AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 30968–30989.
DOI: 10.3934/math.20251359
Received: 16 September 2025
Revised: 15 December 2025
Accepted: 19 December 2025
Published: 31 December 2025

Research article

Picard S-type semi implicit mid-point method for fixed point approximation with applications

Doaa Filali[1], **Mohammad Dilshad**[2,*], **Ibrahim Alraddadi**[3] **and Mohammad Akram**[3,*]

[1]Department of Mathematical Science, College of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia
[3]Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia

* **Correspondence:** Email: akramkhan_20@rediffmail.com, mdilshaad@gmail.com.

Abstract: In this article, our goal is to propose and design a hybrid Picard S-type semi-implicit mid-point method to approximate the fixed point of a contractive-like mapping. The convergence result and stability of the proposed method are established under suitable assumptions. We implemented the newly constructed method to approximate the common element, which is the fixed point of a contractive-like mapping and simultaneously solves a general variational inequality. Finally, the significance of the proposed scheme is illustrated through the study of a fractional diffusion equation.

Keywords: convergence; stability; semi-implicit midpoint method; fixed point; general variational inequality; diffusion equation
Mathematics Subject Classification: 47H05, 47H09, 47H10, 47H22, 47H25, 49J40

1. Introduction

Let $\emptyset \neq \mathbb{C}$ be a closed convex subset (CCS) of a Banach space $\mathbb{W}$, $\mathbb{R}$ be the set of real numbers, and let the set of fixed points of $\mathcal{S}$ is represented by $Fix(\mathcal{S}) = \{\varsigma \in \mathbb{C} : \mathcal{S}\varsigma = \varsigma\}$. A mapping $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ is called a contraction if $\forall \varsigma, \omega \in \mathbb{C}, \exists \varepsilon \in [0, 1)$ such that $\|\mathcal{S}\varsigma - \mathcal{S}\omega\| \leq \varepsilon\|\varsigma - \omega\|$ and for $\varepsilon = 1$, $\mathcal{S}$ is called non-expansive. An indispensable generalization of contraction mappings is given by non-expansive mappings. Non-expansive mappings are closely associated with monotonicity, which displays numerous applications in pure and applied sciences. Unquestionably, non-expansive mappings have a strong connection with fixed point theory. Browder [14] and Gohde [21] showed that a non-expansive mapping on a CCS of uniformly convex Banach spaces (UCBS) [17] possesses a fixed point. This vital fact makes the class of non-expansive mapping more significant.

In fact, the Picard iteration method [12] converges strongly to a fixed point of a contraction mapping in a complete metric space [11]. However, a non-expansive self-mapping may not possess a fixed point in a complete metric space. Moreover, unlike contraction mappings, the sequence generated by Picard iteration may fail to converge to a fixed point of a nonexpansive mapping.

Example 1.1. *Let* $B = \{\theta = (\theta_1, \theta_2, \cdots) : \theta_m \geq 0, \forall m, \sum\limits_{m=1}^{\infty} \theta_m = 1\}$ *be a closed and bounded subset of* $(l^1, \|\cdot\|_1)$*, the Banach space of all real absolutely summable sequences. Then the nonexpansive mapping* $\Phi : B \to B$*, defined by* $\Phi(\theta) = (0, \theta_1, \theta_2, \cdots)$ *does not possess a fixed point.*

This information motivated researchers to explore broader and more generalizes spaces and mappings that exhibit fixed points. In this sequel, several authors have extended non-expansive mappings and established related fixed point results. Notably, Berinde [13] introduced the concept of weak contraction (also known as almost contraction).

Definition 1.1. *A mapping* $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ *is called almost contraction if for some* $l \geq 0, \exists \varepsilon \in (0, 1)$ *so that*

$$\|\mathcal{S}(\varsigma) - \mathcal{S}(\kappa)\| \leq \varepsilon\|\varsigma - \kappa\| + l\|\varsigma - \mathcal{S}(\varsigma)\|, \forall \varsigma, \kappa \in \mathbb{C}. \tag{1.1}$$

Imoru and Olantiwo [23] further extended the mapping defined in (1.1) as follows:

Definition 1.2. *A mapping* $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ *is called contractive-like if there exists a strictly increasing continuous function* $\psi : [0, \infty) \to [0, \infty)$ *with* $\psi(0) = 0$ *and* $\varepsilon \in [0, 1)$ *so that*

$$\|\mathcal{S}(\varsigma) - \mathcal{S}(\kappa)\| \leq \psi(\|\varsigma - \mathcal{S}(\varsigma)\|) + \varepsilon\|\varsigma - \kappa\|, \forall \varsigma, \kappa \in \mathbb{C}. \tag{1.2}$$

These mappings are firmly connected with monotonicity and exhibit applications in nonlinear analysis, including variational inequalities, optimization, economics, engineering, computer science, equilibrium problems, and initial value problems.

On the contrary, after identifying the fixed point, it is desirable to propose an efficient iterative method to analyze the fixed point of the considered mapping. An effective iterative method is essential for exploring fixed points through computation and convergence analysis. Due to its significance, several fixed point iterative methods have been proposed. Some commonly proposed iterative methods include Mann [31], Ishikawa [24], S-iteration [1], and the Noor three-step [32] method, etc. Most fixed point iterative methods build upon the Banach Fixed Point Theorem and have become indispensable tools for examining mathematical models of partial differential equations (PDEs). Third order differential equations have tremendous applications in real-life problems including physics and engineering which can be addressed using the fixed point approach, see, [25, 42]. Recent developments in hybrid methods have improved convergence rate, and making these methods crucial for efficiently studying complex high-dimensional PDEs. Khan [27], introduced a hybrid Picard-Mann iterative method as follows:

$$\begin{cases} \varsigma_{m+1} = \mathcal{S}(\rho_m), \\ \rho_m = (1 - \alpha_m)\varsigma_m + \alpha_m \mathcal{S}\varsigma_m, m \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_m\}$ is in (0,1). The author claimed that this hybrid scheme is independent and more efficient than, the Picard, Mann, and Ishikawa iterations for contraction mappings. Okeke [35], put forward the

Picard-Ishikawa hybrid scheme as follows:

$$\begin{cases} \varsigma_{m+1} = \mathcal{S}(\rho_m), \\ \rho_m = (1-\alpha_m)\varsigma_m + \alpha_m \mathcal{S}\omega_m, \\ \omega_m = (1-\beta_m)\varsigma_m + \beta_m \mathcal{S}\varsigma_m, m \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are in (0,1). By illustrative experiments, it was established that this hybrid method converges faster than the Picard, Krasnoselskii, Mann, Ishikawa, Noor, Picard-Mann, and Picard-Krasnoselskii iterative processes. Following the hybridization procedures, Gursoy and Karakaya [22] and Srivastava [39] independently investigated the hybrid Picard-S method as follows:

$$\begin{cases} \varsigma_{m+1} = \mathcal{S}(\rho_m), \\ \rho_m = (1-\alpha_m)\mathcal{S}\varsigma_m + \alpha_m \mathcal{S}\omega_m, \\ \omega_m = (1-\beta_m)\varsigma_m + \beta_m \mathcal{S}\varsigma_m, m \in \mathbb{N}. \end{cases} \tag{1.5}$$

The authors corroborated that their scheme is more efficient than the Picard, Mann, Ishikawa, Noor, SP, CR, S, and several other iteration methods. They also employed the proposed scheme to explore delay differential equations.

The semi-implicit method is an appealing tool for solving differential equations, algebraic equations and initial value problems (IVPs) of differential equations. Several real-world physical problems can be examined by reformulating them into the IVP of the following form:

$$\frac{d\varsigma}{du} = \zeta(\varsigma); \varsigma(0) = \varsigma_0. \tag{1.6}$$

However, it is burdensome to solve such models if the operator ζ is not continuous. This difficulty can be addressed by setting up a sequence of Lipschitz functions that approximate ς. One of the fundamental techniques to study (1.6) is the implicit midpoint rule (IMR) in which ζ is approximated by the following iterative procedure:

$$\varsigma_{m+1} = \varsigma_m + \kappa\zeta\Big(\frac{\varsigma_{m+1} + \varsigma_m}{2}\Big), \tag{1.7}$$

where $\kappa > 0$ is a step-size. If the self-map ζ on $\mathbb{R}^N$ is smooth and Lipschitz continuous, then the sequence $\{\varsigma_m\}$ generated by (1.7) converges to the exact solution of IVP (1.6) as $\kappa \to 0$, uniformly over $s \in [0, \bar{s})$ for any fixed $\bar{s} > 0$. Song and Pei [38] proposed a semi-implicit midpoint rule based on the viscosity technique for approximating a common fixed point of non-expansive mappings and 2-generalized hybrid mappings in a real Hilbert space. The authors also examined a split feasibility problem by employing their scheme. Luo et al. [30] utilized the viscosity technique to construct the implicit midpoint rule for non-expansive mappings. They applied their main results to determine the fixed point of strict pseudocontractive mappings. By implementing the proposed method, variational inequality problems in Banach spaces and equilibrium problems in Hilbert spaces were examined. Aibinu et al. [3] designed the IMR using a viscosity approximation method for non-expansive mappings. The strong convergence of the proposed method was established, and the obtained results demonstrated that scheme could be implemented to tackle a variational inequality problem. Xu et al. [45] designed an implicit midpoint rule for non-expansive mappings in Banach spaces and reported the weak convergence of their scheme under Opial's property. Furthermore, Xu et al. [44] suggested a class of general semi-implicit iterative methods involving the semi-implicit rule and computational errors. They approximated common fixed points of three different nonexpansive-type operators and employed the proposed iterative scheme to study the Stampacchia variational inequality.

Recently, Akram [4] and Filali [19] independently designed a four-step semi-implicit approximation scheme to determine the fixed point of a contractive-like mapping and an almost contraction mapping, respectively. The convergence analysis and stability of the proposed schemes were discussed, and the proposed schemes implemented to explore a general quasi-variational inequality, a general variational inequality, a nonlinear fractional differential equation, and a nonlinear integral equation.

Numerous problems in applied sciences, including linear programming, monotone inclusions, convex optimization, and elliptic differential equations, can be examined as an equilibrium state model (1.6) which is analogous to the inclusion problem $0 \in \zeta(\varsigma)$, see, Browder [15] and Chidume [16]. If $\zeta(\varsigma) := g(\varsigma) - \varsigma$, then IVP (1.6) coincides with $\varsigma' = g(\varsigma) - \varsigma$ and the equilibrium point of (1.6) is the fixed point of g, i.e., $\varsigma \in Fix(g)$. This noble formulation inspired Alghamdi et al. [6] to set up the implicit iteration method:

$$\varsigma_{m+1} = (1-\alpha_m)\varsigma_m + \alpha_m \mathcal{S}\Big(\frac{\varsigma_{m+1}+\varsigma_m}{2}\Big), \tag{1.8}$$

where $\{\alpha_m\} \subset (0,1)$ and $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ is a non-expansive mapping. The authors established weak convergence result under appropriate conditions. Furthermore, viscocity implicit midpoint scheme was developed by Xu et al. [46] to prove a strong convergence result for non-expansive mappings as follows:

$$\varsigma_{m+1} = \alpha_m \Phi(\varsigma_m) + (1-\alpha_m)\mathcal{S}\Big(\frac{\varsigma_{m+1}+\varsigma_m}{2}\Big), \tag{1.9}$$

where, Φ is contraction and $\mathcal{S}$ is non-expansive. In particular, the following theorem was proved.

