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Research article

The offset octonion linear canonical transform: Differential properties and uncertainty principle

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Abstract: As information technology continues to advance rapidly, the field of signal processing is confronted with increasingly complex high-dimensional data processing. Traditional linear canonical transforms (LCT) show certain limitations when dealing with high-dimensional data. Therefore, researchers have begun to explore hypercomplex number fields, which can represent rotations in three-dimensional space more naturally and accurately, and have demonstrated significant advantages in signal processing tasks. Transformations based on quaternions and octonions, such as the quaternion Fourier transform (QFT) and the octonion Fourier transform (OFT), have been widely applied in areas such as signal detection, pattern recognition, and time-frequency analysis. Against this backdrop, this paper investigated the quaternion offset linear canonical transform (QOLCT) and its extension in the octonion domain—the offset octonion linear canonical transform (OOCLCT). First, the paper established the modulatory and differentiation properties of the QOLCT, which were missing in existing literature, laying a solid theoretical foundation for subsequent research. Then, based on the relationship between the QOLCT and the OOCLCT, multiple classical uncertainty principles within the framework of the OOCLCT were derived. These principles included the Heisenberg uncertainty principle, the sharp Hausdorff-Young inequality, the Matolcsi-Szucs uncertainty principle, and the Benedicks-Amrein-Berthier uncertainty principle. Through the derivation and verification of these uncertainty principles, this paper not only deepened the understanding of the OOCLCT theory but also provided new mathematical tools and methods for high-dimensional signal processing. Finally, we discussed the future research directions of the OOCLCT, providing a reference for subsequent studies.

Keywords: the offset octonion linear canonical transform; differential properties; uncertainty principle; the Heisenberg uncertainty principle; the Benedicks-Amrein-Berthier uncertainty principle
Mathematics Subject Classification: 26D10, 35A22

1. Introduction

The linear canonical transform (LCT) is a general framework that encompasses classical transforms, including the Fourier transform (FT) and the fractional Fourier transform (FrFT) [1]. It enables the transformation of signals between the time and frequency domains through parameterized linear operations. The LCT has been extensively applied in various fields, including optical imaging [2], signal processing, quantum mechanics, and nonstationary signal analysis [3].

With the development of information technology, the information environment has become increasingly complex. Since the LCT is limited to signals in the complex domain, it faces challenges when processing high-dimensional data. For example, it is difficult to naturally represent three-dimensional rotations, multi-channel signal coupling, and so on. This has prompted researchers to explore the hypercomplex domain for the effective processing of such high-dimensional data.

Quaternions, as a type of hypercomplex algebra, can provide a more natural and precise representation for three-dimensional spatial rotations and signal processing. Researchers have expanded their research scope to quaternions and proposed important transformations such as the quaternion Fourier transform (QFT) [4], the quaternion fractional Fourier transform (QFrFT) [5,6], the quaternion linear canonical transform (QLCT) [7,8], the quaternion offset linear canonical transform (QOLCT) [9,10], the quaternion windowed fractional Fourier transform (QWFRFT) [11], and the simplified dual-window quaternion linear canonical transform (RBWLCT) [12]. Octonions are a higher-dimensional and more complex algebraic structure, featuring non-commutativity and non-associativity, and they are more suitable for high-dimensional signal analysis. Researchers have further extended the transformations to the octonion domain [13–16]. These transformations not only offer greater flexibility but also demonstrate stronger adaptability in nonstationary signal analysis, thus showing great potential in applications such as time-frequency representation, signal recognition, and pattern classification.

The uncertainty principle [17] describes the inherent trade-off between temporal and frequency resolution in a signal. Considerable advancements have been achieved in the investigation of the uncertainty principle across multiple transform domains [18], including the QFT [19], the FrFT [20–23], the LCT [24,25], the QLCT [26,27], and the QOLCT [28,29]. Meanwhile, scholars have also examined the formulation of the uncertainty principle in the octonion domain [30–32] and its applications in multidimensional signal processing [33,34]. This extension to higher dimensions introduces more complex constraint forms to the uncertainty principle while also exhibiting unique application potential in areas such as mathematics and engineering, including multidimensional signal processing and high-dimensional time-frequency analysis.

The octonion offset linear canonical transform (OOCLCT) is a further extension of the linear canonical transform within the octonion domain. In recent times, significant advancements have been made in the development of its foundational theory. However, on the one hand, compared with the existing transforms, the octonion Fourier transform (OFT), the octonion linear canonical transform (OCLCT), and the OOCLCT of the offset parameters ‘m’ and ‘n’ can align the different local time-frequency components of nonstationary signals, improve the time-frequency resolution, multi-component separation ability, and the stability of parameter estimation under low signal-to-noise ratio, and have low computational cost while inheriting the advantages of the original algorithms. On the other hand, studies on the uncertainty principle associated with the OOCLCT remain at the preliminary stage. Therefore, establishing an uncertainty theorem for the OOCLCT not only enhances theoretical completeness but also offers novel mathematical tools for high-dimensional signal processing, thereby carrying both profound theoretical significance and practical application value.

The uncertainty principle is a core theory in signal processing, and its research holds significant theoretical and practical value. The OOCLCT has considerable application potential in complex signal processing, yet the research on the uncertainty principle in this domain remains insufficient. This paper aims to investigate the properties of the OOCLCT and its associated uncertainty principles. First, we present the definition of the OOCLCT and elaborate on its relationship with the OCLCT and the OFT. Subsequently, we derive the differential and modulation properties of the OOCLCT. Finally, we establish and verify several uncertainty principles within the framework of the OOCLCT.

The structure of this paper is as follows: In Section 2, we review fundamental concepts of octonion algebra. In Section 3, we define the OOCLCT and prove its modulatory and differential properties. In Section 4, we formulate and verify several uncertainty principles, including the Heisenberg uncertainty principle, the sharp Hausdorff-Young inequality, the Matolcsi-Szucs uncertainty principle, and the Benedicks-Amrein-Berthier uncertainty principle, all in the OOCLCT domain. Section 5 concludes the paper.

2. Preliminaries and definitions

2.1. Octonion algebra

The octonion algebra $\mathbb{O}$ is a nonassociative, noncommutative algebraic structure defined over the field of real numbers. Its standard form consists of one real unit element and seven imaginary units $\{e_1, e_2, \ldots, e_7\}$. This algebra can generally be expressed as:

$$o = k_0 + k_1e_1 + k_2e_2 + k_3e_3 + k_4e_4 + k_5e_5 + k_6e_6 + k_7e_7,$$

where $k_0, k_1, ..., k_7 \in \mathbb{R}$.

Every octonion o can be represented in the form of quaternions as

$$o = m + ne_4,$$

where, $m = k_0 + k_1e_1 + k_2e_2 + k_3e_3, n = k_4 + k_5e_1 + k_6e_2 + k_7e_3 \in \mathbb{H}$.

The conjugate of an octonion o can be formally defined as

$$\bar{o} = k_0 - k_1e_1 - k_2e_2 - k_3e_3 - k_4e_4 - k_5e_5 - k_6e_6 - k_7e_7.$$

For any octonion o, its norm is given by:

$$|o| = \sqrt{o\bar{o}} = \sqrt{\bar{o}o},$$

or

$$|o|^2 = \sum_{i=0}^{7} k_i^2.$$

The octonion-valued function $h(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{O}$ is defined by :

$$h(\mathbf{x}) = h_0(\mathbf{x}) + h_1(\mathbf{x})e_1 + \ldots + h_7(\mathbf{x})e_7,$$

where $h_j(\mathbf{x}) \in \mathbb{R},\ j = 0, 1, ..., 7$ and $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Table 1 summarizes the complete multiplication rules for octonion algebra, defining the products between its basis vectors.

Table 1. Multiplication rules for octonion algebra.

$*$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Note:
(1) Octonion basis vector $\{1, e_1, \cdots, e_7\}$.
(2) Non-commutativity: $e_i \cdot e_j = -e_j \cdot e_i (i \neq j, i, j = 1, \ldots, 7)$.
(3) Non-associativity: $\left(e_q e_p\right) e_l \neq e_q \left(e_p e_l\right) (q, p, l = 1, \ldots, 7)$.

The norm of a real-valued octonionic function on space $\mathcal{L}^p$ can be defined as

$$\|h\|_p = \left(\int_{\mathbb{R}^3} |h(\mathbf{x})|^p \mathrm{d}\mathbf{x}\right)^{1/p}.$$

