators by introducing the $\mathfrak{q}$-differential and $\mathfrak{q}$-integral operators and proving their applicability to the study of geometric functions.

Symmetric quantum (or $\mathfrak{q}$-) calculus applies $\mathfrak{q}$-calculus concepts to define new families of multivalent functions in GFT, which studies the geometric properties of analytic functions. This approach uses symmetric quantum difference operators to generate new subclasses of functions with properties like $\mathfrak{q}$-starlikeness and $\mathfrak{q}$-convexity. Research in this area focuses on using these operators to establish necessary and sufficient conditions for function classes, explore properties like compactness and coefficient bounds, and generalize existing results in GFT for both analytic functions. $\mathfrak{q}$-difference equations are an important aspect of mathematical analysis, particularly in the field known as GFT. Quantum calculus is frequently used in mathematical disciplines because of its numerous possible applications in basic hypergeometric functions [4], orthogonal polynomials [5, 6], combinatorics [7], and number theory [8]. Several fundamental ideas in $\mathfrak{q}$-calculus [9, 10] demonstrate how it is integrated into mathematical ideas. Srivastava's 1989 [11, chapter 25, P. 329] offered the appropriate foundation for integrating the concepts of $\mathfrak{q}$-calculus into GFT. Several researchers have studied different $\mathfrak{q}$-calculus applications for subclasses of analytic functions (see [12–17]).

We say $\mathfrak{f}$ and $\mathfrak{l}$, are analytic functions and are subordinated, then the result is $\mathfrak{f} \prec \mathfrak{l}$, which is defined as

$$\mathfrak{f}(\tau) = \mathfrak{l}(\chi(\tau)),$$

where $\chi(\tau)$ is the Schwartz function in $\mathbb{U}$ (see [18, 19]).

Let $\Sigma$ be the class of meromorphic analytic functions in punctured unit disk

$$\mathbb{U}^* = \mathbb{U} \setminus \{0\} = \{\tau : \tau \in \mathbb{C} \quad \text{and} \quad 0 < |\tau| < 1\},$$

with:

$$\mathfrak{f}(\tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty} a_\kappa \tau^\kappa. \tag{1.1}$$

Let ST, CV, K and $\mathfrak{Q}$ represent the corresponding subclasses from the univalent class $\Sigma$ that are starlike, convex, close-to-convex, and quasi-convex functions.

For $\mathfrak{f}$ given by (1.1) and $\hbar$ given by

$$\hbar(\tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty} b_\kappa \tau^\kappa, \quad \tau \in \mathbb{U}^*,$$

the well-known *convolution product* is

$$(\mathfrak{f} * \hbar)(\tau) := (\hbar * \mathfrak{f})(\tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty} a_\kappa b_\kappa \tau^\kappa =: (\hbar * \mathfrak{f})(\tau).$$

A meromorphic function $\mathfrak{f} \in \Sigma$ in $\mathbb{U}^*$ is a meromorphically starlike function of order $\alpha (0 \leq \alpha < 1)$ if

$$-Re\left\{\frac{\tau\mathfrak{f}'(\tau)}{\mathfrak{f}(\tau)}\right\} > \alpha \quad (\tau \in U^*). \tag{1.2}$$

Meromorphic convex functions $\mathfrak{f} \in \Sigma$ in $\mathbb{U}^*$ is a meromorphically convex function of order $\alpha (0 \leq \alpha < 1)$ if

$$-Re\left[1 + \frac{\tau\mathfrak{f}''(\tau)}{\mathfrak{f}'(\tau)}\right] > \alpha, \ \tau \in \mathbb{U}^*.$$

These classes' foundational concepts began in 1959 when Cluin [20] investigated meromorphic schlicht functions. Meromorphic starlike functions were defined by Pommerenke [21] in 1963. A meromorphic convex function was introduced by Miller [22] in 1970. He also looked at certain generalized coefficient problems and other helpful properties of meromorphic convex functions.

**Definition 1.** [2, 3] The $\mathfrak{q}$-*derivative*, or the *Jackson derivative* of a function $\mathfrak{f}$ is defined by

$$\begin{aligned} D_{\mathfrak{q}}\mathfrak{f}(\tau) &:= \mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\tau) = \frac{\mathfrak{f}(\mathfrak{q}\tau) - \mathfrak{f}(\tau)}{(\mathfrak{q}-1)\tau} \\ &= -\frac{1}{\mathfrak{q}\tau^2} + \sum_{\kappa=1}^{\infty} [\kappa]_{\mathfrak{q}} a_\kappa \tau^{\kappa-1} \ (\mathfrak{q} \in (0,1),\ \tau \in \mathbb{U}^*), \end{aligned}$$

where

$$[\kappa]_{\mathfrak{q}} = \begin{cases} \sum\limits_{j=0}^{\kappa-1} \mathfrak{q}^\kappa = 1 + \mathfrak{q} + \mathfrak{q}^2 + .... + \mathfrak{q}^{\kappa-1}, & \kappa \in \mathbb{N} = \{1,2,3,...\}, \\ 0, & \kappa = 0, \end{cases}$$

$$[\kappa]_{\mathfrak{q}}! = \begin{cases} [\kappa]_{\mathfrak{q}} [\kappa-1]_{\mathfrak{q}} \ldots\ldots\ldots [2]_{\mathfrak{q}} [1]_{\mathfrak{q}}, & \kappa = 1,2,3,..., \\ 1, & \kappa = 0, \end{cases}$$

and

$$\lim_{\mathfrak{q}\to 1-} \mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\tau) = \mathfrak{f}'(\tau).$$

The $\mathfrak{q}$-difference operator is subject to the following basic laws:

$$\begin{aligned} \mathfrak{d}_{\mathfrak{q}}\left(c\varpi_1(\tau) \pm d\varpi_2(\tau)\right) &= c\mathfrak{d}_{\mathfrak{q}}\varpi_1(\tau) \pm d\mathfrak{d}_{\mathfrak{q}}\varpi_2(\tau), \\ \mathfrak{d}_{\mathfrak{q}}\left(\varpi_1(\tau)\varpi_2(\tau)\right) &= \varpi_1(\mathfrak{q}\tau)\mathfrak{d}_{\mathfrak{q}}\left(\varpi_2(\tau)\right) + \varpi_2(\tau)\mathfrak{d}_{\mathfrak{q}}\left(\varpi_1(\tau)\right), \\ \mathfrak{d}_{\mathfrak{q}}\left(\frac{\varpi_1(\tau)}{\varpi_2(\tau)}\right) &= \frac{\mathfrak{d}_{\mathfrak{q}}\left(\varpi_1(\tau)\right)\varpi_2(\tau) - \varpi_1(\tau)\mathfrak{d}_{\mathfrak{q}}\left(\varpi_2(\tau)\right)}{\varpi_2(\mathfrak{q}\tau)\varpi_2(\tau)}, \ \varpi_2(\mathfrak{q}\tau)\varpi_2(\tau) \neq 0, \\ \mathfrak{d}_{\mathfrak{q}}\left(\log\varpi_1(\tau)\right) &= \frac{\ln\mathfrak{q}}{\mathfrak{q}-1}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\varpi_1(\tau)\right)}{\varpi_1(\tau)}, \end{aligned}$$

where $\varpi_1, \varpi_2$ are analytic functions and $c$ and $d$ are real or complex constants.