Theorem 1.1. *Let $\mathbb{C} \neq \emptyset$ be a CCS of a Hilbert space $\mathbb{H}$. Suppose $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ is a nonexpansive mapping with $Fix(\mathcal{S}) \neq \emptyset$ and $\Phi : \mathbb{C} \to \mathbb{C}$ is a contraction. If the sequence $\{\alpha_m\}$ complies with the following assumptions:*
(a_1) $\lim_{m\to\infty} \alpha_m = 0$; (a_2) $\sum_{m=0}^{\infty} \alpha_m = \infty$; (a_3) $\sum_{m=0}^{\infty} |\alpha_{m+1} - \alpha_m| < \infty$.
Then the sequence $\{\varsigma_m\}$ initiated by (1.9) converges to $\varsigma \in Fix(\mathcal{S})$ and ς is the solution of the following variational inequality:

$$\langle (I-\omega)\Phi, \varsigma - \omega\rangle \geq 0, \forall \varsigma \in Fix(\mathcal{S}).$$

Luo et al. [30] further generalized this study in uniformly smooth Banach spaces.

As discussed above, the hybridization of iterative schemes escalates the rate of convergence, and implicit methods are significant as they play a vital role in dealing with mathematical models- including differential equations, integral equations, boundary value problems, and related formulations which are noteworthy for exploring real-life problems. Motivated by prior investigations and the outcomes explained above, this study focuses on designing a Picard-S-type semi implicit mid-point scheme involving a contractive-like mapping in Banach spaces. The convergence of the semi-implicit scheme is proved to yield a fixed point of the contractive-like mapping, and the uniqueness of the solution is also established. A comparative analysis is carried out by considering an illustrative example with different initial guesses, comparing our scheme (2.1) with scheme (1.5). We prove the stability of the constructed scheme. Finally, we showcase the practical application of our proposed scheme and theoretical findings by employing our methodology to explore a general nonlinear variational inequality and a fractional diffusion equation.

2. Iterative scheme and convergence

Next, we construct a Picard-S-type semi implicit mid-point scheme (PSSIMPS) based on the Picard-S-hybrid method (1.5). An analytical demonstration of the convergence of the proposed iterative method is presented. Our new method is defined as follows:

$$\begin{cases} \varsigma_{m+1} = \mathcal{S}(\rho_m), \\ \rho_m = (1-\alpha_m)\mathcal{S}\Big(\frac{\varsigma_m + \rho_m}{2}\Big) + \alpha_m \mathcal{S}\Big(\frac{\omega_m + \rho_m}{2}\Big), \\ \omega_m = (1-\beta_m)\Big(\frac{\varsigma_m + \omega_m}{2}\Big) + \beta_m \mathcal{S}\Big(\frac{\varsigma_m + \omega_m}{2}\Big), m \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are in $(0, 1)$.

Remark 2.1. *It is worth mentioning that the semi-implicit terms in the second and third equations of PSSIMPS (2.1) replace the corresponding terms of the hybrid Picard-S method (1.5).*

Following lemma is crucial in the development of convergence result.

Lemma 2.1. *[43] If the nonnegative real sequences $\{\varrho_k\}_{k=1}^{\infty}$ and $\{\varsigma_k\}_{k=1}^{\infty}$ under the assumptions $\sum\limits_{k=1}^{\infty} p_k = \infty$ and $\lim\limits_{k\to\infty} \frac{\varsigma_k}{p_k} = 0$ satisfy the inequality:*

$$\varrho_{k+1} \leq (1-p_k)\varrho_k + \varsigma_k,$$

where $p_k \in (0, 1)$. Then $\lim\limits_{k\to\infty} \varrho_k = 0$.

Next, we manifest strong convergence of PSSIMPS (2.1).

Theorem 2.1. *Suppose that $\emptyset \neq \mathbb{C} \subseteq \mathbb{W}$ is a CCS and $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ is a contractive-like mapping. If $Fix(\mathcal{S}) \neq \emptyset$ and $\{\varsigma_m\}_{m=1}^{\infty}$ is produced by the scheme (2.1), then $\lim\limits_{m\to\infty} \varsigma_m = \varsigma \in Fix(\mathcal{S})$.*

Proof. Assume that $\varsigma^* \in Fix(\mathcal{S})$ and $\varsigma \in \mathcal{S}$. Then from iterative scheme (2.1) and (1.2), we obtain

$$\begin{aligned} \|\omega_m - \varsigma^*\| &= \Big\|\Big[(1-\beta_m)\Big(\frac{\varsigma_m + \omega_m}{2}\Big) + \beta_m \mathcal{S}\Big(\frac{\varsigma_m + \omega_m}{2}\Big)\Big] - \varsigma^*\Big\| \\ &\leq (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \beta_m\Big\|\mathcal{S}(\varsigma^*) - \mathcal{S}\Big(\frac{\varsigma_m + \omega_m}{2}\Big)\Big\| \\ &\leq (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \beta_m\Big[\psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| \\ &= (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \varepsilon\beta_m\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| \\ &\leq \frac{[1-\beta_m(1-\varepsilon)]}{2}[\|\varsigma_m - \varsigma^*\| + \|\omega_m - \varsigma^*\|], \end{aligned}$$

which turns into

$$\|\omega_m - \varsigma^*\| \leq \frac{\tau_m}{(2-\tau_m)}\|\varsigma_m - \varsigma^*\|, \tag{2.2}$$

where,

$$\tau_m = 1 - \beta_m(1-\varepsilon). \tag{2.3}$$

Again it follows from the second formulation of the scheme (2.1) and (1.2) that

$$\begin{aligned}
\|\rho_m - \varsigma^*\| &= \Big\|\Big[(1-\alpha_m)\mathcal{S}\Big(\frac{\varsigma_m+\rho_m}{2}\Big) + \alpha_m\mathcal{S}\Big(\frac{\omega_m+\rho_m}{2}\Big)\Big] - \varsigma^*\Big\| \\
&\le (1-\alpha_m)\Big\|\mathcal{S}\Big(\frac{\varsigma_m+\rho_m}{2}\Big) - \varsigma^*\Big\| + \alpha_m\Big\|\mathcal{S}\Big(\frac{\omega_m+\rho_m}{2}\Big) - \varsigma^*\Big\| \\
&= (1-\alpha_m)\Big\|\mathcal{S}(\varsigma^*) - \mathcal{S}\Big(\frac{\varsigma_m+\rho_m}{2}\Big)\Big\| + \alpha_m\Big\|\mathcal{S}(\varsigma^*) - \mathcal{S}\Big(\frac{\omega_m+\rho_m}{2}\Big)\Big\| \\
&\le (1-\alpha_m)\Big[\psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\Big\|\frac{\varsigma_m+\rho_m}{2} - \varsigma^*\Big\|\Big] \\
&\quad + \alpha_m\Big[\psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\Big\|\frac{\omega_m+\rho_m}{2} - \varsigma^*\Big\|\Big] \\
&= \varepsilon(1-\alpha_m)\Big\|\frac{\varsigma_m+\rho_m}{2} - \varsigma^*\Big\| + \varepsilon\alpha_m\Big\|\frac{\omega_m+\rho_m}{2} - \varsigma^*\Big\| \\
&\le \frac{\varepsilon(1-\alpha_m)}{2}\|\varsigma_m - \varsigma^*\| + \frac{\varepsilon\alpha_m}{2}\|\omega_m - \varsigma^*\| + \frac{\varepsilon}{2}\|\rho_m - \varsigma^*\|,
\end{aligned}$$

which along with (2.2) yields

$$\|\rho_m - \varsigma^*\| \le \frac{\varepsilon}{2-\varepsilon}\Big[1 - \alpha_m\Big(1 - \frac{\tau_m}{2-\tau_m}\Big)\Big]\|\varsigma_m - \varsigma^*\|, \tag{2.4}$$

and the first relation of scheme (2.1) along with (2.4) turns into

$$\begin{aligned}
\|\varsigma_{m+1} - \varsigma^*\| &= \|\mathcal{S}(\rho_m) - \varsigma^*\| = \|\mathcal{S}(\varsigma^*) - \mathcal{S}(\rho_m)\| \\
&\le \psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\|\rho_m - \varsigma^*\| = \varepsilon\|\rho_m - \varsigma^*\| \\
&\le \frac{\varepsilon^2}{2-\varepsilon}\Big[1 - \alpha_m\Big(1 - \frac{\tau_m}{2-\tau_m}\Big)\Big]\|\varsigma_m - \varsigma^*\|.
\end{aligned} \tag{2.5}$$

Taking the assumption $\{\alpha_m\}, \{\beta_m\} \subseteq (0,1)$ and $\varepsilon \in [0,1)$, we acquire $\tau_m \in (0,1)$ and hence $1 - \alpha_m\Big(1 - \frac{\tau_m}{2-\tau_m}\Big) \le 1$. Thus, we achieve

$$\|\varsigma_{m+1} - \varsigma^*\| \le \frac{\varepsilon^2}{2-\varepsilon}\|\varsigma_m - \varsigma^*\| = (1 - \hat{\iota_m})\|\varsigma_m - \varsigma^*\|, \tag{2.6}$$

where,

$$\hat{\iota_m} = \frac{(2-\varepsilon) - \varepsilon^2}{(2-\varepsilon)} \ge \frac{1}{2}[1 + (1-\varepsilon) - \varepsilon^2] > 0 \ \text{ as } \varepsilon \in [0,1). \tag{2.7}$$

Further, $1 - \hat{\iota}_m = \frac{\varepsilon^2}{2-\varepsilon} \ge 0$ and $\sum_{m=0}^{\infty} \hat{\iota}_m = \infty$. Utilizing Lemma 2.1, it follows from (2.6) that $\lim_{m\to\infty} \|\varsigma_m - \varsigma^*\| = 0$. Next, we prove the uniqueness of the solution. Suppose that $\varsigma_1, \varsigma_2 \in \mathbb{C}$ so that $\varsigma_1 \ne \varsigma_2$ and $\varsigma_1, \varsigma_2 \in Fix(\mathcal{S})$. Then

$$\begin{aligned}
\|\varsigma_1 - \varsigma_2\| &= \|\mathcal{S}(\varsigma_1) - \mathcal{S}(\varsigma_2)\| \\
&\le \psi(\|\varsigma_1 - \mathcal{S}(\varsigma_1)\|) + \varepsilon\|\varsigma_1 - \varsigma_2\| \\
&= \varepsilon\|\varsigma_1 - \varsigma_2\|.
\end{aligned} \tag{2.8}$$

Since $\varepsilon \in [0,1)$, then (2.8) yields $\|\varsigma_1 - \varsigma_2\| = 0$. Thus, $\varsigma_1 = \varsigma_2$. □

Remark 2.2. *The contractive-like condition in (1.2) is much comprehensive which comprise several other contractive conditions. If* $\psi s = ls$*, where* $l \geq 0$ *then (1.2) reduces to (1.1). Further, if* $l = m\varepsilon, m = \frac{1}{(1-\varepsilon)}, 0 < \varepsilon < 1$*, then the contractive condition coincides with that is due to Rhoades [37]. Moreover, if* $\psi s = 0$*, then (1.2) becomes*

$$\|\mathcal{S}(\varsigma) - \mathcal{S}(\kappa)\| \leq \varepsilon\|\varsigma - \kappa\|, \forall \varsigma, \kappa \in \mathbb{C}. \tag{2.9}$$

which is considered by Gursoy and Karakaya [22] and Srivastava [39].