2.2. *Offset octonion linear canonical transform*

Definition 1. [16] For $h \in \mathbb{R}^3, M_j = \begin{bmatrix} a_j & b_j & | & m_j \\ c_j & d_j & | & n_j \end{bmatrix}, j = 1, 2, 3, a_j, b_j, c_j, d_j, m_j, n_j \in \mathbb{R}$ is a matrix parameter satisfying $a_j d_j - b_j c_j = 1$. The three dimensional OOCLCT is defined by

$$O^O_{M_1,M_2,M_3}\{h(\mathbf{x})\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} h(\mathbf{x}) K^{e_1}_{M_1}(x_1, \omega_1) K^{e_2}_{M_2}(x_2, \omega_2) K^{e_4}_{M_3}(x_3, \omega_3) \mathrm{d}\mathbf{x}, \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, x_3), \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$. $K^{e_1}_{M_1}(x_1, \omega_1)$, $K^{e_2}_{M_2}(x_2, \omega_2)$, $K^{e_4}_{M_3}(x_3, \omega_3)$ are kernels, and they can be defined as:

$$K^{e_1}_{M_1}(x_1, \omega_1) = \begin{cases} \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{e_1}{2b_1}\left[a_1 x_1^2 - 2x_1(\omega_1 - m_1) - 2\omega_1(d_1 m_1 - b_1 n_1) + d_1\left(\omega_1^2 + m_1^2\right)\right]}, & b_1 \neq 0, \\ \sqrt{d_1} e^{e_1\left[\frac{c_1 d_1}{2}(\omega_1 - m_1)^2 + \omega_1 m_1\right]} \delta[x_1 - d_1(\omega_1 - m_1)], & b_1 = 0, \end{cases} \tag{2.2}$$

$$K^{e_2}_{M_2}(x_2, \omega_2) = \begin{cases} \frac{1}{\sqrt{2\pi|b_2|}} e^{\frac{e_2}{2b_2}\left[a_2 x_2^2 - 2x_2(\omega_2 - m_2) - 2\omega_2(d_2 m_2 - b_2 n_2) + d_2\left(\omega_2^2 + m_2^2\right)\right]}, & b_2 \neq 0, \\ \sqrt{d_2} e^{e_2\left[\frac{c_2 d_2}{2}(\omega_2 - m_2)^2 + \omega_2 m_2\right]} \delta[x_2 - d_2(\omega_2 - m_2)], & b_2 = 0, \end{cases} \tag{2.3}$$

$$K_{M_3}^{e_4}(x_3,\omega_3)=\begin{cases}\frac{1}{\sqrt{2\pi|b_3|}}e^{\frac{e_4}{2b_3}\left[a_3x_3^2-2x_3(\omega_3-m_3)-2\omega_3(d_3m_3-b_3n_3)+d_3\left(\omega_3^2+m_3^2\right)\right]}, & b_3\neq 0,\\ \sqrt{d_3}e^{e_4\left[\frac{c_3d_3}{2}(\omega_3-m_3)^2+\omega_3m_3\right]\delta[x_3-d_3(\omega_3-m_3)]}, & b_3=0,\end{cases}\tag{2.4}$$

where $\delta(\mathbf{x})$ is the Dirac function.

When $b_j=0$, $j=1,2,3$, the OOCLCT reduces to a chirp multiplication operator with no significant relevance to this work.

The inversion formula can be expressed as follows:

$$h(\mathbf{x})=\int_{\mathbb{R}^3}O^O_{M_1,M_2,M_3}\{h\}(\boldsymbol{\omega})\overline{K_{M_3}^{e_4}(\omega_3,x_3)K_{M_2}^{e_2}(\omega_2,x_2)K_{M_1}^{e_1}(\omega_1,x_1)}\mathrm{d}\boldsymbol{\omega},\tag{2.5}$$

where $M_j=\begin{bmatrix} a_j & b_j & | & m_j\\ c_j & d_j & | & n_j\end{bmatrix}$, $j=1,2,3,a_j,b_j,c_j,d_j,m_j,n_j\in\mathbb{R}$.

Remark 2.1. If $M=\begin{bmatrix} a_j & b_j & | & 0\\ c_j & d_j & | & 0\end{bmatrix}$, the OOCLCT reduces to the OCLCT:

$$\mathcal{L}^O_{M_1,M_2,M_3}\{h\}(\omega)=\int_{\mathbb{R}^3}h(\mathbf{x})K_{M_1}^{e_1}(x_1,\omega_1)K_{M_2}^{e_2}(x_2,\omega_2)K_{M_3}^{e_4}(x_3,\omega_3)\,\mathrm{d}\mathbf{x},$$

and the kernels are:

$$K_{M_j}^{e_i}(x_j,\omega_j)=\frac{1}{\sqrt{2\pi\left|b_j\right|}}e^{e_i(\frac{a_jx_j^2}{2b_j}-\frac{x_j\omega_j}{b_j}+\frac{d_j}{2b_j}\omega_j^2)},b_j\neq 0,$$

where $(j,i)=(1,1),(2,2),(3,4)$.

Remark 2.2. Let $M_j=\begin{bmatrix} a_j & -1 & | & m_j\\ -1 & d_j & | & n_j\end{bmatrix}$, $j=1,2,3$, the OOCLCT reduces to the OFT, and there are the following relations:

$$O^O_{M_1,M_2,M_3}\{h\}(\omega)=\frac{1}{\sqrt{(2\pi)^3}}F^O(h)\left(\frac{\omega_1}{2\pi},\frac{\omega_2}{2\pi},\frac{\omega_3}{2\pi}\right)e_7.$$

For the quaternionic form of the octonion-valued function h, we have[32]

$$\|h\|_2^2=\left\|\tilde{h}+e_4\hat{h}\right\|^2=\left\|\tilde{h}_e\right\|_2^2+\left\|\tilde{h}_o\right\|_2^2+\left\|\hat{h}_e\right\|_2^2+\left\|\hat{h}_o\right\|_2^2.\tag{2.6}$$

The relationship between the OOCLCT and the QOLCT is presented in [16], where the OOCLCT can be separated into four QOLCTs as follows

$$\begin{aligned}\left\|O^O_{M_1,M_2,M_3}\{h\}(\boldsymbol{\omega})\right\|^2=&\frac{1}{2\pi|b_3|}\left(\left\|O^Q_{M_1,M_2}\left\{\hat{h}_e\right\}(\boldsymbol{\omega})\right\|^2+\left\|O^Q_{M_1,M_2}\left\{\hat{h}_o\right\}(\boldsymbol{\omega})\right\|^2\right.\\&\left.+\left\|O^Q_{M_1,M_2}\left\{\tilde{h}_e\right\}(\boldsymbol{\omega})\right\|^2+\left\|O^Q_{M_1,M_2}\left\{\tilde{h}_o\right\}(\boldsymbol{\omega})\right\|^2\right).\end{aligned}\tag{2.7}$$

The Plancherel's equation for the OOCLCT is given as follows [16].

$$\|h\|_2^2=2\pi|b_3|\left\|O^O_{M_1,M_2,M_3}\right\|_2^2.\tag{2.8}$$

3. Properties of the OOCLCT

In this section, we present several properties of the OOCLCT, including its modulatory and differential characteristics in both the time and frequency domains.

Theorem 3.1. (Modulation) Let $h \in L^2\left(\mathbb{R}^3, \mathbb{O}\right)$ be an octonion-valued function. Suppose $\omega_0 \in \mathbb{R}$. The modulation operator is $\mathbb{M}_{\omega_0}^1 h(\mathbf{x}) = h(\mathbf{x})e^{-e_1 2\pi\omega_0 x_1}, \mathbb{M}_{\omega_0}^2 h(\mathbf{x}) = h(\mathbf{x})e^{-e_2 2\pi\omega_0 x_2}, \mathbb{M}_{\omega_0}^3 h(\mathbf{x}) = h(\mathbf{x})e^{-e_4 2\pi\omega_0 x_3}$, then

$$
\begin{aligned}
&O^O_{M_1,M_2,M_3}\left\{\mathbb{M}_{\omega_0}^1 h(\mathbf{x})\right\}(\boldsymbol{\omega})\\
=&\frac{1}{2}\left(O^O_{M_1,M_2,M_3}h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\sin\alpha_1+O^O_{M_1,M_2,M_3}h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\sin\beta_1\right)\\
&+\frac{1}{2}\left(O^O_{M_1,M_2,M_3}h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\cos\alpha_1-O^O_{M_1,M_2,M_3}h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\cos\beta_1\right)\\
&+\frac{1}{2}\left(O^O_{M_1,M_2',M_3'}h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\cos\alpha_1 e_1+O^O_{M_1,M_2',M_3'}h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\cos\beta_1 e_1\right)\\
&-\frac{1}{2}\left(O^O_{M_1,M_2',M_3'}h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\sin\alpha_1 e_1-O^O_{M_1,M_2',M_3'}h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\sin\beta_1 e_1\right),
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&O^O_{M_1,M_2,M_3}\left\{\mathbb{M}_{\omega_0}^2 h(\mathbf{x})\right\}(\boldsymbol{\omega})\\
=&\frac{1}{2}\left(O^O_{M_1,M_2,M_3}h(\mathbf{x})(\omega_1,\omega_2+2\pi b_2\omega_0,\omega_3)\sin\alpha_2+O^O_{M_1,M_2,M_3}h(\mathbf{x})(\omega_1,\omega_2-2\pi b_2\omega_0,\omega_3)\sin\beta_2\right)\\
&-\frac{1}{2}\left(O^O_{M_1',M_2,M_3}h(\mathbf{x})(\omega_1,\omega_2+2\pi b_2\omega_0,\omega_3)\cos\alpha_2-O^O_{M_1',M_2,M_3}h(\mathbf{x})(\omega_1,\omega_2-2\pi b_2\omega_0,\omega_3)\cos\beta_2\right)\\
&-\frac{1}{2}\left(O^O_{M_1,M_2,M_3'}h(\mathbf{x})(\omega_1,\omega_2+2\pi b_2\omega_0,\omega_3)\cos\alpha_2 e_2+O^O_{M_1,M_2,M_3'}h(\mathbf{x})(\omega_1,\omega_2-2\pi b_2\omega_0,\omega_3)\cos\beta_2 e_2\right)\\
&+\frac{1}{2}\left(O^O_{M_1',M_2,M_3'}h(\mathbf{x})(\omega_1,\omega_2+2\pi b_2\omega_0,\omega_3)\sin\alpha_2 e_2-O^O_{M_1',M_2,M_3'}h(\mathbf{x})(\omega_1,\omega_2-2\pi b_2\omega_0,\omega_3)\sin\beta_2 e_2\right),
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
&O^O_{M_1,M_2,M_3}\left\{\mathbb{M}_{\omega_0}^3 h(\mathbf{x})\right\}(\boldsymbol{\omega})\\
=&\frac{1}{2}\left(O^O_{M_1',M_2',M_3}h(\mathbf{x})(\omega_1,\omega_2,\omega_3+2\pi b_3\omega_0)e^{-e_4\alpha_3}-O^O_{M_1',M_2',M_3}h(\mathbf{x})(\omega_1,\omega_2,\omega_3-2\pi b_3\omega_0)e^{-e_4\beta_3}\right)\\
&+\frac{1}{2}\left(O^O_{M_1,M_2,M_3}h(\mathbf{x})(\omega_1,\omega_2,\omega_3+2\pi b_3\omega_0)e^{-e_4\alpha_3}+O^O_{M_1,M_2,M_3}h(\mathbf{x})(\omega_1,\omega_2,\omega_3-2\pi b_3\omega_0)e^{-e_4\beta_3}\right)e_4,
\end{aligned}
\tag{3.3}
$$

where $M_j' = \left(a_j, -b_j; -c_j, d_j\right), \alpha_j = \frac{1}{2b_j}\left[4\pi b_j d_j\omega_0\omega_j + 4\pi^2 b_j^2 d_j\omega_0^2 - 4\pi b_j\omega_0\left(d_j m_j - b_j n_j\right)\right], \beta_j = \frac{1}{2b_j}\left[4\pi^2 b_j^2 d_j\omega_0^2 - 4\pi b_j d_j\omega_0\omega_j + 4\pi b_j\omega_0\left(d_j m_j - b_j n_j\right)\right], j = 1, 2, 3.$