Also, Jackson [2] introduced the $\mathfrak{q}$-integral as

$$\int_0^\tau \mathfrak{f}(\tau) d_{\mathfrak{q}}\tau = (1-\mathfrak{q})\tau \sum_{\mathfrak{n}=0}^{\infty} \mathfrak{q}^{\mathfrak{n}} \mathfrak{f}(\mathfrak{q}^{\mathfrak{n}}\tau), \tag{1.3}$$

we note that

$$\int_0^\tau \mathfrak{f}(\tau) d_{\mathfrak{q}}\tau = \int_0^\tau \tau^{\mathfrak{n}} d_{\mathfrak{q}}\tau = \frac{1}{[\mathfrak{n}+1]_{\mathfrak{q}}}\tau^{\mathfrak{n}+1} \qquad (\mathfrak{n} \neq -1),$$

and

$$\lim_{\mathfrak{q}\to 1-} \int_0^\tau \mathfrak{f}(\tau) d_{\mathfrak{q}}\tau = \lim_{\mathfrak{q}\to 1-} \frac{1}{[\mathfrak{n}+1]_{\mathfrak{q}}}\tau^{\mathfrak{n}+1} = \frac{1}{\mathfrak{n}+1}\tau^{\mathfrak{n}+1} = \int_0^\tau \mathfrak{f}(\tau) d\tau,$$

where $\int_0^\tau \mathrm{f}(\tau) d\tau$ denotes the ordinary integral.

For $\varepsilon > -1$, define the meromorphic $\mathfrak{q}$-analogue of Ruscheweyh operator $\mathfrak{R}_{\mathfrak{q}}^{\varepsilon}\colon \Sigma \to \Sigma$ by Hadmard product (convolution)

$$\mathfrak{R}_{\mathfrak{q}}^{\varepsilon}\mathfrak{f}(\tau) = \mathfrak{f}(\tau) * \phi(\mathfrak{q}, \varepsilon + 1; \tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty} \frac{[\kappa + \varepsilon + 1]_{\mathfrak{q}}}{[\varepsilon]_{\mathfrak{q}}![\kappa + 1]_{\mathfrak{q}}!} a_{\kappa}\tau^{\kappa}, \quad (\varepsilon \geq 0, 0 < \mathfrak{q} < 1),$$

where

$$\phi(\mathfrak{q}, \varepsilon; \tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty} \frac{[\kappa + \varepsilon + 1]_{\mathfrak{q}}}{[\varepsilon]_{\mathfrak{q}}![\kappa + 1]_{\mathfrak{q}}!}\tau^{\kappa},$$

was introduced and studied by Ahmad and Arif [23].

The $\mathfrak{q}$-analogue of the Ruscheweyh operator is a natural extension of the classical Ruscheweyh derivative obtained by replacing ordinary derivatives and convolutions with their $\mathfrak{q}$-calculus counterparts.

This is precisely the $\mathfrak{q}$-extension of the meromorphic Ruscheweyh operator; it reduces to the classical case as $\mathfrak{q} \to 1^-$:

$$\lim_{\mathfrak{q}\to 1^-} \phi(\mathfrak{q}, \varepsilon + 1; \xi) = \frac{1}{\xi(1-\xi)^{\varepsilon+1}}, \qquad \lim_{\mathfrak{q}\to 1^-} (\mathfrak{R}_{\mathfrak{q}}^{\varepsilon}\mathfrak{f}) = \mathfrak{f} * \frac{1}{\xi(1-\xi)^{\varepsilon+1}}.$$

Useful identities include

$$\mathfrak{R}_{\mathfrak{q}}^{0}\mathfrak{f} = \mathfrak{f}, \qquad \mathfrak{R}_{\mathfrak{q}}^{1}\mathfrak{f} - [2]_{\mathfrak{q}}\mathfrak{R}_{\mathfrak{q}}^{0}\mathfrak{f}(\mathfrak{q}\tau) = \tau D_{\mathfrak{q}}\mathfrak{f}(\tau),$$

and, for $m \in \mathbb{N}$,

$$\mathfrak{R}_{\mathfrak{q}}^{m}\mathfrak{f}(\tau) = \frac{\tau^{-1}}{[m]_{\mathfrak{q}}!} D_{\mathfrak{q}}(\tau^{m+1}\mathfrak{f}(\tau)).$$

For $\mathfrak{f}(\tau) \in \Sigma$, $r \in \mathbb{N}_0, \varrho, \varkappa \geq 0$, $0 < \mathfrak{q} < 1$ let:

$$\begin{aligned}
\mathfrak{D}_{\varrho,\varkappa}^{0,\mathfrak{q}}\mathfrak{f}(\tau) &=: \mathfrak{D}_{\varrho,\varkappa}^{\mathfrak{q}}\mathfrak{f}(\tau) = \mathfrak{f}(\tau),\\
\mathfrak{D}_{\varrho,\varkappa}^{1,\mathfrak{q}}\mathfrak{f}(\tau) &= (1-\varkappa)\,\mathfrak{f}(\tau) + \frac{\varkappa}{[\varrho]_{\mathfrak{q}}\tau^{\varrho}}\mathfrak{D}_{\mathfrak{q}}\left(\tau^{\varrho+1}\mathfrak{f}(\tau)\right)\\
&= \frac{1}{\tau} + \sum_{\kappa=1}^{\infty}\left(\frac{[\varrho]_{\mathfrak{q}} + \varkappa([\kappa+\varrho+1]_{\mathfrak{q}} - [\varrho]_{\mathfrak{q}})}{[\varrho]_{\mathfrak{q}}}\right) a_{\kappa}\tau^{\kappa},\\
&\cdots\cdots\cdots\\
\mathfrak{D}_{\varrho,\varkappa}^{r,\mathfrak{q}}\mathfrak{f}(\tau) &= (1-\varkappa)\,\mathfrak{D}_{\varrho,\varkappa}^{r-1,\mathfrak{q}}\mathfrak{f}(\tau) + \frac{\varkappa}{[\varrho]_{\mathfrak{q}}\tau^{\varrho}}\mathfrak{D}_{\mathfrak{q}}\left(\tau^{\varrho+1}\mathfrak{D}_{\varrho,\varkappa}^{r-1,\mathfrak{q}}\mathfrak{f}(\tau)\right),\ r \geq 1,
\end{aligned}$$

and

$$\mathfrak{D}_{\varrho,\varkappa}^{r,\mathfrak{q}}\mathfrak{f}(\tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty}\left(\frac{[\varrho]_{\mathfrak{q}} + \varkappa([\kappa+\varrho+1]_{\mathfrak{q}} - [\varrho]_{\mathfrak{q}})}{[\varrho]_{\mathfrak{q}}}\right)^{r} a_{\kappa}\tau^{\kappa}, \quad (r \in \mathbb{N}_0, \varrho, \varkappa \geq 0, 0 < \mathfrak{q} < 1). \tag{1.4}$$

Setting

$$\mathfrak{f}_{\mathfrak{q},\varkappa,\varrho}^{r}(\tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty}\left(\frac{[\varrho]_{\mathfrak{q}} + \varkappa([\kappa+\varrho+1]_{\mathfrak{q}} - [\varrho]_{\mathfrak{q}})}{[\varrho]_{\mathfrak{q}}}\right)^{r}\tau^{\kappa}.$$

Now we define a new function $\mathfrak{f}^{r,\varepsilon}_{\mathfrak{q},\varkappa,\varrho}(\tau)$ in terms of the Hadamard product (or convolution) by

$$\mathfrak{f}^{r}_{\mathfrak{q},\varkappa,\varrho}(\tau) * \mathfrak{f}^{r,\varepsilon}_{\mathfrak{q},\varkappa,\varrho}(\tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty} \frac{[\kappa+\varepsilon+1]_{\mathfrak{q}}!}{[\varepsilon]_{\mathfrak{q}}![\kappa+1]_{\mathfrak{q}}!}\tau^{\kappa}.$$