Under the assumptions considered in Remark 2.2, we have the convergence result for the scheme (1.5) given in the following corollary.

Corollary 2.1. *Suppose that* $\emptyset \neq \mathbb{C} \subseteq \mathbb{W}$ *is a CCS and* $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ *is a contraction mapping defined in (2.9). If* $Fix(\mathcal{S}) \neq \emptyset$ *and* $\{\varsigma_m\}_{m=1}^{\infty}$ *is produced by the scheme (1.5), then* $\lim_{m\to\infty} \varsigma_m = \varsigma \in Fix(\mathcal{S})$*.*

Next, we put up the following example to set up a comparative analysis of the schemes (1.5) and (2.1).

Example 2.1. *Let* $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ *be a mapping defined by* $\mathcal{S}(\varsigma) = \sqrt{\varsigma^2 - 9\varsigma + 54}, \forall \varsigma \in \mathbb{C} = [0, 20]$*. Consider the sequences* $\{\alpha_m\}$*, and* $\{\beta_m\}$ *in* $(0, 1)$ *defined by* $\alpha_m = \frac{1}{(m+1)}$ *and* $\beta_m = \frac{1}{(m+1)^2}$ *with initial guesses* $\varsigma_0 = 15$ *and* $\varsigma_0 = 8$*. The comparison of convergence and their fixed points for two initial guesses have been presented in Figures 1 and 2.*

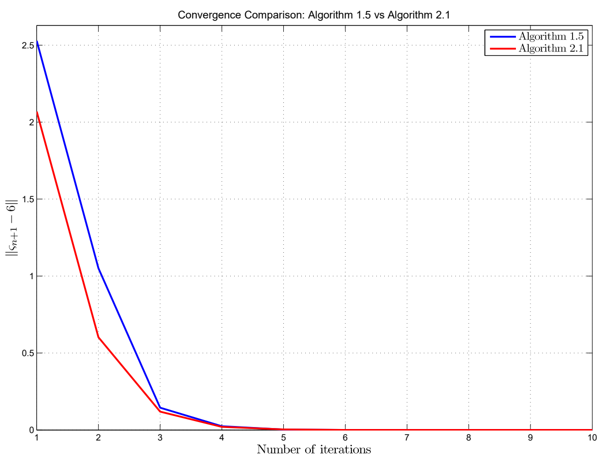

Figure 1. Comparative analysis of the convergence behavior of schemes (2.1), and (1.5) with initial guess $\varsigma_0 = 15$.

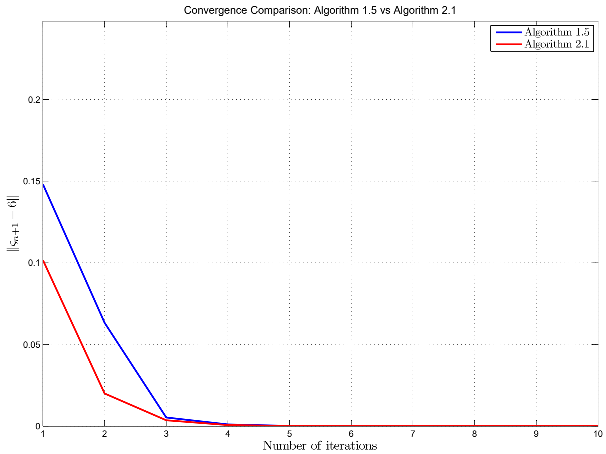

Figure 2. Comparative analysis of the convergence behavior of schemes (2.1), and (1.5) with initial guess $\varsigma_0 = 8$.

Next, we establish the stability theorem for PSSIMPS (2.1).

Definition 2.1. *[12] Let $\{\upsilon_m\} \subset \mathbb{C}$ be an arbitrary sequence. An iterative scheme $\varsigma_{m+1} = \psi(\mathcal{S}, \varsigma_m)$ so that $\{\varsigma_m\} \to \varsigma \in Fix(\mathcal{S})$ is said to be $\mathcal{S}$-stable. If for $\epsilon_m = \|\upsilon_{m+1} - \psi(\mathcal{S}, \varsigma_m)\|$, we have $\lim_{m\to\infty} \epsilon_m = 0$ if and only if $\lim_{m\to\infty} \upsilon_m = \varsigma$.*

Theorem 2.2. *Suppose that $\emptyset \neq \mathbb{C} \subseteq \mathbb{W}$ is a CCS and the mapping $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ satisfies (1.2). If $\varsigma^* \in Fix(\mathcal{S})$, then $\{\varsigma_m\}_{m=1}^{\infty}$ produced by the scheme (2.1) is $\mathcal{S}$-stable.*

Proof. Let $\{\psi_m\} \subset \mathbb{C}$ be an arbitrary sequence equivalent to $\{\varsigma_m\}$, where the sequence $\{\varsigma_m\}$ is generated by $PSSIMPS$ (2.1). Consider the relation $\varsigma_{m+1} = \Xi(\mathcal{S}, \psi_m)$ so that $\{\varsigma_m\} \to \varsigma^* \in Fix(\mathcal{S})$. Assume that $\nu_m = \|\psi_{m+1} - \Xi(\mathcal{S}, \psi_m)\|$, where $\{\psi_m\}$ is generated by the following relation:

$$\begin{cases} \psi_{m+1} = \mathcal{S}(\varphi_m), \\ \varphi_m = (1-\alpha_m)\mathcal{S}\Big(\dfrac{\psi_m + \varphi_m}{2}\Big) + \alpha_m \mathcal{S}\Big(\dfrac{\eta_m + \varphi_m}{2}\Big), \\ \eta_m = (1-\beta_m)\Big(\dfrac{\psi_m + \eta_m}{2}\Big) + \beta_m \mathcal{S}\Big(\dfrac{\psi_m + \eta_m}{2}\Big), m \in \mathbb{N}. \end{cases} \tag{2.10}$$

To demonstrate the $\mathcal{S}$-stability of (2.1), we set forth $\lim_{m\to\infty} \nu_m = 0 \Leftrightarrow \lim_{m\to\infty} \psi_m = \varsigma^*$. Suppose that $\lim_{m\to\infty} \nu_m = 0$, and applying the triangle inequality, we achieve

$$\begin{aligned} \|\psi_{m+1} - \varsigma^*\| &= \|\psi_{m+1} - \Xi(\mathcal{S}, \psi_m) + \Xi(\mathcal{S}, \psi_m) - \varsigma^*\| \\ &\le \|\psi_{m+1} - \Xi(\mathcal{S}, \psi_m)\| + \|\Xi(\mathcal{S}, \psi_m) - \varsigma^*\| \\ &= \nu_m + \|\varsigma_{m+1} - \varsigma^*\| \\ &= \nu_m + \|\mathcal{S}(\varphi_m) - \varsigma^*\| \\ &= \nu_m + \|\mathcal{S}(\varsigma^*) - \mathcal{S}(\varphi_m)\| \end{aligned}$$

$$
\begin{aligned}
&\leq \nu_m + \psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\|\varphi_m - \varsigma^*\| \\
&= \nu_m + \varepsilon\|\varphi_m - \varsigma^*\|.
\end{aligned} \tag{2.11}
$$

From the second relation of (2.10), we obtain

$$
\begin{aligned}
\|\varphi_m - \varsigma^*\| &= \left\|(1-\alpha_m)\mathcal{S}\Big(\frac{\psi_m + \varphi_m}{2}\Big) + \alpha_m\mathcal{S}\Big(\frac{\eta_m + \varphi_m}{2}\Big) - \varsigma^*\right\| \\
&\leq (1-\alpha_m)\left\|\mathcal{S}\Big(\frac{\psi_m + \varphi_m}{2}\Big) - \varsigma^*\right\| + \alpha_m\left\|\mathcal{S}\Big(\frac{\eta_m + \varphi_m}{2}\Big) - \varsigma^*\right\| \\
&= (1-\alpha_m)\left\|\mathcal{S}(\varsigma^*) - \mathcal{S}\Big(\frac{\psi_m + \varphi_m}{2}\Big)\right\| + \alpha_m\left\|\mathcal{S}(\varsigma^*) - \mathcal{S}\Big(\frac{\eta_m + \varphi_m}{2}\Big)\right\| \\
&\leq (1-\alpha_m)\Big[\psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\left\|\frac{\psi_m + \varphi_m}{2} - \varsigma^*\right\|\Big] \\
&\quad + \alpha_m\Big[\psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\left\|\frac{\eta_m + \varphi_m}{2} - \varsigma^*\right\|\Big] \\
&\leq \varepsilon(1-\alpha_m)\left\|\frac{\psi_m + \varphi_m}{2} - \varsigma^*\right\| + \varepsilon\alpha_m\left\|\frac{\eta_m + \varphi_m}{2} - \varsigma^*\right\| \\
&\leq \frac{\varepsilon(1-\alpha_m)}{2}\|\psi_m - \varsigma^*\| + \frac{\varepsilon}{2}\|\varphi_m - \varsigma^*\| + \frac{\varepsilon\alpha_m}{2}\|\eta_m - \varsigma^*\|,
\end{aligned}
$$

which turns into

$$
\|\varphi_m - \varsigma^*\| \leq \frac{\varepsilon(1-\alpha_m)}{2-\varepsilon}\|\psi_m - \varsigma^*\| + \frac{\varepsilon\alpha_m}{2-\varepsilon}\|\eta_m - \varsigma^*\|. \tag{2.12}
$$