Proof. We prove the first equation. According to the definition and Euler's formula , we get

$$
e^{-e_1 2\pi\omega_0 x_1} = \cos\left(2\pi\omega_0 x_1\right) - e_1\sin\left(2\pi\omega_0 x_1\right),
$$

then we have

$$
\begin{aligned}
O^{O}_{M_1,M_2,M_3}\{\mathbb{M}^1_{\omega_0}h(\mathbf{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} h(\mathbf{x})\, e^{-e_1 2\pi\omega_0 x_1} K^{e_1}_{M_1}(x_1,\omega_1)\, K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^3} h(\mathbf{x}) \cos(2\pi\omega_0 x_1)\, K^{e_1}_{M_1}(x_1,\omega_1) K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&\quad - \int_{\mathbb{R}^3} h(\mathbf{x}) \sin(2\pi\omega_0 x_1)\, e_1 K^{e_1}_{M_1}(x_1,\omega_1) K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x}.
\end{aligned}
$$

We prove the previous equation,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} h(\mathbf{x}) \cos(2\pi\omega_0 x_1)\, K^{e_1}_{M_1}(x_1,\omega_1) K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x}) \left(e^{-e_1 2\pi x_1\omega_0} + e^{e_1 2\pi x_1 \omega_0}\right) e_1 K^{e_1}_{M_1}(x_1,\omega_1)\, K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x})\, e^{-e_1 2\pi x_1\omega_0} e_1 \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{e_1}{2b_1}\left[a_1x_1^2 - 2x_1(\omega_1 - m_1) - 2\omega_1(d_1m_1 - b_1n_1) + d_1\left(\omega_1^2 + m_1^2\right)\right]} K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x})\, e^{e_1 2\pi x_1\omega_0} e_1 \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{e_1}{2b_1}\left[a_1x_1^2 - 2x_1(\omega_1 - m_1) - 2\omega_1(d_1m_1 - b_1n_1) + d_1\left(\omega_1^2 + m_1^2\right)\right]} K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x}) e_1 \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{e_1}{2b_1}\left[4\pi b_1\omega_0(d_1m_1 - b_1n_1) - 4\pi^2 {b_1}^2 d_1 {\omega_0}^2 - 4\pi b_1 d_1 \omega_0\omega_1\right]} \\
&\quad \cdot e^{\frac{e_1}{2b_1}\left[a_1x_1^2 - 2x_1[(\omega_1 + 2\pi b_1\omega_0) - m_1] - 2(\omega_1 + 2\pi b_1\omega_0)(d_1m_1 - b_1n_1) + d_1\left[(\omega_1 + 2\pi b_1\omega_0)^2 + m_1^2\right]\right]} K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x}) e_1 \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{e_1}{2b_1}\left[4\pi b_1 d_1 \omega_0\omega_1 - 4\pi^2 {b_1}^2 d_1 {\omega_0}^2 - 4\pi b_1\omega_0(d_1m_1 - b_1n_1)\right]} \\
&\quad \cdot e^{\frac{e_1}{2b_1}\left[a_1x_1^2 - 2x_1[(\omega_1 - 2\pi b_1\omega_0) - m_1] - 2(\omega_1 - 2\pi b_1\omega_0)(d_1m_1 - b_1n_1) + d_1\left[(\omega_1 - 2\pi b_1\omega_0)^2 + m_1^2\right]\right]} K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x}.
\end{aligned}
$$

Let $\alpha_j = \frac{1}{2b_j}\left[4\pi b_j d_j \omega_0\omega_j + 4\pi^2 b_j^2 d_j \omega_0^2 - 4\pi b_j\omega_0\left(d_jm_j - b_jn_j\right)\right]$, $j = 1,2,3, \beta_j = \frac{1}{2b_j}\left[4\pi^2 b_j^2 d_j\omega_0^2 \right.$ $\left.-4\pi b_j d_j\omega_0\omega_j + 4\pi b_j\omega_0\left(d_jm_j - b_jn_j\right)\right]$, and we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} h(\mathbf{x}) \cos(2\pi\omega_0 x_1)\, K^{e_1}_{M_1}(x_1,\omega_1) K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x})\, e^{-e_1\alpha_1} e_1 K^{e_1}_{M_1}(x_1,\omega_1 + 2\pi b_1\omega_0) K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x})\, e^{-e_1\beta_1} e_1 K^{e_1}_{M_1}(x_1,\omega_1 - 2\pi b_1\omega_0) K^{e_2}_{M_2}(x_2,\omega_2)\, K^{e_4}_{M_3}(x_3,\omega_3)\, \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}(\cos\alpha_1 e_1 + \sin\alpha_1)\, O^{O}_{M_1,M_2,M_3} h(\omega_1 + 2\pi b_1\omega_0, \omega_2, \omega_3) \\
&\quad + \frac{1}{2}(\cos\beta_1 e_1 + \sin\beta_1)\, O^{O}_{M_1,M_2,M_3} h(\omega_1 - 2\pi b_1\omega_0, \omega_2, \omega_3) \\
&= \frac{1}{2} e_1 \left(O^{O}_{M_1,M_2,M_3} h(\omega_1 + 2\pi b_1\omega_0, \omega_2, \omega_3)\cos\alpha_1 + O^{O}_{M_1,M_2,M_3} h(\omega_1 - 2\pi b_1\omega_0, \omega_2, \omega_3)\cos\beta_1\right) \\
&\quad + \frac{1}{2}\left(O^{O}_{M_1,M_2,M_3} h(\omega_1 + 2\pi b_1\omega_0, \omega_2, \omega_3)\sin\alpha_1 + O^{O}_{M_1,M_2,M_3} h(\omega_1 - 2\pi b_1\omega_0, \omega_2, \omega_3)\sin\beta_1\right).
\end{aligned}
$$

Then, we prove the latter formula

$$\begin{aligned}
&\int_{\mathbb{R}^3} h(\mathbf{x}) \sin(2\pi\omega_0 x_1) K_{M_1}^{e_1}(x_1,\omega_1) K_{M_2}^{e_2}(x_2,\omega_2) K_{M_3}^{e_4}(x_3,\omega_3)\,\mathrm{d}\mathbf{x}\\
=&\frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x}) \left(e^{-e_1 2\pi\omega_0 x_1} - e^{e_1 2\pi\omega_0 x_1}\right) e_1 K_{M_1}^{e_1}(x_1,\omega_1) K_{M_2}^{e_2}(x_2,\omega_2) K_{M_3}^{e_4}(x_3,\omega_3)\,\mathrm{d}\mathbf{x}\\
=&\frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x}) e^{-e_1\alpha_1} e_1 K_{M_1}^{e_1}(x_1,\omega_1+2\pi b_1\omega_0) K_{M_2}^{e_2}(x_2,\omega_2) K_{M_3}^{e_4}(x_3,\omega_3)\,\mathrm{d}\mathbf{x}\\
&-\frac{1}{2}\int_{\mathbb{R}^3} h(\mathbf{x}) e^{-e_1\beta_1} e_1 K_{M_1}^{e_1}(x_1,\omega_1-2\pi b_1\omega_0) K_{M_2}^{e_2}(x_2,\omega_2) K_{M_3}^{e_4}(x_3,\omega_3)\,\mathrm{d}\mathbf{x}\\
=&\frac{1}{2}(\cos\alpha_1 e_1 + \sin\alpha_1) O_{M_1,M_2,M_3}^{O} h(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\\
&-\frac{1}{2}(\cos\beta_1 e_1 + \sin\beta_1) O_{M_1,M_2,M_3}^{O} h(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\\
=&\frac{1}{2}e_1\left(O_{M_1,M_2,M_3}^{O} h(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\cos\alpha_1 - O_{M_1,M_2,M_3}^{O} h(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\cos\beta_1\right)\\
&+\frac{1}{2}\left(O_{M_1,M_2,M_3}^{O} h(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\sin\alpha_1 - O_{M_1,M_2,M_3}^{O} h(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\sin\beta_1\right).
\end{aligned}$$

From Table 1, we get

$$e_1 K_{M_1}^{e_1}(x_1,\omega_1) K_{M_2}^{e_2}(x_2,\omega_2) K_{M_3}^{e_4}(x_3,\omega_3) = K_{M_1}^{e_1}(x_1,\omega_1) K_{M_{2'}}^{e_2}(x_2,\omega_2) K_{M_{3'}}^{e_4}(x_3,\omega_3) e_1.$$