Next, we provide the operator

$$\mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}(\varkappa,\varrho)f(\tau) : \Sigma \to \Sigma,$$

which is primarily inspired by the $\mathfrak{q}$-analogue of the Ŕuscheweyh operator and the $\mathfrak{q}$-analogue multiplier defined by

$$\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) = \mathfrak{f}^{r,\varepsilon}_{\mathfrak{q},\varkappa,\varrho}(\tau) * \mathfrak{f}(\tau), \tag{1.5}$$

where $r \in \mathbb{N}_0, \varrho, \varkappa, \varepsilon \geq 0, 0 < \mathfrak{q} < 1$. For $\mathfrak{f} \in \Sigma$; and (1.5) then

$$\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) = \frac{1}{\tau} + \sum_{\kappa=1}^{\infty} \left(\frac{[\varrho]_{\mathfrak{q}}}{[\varrho]_{\mathfrak{q}} + \varkappa([\kappa+\varrho+1]_{\mathfrak{q}} - [\varrho]_{\mathfrak{q}})}\right)^{r} \frac{[\kappa+\varepsilon+1]_{\mathfrak{q}}!}{[\varepsilon]_{\mathfrak{q}}![\kappa+1]_{\mathfrak{q}}!} a_{\kappa}\tau^{\kappa}. \tag{1.6}$$

By using (1.6) we get

$$\varkappa\mathfrak{q}^{\varrho+1}\tau\partial_{\mathfrak{q}}(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)) = [\varrho]_{\mathfrak{q}}\,\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) - (\varkappa\mathfrak{q}^{\varrho} + [\varrho]_{\mathfrak{q}})\,\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau), \quad \varkappa > 0, \tag{1.7}$$

$$\mathfrak{q}^{\varepsilon+2}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right) = [\varepsilon+1]_{\mathfrak{q}}\,\mathfrak{D}^{r}_{\varepsilon+1,\mathfrak{q}}(\varkappa,\varrho)\mathfrak{f}(\tau) - [\varepsilon+2]_{\mathfrak{q}}\,\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau). \tag{1.8}$$

We note that:

($i$) If $r = 0$ and $\mathfrak{q} \to 1^{-}$ we obtain $\mathfrak{R}^{\varepsilon}\mathrm{f}(\tau)$ is Ŕusscheweyh differential operator [24];

($ii$) If we set $\varepsilon = 0$ and $\mathfrak{q} \to 1^{-}$ we get $\mathfrak{D}_{r}(x,\varrho)\mathfrak{f}(\tau)$. It was introduced by Bulboaca et al [25] and El-Ashwah [26] with ($p = 1$).

We also see:

($i$) $\mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}(1,\varrho)\mathfrak{f}(\tau) = \mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}(\varrho)\mathfrak{f}(\tau)$,

$$\mathfrak{f}(\tau) \in \mathbf{\Sigma} : \mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}(\varrho)\mathfrak{f}(\tau) = \frac{1}{\tau} + \sum_{\kappa=2}^{\infty} \left(\frac{[\varrho]_{\mathfrak{q}}}{[\kappa+\varrho+1]_{\mathfrak{q}}}\right)^{r} \frac{[\kappa+\varepsilon+1]_{\mathfrak{q}}!}{[\varepsilon]_{\mathfrak{q}}![\kappa+1]_{\mathfrak{q}}!} a_{\kappa}\tau^{\kappa},$$
$$r \in \mathbb{N}_0,\ \varepsilon \geq 0, 0 < \mathfrak{q} < 1, \tau \in \mathbb{U}^{*}.$$

($ii$) $\mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}(1,1)\mathfrak{f}(\tau) = \mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}\mathfrak{f}(\tau)$,

$$\mathfrak{f}(\tau) \in \mathbf{\Sigma} : \mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}\mathfrak{f}(\tau) = \frac{1}{\tau} + \sum_{\kappa=2}^{\infty} \left(\frac{1}{[\kappa+2]_{\mathfrak{q}}}\right)^{r} \frac{[\kappa+\varepsilon+1]_{\mathfrak{q}}!}{[\varepsilon]_{\mathfrak{q}}![\kappa+1]_{\mathfrak{q}}!} a_{\kappa}\tau^{\kappa},$$
$$r \in \mathbb{N}_0, \varrho > 0, \varepsilon \geq 0, 0 < \mathfrak{q} < 1, \tau \in \mathbb{U}^{*}.$$

($iii$) $\mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}(\varkappa,1)\mathfrak{f}(\tau) = \mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}(\varkappa)\mathfrak{f}(\tau)$,

$$\mathfrak{f}(\tau) \in \mathbf{\Sigma} : \mathfrak{D}^{r}_{\varepsilon,\mathfrak{q}}(\varkappa)\mathfrak{f}(\tau) = \frac{1}{\tau} + \sum_{\kappa=2}^{\infty} \left(\frac{1}{1 + \varkappa([\kappa+2]_{\mathfrak{q}} - 1)}\right)^{r} \frac{[\kappa+\varepsilon+1]_{\mathfrak{q}}!}{[\varepsilon]_{\mathfrak{q}}![\kappa+1]_{\mathfrak{q}}!} a_{\kappa}\tau^{\kappa},$$
$$r \in \mathbb{N}_0, \varkappa > 0, \varepsilon \geq 0, 0 < \mathfrak{q} < 1, \tau \in \mathbb{U}^{*}.$$

Let $\Phi$ be the class of analytic and univalent convex functions $\varphi$, with $\varphi(0) = 1$, and $Re\varphi(\tau) > 0$ in $\mathbb{U}$.

**Definition 2.** $\mathfrak{f} \in \Sigma$ is definitely in the class $ST_{\mathfrak{q}}(\varphi)$ if it satisfies

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}(\tau)\right)}{\mathfrak{f}(\tau)} \prec \varphi(\tau),$$

*where* $\mathfrak{d}_{\mathfrak{q}}$ *is the* $\mathfrak{q}$*-difference operator.*

Analogously, $\mathfrak{f} \in \Sigma$ is definitely in the class $\mathrm{CV}_{\mathfrak{q}}(\varphi)$ if

$$-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}(\tau)\right) \in \mathrm{ST}_{\mathfrak{q}}(\varphi). \tag{1.9}$$

For $\mathfrak{q} \to 1^-$ we obtain a class of starlike meromorphic functions (see [21]), and a class of convex meromorphic functions (see [22]).

By using the operators defined above, we determine the next part.

**Definition 3.** Suppose that $\mathfrak{f} \in \Sigma$, $r \in \mathbb{N}_0$, and $\varkappa, \varrho > 0, \varepsilon \geq 0,\ 0 < \mathfrak{q} < 1$. Then

$$\mathfrak{f} \in \mathrm{ST}^r_{\varepsilon,\mathfrak{q}}(\varkappa,\varrho)\left(\varphi\right) \Leftrightarrow \mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) \in \mathrm{ST}_{\mathfrak{q}}\left(\varphi\right),$$

and

$$\mathfrak{f} \in \mathrm{CV}^r_{\varepsilon,\mathfrak{q}}(\varkappa,\varrho)\left(\varphi\right) \Leftrightarrow \mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) \in \mathrm{CV}_{\mathfrak{q}}\left(\varphi\right). \tag{1.10}$$

It is known that

$$\mathfrak{f} \in \mathrm{CV}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\varphi\right) \Leftrightarrow -\mathfrak{q}\tau(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in \mathrm{ST}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\varphi\right). \tag{1.11}$$

**Definition 4.** $\mathfrak{f} \in \Sigma$, $\varphi \in \Phi$, and $\mathfrak{q} \in (0,1)$. Then $\mathfrak{f} \in \mathrm{K}_{\mathfrak{q}}(\varphi)$ if

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}\left(\tau\right)}{\mathfrak{g}\left(\tau\right)} \prec \varphi\left(\tau\right),$$

for some $\mathfrak{g} \in \mathrm{ST}_{\mathfrak{q}}\left(\psi\right),\ \psi \in \Phi$.