$$
\begin{aligned}
\|\eta_m - \varsigma^*\| &= \left\|(1-\beta_m)\Big(\frac{\psi_m + \eta_m}{2}\Big) + \beta_m\mathcal{S}\Big(\frac{\psi_m + \eta_m}{2}\Big) - \varsigma^*\right\| \\
&\leq (1-\beta_m)\left\|\frac{\psi_m + \eta_m}{2} - \varsigma^*\right\| + \beta_m\left\|\mathcal{S}\Big(\frac{\psi_m + \eta_m}{2}\Big) - \varsigma^*\right\| \\
&\leq (1-\beta_m)\left\|\frac{\psi_m + \eta_m}{2} - \varsigma^*\right\| + \beta_m\Big[\psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\left\|\frac{\psi_m + \eta_m}{2} - \varsigma^*\right\| \\
&= [1-\beta_m(1-\varepsilon)]\left\|\frac{\psi_m + \eta_m}{2} - \varsigma^*\right\| \\
&= \frac{[1-\beta_m(1-\varepsilon)]}{2}[\|\psi_m - \varsigma^*\| + \|\eta_m - \varsigma^*\|],
\end{aligned}
$$

which turns into

$$
\|\eta_m - \varsigma^*\| \leq \frac{\tau_m}{(2-\tau_m)}\|\psi_m - \varsigma^*\|, \tag{2.13}
$$

where, τ_m is identical as defined in (2.3). Also, by the assumption of parameters, we obtain $1 - \alpha_m\Big(1 - \frac{\tau_m}{2-\tau_m}\Big) < 1$ and performing the back substitution from (2.12) and (2.13), (2.11) becomes

$$
\|\psi_{m+1} - \varsigma^*\| \leq \nu_m + (1 - \hat{\ell}_m)\|\psi_m - \varsigma^*\|, \tag{2.14}
$$

where $\hat{\ell}_m$ is defined in (2.7). By recalling the assumption $\lim_{m\to\infty} \nu_m = 0$ and implementing the Lemma 2.1, we obtain $\lim_{m\to\infty} \psi_m = \varsigma^*$. On the other hand, assume that $\lim_{m\to\infty} \psi_m = \varsigma^*$, and repeating the same procedure, we achieve

$$\begin{aligned}\nu_m &= \|\psi_{m+1} - \Xi(\mathcal{S}, \psi_m)\| \\ &= \|\psi_{m+1} - \varsigma^* + \varsigma^* - \Xi(\mathcal{S}, \psi_m)\| \\ &\leq \|\psi_{m+1} - \varsigma^*\| + \|\Xi(\mathcal{S}, \psi_m) - \varsigma^*\| \\ &= \|\psi_{m+1} - \varsigma^*\| + \|\psi_{m+1} - \varsigma^*\| \\ &\leq \|\psi_{m+1} - \varsigma^*\| + (1 - \hat{\ell}_m)\|\psi_m - \varsigma^*\|.\end{aligned}$$

The assumption $\lim_{m\to\infty} \psi_m = \varsigma^*$ yields $\lim_{m\to\infty} \nu_m = 0$. Thus, the scheme (2.1) is $\mathcal{S}$-stable. □

3. Applications

Now, we shall implement our hybrid Picard S-type semi-implicit midpoint iterative method to investigate a general variational inequality and a fractional diffusion equation.

3.1. Variational inequality problem

Let $\emptyset \neq \mathbb{C} \subseteq \mathbb{X}$ be a CCS and $\mathcal{G} : \mathbb{C} \to \mathbb{C}$ be a nonlinear mapping. Stampacchia [41] introduced the variational inequality problem (VI($\mathbb{C}, \mathcal{G}$)) which seeks an element $\varsigma \in \mathbb{C}$ so that

$$(\forall \kappa \in \mathbb{C}), \quad \langle \mathcal{G}(\varsigma), \kappa - \varsigma \rangle \geq 0. \tag{3.1}$$

The obstacle problem is a fundamental example closely connected to the mathematical study of variational inequalities and free boundary problems. Obstacle problems are a special class of variational problems arising in the calculus of variations, which minimize functionals $\mathcal{F} : \mathbb{X} \to \mathbb{R}$ on infinite-dimensional function space $\mathbb{X}$. That is, find $\varsigma_0 \in \mathbb{X}$, such that $\mathcal{F}(\varsigma_0) = \min_{\varsigma\in\mathbb{X}} \mathcal{F}(\varsigma)$, where ς_0 is called a minimizer. Such variational problems occur in physics, where the functional $\mathcal{F}$ which is to be minimized represents physical quantities such as time or energy.

The notion of finite-dimensional VI was introduced in 1980 by Dafermos while modeling traffic network equilibrium. Cottle [18] recognized that finite-dimensional VI is a subsequent expansion of the complementarity problem. Moreover, problems encountered in everyday contexts such as economics, management science, operations research, and engineering are based on the key concept of equilibrium. Approaches used to formulate, analyze and compute equilibrium problems include systems of equations, optimization, complementarity, and fixed point theory. All these problems can be structured within a unified model of variational inequalities.

Since its inception, variational inequality theory has experienced remarkable growth. This area of research has been enriched by contemporary techniques and methodologies that address previously inaccessible fundamental problems. It has become a practical and effective tool for solving mathematical models in interdisciplinary sciences. Simultaneously, this development has interconnected areas of mathematics and applied sciences, including nonlinear programming, elasticity, transportation, operations research, economics, frictional contact problems, traffic equilibrium, and wireless and wireline systems, see, [5, 20, 36, 37]. Due to its versatility, the theory of variational inequality has been expanded by researchers by advancing theoretical insights and introducing diverse computational techniques. The general nonlinear variational inequality $GNVI(\mathbb{C},\mathcal{G},\zeta)$, a significant generalization of VI$(\mathbb{C},\mathcal{G})$ was introduced by Noor [33] as follows: Find $\varsigma\in\mathbb{C}$ such that

$$(\forall\kappa\in\mathbb{X},\zeta(\kappa),\zeta(\varsigma)\in\mathbb{C})\ \ \langle\mathcal{G}(\varsigma),\zeta(\kappa)-\zeta(\varsigma)\rangle\geq 0, \tag{3.2}$$

where $\mathcal{G},\zeta:\mathbb{X}\to\mathbb{X}$ are nonlinear mappings. We signify the solution set of $GNVI(\mathbb{C},\mathcal{G},\zeta)$ by $\Sigma(\mathbb{C},\mathcal{G},\zeta)=\{\varsigma\in\mathbb{C}:\langle\mathcal{G}(\varsigma),\zeta(\kappa)-\zeta(\varsigma)\rangle\geq 0,\forall\kappa\in\mathbb{X},\zeta(\kappa),\zeta(\varsigma)\in\mathbb{C}\}$. If $\zeta=I$ then $GNVI(\mathbb{C},\mathcal{G},\zeta)$ reduces to VI$(\mathbb{C},\mathcal{G})$. $GNVI(\mathbb{C},\mathcal{G},\zeta)$ can be structured as a general nonlinear complementarity problem as under: Find $\varsigma\in\mathbb{C}$ so that

$$\langle\mathcal{G}(\varsigma),\zeta(\varsigma)\rangle=0,\zeta(\varsigma)\in\mathbb{C},\mathcal{G}(\varsigma)\in\mathbb{C}^*, \tag{3.3}$$

where the dual cone $\mathbb{C}^*$ of $\mathbb{C}$ is given by $\mathbb{C}^*=\{\varsigma\in\mathbb{X}:\langle\varsigma,\kappa\rangle\geq 0,\forall\kappa\in\mathbb{C}\}$.

Now, we enumerate the following prominent definitions and results to accomplish the desired goal.

Definition 3.1. *[34] A mapping $\mathcal{G}:\mathbb{C}\subset\mathbb{X}\to\mathbb{X}$ is called*

(i) μ-inverse strongly monotone if $\exists\mu>0$, such that

$$(\forall\varsigma,\kappa\in\mathbb{C})\ \ \langle\mathcal{G}(\varsigma)-\mathcal{G}(\kappa),\varsigma-\kappa\rangle\geq\mu\|\mathcal{G}(\varsigma)-\mathcal{G}(\kappa)\|^2,$$

(i) relaxed (τ,ι)-cocoercive if $\exists\tau,\iota>0$, such that

$$(\forall\varsigma,\kappa\in\mathbb{C})\ \ \langle\mathcal{G}(\varsigma)-\mathcal{G}(\kappa),\varsigma-\kappa\rangle\geq(-\tau)\|\mathcal{G}(\varsigma)-\mathcal{G}(\kappa)\|^2+\iota\|\varsigma-\kappa\|^2,$$

(ii) L-Lipschitzian if $\exists L>0$, such that

$$(\forall\varsigma,\kappa\in\mathbb{C})\ \ \|\mathcal{G}(\varsigma)-\mathcal{G}(\kappa)\|\leq L\|\varsigma-\kappa\|.$$

$\mathcal{G}$ is non-expansive for $L=1$, and contraction if $L\in(0,1)$. Note that μ-inverse strongly monotone mapping is $\frac{1}{\mu}$-Lipschitzian. The class of relaxed (τ,ι)-cocoercive is a larger class that includes the classes of inverse strongly monotone and strongly monotone mappings. However, the converse statement is not true in general.

Lemma 3.1. *[9] Let $\mathcal{P}_{\mathbb{C}}:\mathbb{X}\to\mathbb{C}$ be a projection mapping. For any $\varsigma\in\mathbb{X}$ and $\kappa\in\mathbb{C}$,*

$$\mathcal{P}_{\mathbb{C}}[\varsigma]=\kappa\Leftrightarrow\langle\kappa-\varsigma,\rho-\kappa\rangle\geq 0,\forall\rho\in\mathbb{C}.$$

Clearly, $\mathcal{P}_{\mathbb{C}}$ is non-expansive. Next, by utilizing Lemma 3.1, we shall set up a fixed point problem analogous to $GNVI(\mathbb{C},\mathcal{G},\zeta)$.