Then,

$$\begin{aligned}
&O_{M_1,M_2,M_3}^{O}\{\mathbb{M}_{\omega_0}^{1} h(\mathbf{x})\}(\boldsymbol{\omega})\\
=&\frac{1}{2}e_1\left(O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\cos\alpha_1 + O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\cos\beta_1\right)\\
&+\frac{1}{2}\left(O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\sin\alpha_1 + O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\sin\beta_1\right)\\
&-\frac{1}{2}e_1 e_1\left(O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\cos\alpha_1 - O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\cos\beta_1\right)\\
&-\frac{1}{2}e_1\left(O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\sin\alpha_1 - O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\sin\beta_1\right)\\
=&\frac{1}{2}\left(O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\sin\alpha_1 + O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\sin\beta_1\right)\\
&+\frac{1}{2}\left(O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\cos\alpha_1 - O_{M_1,M_2,M_3}^{O} h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\cos\beta_1\right)\\
&+\frac{1}{2}\left(O_{M_1,M_{2'},M_{3'}}^{O} h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\cos\alpha_1 e_1 + O_{M_1,M_{2'},M_{3'}}^{O} h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\cos\beta_1 e_1\right)\\
&-\frac{1}{2}\left(O_{M_1,M_{2'},M_{3'}}^{O} h(\mathbf{x})(\omega_1+2\pi b_1\omega_0,\omega_2,\omega_3)\sin\alpha_1 e_1 - O_{M_1,M_{2'},M_{3'}}^{O} h(\mathbf{x})(\omega_1-2\pi b_1\omega_0,\omega_2,\omega_3)\sin\beta_1 e_1\right).
\end{aligned}$$

The proof has been completed. The other two can be derived in the same way.

Next, we prove the differential properties of the OOCLCT.

Theorem 3.2. Let $h \in S\left(\mathbb{R}^3;\mathbb{O}\right)$, and the differential properties of the OOCLCT are as follows:

$$O_{M_1,M_2,M_3}^{O}\left\{\frac{\partial h}{\partial x_1}\right\}(\boldsymbol{\omega}) = \left[\frac{a_1}{b_1}x_1 - \frac{\omega_1 - m_1}{b_1}\right]\hat{O}_{M_1,M_2,M_3}^{O}\{h\}(\boldsymbol{\omega})(-e_1), \tag{3.4}$$

$$O^{O}_{M_1,M_2,M_3}\left\{\frac{\partial h}{\partial x_2}\right\}(\omega)=\left[\frac{a_2}{b_2}x_2-\frac{\omega_2-m_2}{b_2}\right]\hat{O}^{O}_{M_1,M_2,M_3}\{h\}(\omega)(-e_2), \tag{3.5}$$

$$O^{O}_{M_1,M_2,M_3}\left\{\frac{\partial h}{\partial x_3}\right\}(\omega)=\left[\frac{a_3}{b_3}x_3-\frac{\omega_3-m_3}{b_3}\right]\hat{O}^{O}_{M_1,M_2,M_3}\{h\}(\omega)(-e_4), \tag{3.6}$$

where

$$\begin{aligned}\hat{O}^{O}_{M_1,M_2,M_3}\{h\}(\omega)=&\Psi_0\{h\}(\omega)-\Psi_1\{h\}(\omega)e_1+\Psi_2\{h\}(\omega)e_2-\Psi_3\{h\}(\omega)e_3\\&+\Psi_4\{h\}(\omega)e_4-\Psi_5\{h\}(\omega)e_5+\Psi_6\{h\}(\omega)e_6-\Psi_7\{h\}(\omega)e_7.\end{aligned} \tag{3.7}$$

Proof. We prove the second equation. The OOCLCT of $h:\mathbb{R}^3\to\mathbb{O}$ can be written as [31]:

$$\begin{aligned}O^{O}_{M_1,M_2,M_3}\{h\}=&\Psi_0(\omega)+\Psi_1(\omega)e_1+\Psi_2(\omega)e_2+\Psi_3(\omega)e_3\\&+\Psi_4(\omega)e_4+\Psi_5(\omega)e_5+\Psi_6(\omega)e_6+\Psi_7(\omega)e_7,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\Psi_0(\omega)=&\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}h_{eee}(\mathbf{x})\cos\zeta_1\cos\zeta_2\cos\zeta_3\mathrm{d}\mathbf{x},\\
\Psi_1(\omega)=&\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}h_{oee}(\mathbf{x})\sin\zeta_1\cos\zeta_2\cos\zeta_3\mathrm{d}\mathbf{x},\\
\Psi_2(\omega)=&\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}h_{eoe}(\mathbf{x})\cos\zeta_1\sin\zeta_2\cos\zeta_3\mathrm{d}\mathbf{x},\\
\Psi_3(\omega)=&\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}h_{ooe}(\mathbf{x})\sin\zeta_1\sin\zeta_2\cos\zeta_3\mathrm{d}\mathbf{x},\\
\Psi_4(\omega)=&\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}h_{eeo}(\mathbf{x})\cos\zeta_1\cos\zeta_2\sin\zeta_3\mathrm{d}\mathbf{x},\\
\Psi_5(\omega)=&\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}h_{oeo}(\mathbf{x})\sin\zeta_1\cos\zeta_2\sin\zeta_3\mathrm{d}\mathbf{x},\\
\Psi_6(\omega)=&\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}h_{eoo}(\mathbf{x})\cos\zeta_1\sin\zeta_2\sin\zeta_3\mathrm{d}\mathbf{x},\\
\Psi_7(\omega)=&\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}h_{ooo}(\mathbf{x})\sin\zeta_1\sin\zeta_2\sin\zeta_3\mathrm{d}\mathbf{x},
\end{aligned}$$

where $\zeta_j=\frac{1}{2b_j}\left[a_jx_j^2-2x_j\left(\omega_j-m_j\right)-2\omega_j\left(d_jm_j-b_jn_j\right)+d_j\left(\omega_j^2+m_j^2\right)\right]$, $j=1,2,3$.

The subscripts ‘e’ and ‘o’ are used to indicate that a function is either even (‘e’) or odd (‘o’) with respect to an appropriate variable, such as $h_{oee}(\mathbf{x})$ is odd for x_1 and even for x_2 and x_3.

According to Eq (3.8), $O^{O}_{M_1,M_2,M_3}\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)$ can be expressed in the following form.

$$\begin{aligned}O^{O}_{M_1,M_2,M_3}\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=&\Psi_0\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)+\Psi_1\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)e_1+\Psi_2\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)e_2+\Psi_3\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)e_3\\&+\Psi_4\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)e_4+\Psi_5\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)e_5+\Psi_6\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)e_6+\Psi_7\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)e_7,\end{aligned}$$

where

$$\Psi_0\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}\frac{\partial h(\mathbf{x})}{\partial x_1}\cos\zeta_1\cos\zeta_2\cos\zeta_3\mathrm{d}\mathbf{x},$$
$$\Psi_1\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}\frac{\partial h(\mathbf{x})}{\partial x_1}\sin\zeta_1\cos\zeta_2\cos\zeta_3\mathrm{d}\mathbf{x},$$
$$\Psi_2\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}\frac{\partial h(\mathbf{x})}{\partial x_1}\cos\zeta_1\sin\zeta_2\cos\zeta_3\mathrm{d}\mathbf{x},$$
$$\Psi_3\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}\frac{\partial h(\mathbf{x})}{\partial x_1}\sin\zeta_1\sin\zeta_2\cos\zeta_3\mathrm{d}\mathbf{x},$$

$$\Psi_4\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}\frac{\partial h(\mathbf{x})}{\partial x_1}\cos\zeta_1\cos\zeta_2\sin\zeta_3\mathrm{d}\mathbf{x},$$

$$\Psi_5\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}\frac{\partial h(\mathbf{x})}{\partial x_1}\sin\zeta_1\cos\zeta_2\sin\zeta_3\mathrm{d}\mathbf{x},$$

$$\Psi_6\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}\frac{\partial h(\mathbf{x})}{\partial x_1}\cos\zeta_1\sin\zeta_2\sin\zeta_3\mathrm{d}\mathbf{x},$$

$$\Psi_7\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\frac{1}{\sqrt{(2\pi)^3|b_1b_2b_3|}}\int_{\mathbb{R}^3}\frac{\partial h(\mathbf{x})}{\partial x_1}\sin\zeta_1\sin\zeta_2\sin\zeta_3\mathrm{d}\mathbf{x}.$$

Apply the step-by-step integration method and $\lim\limits_{x_1\to\pm\infty}h(\mathbf{x})=0$, and we have

$$\Psi_0\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)\Psi_1\{h\}(\omega),$$
$$\Psi_1\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=-\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)\Psi_0\{h\}(\omega),$$
$$\Psi_2\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)\Psi_3\{h\}(\omega),$$
$$\Psi_3\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=-\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)\Psi_2\{h\}(\omega),$$
$$\Psi_4\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)\Psi_5\{h\}(\omega),$$
$$\Psi_5\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=-\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)\Psi_4\{h\}(\omega),$$
$$\Psi_6\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)\Psi_7\{h\}(\omega),$$
$$\Psi_7\left\{\frac{\partial h}{\partial x_1}\right\}(\omega)=-\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)\Psi_6\{h\}(\omega).$$

Then, we have

$$
\begin{aligned}
O^{O}_{M_1,M_2,M_3}\left\{\frac{\partial h}{\partial x_1}\right\}(\omega) =&\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)(\Psi_1\{h\}(\omega)-\Psi_0\{h\}(\omega)e_1+\Psi_3\{h\}(\omega)e_2\\
&-\Psi_2\{h\}(\omega)e_3+\Psi_5\{h\}(\omega)e_4-\Psi_4\{h\}(\omega)e_5+\Psi_7\{h\}(\omega)e_6-\Psi_6\{h\}(\omega)e_7)\\
=&\left(\frac{a_1}{b_1}x_1-\frac{(\omega_1-m_1)}{b_1}\right)(\Psi_0\{h\}(\omega)-\Psi_1\{h\}(\omega)e_1+\Psi_2\{h\}(\omega)e_2-\Psi_3\{h\}(\omega)e_3\\
&+\Psi_4\{h\}(\omega)e_4-\Psi_5\{h\}(\omega)e_5+\Psi_6\{h\}(\omega)e_6-\Psi_7\{h\}(\omega)e_7)(-e_1).
\end{aligned}
$$

The proof has been completed. The other two can be proven using the same method.