For $\mathfrak{q} \to 1^-$ we obtain a class of close to convex meromorphic functions (see [27]).

Like the previously described classes, we define

$$\mathfrak{f} \in \mathrm{K}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\varphi\right) \Leftrightarrow \mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) \in \mathrm{K}_{\mathfrak{q}}\left(\varphi\right),$$

and

$$\mathfrak{f} \in \mathfrak{Q}^r_{\varepsilon,\mathfrak{q}}(\varkappa,\varrho)\left(\psi\right) \Leftrightarrow \mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) \in \mathfrak{Q}_{\mathfrak{q}}\left(\psi\right).$$

It is known that

$$\mathfrak{f} \in \mathfrak{Q}^r_{\varepsilon,\mathfrak{q}}(\varkappa,\varrho)\left(\psi\right) \Leftrightarrow -\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}\left(\tau\right) \in \mathrm{K}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\varphi\right).$$

**Definition 5.** $\mathfrak{f} \in \Sigma$, $\varphi \in \Phi$, and $\mathfrak{q} \in (0,1)$. Then $\mathfrak{f} \in \mathrm{ST}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$ if

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}{\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}\left(\tau\right)} \prec \varphi\left(\tau\right),$$

and $\mathfrak{f} \in \Sigma$, $\varphi \in \Phi$, and $\mathfrak{q} \in (0,1)$. Then $\mathfrak{f} \in \mathrm{K}^r_{\varepsilon,\mathfrak{q}}(\varkappa,\varrho)(\varphi)$ if

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}{\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}\left(\tau\right)} \prec \varphi\left(\tau\right),$$

for some $\mathfrak{g} \in \mathrm{ST}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\varphi\right)$ with $r \in \mathbb{N}_0, \varkappa, \varrho \geq 0, 0 < \mathfrak{q} < 1$.

**Remark 1.** (*i*)

$$\mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\frac{1+(1-2\alpha)\tau}{1-\tau}) = \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\alpha)$$
$$= \left\{ \mathfrak{f} \in \Sigma : Re\left(-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}\right) > \alpha;\ 0 \le \alpha < 1, \tau \in \mathbb{U} \right\},$$

*and*

$$\mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\frac{1+(1-2\alpha)\tau}{1-\tau}) = \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\alpha)$$
$$= \left\{ \mathfrak{f} \in \Sigma : Re\left(-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{d}_{\mathfrak{q}}\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}\right) > \alpha,\ 0 \le \alpha < 1, \tau \in \mathbb{U} \right\},$$

*the subclasses related to meromorphic* $\mathfrak{q}$-starlike and $\mathfrak{q}$-*convex respectively;*

(*ii*)

$$\lim_{\mathfrak{q}\to 1^-} \mathrm{ST}^{0}_{\mathfrak{q},0}(\frac{1+(1-2\alpha)\tau}{1-\tau}) = \mathrm{ST}(\alpha)$$
$$= \left\{ \mathfrak{f} \in \Sigma : Re\left(-\frac{\tau\mathfrak{f}'(\tau)}{\mathfrak{f}(\tau)}\right) > \alpha;\ 0 \le \alpha < 1, \tau \in \mathbb{U} \right\},$$

*and*

$$\lim_{\mathfrak{q}\to 1^-} \mathrm{CV}^{0}_{\mathfrak{q},0}(\frac{1+(1-2\alpha)\tau}{1-\tau}) = \mathrm{CV}(\alpha)$$
$$= \left\{ \mathfrak{f} \in \Sigma : Re\left(-1-\frac{\tau\mathfrak{f}''(\tau)}{\mathfrak{f}'(\tau)}\right) > \alpha,\ 0 \le \alpha < 1,\ \tau \in \mathbb{U} \right\},$$

*were investigated by Kaczmarski [28];*

(*iii*)

$$\lim_{\mathfrak{q}\to 1^-} \mathrm{ST}^{0}_{\mathfrak{q},0}(1,-1) = \mathrm{ST},$$

*and*

$$\lim_{\mathfrak{q}\to 1^-} \mathrm{CV}^{0}_{\mathfrak{q},0}(1,-1) = \mathrm{CV},$$

*which are well-known classes of starlike and convex meromorphic functions, respectively; see the researches [20, 21].*

## 2. Main results

To illustrate our conclusions, the following lemma is necessary:

**Lemma 1.** *[29] Considering $\gamma$ and $\delta$ are complex numbers with $\gamma \neq 0$, and let $\hbar(\tau)$ be regular in $\mathbb{U}$ with*

$$\hbar(0) = 1 \quad and \quad Re\{\gamma\hbar(\tau) + \delta\} > 0.$$

*If*

$$\omega(\tau) = 1 + \omega_1\tau + \omega_2\tau^2 + ....$$

*is analytic in* $\mathbb{U}$, *then*

$$\omega(\tau) + \frac{\tau \eth_{\mathfrak{q}}\omega(\tau)}{\gamma\omega(\tau) + \delta} \prec \hbar(\tau),$$

*then* $\omega(\tau) \prec \hbar(\tau)$.

**Lemma 2.** *[30] Let* $\pi(\tau)$ *be convex in* $\mathbb{U}$ *with* $\pi(0) = 1$ *and let*

$$Y : \mathbb{U} \rightarrow \mathbb{C},$$

*with*

$$Re\,(Y(\tau)) > 0,$$

*in* $\mathbb{U}$. *If*

$$y(\tau) = 1 + y_1\tau + y_2\tau^2...,$$

*is analytic in* $\mathbb{U}$, *then*

$$y(\tau) + Y(\tau).\tau\eth_{\mathfrak{q}} y(\tau) \prec \pi(\tau),$$

*implies that* $y(\tau) \prec \pi(\tau)$.

**Theorem 1.** *Assume that* $\varphi(\tau)$ *is a regular and convex univalent function with*

$$\varphi(0) = 1 \quad and \quad Re(\varphi(\tau)) > 0,$$

*for* $\tau \in \mathbb{U}$. *Then, for* $r \in \mathbb{N}_0$, *and* $\varrho, \varepsilon \geq 0, \varkappa > 0, 0 < \mathfrak{q} < 1$ *with*

$$Re\left\{-\frac{1}{\mathfrak{q}}\varphi + \frac{\eta_{\mathfrak{q}}}{\mathfrak{q}}\right\} > 0,$$

*and*

$$\mathrm{ST}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa,\varrho)\,(\varphi) \subset \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\,(\varphi) \subset \mathrm{ST}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\,(\varphi).$$

*Proof.* Let $\mathfrak{f} \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\,(\varphi)$ satisfying

$$\frac{-\mathfrak{q}\tau\eth_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} \prec \varphi(\tau),$$

consider

$$\omega(\tau) = \frac{-\mathfrak{q}\tau\eth_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}, \tag{2.1}$$

where $\omega(\tau)$ is analytic in $\mathbb{U}$, $\omega(0) = 1$.