Lemma 3.2. *An element $\varsigma^*\in\Sigma(\mathbb{C},\mathcal{G},\zeta)$ if and only if $\varsigma^*\in Fix(\mathcal{U})$, where $\mathcal{U}(\varsigma^*)=\varsigma^*-\zeta(\varsigma^*)+\mathcal{P}_{\mathbb{C}}[\zeta(\varsigma^*)-\mathcal{G}(\varsigma^*)]$.*

Proof. Suppose that $\varsigma^*\in\Sigma(\mathbb{C},\mathcal{G},\zeta)$ then for all $\kappa\in\mathbb{X},\zeta(\kappa),\zeta(\varsigma^*)\in\mathbb{C}$, we have $\langle\mathcal{G}(\varsigma^*),\zeta(\kappa)-\zeta(\varsigma^*)\rangle\geq 0$. Therefore, by Lemma 3.1, we obtain $\mathcal{P}_{\mathbb{C}}[\zeta(\varsigma^*)-\mathcal{G}(\varsigma^*)]=\zeta(\varsigma^*)$, which yields $\mathcal{U}(\varsigma^*)=\varsigma^*$. Conversely, suppose that $\varsigma^*\in Fix(\mathcal{U})$, i.e., $\mathcal{U}(\varsigma^*)=\varsigma^*$. Then, we obtain $\zeta(\varsigma^*)=\mathcal{P}_{\mathbb{C}}[\zeta(\varsigma^*)-\mathcal{G}(\varsigma^*)]$. Again, taking Lemma 3.1 into account, we obtain $\langle\mathcal{G}(\varsigma^*),\zeta(\kappa)-\zeta(\varsigma^*)\rangle\geq 0,\forall\kappa\in\mathbb{X},\zeta(\kappa),\zeta(\varsigma^*)\in\mathbb{C}$. Thus, we get $\varsigma^*\in\Sigma(\mathbb{C},\mathcal{G},\zeta)$. □

Now, based on the scheme (2.1) and taking advantage of Lemma 3.1, we shall re-design PSSIMPS (2.1) to observe $\varsigma \in \mathbb{C}$ such that $\varsigma \in Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta)$, where $\mathcal{S}$ is a contractive-like mapping defined in (1.2) and $\mathcal{G}, \zeta : \mathbb{X} \to \mathbb{X}$ are nonlinear mappings.

Algorithm 3.1. *For given $\varsigma_0 \in \mathbb{C}$, the sequence $\{\varsigma_m\}_{m=1}^{\infty}$ is generated by the following implicit iterative scheme:*

$$\begin{cases} \varsigma_{m+1} = \mathcal{S}(\rho_m - \zeta(\rho_m) + \mathcal{P}_{\mathbb{C}}[\zeta(\rho_m) - \lambda \mathcal{G}(\rho_m)]), \\ \rho_m = (1-\alpha_m)\mathcal{S}\Big(\frac{\varsigma_m + \rho_m}{2}\Big) + \alpha_m \mathcal{S}\Big(\frac{\omega_m + \rho_m}{2}\Big), \\ \omega_m = (1-\beta_m)\Big(\frac{\varsigma_m + \omega_m}{2}\Big) + \beta_m \mathcal{S}\Big(\frac{\varsigma_m + \omega_m}{2}\Big), m \in \mathbb{N}, \end{cases} \tag{3.4}$$

where $\{\alpha_m\}, \{\beta_m\} \subseteq (0,1)$.

Theorem 3.1. *Let $\emptyset \neq \mathbb{C} \subset \mathbb{X}$ be a closed, and bounded set. Let $\mathcal{G} : \mathbb{C} \to \mathbb{C}$ be a relaxed (τ, ι)-cocoercive and L-Lipschizian mapping, $\zeta : \mathbb{C} \to \mathbb{C}$ be η-inverse strongly monotone. If $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ satisfies (1.2) such that $Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta) \neq \emptyset$. In addition, the parameters satisfy the inequality:*

$$1 - \frac{1}{\eta} < \frac{1-\mathbb{Q}}{2}, \quad \mathbb{Q} = \sqrt{1 - 2\lambda(\iota - \tau L^2) + \lambda^2 L^2}, \lambda > 0. \tag{3.5}$$

Then $\{\varsigma_m\}$ generated by (3.4) converges strongly to $\varsigma^* \in Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta)$.

Proof. Since ζ is η-inverse strongly monotone and hence $\frac{1}{\eta}$-Lipschitz continuous, therefore

$$\begin{aligned} &\|\rho_m - \varsigma^* - (\zeta(\rho_m) - \zeta(\varsigma^*))\|^2 \\ &= \|\rho_m - \varsigma^*\|^2 - 2\langle \zeta(\rho_m) - \zeta(\varsigma^*), \rho_m - \varsigma^* \rangle + \|\zeta(\rho_m) - \zeta(\varsigma^*)\|^2 \\ &\leq \|\rho_m - \varsigma^*\|^2 - 2\eta\|\zeta(\rho_m) - \zeta(\varsigma^*)\|^2 + \|\zeta(\rho_m) - \zeta(\varsigma^*)\|^2 \\ &= \Big(\frac{\eta - 1}{\eta}\Big)^2 \|\rho_m - \varsigma^*\|^2 = \mathbb{P}^2 \|\rho_m - \varsigma^*\|^2. \end{aligned} \tag{3.6}$$

By the assumptions, observe that $\mathcal{G}$ is relaxed (τ, ι)-cocoercive and L-Lipschizian mapping, thus

$$\begin{aligned} &\|\rho_m - \varsigma^* - \lambda(\mathcal{G}(\rho_m) - \mathcal{G}(\varsigma^*))\|^2 \\ &= \|\rho_m - \varsigma^*\|^2 - 2\lambda\langle \mathcal{G}(\rho_m) - \mathcal{G}(\varsigma^*), \rho_m - \varsigma^* \rangle + \lambda^2\|\mathcal{G}(\rho_m) - \mathcal{G}(\varsigma^*)\|^2 \\ &\leq \|\rho_m - \varsigma^*\|^2 + 2\lambda\tau\|\mathcal{G}(\rho_m) - \mathcal{G}(\varsigma^*)\|^2 - 2\lambda\iota\|\rho_m - \varsigma^*\|^2 + \lambda^2\|\mathcal{G}(\rho_m) - \mathcal{G}(\varsigma^*)\|^2 \\ &\leq [1 - 2\lambda(\iota - \tau L^2) + \lambda^2 L^2]\|\rho_m - \varsigma^*\|^2 = \mathbb{Q}^2\|\rho_m - \varsigma^*\|^2. \end{aligned} \tag{3.7}$$

Now, we shall exhibit that $\varsigma_m \to \varsigma^* \in Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta)$. Then $\varsigma^* = \varsigma^* - \zeta(\varsigma^*) + \mathcal{P}_{\mathbb{C}}[\zeta(\varsigma^*) - \lambda\mathcal{G}(\varsigma^*)]$. It follows from the first formulation of (3.4) together with (3.6) and (3.7) that

$$\begin{aligned} \|\varsigma_{m+1} - \varsigma^*\| &= \|\mathcal{S}(\rho_m - \zeta(\rho_m) + \mathcal{P}_{\mathbb{C}}[\zeta(\rho_m) - \lambda\mathcal{G}(\rho_m)]) - \varsigma^*\| \\ &= \|\mathcal{S}(\varsigma^*) - \mathcal{S}[\rho_m - \zeta(\rho_m) + \mathcal{P}_{\mathbb{C}}[\zeta(\rho_m) - \lambda\mathcal{G}(\rho_m)]]\| \\ &\leq \psi(\|\varsigma^* - \mathcal{S}(\varsigma^*)\|) + \varepsilon\|\rho_m - \zeta(\rho_m) + \mathcal{P}_{\mathbb{C}}[\zeta(\rho_m) - \lambda\mathcal{G}(\rho_m)] - \varsigma^*\| \\ &= \varepsilon\|\rho_m - \zeta(\rho_m) + \mathcal{P}_{\mathbb{C}}[\zeta(\rho_m) - \lambda\mathcal{G}(\rho_m)] \\ &\quad - [\varsigma^* - \zeta(\varsigma^*) + \mathcal{P}_{\mathbb{C}}[\zeta(\varsigma^*) - \lambda\mathcal{G}(\varsigma^*)]]\| \\ &\leq 2\varepsilon\|\rho_m - \varsigma^* - (\zeta(\rho_m) - \zeta(\varsigma^*))\| + \varepsilon\|\rho_m - \varsigma^* - \lambda(\mathcal{G}(\rho_m) - \mathcal{G}(\varsigma^*))\| \\ &\leq \varepsilon(2\mathbb{P} + \mathbb{Q})\|\rho_m - \varsigma^*\|. \end{aligned} \tag{3.8}$$

Duplicating the procedures as from (2.2)–(2.6), and combining with (3.8), we obtain

$$\|\varsigma_{m+1} - \varsigma^*\| \leq (1 - \hat{\iota}_m)\varepsilon(2\mathbb{P} + \mathbb{Q})\|\varsigma_m - \varsigma^*\|, \tag{3.9}$$

where $\hat{\iota}_m = \dfrac{(2-\varepsilon) - \varepsilon^2}{(2-\varepsilon)}$. Since $\varepsilon \in [0, 1)$ and by the relation (3.5), we acquire $\varepsilon(2\mathbb{P} + \mathbb{Q}) < 1$. Consequently, (3.9) turns into

$$\|\varsigma_{m+1} - \varsigma^*\| \leq (1 - \hat{\iota}_m)\|\varsigma_m - \varsigma^*\|. \tag{3.10}$$

Utilizing Lemma 2.1 in (3.10), we achieve $\lim_{m\to\infty} \|\varsigma_m - \varsigma^*\| = 0$, i.e., $\{\varsigma_m\} \to \varsigma^* \in Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta)$. □

Note that one can take Lemma 3.1 into consideration to express the scheme (1.5) in the following form.

$$\begin{cases} \varsigma_{m+1} = \mathcal{S}(\rho_m - \zeta(\rho_m) + \mathcal{P}_{\mathbb{C}}[\zeta(\rho_m) - \lambda\mathcal{G}(\rho_m)]), \\ \rho_m = (1 - \alpha_m)\mathcal{S}\varsigma_m + \alpha_m\mathcal{S}\omega_m, \\ \omega_m = (1 - \beta_m)\varsigma_m + \beta_m\mathcal{S}\varsigma_m, m \in \mathbb{N}, \end{cases} \tag{3.11}$$

where $\{\alpha_m\}, \{\beta_m\} \subseteq (0, 1)$ and $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ is a contraction mapping defined in (2.9). Next, we can utilize the iterative scheme (3.11), to find an element $\varsigma \in \mathbb{C}$ so that $\varsigma \in Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta)$.

Corollary 3.1. *Let $\emptyset \neq \mathbb{C} \subset \mathbb{X}$ be a closed, and bounded set. Let $\mathcal{G} : \mathbb{C} \to \mathbb{C}$ be a relaxed (τ, ι)-cocoercive and L-Lipschizian mapping, $\zeta : \mathbb{C} \to \mathbb{C}$ be η-inverse strongly monotone. If $\mathcal{S} : \mathbb{C} \to \mathbb{C}$ satisfies (2.9) such that $Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta) \neq \emptyset$. In addition, the parameters satisfy the inequality:*

$$1 - \frac{1}{\eta} < \frac{1 - \mathbb{Q}}{2}, \quad \mathbb{Q} = \sqrt{1 - 2\lambda(\iota - \tau L^2) + \lambda^2 L^2}, \lambda > 0. \tag{3.12}$$

Then $\{\varsigma_m\}$ generated by (3.11) converges strongly to $\varsigma^* \in Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta)$.