Corollary 3.1. Let $O^{O}_{M_1M_2M_3}\{\frac{\partial^2 h}{\partial x_j^2}\}$ denote the OOCLCT of $\frac{\partial^2 h}{\partial x_j^2}$. Then, we obtain

$$
O^{O}_{M_1,M_2,M_3}\{\frac{\partial^2 h}{\partial x_1^2}\}(\omega)=\left[-\left(\frac{a_1}{b_1}\right)^2x_1^2+\frac{2a_1(\omega_1-m_1)}{b_1}x_1-\left(\frac{\omega_1-m_1}{b_1}\right)^2\right]O^{O}_{M_1,M_2,M_3}\{h\}(\omega), \tag{3.9}
$$

$$
O^{O}_{M_1,M_2,M_3}\{\frac{\partial^2 h}{\partial x_2^2}\}(\omega)=\left[-\left(\frac{a_2}{b_2}\right)^2x_2^2+\frac{2a_2(\omega_2-m_2)}{b_2}x_2-\left(\frac{\omega_2-m_2}{b_2}\right)^2\right]O^{O}_{M_1,M_2,M_3}\{h\}(\omega), \tag{3.10}
$$

$$
O^{O}_{M_1,M_2,M_3}\{\frac{\partial^3 h}{\partial x_3^2}\}(\omega)=\left[-\left(\frac{a_3}{b_3}\right)^2x_3^2+\frac{2a_3(\omega_3-m_3)}{b_3}x_3-\left(\frac{\omega_3-m_3}{b_3}\right)^2\right]O^{O}_{M_1,M_2,M_3}\{h\}(\omega). \tag{3.11}
$$

Corollary 3.2. Let $O^{O}_{M_1M_2M_3}\{\frac{\partial^2 h}{\partial x_i\partial x_j}\}$ denote the OOCLCT of $\frac{\partial^2 h}{\partial x_i\partial x_j}$. Then, we obtain

$$
\begin{aligned}
&O^{O}_{M_1,M_2,M_3}\{\frac{\partial^2 h}{\partial x_1\partial x_2}\}(\omega)\\
=&\left(\frac{a_1a_2}{b_1b_2}x_1x_2-\frac{a_1(\omega_2-m_2)}{b_1b_2}x_1-\frac{a_2(\omega_1-m_1)}{b_1b_2}x_2+\frac{(\omega_1-m_1)(\omega_2-m_2)}{b_1b_2}\right)O^{O}_{M_1,M_2,M_3}\{h\}(\omega),
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
&O^{O}_{M_1,M_2,M_3}\{\frac{\partial^2 h}{\partial x_1\partial x_3}\}(\omega)\\
=&\left(\frac{a_1a_3}{b_1b_3}x_1x_3-\frac{a_1(\omega_3-m_3)}{b_1b_3}x_1-\frac{a_3(\omega_1-m_1)}{b_1b_3}x_3+\frac{(\omega_1-m_1)(\omega_3-m_3)}{b_1b_3}\right)O^{O}_{M_1,M_2,M_3}\{h\}(\omega),
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
&O^{O}_{M_1,M_2,M_3}\{\frac{\partial^2 h}{\partial x_2\partial x_3}\}(\omega)\\
=&\left(\frac{a_2a_3}{b_2b_3}x_2x_3-\frac{a_2(\omega_3-m_3)}{b_2b_3}x_2-\frac{a_3(\omega_2-m_2)}{b_2b_3}x_3+\frac{(\omega_2-m_2)(\omega_3-m_3)}{b_2b_3}\right)O^{O}_{M_1,M_2,M_3}\{h\}(\omega).
\end{aligned} \tag{3.14}
$$

Theorem 3.3. Let $h\in S(\mathbb{R}^3;\mathbb{O})$. The differential properties of the OOCLCT in frequency domain are given as follows:

$$
\frac{\partial\{O^{O}_{M_1,M_2,M_3}\{h\}(\omega)\}}{\partial\omega_1}=-\frac{e_1}{b_1}(x_1+((d_1m_1-b_1n_1)-d_1\omega_1))O^{O}_{M_1,M_2,M_3}\{h\}(\omega), \tag{3.15}
$$

$$
\frac{\partial\{O^{O}_{M_1,M_2,M_3}\{h\}(\omega)\}}{\partial\omega_1}=-\frac{e_4}{b_3}(x_3+((d_3m_3-b_3n_3)-d_3\omega_3))O^{O}_{M_1,M_2,M_3}\{h\}(\omega). \tag{3.16}
$$

Proof. We will prove the first equation.

$$\begin{aligned}\frac{\partial\{O^{O}_{M_1,M_2,M_3}\{h\}(\boldsymbol{\omega})\}}{\partial\omega_1} &= \int_{\mathbb{R}^3} h(\mathbf{x})\frac{\partial K^{e_1}_{M_1}(x_1,\omega_1)}{\partial\omega_1}K^{e_2}_{M_2}(x_2,\omega_2)K^{e_4}_{M_3}(x_3,\omega_3)\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^3} h(\mathbf{x})\frac{1}{\sqrt{2\pi|b_1|}}e^{\frac{e_1}{2b_1}\left(a_1x_1^2-2x_1(\omega_1-m_1)-2\omega_1(d_1m_1-b_1n_1)+d_1\left(\omega_1^2+m_1^2\right)\right)}\\ &\cdot\left(-\frac{e_1}{b_1}\right)(x_1+(d_1m_1-b_1n_1)-d_1\omega_1)\,K^{e_2}_{M_2}(x_2,\omega_2)K^{e_4}_{M_3}(x_3,\omega_3)\mathrm{d}\mathbf{x}\\ &= -\left(\frac{e_1}{b_1}x_1+\frac{e_1}{b_1}\left((d_1m_1-b_1n_1)-d_1\omega_1\right)\right)O^{O}_{M_1,M_2,M_3}\{h\}(\boldsymbol{\omega}).\end{aligned}$$

We have completed the proof. The proof for the other formula is similar.

Theorem 3.4. Let $h\in S(\mathbb{R}^3;\mathbb{O})$, $\Delta_{x_j}h=\frac{\partial h}{\partial x_j}-\frac{a_j}{b_j}e_ix_jh$, $j=1,2,3$. Where e_i denotes the octonion imaginary unit, $i=1,2,4$. For any $l\in Z^+$, we get

$$\Delta^l_{x_j}K_{M_j}\left(x_j,\omega_j\right)=\left(-\frac{e_i\left(\omega_j-m_j\right)}{b_j}\right)^l K_{M_j}\left(x_j,\omega_j\right). \tag{3.17}$$

Proof. For $l=1$, we have

$$\begin{aligned}\Delta_{x_j}K_{M_j}\left(x_j,\omega_j\right) &= \frac{\partial K_{M_j}\left(x_j,\omega_j\right)}{\partial x_j}-\frac{a_j}{b_j}e_ix_jK_{M_j}\left(x_j,\omega_j\right)\\ &= \frac{e_i}{b_j}\left(a_jx_j-\left(\omega_j-m_j\right)\right)K_{M_j}\left(x_j,\omega_j\right)-\frac{a_j}{b_j}e_ix_jK_{M_j}\left(x_j,\omega_j\right)\\ &= \left(-\frac{e_i\left(\omega_j-m_j\right)}{b_j}\right)K_{M_j}\left(x_j,\omega_j\right).\end{aligned}$$

Supposing $\Delta^{l-1}_{x_j}K_{M_j}\left(x_j,\omega_j\right)=\left(-\frac{e_i(\omega_j-m_j)}{b_j}\right)^{l-1}K_{M_j}\left(x_j,\omega_j\right)$, then we have

$$\begin{aligned}\Delta^l_{x_j}K_{M_j}\left(x_j,\omega_j\right) &= \Delta_{x_j}[\Delta^{l-1}_{x_j}K_{M_j}(x_j,\omega_j)]\\ &= \frac{\partial\left(-\frac{e_i(\omega_j-m_j)}{b_j}\right)^{l-1}K_{M_j}\left(x_j,\omega_j\right)}{\partial x_j}-\frac{a_j}{b_j}e_ix_j\left(-\frac{e_i\left(\omega_j-m_j\right)}{b_j}\right)^{l-1}K_{M_j}\left(x_j,\omega_j\right)\\ &= \left(-\frac{e_i\left(\omega_j-m_j\right)}{b_j}\right)^{l-1}\left[\frac{e_i}{b_j}\left(a_jx_j-\left(\omega_j-m_j\right)\right)K_{M_j}\left(x_j,\omega_j\right)\right]\\ &\quad-\frac{a_j}{b_j}e_ix_j\left(-\frac{e_i\left(\omega_j-m_j\right)}{b_j}\right)^{l-1}K_{M_j}\left(x_j,\omega_j\right)\\ &= \left(-\frac{e_i\left(\omega_j-m_j\right)}{b_j}\right)^{l}K_{M_j}\left(x_j,\omega_j\right).\end{aligned}$$

The proof of the theorem is complete.