From (1.7) we have

$$-\frac{\mathfrak{q}\tau\eth_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} = -\frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho}}\frac{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} + \left(1 + \frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho}}\right), \ \ \varkappa > 0,$$

by using (2.1) we obtain

$$-\frac{[\varrho]_{\mathfrak{q}}}{\varkappa \mathfrak{q}^{\varrho}} \frac{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} = \omega(\tau) - \left(1 + \frac{[\varrho]_{\mathfrak{q}}}{\varkappa \mathfrak{q}^{\varrho}}\right). \tag{2.2}$$

On $\mathfrak{q}$-logarithmic differentiation of (2.2), we have

$$\frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} = \omega(\tau) + \frac{\tau\mathfrak{d}_{\mathfrak{q}}\omega(\tau)}{\frac{-1}{\mathfrak{q}}\omega(\tau) + \frac{\eta_{\mathfrak{q}}}{\mathfrak{q}}}, \tag{2.3}$$

where

$$\eta_{\mathfrak{q}} = \left(1 + \frac{[\varrho]_{\mathfrak{q}}}{\varkappa \mathfrak{q}^{\varrho}}\right).$$

Since $\mathfrak{f} \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$, from (2.3) we have

$$\omega(\tau) + \frac{\tau\mathfrak{d}_{\mathfrak{q}}\omega(\tau)}{\frac{-1}{\mathfrak{q}}\omega(\tau) + \frac{\eta_{\mathfrak{q}}}{\mathfrak{q}}} \prec \varphi(\tau).$$

From Lemma 1, we get $\omega(\tau) \prec \varphi(\tau)$. Consequently,

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} \prec \varphi(\tau),$$

then $\mathfrak{f} \in \mathrm{ST}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$. To prove the first part, let $\mathfrak{f} \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa,\varrho)(\varphi)$ and set

$$\chi(\tau) = \frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}, \tag{2.4}$$

where $\chi$ is analytic in $\mathbb{U}$, $\chi(0) = 1$.

From (1.8) we have

$$\frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} = -\frac{[\varepsilon+1]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+1}} \frac{\mathfrak{D}^{r}_{\varepsilon+1,\mathfrak{q}}(\varkappa,\varrho)\mathfrak{f}(\tau)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} + \frac{[\varepsilon+2]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+1}},$$

by using (2.4) and $\mathfrak{q}$-logarithmic differentiation we obtain

$$\frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} \prec \varphi,$$

with

$$Re\left\{-\frac{1}{\mathfrak{q}}\varphi + \frac{[\varepsilon+2]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+2}}\right\} > 0.$$

The proof is now finished. □

**Theorem 2.** *Let $\varphi(\tau)$ be regular and convex univalent function with*

$$\varphi(0) = 1 \quad and \quad Re(\varphi(\tau)) > 0,$$

*for* $\tau \in \mathbb{U}$. *Then, for* $r \in \mathbb{N}_0$ *and* $\varrho, \varepsilon \geq 0, \varkappa > 0, 0 < \mathfrak{q} < 1$, *with*

$$Re\left\{-\frac{1}{\mathfrak{q}}\varphi + \frac{\eta_{\mathfrak{q}}}{\mathfrak{q}}\right\} > 0,$$

*and*

$$Re\left\{-\frac{1}{\mathfrak{q}}\varphi + \frac{[\varepsilon+2]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+2}}\right\} > 0,$$

*and*

$$\mathrm{CV}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa,\varrho)(\varphi) \subset \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi) \subset \mathrm{CV}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi).$$

*Proof.* Let $\mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$. Applying (1.11), we show that

$$\begin{aligned}
\mathfrak{f} \in \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi) &\Leftrightarrow \mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) \in \mathrm{CV}_{\mathfrak{q}}(\varphi)\\
&\Leftrightarrow -\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right) \in \mathrm{ST}_{\mathfrak{q}}(\varphi)\\
&\Leftrightarrow -\mathfrak{q}\tau(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)\\
&\Leftrightarrow -\mathfrak{q}\tau(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in \mathrm{ST}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)\\
&\Leftrightarrow -\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right) \in \mathrm{ST}_{\mathfrak{q}}(\varphi)\\
&\Leftrightarrow \mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(-\mathfrak{q}\tau(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f})) \in \mathrm{ST}_{\mathfrak{q}}(\varphi)\\
&\Leftrightarrow \mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) \in \mathrm{CV}_{\mathfrak{q}}(\varphi)\\
&\Leftrightarrow \mathfrak{f} \in \mathrm{CV}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi).
\end{aligned}$$

We can use arguments like the ones mentioned above to illustrate the first part.

The proof is now finished. □

**Example 1.** We can expand the inclusions according to using Theorems 1 and 2

(*i*)

$$\begin{gathered}
\mathrm{ST}^{r}_{\mathfrak{q},\varepsilon+m}(\varkappa,\varrho)(\varphi) \subset \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon+m-1}(\varkappa,\varrho)(\varphi) \subset \ldots\ldots. \subset \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi),\\
\mathrm{CV}^{r}_{\mathfrak{q},\varepsilon+m}(\varkappa,\varrho)(\varphi) \subset \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon+m-1}(\varkappa,\varrho)(\varphi) \subset \ldots\ldots. \subset \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi);
\end{gathered}$$

(*ii*)

$$\begin{gathered}
\mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi) \subset \mathrm{ST}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi) \subset \ldots\ldots. \subset \mathrm{ST}^{r+s}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi),\\
\mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi) \subset \mathrm{CV}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi) \subset \ldots\ldots. \subset \mathrm{CV}^{r+s}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi).
\end{gathered}$$

**Corollary 1.** *Suppose that* $r \in \mathbb{N}_0$ *and* $\varrho, \varkappa, \varepsilon \geq 0, 0 < \mathfrak{q} < 1$. *Then, for*

$$\varphi(\tau) = \frac{1 + (1-2\alpha)\tau}{1-\tau}, \quad \eta_{\mathfrak{q}} > \frac{\alpha}{\mathfrak{q}},$$

*and*

$$\frac{[\varepsilon+2]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+1}} > \alpha,$$

*we obtain*

$$\begin{aligned}\mathrm{ST}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa,\varrho)\left(\frac{1+(1-2\alpha)\tau}{1-\tau}\right) &\subset \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\frac{1+(1-2\alpha)\tau}{1-\tau}\right)\\ &\subset \mathrm{ST}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\frac{1+(1-2\alpha)\tau}{1-\tau}\right),\\ \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa,\varrho)\left(\frac{1+(1-2\alpha)\tau}{1-\tau}\right) &\subset \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\frac{1+(1-2\alpha)\tau}{1-\tau}\right)\\ &\subset \mathrm{CV}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\frac{1+(1-2\alpha)\tau}{1-\tau}\right),\end{aligned}$$

*respectively.*

**Example 2.** Suppose that $r \in \mathbb{N}_0$ and $\varrho, \varkappa, \varepsilon \geq 0, 0 < \mathfrak{q} < 1$. For

$$\varphi(\tau) = \frac{1}{1-\mathfrak{q}\tau},$$

we obtain

$$\begin{aligned}\mathrm{ST}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa,\varrho)\left(\frac{1}{1-\mathfrak{q}\tau}\right) \subset \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\frac{1}{1-\mathfrak{q}\tau}\right) \subset \mathrm{ST}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\frac{1}{1-\mathfrak{q}\tau}\right),\\ \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa,\varrho)\left(\frac{1}{1-\mathfrak{q}\tau}\right) \subset \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\frac{1}{1-\mathfrak{q}\tau}\right) \subset \mathrm{CV}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\left(\frac{1}{1-\mathfrak{q}\tau}\right).\end{aligned}$$

The following conclusions can be shown by using the same arguments as before.