Example 3.1. *Let $\mathbb{C} = [1, 2]$ with the usual norm on $\mathbb{R}$ and inner product $\langle\kappa, \varsigma\rangle = \kappa.\varsigma$. Define $\mathcal{G}(\varsigma) = \dfrac{\varsigma^2 + \varsigma}{5}, \zeta(\varsigma) = \dfrac{\varsigma^3}{11}$ and $\mathcal{S}(\varsigma) = \dfrac{\varsigma}{4} + \dfrac{3}{4}, \forall\varsigma \in \mathbb{C}$. Apparently, $1 \in Fix(\mathcal{S})$. Next, we corroborate that $\mathcal{S}$ satisfies (1.2). To manifest this, consider $\varepsilon = \dfrac{1}{4}$ and a strictly continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$, we estimate*

$$\begin{aligned} &\|\mathcal{S}(\varsigma) - \mathcal{S}(\kappa)\| - \varepsilon\|\varsigma - \kappa\| - \psi(\|\varsigma - \mathcal{S}(\varsigma)\|) \\ &= \frac{1}{4}|\varsigma - \kappa| - \frac{1}{4}|\varsigma - \kappa| - \psi(|\varsigma - \frac{\varsigma}{4} - \frac{3}{4}|) \\ &= -\psi(\frac{3}{4}|\varsigma - 1|) \leq 0, \end{aligned}$$

i.e., $\|\mathcal{S}(\varsigma) - \mathcal{S}(\kappa)\| \leq \psi(\|\varsigma - \mathcal{S}(\varsigma)\|) + \varepsilon|\varsigma - \kappa|$. Thus, $\mathcal{S}$ satisfies (1.2). Next, we shall estimate a common element ς^* such that $\varsigma^* \in Fix(\mathcal{S}) \cap \Sigma(\mathbb{C}, \mathcal{G}, \zeta)$. Now for all $\varsigma, \kappa \in \mathbb{C}$, we find

$$\begin{aligned} \langle\mathcal{G}(\varsigma) - \mathcal{G}(\kappa), \varsigma - \kappa\rangle &= \langle\frac{\varsigma^2 + \varsigma}{5} - \frac{\kappa^2 + \kappa}{5}, \varsigma - \kappa\rangle \\ &= \frac{(\varsigma - \kappa)^2}{5}(\varsigma + \kappa + 1), \\ &= \frac{(\varsigma - \kappa)^2}{5}(\varsigma + \kappa) + \frac{(\varsigma - \kappa)^2}{5}, \\ &\geq \frac{-1}{15}\|\mathcal{G}(\varsigma) - \mathcal{G}(\kappa)\|^2 + \|\varsigma - \kappa\|^2, \end{aligned}$$

also,

$$\|\mathcal{G}(\varsigma)-\mathcal{G}(\kappa)\| = \left|\frac{\varsigma^2+\varsigma}{5}-\frac{\kappa^2+\kappa}{5}\right| = \left|\frac{(\varsigma-\kappa)}{5}(\varsigma+\kappa+1)\right| \leq \|\varsigma-\kappa\|.$$

Thus, $\mathcal{G}$ is relaxed $(\frac{1}{15},1)$-cocoercive and 1-Lipschitz continuous.

$$\langle\zeta(\varsigma)-\zeta(\kappa),\varsigma-\kappa\rangle = \langle\frac{\varsigma^3}{11}-\frac{\kappa^3}{11},\varsigma-\kappa\rangle = \frac{(\varsigma-\kappa)^2}{11}(\varsigma^2+\kappa^2+\varsigma\kappa),$$
$$\|\zeta(\varsigma)-\zeta(\kappa)\|^2 = \left|\frac{(\varsigma-\kappa)}{11}(\varsigma^2+\kappa^2+\varsigma\kappa)\right|^2 = \frac{(\varsigma-\kappa)^2}{121}(\varsigma^2+\kappa^2+\varsigma\kappa)^2.$$

Thus, $\langle\zeta(\varsigma)-\zeta(\kappa),\varsigma-\kappa\rangle \geq \frac{11}{12}\|\zeta(\varsigma)-\zeta(\kappa)\|^2$, i.e., ζ is $\frac{11}{12}$-inverse strongly monotone. Also, for $\lambda=1$, the constants $\tau=\frac{1}{15}, \iota=1, L=1$ and $\eta=\frac{11}{12}$ fulfills the relation $1-\frac{1}{\eta}<\frac{1-\mathbb{Q}}{2}$ of Theorem 3.1, where $\mathbb{Q}=\sqrt{1-2\lambda(\iota-\tau L^2)+\lambda^2L^2}$. Finally, it remains to verify that $\varsigma^*=1\in\Sigma(\mathbb{C},\mathcal{G},\zeta)$. Thus, for $\varsigma^*=1\in\mathbb{C}$, we estimate

$$\begin{aligned}\langle\mathcal{G}(\varsigma^*),\zeta(\kappa)-\zeta(\varsigma^*)\rangle &= \langle\frac{\varsigma^{*2}+\varsigma^*}{5},\frac{\kappa^3}{11}-\frac{\varsigma^{*3}}{11}\rangle = \langle\frac{2}{5},\frac{\kappa^3}{11}-\frac{1}{11}\rangle\\ &= \frac{2}{55}(\kappa^3-1)\geq 0, \forall\kappa\in\mathbb{C}.\end{aligned}$$

Thus, $1\in Fix(\mathcal{S})\cap\Sigma(\mathbb{C},\mathcal{G},\zeta)$.

3.2. Fractional diffusion equation

The notion of fractional calculus was introduced by Niels Henrik Abel in 1823. The theory and its applications were widely expanded during the 19th and 20th centuries with different definitions of fractional derivatives and integrals introduced by various researchers. An independent foundation of the subject was laid by Liouville in 1832. Oliver Heaviside applied the fractional differential operator in real-life contexts to analyze electrical transmission. For substantial applications of the subject, see [2, 7,8,10,26]. Fractional derivatives are significant tools for analyzing the behavior of numerous physical phenomena. They are particularly useful in explaining the dynamics of oxygen diffusion. Fractional order derivatives can provide insight into the time-dependent behavior of oxygen concentration in tissues. Oxygen diffusion within cells and tissues is a vital process in the human body. It not only provides the energy to every cell but also has noteworthy physiological implications. Oxygen diffuses from the blood into cells and is utilized in producing energy in the mitochondria. It is also vital for balancing cellular metabolism.

The oxygen distribution process involves a series of diffusive and convective mechanisms. Convective oxygen transport is an active circulation process necessary for aerobic respiration and energy production in the body. Diffusion transport is the passive movement of oxygen from areas of higher concentration to areas of lower concentration. In reality, the oxygen that diffuses into the blood from air through the lungs has varying rates of consumption, see, [29]. Oxygen binds to haemoglobin in far greater amounts than it dissolves in blood plasma, and oxygenated blood is pumped to the capillaries via arteries, a process explained through haemoglobin-oxygen kinetics. The theory of energy transportation was introduced by Krogh [28], which is referred to as the Krogh tissue cylinder. This model explains that the oxygen tension (PO2) drives the transportation of oxygen in tissues. The mathematical model, involving a differential equation to describe oxygen diffusion, was jointly presented by the mathematician Erlang and Krogh. The solution of the model expresses oxygen tension in the tissue as a function of spatial position within the tissue cylinder.

Herein, we are interested in implementing our hybrid Picard S-type semi-implicit mid point scheme (2.1) to investigate the fractional diffusion model introduced by Srivastava and Rai [40]. The fractional diffusion equation involving the concentration of oxygen $\Xi(r,z,u)$ and the diffusion coefficient of oxygen d is given by the following relation:

$$\frac{\partial^{\xi}\Xi}{\partial t^{\xi}} - \lambda\frac{\partial^{\mu}\Xi}{\partial t^{\mu}} = \nabla(d\cdot\nabla\Xi) - \Theta, \xi, \mu \in (0,1], \tag{3.13}$$

where, $\Theta(r,z,u)$ is the rate of consumption per volume of tissue and $\frac{\partial^{\xi}\Xi}{\partial t^{\xi}}, 0 < \xi < 1$ determines the sub-diffusion process. The net amount of oxygen diffused into tissue is $\frac{\partial^{\xi}\Xi}{\partial t^{\xi}} - \lambda\frac{\partial^{\mu}\Xi}{\partial t^{\mu}}$ and λ is the time lag in the concentration of oxygen Ξ along the z-axis.

An analogous relation to Eq (3.13) is as under:

$$\Xi(r,z,t) = \Xi(r,z,0)\Big(1 - \lambda\frac{t^{\xi-\mu}}{\Gamma(\xi-\mu+1)}\Big) + \lambda\mathcal{D}_t^{-(\xi-\mu)}\Xi + \mathcal{D}_t^{-\xi}(\nabla(d\cdot\nabla\Xi)) - \mathcal{D}_t^{-\xi}\Theta. \tag{3.14}$$

Alternative formulation of (3.14) is as under:

$$\Psi(r,z,t) = \Psi(r,z,0)\Big(1 - \lambda\frac{t^{\xi-\mu}}{\Gamma(\xi-\mu+1)}\Big) + \lambda\mathcal{D}_t^{-(\xi-\mu)}\Psi + \mathcal{D}_t^{-\xi}(\nabla(d\cdot\nabla\Psi) - \Theta), \tag{3.15}$$

or equivalently,

$$\Psi(r,z,t) = \Omega(\Psi_0) + \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,\Psi(s),\Theta)ds, \tag{3.16}$$

where, $\Omega(\Psi_0) = \Psi(r,z,0)\Big[1 - \lambda\frac{t^{\xi-\mu}}{\Gamma(\xi-\mu+1)}\Big]$ and $\mathcal{H}(s,\Psi,\Theta) = \lambda\frac{\partial^{\mu}\Psi}{\partial t^{\mu}} + (\nabla(d\cdot\nabla\Psi) - \Theta)$.

Define the integral operator $\mathcal{S}$ by

$$\mathcal{S}\Psi(r,z,t) = \Omega(\Psi_0) + \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,\Psi(s),\Theta)ds, \tag{3.17}$$

and $\|\Psi\| = \sup_{t\in[0,T]}\{|\Psi(t)| : \Psi \in \mathbb{J}\}$, where $\mathbb{J} = ([0,T],\mathbb{R})$.

Proposition 3.1. *Suppose that the conditions given below are fulfilled:*

(a_1) There exists a constant $L_{\mathcal{H}} > 0$ so that

$$(\forall \pi_1, \pi_2 \in \mathbb{J}, t \in [0,T]), \quad |\mathcal{H}(t,\pi_1(t),\Theta) - \mathcal{H}(t,\pi_2(t),\Theta)| \le L_{\mathcal{H}}|\pi_1 - \pi_2|;$$

(a_2) $\frac{L_{\mathcal{H}}T}{\Gamma(\xi)} < 1.$

Then, the fractional diffusion Eq (3.13) has a unique solution.