Theorem 3.5. Let $h \in L^2\left(\mathbb{R}^3, \mathbb{O}\right)$, $\Delta_{\omega_j} h = \frac{\partial h}{\partial \omega_j} - \frac{d_j}{b_j} e_i \omega_j h$, $j = 1, 2, 3$. For any $l \in Z^+$, we have

$$\Delta_{\omega_j}^l \left\{O_{M_1,M_2,M_3}^O \{h\}(\omega)\right\} = O_{M_1,M_2,M_3}^O \left\{\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right)^l h\right\}(\omega). \tag{3.18}$$

Proof. For $l = 1$,

$$\begin{aligned}
\Delta_{\omega_j} \left\{O_{M_1,M_2,M_3}^O \{h\}(\omega)\right\} =& \frac{\partial O_{M_1,M_2,M_3}^O \{h\}(\omega)}{\partial \omega_j} - \frac{d_j}{b_j} e_i \omega_j O_{M_1,M_2,M_3}^O \{h\}(\omega) \\
=& -\frac{e_i}{b_j} O_{M_1,M_2,M_3}^O \left\{x_j h\right\}(\omega) - \frac{e_i}{b_j}\left(\left(d_j m_j - b_j n_j\right) - d_j \omega_j\right) O_{M_1,M_2,M_3}^O \{h\}(\omega) \\
& - \frac{d_j}{b_j} e_i \omega_j O_{M_1,M_2,M_3}^O \{h\}(\omega) \\
=& O_{M_1,M_2,M_3}^O \left\{\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right) h\right\}(\omega).
\end{aligned} \tag{3.19}$$

Supposing $\Delta_{\omega_j}^{l-1} \left\{O_{M_1,M_2,M_3}^O \{h\}(\omega)\right\} = O_{M_1,M_2,M_3}^O \left\{\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right)^{l-1} h\right\}(\omega)$, then we get

$$\begin{aligned}
\Delta_{\omega_j}^l \left\{O_{M_1,M_2,M_3}^O \{h\}(\omega)\right\} =& \Delta_{\omega_j}\left[\Delta_{\omega_j}^{l-1} \left\{O_{M_1,M_2,M_3}^O \{h\}(\omega)\right\}\right] \\
=& \Delta_{\omega_j}\left[O_{M_1,M_2,M_3}^O \left\{\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right)^{l-1} h\right\}(\omega)\right] \\
=& \frac{\partial O_{M_1,M_2,M_3}^O \left\{\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right)^{l-1} h\right\}(\omega)}{\partial \omega_j} \\
& - \frac{d_j}{b_j} e_i \omega_j O_{M_1,M_2,M_3}^O \left\{\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right)^{l-1} h\right\}(\omega) \\
=& \frac{e_i}{b_j} O_{M_1,M_2,M_3}^O \left\{\left(-x_j - \left(d_j m_j - b_j n_j\right) + d_j \omega_j\right)\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right)^{l-1} h\right\}(\omega) \\
& - \frac{d_j}{b_j} e_i \omega_j O_{M_1,M_2,M_3}^O \left\{\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right)^{l-1} h\right\}(\omega) \\
=& O_{M_1,M_2,M_3}^O \left\{\left(-\frac{e_i\left(x_j + \left(d_j m_j - b_j n_j\right)\right)}{b_j}\right)^{l} h\right\}(\omega).
\end{aligned} \tag{3.20}$$

The proof is completed.

4. The uncertainty principle of the OOCLCT

In this section, we will formulate and prove the Heisenberg uncertainty principle, the Sharp Hausdorff-Young inequality, the Matolcsi-Szucs uncertainty principle, and the Benedicks-Amrein-Berthier uncertainty principle for the OOCLCT. The lemma will be used in the proof of the next theorem.

Lemma 4.1. (Heisenberg uncertainty principle for the QOLCT [28]) For every $h \in L^2\left(\mathbb{R}^2;\mathbb{H}\right)$ with $O^{Q}_{M_1,M_2}\{h\} \in L^2\left(\mathbb{R}^2;\mathbb{H}\right)$, we have

$$\int_{\mathbb{R}^2} |\mathbf{x}|^2 |h(\mathbf{x})|^2 \mathrm{d}x \int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\{h\}(\omega)\right|^2 \mathrm{d}\omega \geq |b_1 b_2|^2 \left(\int_{\mathbb{R}^2} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right)^2. \tag{4.1}$$

Theorem 4.1. (Heisenberg uncertainty principle for the OOCLCT) For every $h \in L^2\left(\mathbb{R}^3;\mathbb{O}\right)$ with $O^{O}_{M_1,M_2,M_3}\{h\} \in L^2\left(\mathbb{R}^3;\mathbb{O}\right)$, we have

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^3} |\omega|^2 \left|O^{O}_{M_1,M_2,M_3}\{h\}(\omega)\right|^2 \mathrm{d}\omega \geq \frac{|b_1 b_2|^2}{2\pi|b_3|} \left(\int_{\mathbb{R}^3} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right)^2. \tag{4.2}$$

Proof. According to Eq (2.7), we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\omega|^2 \left|O^{O}_{M_1,M_2,M_3}\{h\}(\omega)\right|^2 \mathrm{d}\omega =& \frac{1}{2\pi|b_3|}\left(\int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\{\tilde{h}_e\}(\omega)\right|^2 \mathrm{d}\omega + \int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\{\tilde{h}_o\}(\omega)\right|^2 \mathrm{d}\omega\right.\\ &\left. + \int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\{\hat{h}_e\}(\omega)\right|^2 \mathrm{d}\omega + \int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\{\hat{h}_o\}(\omega)\right|^2 \mathrm{d}\omega\right),\end{aligned}$$

Using Eq (2.6), we obtain

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2} |\mathbf{x}|^2 \left|\tilde{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} |\mathbf{x}|^2 \left|\tilde{h}_o(x)\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} |\mathbf{x}|^2 \left|\hat{h}_e(x)\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} |\mathbf{x}|^2 \left|\hat{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}.$$

By the Lemma 4.1, we get

$$\int_{\mathbb{R}^2} |\mathbf{x}|^2 \left|\tilde{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\left\{\tilde{h}_e\right\}(\omega)\right|^2 \mathrm{d}\omega \geq |b_1 b_2|^2 \left(\int_{\mathbb{R}^2} \left|\tilde{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right)^2,$$

$$\int_{\mathbb{R}^2} |\mathbf{x}|^2 \left|\tilde{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\left\{\tilde{h}_o\right\}(\omega)\right|^2 \mathrm{d}\omega \geq |b_1 b_2|^2 \left(\int_{\mathbb{R}^2} \left|\tilde{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right)^2,$$

$$\int_{\mathbb{R}^2} |\mathbf{x}|^2 \left|\hat{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\left\{\hat{h}_e\right\}(\omega)\right|^2 \mathrm{d}\omega \geq |b_1 b_2|^2 \left(\int_{\mathbb{R}^2} \left|\hat{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right)^2,$$

$$\int_{\mathbb{R}^2} |\mathbf{x}|^2 \left|\hat{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^2} |\omega|^2 \left|O^{Q}_{M_1,M_2}\left\{\hat{h}_o\right\}(\omega)\right|^2 \mathrm{d}\omega \geq |b_1 b_2|^2 \left(\int_{\mathbb{R}^2} \left|\hat{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right)^2,$$

thus,

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^3} |\omega|^2 \left|O^{O}_{M_1,M_2,M_3}\{h\}(\omega)\right|^2 \mathrm{d}\omega \geq \frac{|b_1 b_2|^2}{2\pi|b_3|} \left(\int_{\mathbb{R}^3} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right)^2.$$

This completes the proof of Theorem 4.1.

Corollary 4.1. If we choose $M_j = \begin{bmatrix} 0 & 1 & | & 0 \\ -1 & 0 & | & 0 \end{bmatrix}$, $j = 1, 2, 3$, then Theorem 4.1 will be reduced to

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^3} |\omega|^2 |\mathcal{F}_O\{h\}(\omega)|^2 \mathrm{d}\omega \geq \frac{1}{2\pi} \left(\int_{\mathbb{R}^3} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right)^2.$$

Corollary 4.2. If $M_j = \begin{bmatrix} \cos\theta_j & \sin\theta_j & | & 0 \\ -\sin\theta_j & \cos\theta_j & | & 0 \end{bmatrix}$, $j = 1, 2, 3$, then Theorem 4.1 will be reduced to

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^3} |\omega|^2 |\mathcal{F}_O\{h\}(\omega)|^2 \mathrm{d}\omega \geq \frac{|\sin\theta_1 \sin\theta_2|^2}{2\pi |\sin\theta_3|} \left(\int_{\mathbb{R}^3} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right)^2.$$

Lemma 4.2. (Sharp Hausdorff-Young inequality for the QOLCT [29]) For any $1 \leq r \leq 2$, $s \geq 2$ such that $\frac{1}{s} + \frac{1}{r} = 1$, then for every function $h \in L^r(\mathbb{R})$, the following inequality is satisfied:

$$\left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{h\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} \leq |b_1 b_2|^{\frac{1}{s} - \frac{1}{2}} (r^{\frac{1}{r}} s^{-\frac{1}{s}}) (2\pi)^{\frac{1}{s} - \frac{1}{r}} \left(\int_{\mathbb{R}^2} |h(\mathbf{x})|^r \mathrm{d}\mathbf{x} \right)^{\frac{1}{r}}. \tag{4.3}$$