**Theorem 3.** *Consider $\varphi(\tau)$ be regular and convex univalent function with*

$$\varphi(0) = 1 \quad \textit{and} \quad Re(\varphi(\tau)) > 0,$$

*for $\tau \in \mathbb{U}$. Then for $r \in \mathbb{N}_0$ and $\varrho, \varepsilon \geq 0, \varkappa > 0, 0 < \mathfrak{q} < 1$, with*

$$Re\left(\frac{1}{\frac{-1}{\mathfrak{q}}\psi(\tau) + \zeta_{\mathfrak{q}}}\right) > 0,$$

*and*

$$\mathrm{K}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa,\varrho)(\varphi) \subset \mathrm{K}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi) \subset \mathrm{K}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi).$$

*Proof.* Let $\mathfrak{f} \in \mathrm{K}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$. Then, by definition, There is $\mathfrak{g} \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\psi)$ satisfying

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} \prec \varphi(\tau). \tag{2.5}$$

Consider

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} = \mathfrak{p}(\tau), \tag{2.6}$$

where $\mathfrak{p}(\tau)$ is regular in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$,

$$-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right) = \mathfrak{p}(\tau)\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau),$$

by using (1.7) we have

$$-\frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho}}\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) + \left(1 + \frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho}}\right)\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) = \mathfrak{p}(\tau)\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau).$$

On $\mathfrak{q}$-differentiating with respect to $\tau$, and dividing by $\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)$ we get

$$\begin{aligned} &-\frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho+1}}\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} + \left(\frac{1}{\mathfrak{q}} + \frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho+1}}\right)\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} \\ &= \frac{\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\tau)\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)\right) + \mathfrak{p}(\tau)\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)}, \end{aligned}$$

using simple calculation we obtain

$$-\frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho+1}}\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} = \frac{\frac{\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\tau)\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)\right)+\mathfrak{p}(\tau)\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)\right)-\left(\frac{1}{\mathfrak{q}}+\frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho+1}}\right)\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)}}{\frac{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)}}.$$

Applying identity (1.7) we have

$$\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\varepsilon,\mathfrak{q}}(\varkappa,\varrho)\mathfrak{g}(\tau)\right) = \frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho+1}}\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau) - \left(\frac{1}{\mathfrak{q}} + \frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho+1}}\right)\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau),$$

then

$$\frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} = \frac{-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} - \zeta_{\mathfrak{q}}\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)}}{\frac{\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} + \zeta_{\mathfrak{q}}}, \tag{2.7}$$

where

$$\zeta_{\mathfrak{q}} = \left(\frac{1}{\mathfrak{q}} + \frac{[\varrho]_{\mathfrak{q}}}{\varkappa\mathfrak{q}^{\varrho+1}}\right).$$

On $\mathfrak{q}$-differentiation of (2.6), we have

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} = -\mathfrak{p}(\tau)\frac{\psi(\tau)}{\mathfrak{q}} + \tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\tau), \tag{2.8}$$

where

$$\frac{-1}{\mathfrak{q}}\psi(\tau) = \frac{\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)\right)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)}.$$

From (2.7) and (2.8), we get

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{g}(\tau)} = \mathfrak{p}(\tau) + \frac{\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\tau)}{\frac{-1}{\mathfrak{q}}\psi(\tau) + \zeta_{\mathfrak{q}}}. \tag{2.9}$$

Consequently, from (2.5)

$$\mathfrak{p}(\tau) + \frac{\tau \mathfrak{d}_{\mathfrak{q}} \mathfrak{p}(\tau)}{\frac{-1}{\mathfrak{q}}\psi(\tau) + \zeta_{\mathfrak{q}}} \prec \varphi(\tau). \tag{2.10}$$

Since $\mathfrak{g} \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \tau)(\psi)$, by Theorem 1, we conclude

$$\mathfrak{g} \in \mathrm{ST}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa, \tau)(\psi),$$

since

$$Re(\frac{1}{\frac{-1}{\mathfrak{q}}\psi(\tau) + \zeta_{\mathfrak{q}}}) > 0,$$

in $\mathbb{U}$. Lemma 2 now produces the intended result.

To prove the first part, let $\mathfrak{f} \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa, \varrho)(\varphi)$ and set

$$\chi(\tau) = -\frac{\mathfrak{q}\tau \mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{g}(\tau)},$$

where $\chi$ is analytic in $\mathbb{U}$, $\chi(0) = 1$.

$$-\mathfrak{q}\tau \mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{f}(\tau)\right) = \chi(\tau)\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{g}(\tau),$$

by using (1.8)

$$-\frac{[\varepsilon+1]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+1}}\mathfrak{D}^{r}_{\varepsilon+1,\mathfrak{q}}(\varkappa, \varrho)\mathfrak{f}(\tau) + \frac{[\varepsilon+2]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+1}}\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{f}(\tau) = \chi(\tau)\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{g}(\tau).$$

On $\mathfrak{q}$-differentiating with respect to $\tau$, dividing by $\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{g}(\tau)$, and using simple calculation it follows that

$$-\frac{\mathfrak{q}\tau \mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{f}(\tau)\right)}{\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{g}(\tau)} \prec \varphi(\tau),$$

with

$$Re\left(\frac{1}{\frac{-1}{\mathfrak{q}}\psi(\tau) + \frac{[\varepsilon+2]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+2}}}\right) > 0.$$

The proof is complete. □

**Theorem 4.** *Let $\varphi(\tau)$ is analytic and convex univalent function with*

$$\varphi(0) = 1 \quad and \quad Re(\varphi(\tau)) > 0,$$

*for $\tau \in \mathbb{U}$. Then for $r \in \mathbb{N}_0$ and $\varrho, \varepsilon \geq 0, \varkappa > 0, 0 < \mathfrak{q} < 1$, with*

$$Re(\frac{1}{\frac{-1}{\mathfrak{q}}\psi(\tau) + \zeta_{\mathfrak{q}}}) > 0,$$

*and*

$$Re\left(\frac{1}{\frac{-1}{\mathfrak{q}}\psi(\tau) + \frac{[\varepsilon+2]_{\mathfrak{q}}}{\mathfrak{q}^{\varepsilon+2}}}\right) > 0,$$

*and*

$$\mathfrak{Q}^{r}_{\mathfrak{q},\varepsilon+1}(\varkappa, \varrho)(\varphi) \subset \mathfrak{Q}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)(\varphi) \subset \mathfrak{Q}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)(\varphi).$$

*Proof.* Let $\mathfrak{f} \in \mathfrak{Q}^r_{\mathfrak{q},\varepsilon}(\varkappa, \tau)\,(\varphi)$. We have

$$\begin{aligned}
&\mathfrak{f} \in \mathfrak{Q}^r_{\mathfrak{q},\rho}(\varkappa, \varrho)\,(\varphi) \Leftrightarrow \mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{f}(\tau) \in \mathfrak{Q}_{\mathfrak{q}}\,(\varphi)\\
&\quad\Leftrightarrow -\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{f}(\tau)\right) \in \mathrm{K}_{\mathfrak{q}}\,(\varphi)\\
&\quad\Leftrightarrow -\mathfrak{q}\tau(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in \mathrm{K}^r_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\\
&\quad\Leftrightarrow -\mathfrak{q}\tau(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in \mathrm{K}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\\
&\quad\Leftrightarrow -\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{f}(\tau)\right) \in \mathrm{K}_{\mathfrak{q}}\,(\varphi)\\
&\quad\Leftrightarrow \mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)(-\mathfrak{q}\tau(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f})) \in \mathrm{K}_{\mathfrak{q}}\,(\varphi)\\
&\quad\Leftrightarrow \mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\mathfrak{f}(\tau) \in \mathfrak{Q}_{\mathfrak{q}}\,(\varphi)\\
&\quad\Leftrightarrow \mathfrak{f} \in \mathfrak{Q}^{r+1}_{\mathfrak{q},\rho}(\varkappa, \varrho)\,(\varphi)\,.
\end{aligned}$$

To demonstrate the first section, we can use arguments similar to the ones listed above.