Now, we are ready to accomplish the key result of this section.

Theorem 3.2. *Suppose that the assumptions (a_1)-(a_2) of the Proposition 3.1 hold. If the sequence $\{\varsigma_m\}_{m=1}^{\infty}$ is produced by PSSIMPS (2.1) and $\alpha_m \in (0,1)$ satisfies $\sum_{m=0}^{\infty} \alpha_m = \infty$. Then, the fractional diffusion model (3.13) has a unique solution ς^* and $\varsigma_m \to \varsigma^*$.*

Proof. Suppose that $\{\varsigma_m\}_{m=1}^{\infty}$ is generated by (2.1) and $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ is expressed as

$$\mathcal{S}\Psi(t) = \Omega(\Psi_0) + \frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}(s, \Psi(s), \Theta) ds.$$

We intend to show that $\varsigma_m \to \varsigma^*$ as $m \to \infty$. Utilizing the assumptions (a_1) and (a_2) of Proposition 3.1, it follows from (2.1) and (3.17) that

$$\begin{aligned}
\|\omega_m - \varsigma^*\| &= \Big\|\Big[(1-\beta_m)\Big(\frac{\varsigma_m + \omega_m}{2}\Big) + \beta_m \mathcal{S}\Big(\frac{\varsigma_m + \omega_m}{2}\Big)\Big] - \varsigma^*\Big\| \\
&\le (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \beta_m \Big\|\mathcal{S}(\varsigma^*) - \mathcal{S}\Big(\frac{\varsigma_m + \omega_m}{2}\Big)\Big\| \\
&\le (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \beta_m \sup_{t\in[0,T]} \Big|\mathcal{S}\Big(\frac{\varsigma_m(t) + \omega_m(t)}{2}\Big) - \mathcal{S}\varsigma^*(t)\Big| \\
&\le (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \beta_m \sup_{t\in[0,T]} \Big|\Omega(\Psi_0) + \frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}\Big(s, \Big(\frac{\varsigma_m(s) + \omega_m(s)}{2}\Big), \Theta\Big) ds \\
&\quad - \Omega(\Psi_0) - \frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}(s, \varsigma^*, \Theta) ds\Big| \\
&\le (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \frac{\beta_m}{\Gamma(\xi)} \sup_{t\in[0,T]} \Big| \int_0^t \mathcal{H}\Big(s, \Big(\frac{\varsigma_m(s) + \omega_m(s)}{2}\Big), \Theta\Big) \\
&\quad - \mathcal{H}(s, \varsigma^*, \Theta)\Big| ds \\
&\le (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \frac{\beta_m L_{\mathcal{H}}}{\Gamma(\xi)} \sup_{t\in[0,T]} \int_0^t \Big|\Big(\frac{\varsigma_m(s) + \omega_m(s)}{2}\Big) - \varsigma^*(s)\Big| ds \\
&\le (1-\beta_m)\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| + \frac{\beta_m L_{\mathcal{H}} T}{\Gamma(\xi)} \Big\|\Big(\frac{\varsigma_m + \omega_m}{2}\Big) - \varsigma^*\Big\| \\
&= \Big[1 - \beta_m\Big(1 - \frac{L_{\mathcal{H}} T}{\Gamma(\xi)}\Big)\Big]\Big\|\frac{\varsigma_m + \omega_m}{2} - \varsigma^*\Big\| \\
&= \frac{\varpi_m}{2}[\|\varsigma_m - \varsigma^*\| + \|\omega_m - \varsigma^*\|],
\end{aligned}$$

which implies that

$$\|\omega_m - \varsigma^*\| \le \frac{\varpi_m}{2 - \varpi_m} \|\varsigma_m - \varsigma^*\|, \tag{3.18}$$

where, $\varpi_m = 1 - \beta_m(1 - \nu)$ and $\nu = \frac{L_{\mathcal{H}} T}{\Gamma(\xi)}$. Again, from the second formulation of the scheme (2.1), (3.17) and employing the assumption of Proposition 3.1, we acquire

$$
\begin{aligned}
\|\rho_m - \varsigma^*\| &= \Big\|\Big[(1-\alpha_m)\mathcal{S}\Big(\frac{\varsigma_m + \rho_m}{2}\Big) + \alpha_m \mathcal{S}\Big(\frac{\omega_m + \rho_m}{2}\Big)\Big] - \varsigma^*\Big\| \\
&\leq (1-\alpha_m)\Big\|\mathcal{S}\Big(\frac{\varsigma_m + \rho_m}{2}\Big) - \varsigma^*\Big\| + \alpha_m\Big\|\mathcal{S}\Big(\frac{\omega_m + \rho_m}{2}\Big)\Big] - \varsigma^*\Big\| \\
&\leq (1-\alpha_m)\sup_{t\in[0,T]}\Big|\mathcal{S}\Big(\frac{\varsigma_m(t) + \rho_m(t)}{2}\Big) - \varsigma^*(t)\Big| + \alpha_m \sup_{t\in[0,T]}\Big|\mathcal{S}\Big(\frac{\omega_m(t) + \rho_m(t)}{2}\Big)\Big] - \varsigma^*(t)\Big| \\
&\leq (1-\alpha_m)\sup_{t\in[0,T]}\Big|\Omega(\Psi_0) + \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}\Big(s, \Big(\frac{\varsigma_m(s) + \rho_m(s)}{2}\Big), \Theta\Big)ds \\
&\quad - \Omega(\Psi_0) - \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s, \varsigma^*, \Theta)ds\Big| \\
&\quad + \alpha_m \sup_{t\in[0,T]}\Big|\Omega(\Psi_0) + \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}\Big(s, \Big(\frac{\omega_m(s) + \rho_m(s)}{2}\Big), \Theta\Big)ds \\
&\quad - \Omega(\Psi_0) - \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s, \varsigma^*, \Theta)ds\Big| \\
&\leq \frac{(1-\alpha_m)}{\Gamma(\xi)}\sup_{t\in[0,T]}\Big|\int_0^t \mathcal{H}\Big(s, \Big(\frac{\varsigma_m(s) + \rho_m(s)}{2}\Big), \Theta\Big) - \mathcal{H}(s, \varsigma^*, \Theta)\Big|ds \\
&\quad + \frac{\alpha_m}{\Gamma(\xi)}\sup_{t\in[0,T]}\Big|\int_0^t \mathcal{H}\Big(s, \Big(\frac{\omega_m(s) + \rho_m(s)}{2}\Big), \Theta\Big) - \mathcal{H}(s, \varsigma^*, \Theta)\Big|ds \\
&\leq \frac{(1-\alpha_m)L_{\mathcal{H}}}{\Gamma(\xi)}\sup_{t\in[0,T]}\int_0^t \Big|\Big(\frac{\varsigma_m(s) + \rho_m(s)}{2}\Big) - \varsigma^*(s)\Big|ds \\
&\quad + \frac{\alpha_m L_{\mathcal{H}}}{\Gamma(\xi)}\sup_{t\in[0,T]}\int_0^t \Big|\Big(\frac{\omega_m(s) + \rho_m(s)}{2}\Big) - \varsigma^*(s)\Big|ds \\
&\leq \frac{(1-\alpha_m)L_{\mathcal{H}}T}{\Gamma(\xi)}\Big\|\Big(\frac{\varsigma_m + \rho_m}{2}\Big) - \varsigma^*\Big\| + \frac{\alpha_m L_{\mathcal{H}}T}{\Gamma(\xi)}\Big\|\Big(\frac{\omega_m + \rho_m}{2}\Big) - \varsigma^*\Big\| \\
&\leq \frac{(1-\alpha_m)L_{\mathcal{H}}T}{2\Gamma(\xi)}\|\varsigma_m - \varsigma^*\| + \frac{\alpha_m L_{\mathcal{H}}T}{2\Gamma(\xi)}\|\omega_m - \varsigma^*\| + \frac{L_{\mathcal{H}}T}{2\Gamma(\xi)}\|\rho_m - \varsigma^*\| \\
&\leq \frac{(1-\alpha_m)\nu}{2}\|\varsigma_m - \varsigma^*\| + \frac{\alpha_m \nu \varpi_m}{2(2-\varpi_m)}\|\varsigma_m - \varsigma^*\| + \frac{\nu}{2}\|\rho_m - \varsigma^*\|.
\end{aligned}
$$

After simplification, it becomes

$$
\|\rho_m - \varsigma^*\| \leq \frac{\nu}{2-\nu}\Big[1 - \alpha_m\Big(1 - \frac{\varpi_m}{2-\varpi_m}\Big)\Big]\|\varsigma_m - \varsigma^*\|. \tag{3.19}
$$

Finally, the first relation of (2.1) along with (3.17) and the assumptions (a_1) and (a_2) of the Proposition

3.1, relation (3.19) yields

$$
\begin{aligned}
\|\varsigma_{m+1}-\varsigma^*\| &= \|\mathcal{S}(\rho_m)-\varsigma^*\| \\
&\leq \sup_{t\in[0,T]} |\mathcal{S}(\rho_m)(t)-\varsigma^*(t)| \\
&\leq \sup_{t\in[0,T]} \Big|\Omega(\Psi_0)+\frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,\rho_m(s),\Theta)ds \\
&\quad -\Omega(\Psi_0)-\frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,\varsigma^*,\Theta)ds\Big| \\
&\leq \frac{1}{\Gamma(\xi)} \sup_{t\in[0,T]} \Big|\frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,\rho_m(s),\Theta)-\mathcal{H}(s,\varsigma^*,\Theta)\Big|ds \\
&\leq \frac{L_{\mathcal{H}}}{\Gamma(\xi)} \sup_{t\in[0,T]} \int_0^t |\rho_m(s)-\varsigma^*| \\
&\leq \nu\|\rho_m(s)-\varsigma^*\| \\
&\leq \frac{\nu^2}{2-\nu}\Big[1-\alpha_m\Big(1-\frac{\varpi_m}{2-\varpi_m}\Big)\Big]\|\varsigma_m-\varsigma^*\|.
\end{aligned}
\tag{3.20}
$$

Assumption (a_2) guarantees that $\nu < 1$ and hence (3.20) turns into

$$\|\varsigma_{m+1}-\varsigma^*\| \leq \Big[1-\alpha_m\Big(1-\frac{\varpi_m}{2-\varpi_m}\Big)\Big]\|\varsigma_m-\varsigma^*\|. \tag{3.21}$$

By induction, we obtain

$$\|\varsigma_{m+1}-\varsigma^*\| \leq \|\varsigma_0-\varsigma^*\| \prod_{u=0}^{m} \Big[1-\alpha_u\Big(1-\frac{\varpi_m}{2-\varpi_m}\Big)\Big]. \tag{3.22}$$

Recalling the fact $1-s \leq e^{-s}$ for all $0 \leq s \leq 1$, we acquire

$$\|\varsigma_{m+1}-\varsigma^*\| \leq \|\varsigma_0-\varsigma^*\| \prod_{u=0}^{m} e^{-(1-\frac{\varpi_m}{2-\varpi_m})\sum_{u=0}^{m}\alpha_u}. \tag{3.23}$$

Taking the limit $m \to \infty$, we obtain $\lim_{m\to\infty} \|\varsigma_m-\varsigma^*\| = 0$. □

4. Concluding remarks

A hybrid Picard S-type semi-implicit midpoint iterative algorithm is developed and utilized to approximate the fixed point of a contractive-like mapping under suitable assumptions. A convergence theorem and the stability of the constructed method are presented. A comparative analysis is carried out by considering an illustrative example with different initial guesses, comparing our scheme (2.1) with scheme (1.5) which proves more efficient than many existing iterative schemes. Furthermore, a general variational inequality is studied. The proposed scheme is employed to examine a general variational inequality, and the newly constructed method is implemented to approximate the common element, which is the fixed point of a contractive-like mapping and the solution to a general variational inequality. In addition, a fractional diffusion equation is explored by utilizing our scheme (2.1).