Theorem 4.2. (Sharp Hausdorff-Young inequality for the OOCLCT) Under the assumptions of Lemma 4.2, the following inequality is satisfied:

$$\left(\int_{\mathbb{R}^3} \left| O^Q_{M_1,M_2,M_3}\{h\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} \leq |b_3|^{-\frac{1}{2s}} |b_1 b_2|^{\frac{1}{s} - \frac{1}{2}} (r^{\frac{1}{r}} s^{-\frac{1}{s}}) (2\pi)^{\frac{1}{2s} - \frac{1}{r}} \left(\int_{\mathbb{R}^3} |h(\mathbf{x})|^r \mathrm{d}\mathbf{x} \right)^{\frac{1}{r}}. \tag{4.4}$$

Proof. According to Eq (2.7), we have

$$\begin{aligned} \left(\int_{\mathbb{R}^3} \left| O^Q_{M_1,M_2,M_3}\{h\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} = & \frac{1}{(2\pi |b_3|)^{\frac{1}{2s}}} \left(\left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{\tilde{h}_e\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} + \left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{\tilde{h}_o\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} \right. \\ & \left. + \left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{\hat{h}_e\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} + \left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{\hat{h}_o\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} \right). \end{aligned}$$

Now use the above equation to Lemma 4.2, and we obtain

$$\left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{\tilde{h}_e\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} \leq |b_1 b_2|^{\frac{1}{s} - \frac{1}{2}} (r^{\frac{1}{r}} s^{-\frac{1}{s}}) (2\pi)^{\frac{1}{s} - \frac{1}{r}} \left(\int_{\mathbb{R}^2} \left| \tilde{h}_e(\mathbf{x}) \right|^r \mathrm{d}\mathbf{x} \right)^{\frac{1}{r}},$$

$$\left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{\tilde{h}_o\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} \leq |b_1 b_2|^{\frac{1}{s} - \frac{1}{2}} (r^{\frac{1}{r}} s^{-\frac{1}{s}}) (2\pi)^{\frac{1}{s} - \frac{1}{r}} \left(\int_{\mathbb{R}^2} \left| \tilde{h}_o(\mathbf{x}) \right|^r \mathrm{d}\mathbf{x} \right)^{\frac{1}{r}},$$

$$\left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{\hat{h}_e\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} \leq |b_1 b_2|^{\frac{1}{s} - \frac{1}{2}} (r^{\frac{1}{r}} s^{-\frac{1}{s}}) (2\pi)^{\frac{1}{s} - \frac{1}{r}} \left(\int_{\mathbb{R}^2} \left| \hat{h}_e(\mathbf{x}) \right|^r \mathrm{d}\mathbf{x} \right)^{\frac{1}{r}},$$

$$\left(\int_{\mathbb{R}^2} \left| O^Q_{M_1,M_2}\{\hat{h}_o\}(\omega) \right|^s \mathrm{d}\omega \right)^{\frac{1}{s}} \leq |b_1 b_2|^{\frac{1}{s} - \frac{1}{2}} (r^{\frac{1}{r}} s^{-\frac{1}{s}}) (2\pi)^{\frac{1}{s} - \frac{1}{r}} \left(\int_{\mathbb{R}^2} \left| \hat{h}_o(\mathbf{x}) \right|^r \mathrm{d}\mathbf{x} \right)^{\frac{1}{r}},$$

thus,

$$\begin{aligned}\left(\int_{\mathbb{R}^3}\left|O^O_{M_1,M_2,M_3}\{h\}(\omega)\right|^s \mathrm{d}\omega\right)^{\frac{1}{s}} \leq & \frac{1}{(2\pi\,|b_3|)^{\frac{1}{2s}}}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}(r^{\frac{1}{r}}s^{-\frac{1}{s}})(2\pi)^{\frac{1}{s}-\frac{1}{r}}\cdot\left(\left(\int_{\mathbb{R}^2}\left|\tilde{h}_e(\omega)\right|^s\mathrm{d}\omega\right)^{\frac{1}{s}}\right.\\ & \left.+\left(\int_{\mathbb{R}^2}\left|\tilde{h}_o(\omega)\right|^s\mathrm{d}\omega\right)^{\frac{1}{s}}+\left(\int_{\mathbb{R}^2}\left|\hat{h}_e(\omega)\right|^s\mathrm{d}\omega\right)^{\frac{1}{s}}+\left(\int_{\mathbb{R}^2}\left|\hat{h}_o(\omega)\right|^s\mathrm{d}\omega\right)^{\frac{1}{s}}\right)\\ =&\frac{1}{(2\pi\,|b_3|)^{\frac{1}{2s}}}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}(r^{\frac{1}{r}}s^{-\frac{1}{s}})(2\pi)^{\frac{1}{s}-\frac{1}{r}}\left(\int_{\mathbb{R}^3}|h(\omega)|^s\mathrm{d}\omega\right)^{\frac{1}{s}}\\ =&|b_3|^{-\frac{1}{2s}}|b_1b_2|^{\frac{1}{s}-\frac{1}{2}}(r^{\frac{1}{r}}s^{-\frac{1}{s}})(2\pi)^{\frac{1}{2s}-\frac{1}{r}}\left(\int_{\mathbb{R}^3}|h(\omega)|^s\mathrm{d}\omega\right)^{\frac{1}{s}}.\end{aligned}$$

The second step used Eq (2.6), and we have completed the proof.

Corollary 4.3. If $M_j=\left[\begin{array}{cc|c}0 & 1 & 0\\ -1 & 0 & 0\end{array}\right]$, $j=1,2,3$, then Theorem 4.2 will be reduced to

$$\left(\int_{\mathbb{R}^3}|\mathcal{F}_O\{h\}(\omega)|^s\mathrm{d}\omega\right)^{\frac{1}{s}}\leq(r^{\frac{1}{r}}s^{-\frac{1}{s}})(2\pi)^{\frac{1}{2s}-\frac{1}{r}}\left(\int_{\mathbb{R}^3}|h\,(\mathbf{x})|^r\mathrm{d}\mathbf{x}\right)^{\frac{1}{r}}.$$

Corollary 4.4. If $M_j=\left[\begin{array}{cc|c}\cos\theta_j & \sin\theta_j & 0\\ -\sin\theta_j & \cos\theta_j & 0\end{array}\right]$, $j=1,2,3$, then Theorem 4.2 will be reduced to

$$\left(\int_{\mathbb{R}^3}|\mathcal{F}_O\{h\}(\omega)|^s\mathrm{d}\omega\right)^{\frac{1}{s}}\leq|\sin\theta_3|^{-\frac{1}{2s}}|\sin\theta_1\,\sin\theta_2|^{\frac{1}{s}-\frac{1}{2}}(r^{\frac{1}{r}}s^{-\frac{1}{s}})(2\pi)^{\frac{1}{2s}-\frac{1}{r}}\left(\int_{\mathbb{R}^3}|h\,(\mathbf{x})|^r\mathrm{d}\mathbf{x}\right).$$

Lemma 4.3. (Matolcsi-Szucs uncertainty principle for the QOLCT [29]) For every quaternion function $h\in L^1\left(\mathbb{R}^2;\mathbb{H}\right)\cap L^2\left(\mathbb{R}^2;\mathbb{H}\right)$ with $1\leq r_1\leq r_2\leq 2$, one has

$$\begin{aligned}\left\|O^Q_{M_1,M_2}\{h\}\right\|_{L^{s_2}\left(\mathbb{R}^2;\mathbb{H}\right)}\leq&\left|\mathrm{supp}\left(O^Q_{A_1,A_2}\{h\}\right)\right|^{\frac{s_1-s_2}{s_1s_2}}|b_1b_2|^{\frac{1}{s_1}-\frac{1}{2}}\left(r_1{}^{\frac{1}{r_1}}s_1{}^{-\frac{1}{s_1}}\right)\\ &\cdot(2\pi)^{\frac{1}{s_1}-\frac{1}{r_1}}\left|\mathrm{supp}\,(f)\right|^{\frac{r_2-r_1}{r_1r_2}}\|h\|_{L^{r_2}\left(\mathbb{R}^2;\mathbb{H}\right)},\end{aligned}\tag{4.5}$$

where $\frac{1}{r_1}+\frac{1}{s_1}=1$ and $\frac{1}{r_2}+\frac{1}{s_2}=1$.

Theorem 4.3. (Matolcsi-Szucs uncertainty principle for the OOCLCT) For every octonion function $h\in L^1\left(\mathbb{R}^3;\mathbb{O}\right)\cap L^2\left(\mathbb{R}^3;\mathbb{O}\right)$ under assumptions as in Lemma 4.3, one has

$$\begin{aligned}\left\|O^O_{A_1,A_2,A_3}\{h\}\right\|_{L^{s_2}\left(\mathbb{R}^3;\mathbb{O}\right)}\leq&\left|\mathrm{supp}\left(O^O_{A_1,A_2,A_3}\{h\}\right)\right|^{\frac{s_1-s_2}{s_1s_2}}|b_3|^{-\frac{1}{2s_1}}|b_1b_2|^{\frac{1}{s_1}-\frac{1}{2}}\left(r_1{}^{\frac{1}{r_1}}s_1{}^{-\frac{1}{s_1}}\right)\\ &\cdot(2\pi)^{\frac{1}{2s_1}-\frac{1}{r_1}}\left|\mathrm{supp}\,(h)\right|^{\frac{r_2-r_1}{r_1r_2}}\|h\|_{L^{r_2}\left(\mathbb{R}^3;\mathbb{O}\right)},\end{aligned}\tag{4.6}$$

where $\frac{1}{r_1}+\frac{1}{s_1}=1$ and $\frac{1}{r_2}+\frac{1}{s_2}=1$.
Proof. Let F = $\operatorname{supp}\left(\mathbb{O}_{M_1,M_2,M_3}^{O}\{h\}\right)$, and we get