The proof is now complete. □

**Remark 2.** Based on Theorems 3 and 4, we can infer the following inclusion relations:

$$\begin{aligned}
\mathrm{K}^r_{\mathfrak{q},\varepsilon+m}(\varkappa, \varrho)\,(\varphi) \subset \mathrm{K}^r_{\mathfrak{q},\varepsilon+m-1}(\varkappa, \varrho)\,(\varphi) \subset \ldots.. \subset \mathrm{K}^r_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,,\\
\mathrm{K}^r_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi) \subset \mathrm{K}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi) \subset \ldots.. \subset \mathrm{K}^{r+m}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{Q}^r_{\mathfrak{q},\varepsilon+n}(\varkappa, \varrho)\,(\varphi) \subset \mathfrak{Q}^r_{\mathfrak{q},\varepsilon+n-1}(\varkappa, \varrho)\,(\varphi) \subset \ldots. \subset \mathfrak{Q}^r_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,,\\
\mathfrak{Q}^r_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi) \subset \mathfrak{Q}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi) \subset \ldots. \subset \mathfrak{Q}^{r+n}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,, \quad (m, n \in \mathbb{N}).
\end{aligned}$$

The same reasoning as before can be used to demonstrate the following conclusions.

## 3. Invariance of the classes under $\mathfrak{q}$-Bernardi integral operator

Bernardi $\mathfrak{q}$-meromorphic refers to a class of complex functions that are both meromorphic (analytic except for poles) and exhibit $\mathfrak{q}$-analogue properties, often studied in the context of $\mathfrak{q}$-difference equations or $\mathfrak{q}$-integral operators like the generalized $\mathfrak{q}$-Bernardi integral operator. Researchers use these operators to define and study new subclasses of these functions, such as meromorphic $\mathfrak{q}$-starlike and $\mathfrak{q}$-convex functions, and examine their coefficient estimates, inclusion properties, and other analytical characteristics.

The $\mathfrak{q}$-Bernardi integral operator generalizes the classical Bernardi operator into the framework of $\mathfrak{q}$-calculus, acting as a linear coefficient multiplier operator on analytic functions, with wide applications in $\mathfrak{q}$-versions of GFT.

For a function $\mathfrak{f} \in \Sigma$, we denote by $\mathfrak{I}_{\rho,\mathfrak{q}}$ the $\mathfrak{q}$-Bernardi integral operator $\mathfrak{I}_{\rho,\mathfrak{q}}$ defined by (see the researches [14, 31, 32])

$$\mathfrak{F}_{\mathfrak{q}}(\tau) = \mathfrak{I}_{\rho,\mathfrak{q}}[\mathfrak{f}(\tau)] = \frac{[\rho]_{\mathfrak{q}}}{\tau^{\rho+1}} \int_0^\tau t^\rho\, \mathfrak{f}(t)\, d_{\mathfrak{q}}t \qquad (\rho \in \mathbb{N}). \tag{3.1}$$

For $\mathfrak{q} \to 1^-$ we have

$$\mathfrak{F}(\tau) = \mathfrak{I}_\rho[\mathfrak{f}(\tau)] = \frac{\rho}{\tau^{\rho+1}} \int_0^\tau t^\rho \, \mathfrak{f}(t)\, dt \qquad (\rho \in \mathbb{N}),$$

which are defined in [33].

The $\mathfrak{q}$-Bernardi integral operator $\mathfrak{I}_{\rho,\mathfrak{q}}$: $\Sigma \to \Sigma$, defined in (3.1), satisfies the following relationship:

$$\mathfrak{q}^{\rho+1}\tau\mathfrak{d}_\mathfrak{q}\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{F}_\mathfrak{q}(\tau) = [\rho]_\mathfrak{q}\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau) - [\rho+1]_\mathfrak{q}\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{F}_\mathfrak{q}(\tau). \tag{3.2}$$

The following result is now stated and demonstrated.

**Theorem 5.** *If* $\mathfrak{f} \in \Sigma$ *defined by (1.1) is in the function class* $\mathrm{ST}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$ , *and*

$$Re\{\frac{-1}{\mathfrak{q}}\varphi + \frac{[\rho+1]_\mathfrak{q}}{\mathfrak{q}^{\rho+1}}\} > 0,$$

*then* $\mathfrak{F}_\mathfrak{q}(\tau)$ *defined by (3.1) also belongs to the class* $\mathrm{ST}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$.

*Proof.* Let $\mathfrak{f} \in \mathrm{ST}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$ , we put

$$\omega(\tau) = -\frac{\mathfrak{q}\tau\mathfrak{d}_\mathfrak{q}(\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{F}_\mathfrak{q}(\tau))}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{F}_\mathfrak{q}(\tau)}, \tag{3.3}$$

where $\omega(\tau)$ is analytic in $\mathbb{U}$ with $\omega(0) = 1$.

From (3.2), we show that

$$\omega(\tau) = -\frac{[\rho]_\mathfrak{q}}{\mathfrak{q}^\rho}\frac{\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{F}_\mathfrak{q}(\tau)} + \frac{[\rho+1]_\mathfrak{q}}{\mathfrak{q}^\rho}.$$

On $\mathfrak{q}$-logarithmic differentiation, we get

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_\mathfrak{q}(\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau))}{\mathfrak{D}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{f}(\tau)} = \omega(\tau) + \frac{\tau\mathfrak{d}_\mathfrak{q}\omega(\tau)}{\frac{-1}{\mathfrak{q}}\omega(\tau) + \frac{[\rho+1]_\mathfrak{q}}{\mathfrak{q}^{\rho+1}}}. \tag{3.4}$$

Since $\mathfrak{f} \in \mathrm{ST}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi)$, we can revise (3.4) as

$$\omega(\tau) + \frac{\tau\mathfrak{d}_\mathfrak{q}\omega(\tau)}{\frac{-1}{\mathfrak{q}}\omega(\tau) + \frac{[\rho+1]_\mathfrak{q}}{\mathfrak{q}^{\rho+1}}} \prec \varphi(\tau).$$

Using Lemma 1, we get

$$\frac{-1}{\mathfrak{q}}\omega(\tau) \prec \varphi(\tau).$$

Consequently

$$\frac{-\mathfrak{q}\tau\mathfrak{d}_\mathfrak{q}(\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{F}_\mathfrak{q}(\tau))}{\mathfrak{D}^{r+1}_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)\mathfrak{F}_\mathfrak{q}(\tau)} \prec \varphi(\tau).$$

Hence

$$\mathfrak{F}_\mathfrak{q}(\tau) \in \mathrm{ST}^r_{\mathfrak{q},\varepsilon}(\varkappa,\varrho)(\varphi).$$

□

The conclusion that follows can be demonstrated using arguments that are similar to those in Theorem 5.