The approximation of fixed points of generalized mappings-including enriched non-expansive mappings, Suzuki's generalized non-expansive mappings, asymptotically non-expansive mappings and totally asymptotically non-expansive mappings by implementing the hybrid Picard S-type semi-implicit midpoint iterative algorithm constitutes potential directions for future research.

Author contributions

D. F.: funding, writing review and editing; M. D.: conceptualization, writing review and editing; I. Alraddadi: supervision, M. A.: conceptualization, writing original draft preparation, writing review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

Acknowledgements

The first author acknowledges the Princess Nourah bint Abdulrahman University Researchers Supporting Project (Project Number PNURSP2025R174), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

Authors declare no conflicts of interest in this paper.

References

1. R. P. Agarwal, D. O'Regan, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, **8** (2007), 61–79.
2. R. Agarwal, S. Hristova, D. O'Regan, Basic concepts on Riemann-Liouville fractional differential equations with non-instantaneous impulses, *Symmetry*, **11** (2019), 614. https://doi.org/10.3390/sym11050614
3. M. O. Aibinu, P. Pillay, J. O. Olaleru, O. T. Mewomo, The implicit midpoint rule of nonexpansive mappings and applications in uniformly smooth Banach spaces, *J. Nonlinear Sci. Appl.*, **11** (2018), 1374–1391. https://doi.org/10.22436/jnsa.011.12.08
4. M. Akram, A four-step semi-implicit midpoint approximation scheme for fixed point with applications, *Eur. J. Pure Appl. Math.*, **18** (2025), 5744. https://doi.org/10.29020/nybg.ejpam.v18i1.5744
5. T. Alpcan, T. Başar, Distributed algorithms for Nash equilibria of flow control games, In: *Advances in dynamic games*, Boston, 2005, 473–498. https://doi.org/10.1007/0-8176-4429-6_26
6. M. A. Alghamdi, M. A. Alghamdi, N. Shahzad, H. K. Xu, The implicit midpoint rule for nonexpansive mappings, *Fixed Point Theory Appl.*, **2014** (2014), 96. https://doi.org/10.1186/1687-1812-2014-96

7. A. S. Alshehry, S. Mukhtar, H. S. Khan, R. Shah, Fixed-point theory and numerical analysis of an epidemic model with fractional calculus: Exploring dynamical behavior, *Open Phys.*, **21** (2023), 20230121. https://doi.org/10.1515/phys-2023-0121

8. R. T. Alqahtani, G. A. Okeke, C. I. Ugwuogor, A novel fixed point iterative process for multivalued mappings applied in solving a HIV model of fractional order, *Mathematics*, **13** (2025), 739. https://doi.org/10.3390/math13050739

9. C. Baiocchi, *Variational and quasivariational inequalities*, 1984.

10. D. Baleanu, S. Arshad, A. Jajarmi, W. Shokat, F. A. Ghassabzade, M. Wali, Dynamical behaviors and stability analysis of a generalied fractional model with a real case study, *J. Adv. Res.*, **48** (2023), 157–173. https://doi.org/10.1016/j.jare.2022.08.010

11. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fund. Math.*, **3** (1922), 133–181.

12. V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasi contractive operators, *Fixed Point Theory Appl.*, **2004** (2004), 716359. https://doi.org/10.1155/S1687182004311058

13. V. Berinde, On the approximation of fixed points of weak contractive mappings, *Carpath J. Math.*, **19** (2003), 7–22.

14. F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci. USA*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041

15. F. E. Browder, Nonlinear mappings of nonexpansive and accretive-type in Banach spaces, *Bull. Amer. Math. Soc.*, **73** (1967), 875–882.

16. C. E. Chidume, *Geometric properties of banach spaces and nonlinear iterations*, London: Springer, 2009.

17. J. A. Clarkson, Uniformly convex spaces, *T. Amer. Math. Soc.*, **40** (1936), 396–414. https://doi.org/10.2307/1989630

18. R. W. Cottle, J. S. Pang, R. E. Stone, The linear complementarity problem, SIAM, 1992.

19. D. Filali, M. Akram, M. Dilshad, A. A. Khidir, General semi-implicit midpoint approximation for fixed point of almost contraction mapping and applications, *Appl. Math. Sci. Eng.*, **32** (2024), 2365687. https://doi.org/10.1080/27690911.2024.2365687

20. F. Giannessi, A. Maugeri, *Variational inequalities and network equilibrium problems*, New York: Springer, 1995. https://doi.org/10.1007/978-1-4899-1358-6

21. D. Gohde, Zum Prinzip der contraktiven abbildung, *Math. Nachr.*, **30** (1965), 251–258. https://doi.org/10.1002/mana.19650300312

22. F. Gursoy, V. Karakaya, A Picard-S hybrid type iteration method for solving a differential equation with retarded argument, 2014. https://doi.org/10.48550/arXiv.1403.2546

23. C. O. Imoru, M. O. Olatinwo, On the stability of Picard and Mann iteration processes, *Carpathian J. Math.*, **19** (2003), 155–160.

24. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147–150. https://doi.org/10.1090/S0002-9939-1974-0336469-5

25. P. Jana, Anisotropic p-Laplace equations on long cylindrical domain, *Opuscula Math.*, **44** (2024), 249–265. https://doi.org/10.7494/OpMath.2024.44.2.249

26. A. Kajouni, A. Chafiki, K. Hilal, M. Oukessou, A new conformable fractional derivative and applications, *Int. J. Differ. Equ.*, **2021** (2021), 6245435. https://doi.org/10.1155/2021/6245435

27. S. H. Khan, A Picard-Mann hybrid iterative process, *Fixed Point Theory Appl.*, **2013** (2013), 69. https://doi.org/10.1186/1687-1812-2013-69

28. A. Krogh, The number and distribution of capillaries in muscles with calculations of the oxygen pressure head necessary for supplying the tissue, *J. Physiol.*, **52** (1919), 409–415. https://doi.org/10.1113/jphysiol.1919.sp001839

29. R. M. Leach, D. F. Treacher, ABC of oxygen, Oxygen transport-2. Tissue hypoxia, *BMJ*, **317** (1998), 1370–1373. https://doi.org/10.1136/bmj.317.7169.1370

30. P. Luo, G. Cai, Y. Shehu, The viscosity iterative algorithms for the implicit midpoint rule of nonexpansive mappings in uniformly smooth Banach spaces, *J. Inequal Appl.*, **2017** (2017), 154. https://doi.org/10.1186/s13660-017-1426-8

31. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510. https://doi.org/10.2307/2032162

32. M. A. Noor, Y. Yao, Three-step iterations for variational inequalities and nonexpansive mappings, *Appl. Math Comput.*, **190** (2007), 1312–1321. https://doi.org/10.1016/j.amc.2007.02.013

33. M. A. Noor, General variational inequalities, *Appl. Math. Lett.*, **1** (1988), 119–122.

34. M. A. Noor, General variational inequalities and nonexpansive mappings, *J. Math. Anal. Appl.*, **331** (2007), 810–822. https://doi.org/10.1016/j.jmaa.2006.09.039

35. G. A. Okeke, Convergence analysis of the Picard-Ishikawa hybrid iterative process with applications, *Afr. Mat.*, **30** (2019), 817–835. https://doi.org/10.1007/s13370-019-00686-z

36. H. E. Scarf, T. B. Hansen, *The computation of economic equilibria*, 1973.

37. G. Scutari, D. P. Palomar, F. Facchinei, J. S. Pang, Monotone games for cognitive radio systems, In: *Distributed decision making and control*, London: Springer, **417** (2012). 83–112. https://doi.org/10.1007/978-1-4471-2265-4_4

38. Y. Song, Y. Pei, A new modified semi-implicit midpoint rule for nonexpansive mappings and 2-generalized hybrid mappings, *J. Nonlinear Sci. Appl.*, **9** (2016), 6348–6363. https://doi.org/10.22436/jnsa.009.12.35

39. J. Srivastava, Introduction of new Picard-S hybrid iteration with application and some results for nonexpansive mappings, *Arab J. Math. Sci.*, **28** (2022), 61–76. https://doi.org/10.1108/AJMS-08-2020-0044

40. V. Srivastava, K. N. Rai, A multi-term fractional diffusion equation for oxygen delivery through a capillary to tissues, *Math. Comput. Model.*, **51** (2010), 616–624. https://doi.org/10.1016/j.mcm.2009.11.002

41. G. Stampacchia, Formes bilineaires coercivites sur les ensembles convexes, *C. R. Hebdomadaires Des Seances De L Academie Des Sci.*, **258** (1964), 4413–4416.

42. G. Szajnowska, M. Zima, Positive solutions to a third order nonlocal boundary value problem with a parameter, *Opuscula Math.*, **44** (2024), 267–283. https://doi.org/10.7494/OpMath.2024.44.2.267

43. X. Weng, Fixed point iteration for local strictly pseudo-contractive mapping, *Proc. Amer. Math. Soc.*, **113** (1991), 727–731. https://doi.org/10.1090/S0002-9939-1991-1086345-8

44. H. Y. Xu, H. Y. Lan, F. Zhang, General semi-implicit approximations with errors for common fixed points of nonexpansive type operators and applications to Stampacchia variational inequality, *Comp. Appl. Math.*, **41** (2022), 190. https://doi.org/10.1007/s40314-022-01890-7

45. H. K. Xu, M. A. Alghamdi, N. Shahzad, The implicit midpoint rule for nonexpansive mappings in Banach spaces, *Fixed Point Theory*, **17** (2016), 509–517.

46. H. K. Xu, M. A. Alghamdi, N. Shahzad, The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, **2015** (2015), 41. https://doi.org/10.1186/s13663-015-0282-9