**Theorem 6.** *Assume that*

$$\mathfrak{f} \in \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,.$$

*Then*

$$\mathfrak{F}_{\mathfrak{q}}(\tau) \in \mathrm{CV}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,,$$

*where* $\mathfrak{F}_{\mathfrak{q}}(\tau)$ *be defined by* (3.1) *with*

$$Re\{\frac{-1}{\mathfrak{q}}\varphi + \frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}\} > 0.$$

**Theorem 7.** *Let* $\mathfrak{f} \in \mathrm{K}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)$, $\varphi(0) = 1$, *and*

$$Re\left(\frac{1}{\frac{-1}{\mathfrak{q}}\mathfrak{p}_1(\tau) + \frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}}\right) > 0.$$

*Then*

$$\mathfrak{F}_{\mathfrak{q}}(\tau) \in \mathrm{K}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,,$$

*where* $\mathfrak{F}_{\mathfrak{q}}(\tau)$ *is called* $\mathfrak{q}$*-Bernardi integral operator defined in (3.1).*

*Proof.* Consider

$$\mathfrak{f} \in \mathrm{K}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,.$$

Then we want to show that

$$\mathfrak{F}_{\mathfrak{q}}(\tau) \in \mathrm{K}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,,$$

where $\mathfrak{F}_{\mathfrak{q}}(\tau)$ defined in (3.1), for $\mathfrak{g} \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)$

$$\mathfrak{g}_{\mathfrak{q}}(\tau) = \frac{[\rho]_{\mathfrak{q}}}{\tau^{\rho+1}} \int_0^{\tau} t^{\rho}\mathfrak{g}(t)\mathfrak{d}_{\mathfrak{q}}t \in \mathrm{ST}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)\,(\varphi)\,. \tag{3.5}$$

Consider

$$-\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{F}_{\mathfrak{q}}(\tau)}{\mathfrak{g}_{\mathfrak{q}}(\tau)} = \mathfrak{p}(\tau), \tag{3.6}$$

where $\mathfrak{p}(\tau)$ is regular in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$.

Similarly, from (3.2), we've

$$\mathfrak{q}^{\rho+1}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}(\tau) = [\rho]_{\mathfrak{q}}\mathfrak{g}(\tau) - [\rho+1]_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}(\tau). \tag{3.7}$$

From (3.2) and (3.7), we obtain

$$\frac{\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\tau)}{\mathfrak{g}(\tau)} = \frac{\mathfrak{d}_{\mathfrak{q}}\left(\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{F}_{\mathfrak{q}}\mathfrak{f}(\tau)\right) + \frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{F}_{\mathfrak{q}}\mathfrak{f}(\tau)\right)}{\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}(\tau) + \frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}\mathfrak{g}_{\mathfrak{q}}(\tau)},$$

equivalently

$$\frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\tau)}{\mathfrak{g}(\tau)}=\frac{\frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{F}_{\mathfrak{q}}\mathfrak{f}(\tau)\right)}{\mathfrak{g}_{\mathfrak{q}}(\tau)}-\frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}\frac{\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{F}_{\mathfrak{q}}\mathfrak{f}(\tau)\right)}{\mathfrak{g}_{\mathfrak{q}}(\tau)}}{\frac{\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}(\tau)}{\mathfrak{g}_{\mathfrak{q}}(\tau)}+\frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}}. \tag{3.8}$$

On $\mathfrak{q}$-differentiation of (3.6) and simple calculation implies

$$\frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\left(\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{F}_{\mathfrak{q}}\mathfrak{f}(\tau)\right)}{\mathfrak{g}_{\mathfrak{q}}(\tau)}=-\mathfrak{p}(\tau).\frac{\mathfrak{p}_1(\tau)}{\mathfrak{q}}+\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\tau), \tag{3.9}$$

where

$$\frac{-1}{\mathfrak{q}}\mathfrak{p}_1(\tau)=\frac{\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}_{\mathfrak{q}}(\tau)}{\mathfrak{g}_{\mathfrak{q}}(\tau)}.$$

Substituting (3.9) in (3.8), we obtain

$$\frac{-\mathfrak{q}\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\tau)}{\mathfrak{g}(\tau)}=\mathfrak{p}(\tau)+\frac{\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\tau)}{\frac{-1}{\mathfrak{q}}\mathfrak{p}_1(\tau)+\frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}}. \tag{3.10}$$

Since $\mathfrak{f}\in K_{\mathfrak{q}}(\varphi)$, we can rewrite (3.10) as

$$\mathfrak{p}(\tau)+\frac{\tau\mathfrak{d}_{\mathfrak{q}}\mathfrak{p}(\tau)}{\frac{-1}{\mathfrak{q}}\mathfrak{p}_1(\tau)+\frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}}\prec\varphi(\tau).$$

From (3.5), we determine that

$$Re\left(\frac{-1}{\mathfrak{q}}\mathfrak{p}_1(\tau)\right)>0,$$

in $\mathbb{U}$ indicates

$$Re\left(\frac{1}{\frac{-1}{\mathfrak{q}}\mathfrak{p}_1(\tau)+\frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}}\right)>0,$$

in $\mathbb{U}$. Using Lemma 2, hence $\mathfrak{F}_{\mathfrak{q}}(\tau)\in K_{\mathfrak{q}}(\varphi)$. □

The same arguments are used to support the following theorem.

**Theorem 8.** *Let*

$$\mathfrak{f}\in\mathfrak{Q}^{r}_{\mathfrak{q},\rho}(\varkappa,\varrho)(\varphi).$$

*Then*

$$\mathfrak{F}_{\mathfrak{q}}\mathfrak{f}(\tau)\in\mathfrak{Q}^{r}_{\mathfrak{q},\rho}(\varkappa,\varrho)(\varphi),$$

*where* $\mathfrak{F}_{\mathfrak{q}}\mathfrak{f}(\tau)$ *be defined by* (3.1) *with*

$$Re\left(\frac{1}{\frac{-1}{\mathfrak{q}}\mathfrak{p}_1(\tau)+\frac{[\rho+1]_{\mathfrak{q}}}{\mathfrak{q}^{\rho+1}}}\right)>0.$$

## 4. Conclusions

This work introduces novel classes of analytic normalized functions in the unit disk $\mathbb{U}$ and establishes new results in the theory of meromorphic functions. By employing the concept of a $\mathfrak{q}$-difference operator, we defined the $\mathfrak{q}$-analogue multiplier-Ruscheweyh operator $\mathfrak{D}^{r}_{\mathfrak{q},\varepsilon}(\varkappa, \varrho)$ to explore various subclasses of meromorphic functions. Using this operator, several new subclasses were introduced and systematically analyzed. For these classes, we investigated inclusion relations and demonstrated the integral preservation property, highlighting the operator's utility in geometric function theory.

The framework developed here provides a foundation for further research in several directions. Future work could focus on:

• Extending the study to more generalized $\mathfrak{q}$-operators or hybrid operators combining $\mathfrak{q}$-calculus with other fractional or integral operators.

• Exploring applications in related areas, such as multivalent meromorphic functions, as well as potential connections with complex dynamical systems.

Overall, this study lays the groundwork for a broad spectrum of investigations into $\mathfrak{q}$-analogues of classical operators and their applications in geometric function theory.

## Author contributions

Ekram E. Ali: conceptualization, writing–original draft and editing, formal analysis, supervision, funding acquisition; Rabha M. El-Ashwah: conceptualization, writing–original draft and editing, formal analysis, supervision; Abeer M. Albalahi: conceptualization, writing–original draft and editing, formal analysis. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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