$$\begin{aligned}
\left\|O_{M_1,M_2,M_3}^{O}\{h\}(\omega)\right\|_{L^{s_2}(\mathbb{R}^3;\mathbb{O})} &= \left(\int_{\mathbb{R}^3}\left|O_{M_1,M_2,M_3}^{O}\{h\}(\omega)\right|^{s_2}\mathrm{d}\omega\right)^{\frac{1}{s_2}} \\
&= \left(\int_{\mathbb{R}^3}\left|\mathcal{X}_F O_{M_1,M_2,M_3}^{O}\{h\}(\omega)\right|^{s_2}\mathrm{d}\omega\right)^{\frac{1}{s_2}} \\
&\leq \left(\left(\int_{\mathbb{R}^3}(|\mathcal{X}_F(\omega)|^{s_2})^{\frac{s_1}{s_1-s_2}}\mathrm{d}\omega\right)^{\frac{1}{\frac{s_1}{s_1-s_2}}}\left(\int_{\mathbb{R}^3}\left(\left|O_{M_1,M_2,M_3}^{O}\{h\}(\omega)\right|^{s_2}\right)^{\frac{s_1}{s_2}}\mathrm{d}\omega\right)^{\frac{1}{\frac{s_1}{s_2}}}\right)^{\frac{1}{s_2}} \\
&= \left(\left(\int_{\mathbb{R}^3}\left(|\mathcal{X}_F(\omega)|^{\frac{s_1}{s_1-s_2}}\right)^{s_2}\mathrm{d}\omega\right)^{\frac{1}{\frac{s_1}{s_1-s_2}}}\left(\int_{\mathbb{R}^3}\left(\left|O_{M_1,M_2,M_3}^{O}\{h\}(\omega)\right|^{\frac{s_1}{s_2}}\right)^{s_2}\mathrm{d}\omega\right)^{\frac{1}{\frac{s_1}{s_2}}}\right)^{\frac{1}{s_2}} \\
&= \left(\int_{\mathbb{R}^3}|\mathcal{X}_F(\omega)|^{\frac{s_1s_2}{s_1-s_2}}\mathrm{d}\omega\right)^{\frac{s_1-s_2}{s_1s_2}}\left(\int_{\mathbb{R}^3}\left|O_{M_1,M_2,M_3}^{O}\{h\}(\omega)\right|^{s_1}\mathrm{d}\omega\right)^{\frac{1}{s_1}},
\end{aligned}$$

where the operation uses *Holdër inequality*, and $\mathcal{X}_F$ is the indicator function of the set F. Using Theorem 4.2 and $|\mathcal{X}_F(\mathbf{x})|^{\frac{s_1s_2}{s_1-s_2}}=|\mathcal{X}_F(\mathbf{x})|$, thus,

$$\begin{aligned}
\left\|O_{M_1,M_2,M_3}^{O}\{h\}(\omega)\right\|_{L^{s_2}(\mathbb{R}^3;\mathbb{O})} \leq&\left(\int_{\mathbb{R}^3}|\mathcal{X}_F(\omega)|\,\mathrm{d}\omega\right)^{\frac{s_1-s_2}{s_1s_2}}\left\|O_{M_1,M_2,M_3}^{O}\{h\}\right\|_{L^{s_1}(\mathbb{R}^3;\mathbb{O})} \\
=&\left|\operatorname{supp}\left(O_{M_1,M_2,M_3}^{O}\{h\}\right)\right|^{\frac{s_1-s_2}{s_1s_2}}\left\|O_{M_1,M_2,M_3}^{O}\{h\}\right\|_{L^{s_1}(\mathbb{R}^3;\mathbb{O})} \\
\leq&\left|\operatorname{supp}\left(O_{M_1,M_2,M_3}^{O}\{h\}\right)\right|^{\frac{s_1-s_2}{s_1s_2}}|b_3|^{-\frac{1}{2s_1}}|b_1b_2|^{\frac{1}{s_1}-\frac{1}{2}}(r_1^{\frac{1}{r_1}}s_1^{-\frac{1}{s_1}})(2\pi)^{\frac{1}{2s_1}-\frac{1}{r_1}}\|h\|_{L^{r_1}(\mathbb{R}^3;\mathbb{O})},
\end{aligned}$$

then let $J=\operatorname{supp}(h)$,and use the method similar to the first equation. We get

$$\begin{aligned}
\|h\|_{L^{r_1}(\mathbb{R}^3;\mathbb{O})} &= \|\mathcal{X}_J h\|_{L^{r_1}(\mathbb{R}^3;\mathbb{O})} \\
&\leq \left(\int_{\mathbb{R}^3}|\mathcal{X}_J(\mathbf{x})|\,\mathrm{d}\mathbf{x}\right)^{\frac{r_1-r_2}{r_1r_2}}\left(\int_{R^3}|h(\mathbf{x})|^{\frac{r_1r_2}{r_1}}\mathrm{d}\mathbf{x}\right)^{\frac{r_1}{r_1r_2}} \\
&= |J|^{\frac{r_1-r_2}{r_1r_2}}\left(\int_{\mathbb{R}^3}|h(\mathbf{x})|^{r_2}\mathrm{d}\mathbf{x}\right)^{\frac{1}{r_2}} \\
&= \left|\operatorname{supp}(h)\right|^{\frac{r_1-r_2}{r_1r_2}}\|h\|_{L^{r_2}(\mathbb{R}^3;\mathbb{O})}.
\end{aligned}$$

In conclusion, we have obtained

$$\begin{aligned}
\left\|O_{M_1,M_2,M_3}^{O}\{h\}(\omega)\right\|_{L^{s_2}(\mathbb{R}^3;\mathbb{O})} \leq&\left|\operatorname{supp}\left(O_{M_1,M_2,M_3}^{O}\{h\}\right)\right|^{\frac{s_1-s_2}{s_1s_2}}|b_3|^{-\frac{1}{2s_1}}|b_1b_2|^{\frac{1}{s_1}-\frac{1}{2}}(r_1^{\frac{1}{r_1}}s_1^{-\frac{1}{s_1}}) \\
&\cdot(2\pi)^{\frac{1}{2s_1}-\frac{1}{r_1}}\left|\operatorname{supp}(h)\right|^{\frac{r_1-r_2}{r_1r_2}}\|h\|_{L^{r_2}(\mathbb{R}^3;\mathbb{O})}.
\end{aligned}$$

We have finished the proof.
Lemma 4.4. (Benedicks-Amrein-Berthier uncertainty principle for the QOLCT [29]) Let S,V be

measurable subsets of $\mathbb{R}^2$ with $\mu(S), \mu(V) < \infty$. There exists a constant $C = C(S, V)$ such that for all $h \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$,

$$\int_{\mathbb{R}^2} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \leq C\left(\int_{\mathbb{R}^2\setminus S} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{h\}(\omega)\right|^2 \mathrm{d}\omega\right). \tag{4.7}$$

Theorem 4.4. (Benedicks-Amrein-Berthier uncertainty principle for the OOCLCT) Under assumptions as in Lemma 4.4, let S and V be measurable subsets of $\mathbb{R}^2$, and we have:

$$\int_{\mathbb{R}^3} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \leq C\left(\int_{\mathbb{R}^3\setminus S} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + 2\pi|b_3| \int_{\mathbb{R}^3\setminus V} \left|O^O_{M_1,M_2,M_3}\{h\}(\omega)\right|^2 \mathrm{d}\omega\right). \tag{4.8}$$

Proof. According to Eq (2.7), we have

$$\begin{aligned}\int_{\mathbb{R}^3\setminus V} \left|O^O_{M_1,M_2,M_3}\{h\}(\omega)\right|^2 \mathrm{d}\omega =& \frac{1}{2\pi|b_3|}\left(\int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{\tilde{h}_e\}(\omega)\right|^2 \mathrm{d}\omega + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{\tilde{h}_o\}(\omega)\right|^2 \mathrm{d}\omega\right.\\ &\left.+ \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{\hat{h}_e\}(\omega)\right|^2 \mathrm{d}\omega + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{\hat{h}_o\}(\omega)\right|^2 \mathrm{d}\omega\right),\end{aligned}$$

then

$$\begin{aligned}2\pi|b_3| \int_{\mathbb{R}^3\setminus V} \left|O^O_{M_1,M_2,M_3}\{h\}(\omega)\right|^2 \mathrm{d}\omega =& \left(\int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{\tilde{h}_e\}(\omega)\right|^2 \mathrm{d}\omega + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{\tilde{h}_o\}(\omega)\right|^2 \mathrm{d}\omega\right.\\ &\left.+ \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{\hat{h}_e\}(\omega)\right|^2 \mathrm{d}\omega + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\{\hat{h}_o\}(\omega)\right|^2 \mathrm{d}\omega\right).\end{aligned}$$

Using Eq (2.6), we obtain

$$\int_{\mathbb{R}^3\setminus S} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2\setminus S} \left|\tilde{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus S} \left|\tilde{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus S} \left|\hat{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus S} \left|\hat{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}.$$

By virtue of Lemma 4.4, we have

$$\int_{\mathbb{R}^2} \left|\tilde{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \leq C\left(\int_{\mathbb{R}^2\setminus S} \left|\tilde{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\left\{\tilde{h}_e\right\}(\omega)\right|^2 \mathrm{d}\omega\right),$$

$$\int_{\mathbb{R}^2} \left|\tilde{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \leq C\left(\int_{\mathbb{R}^2\setminus S} \left|\tilde{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\left\{\tilde{h}_o\right\}(\omega)\right|^2 \mathrm{d}\omega\right),$$

$$\int_{\mathbb{R}^2} \left|\hat{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \leq C\left(\int_{\mathbb{R}^2\setminus S} \left|\hat{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\left\{\hat{h}_e\right\}(\omega)\right|^2 \mathrm{d}\omega\right),$$

$$\int_{\mathbb{R}^2} \left|\hat{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \leq C\left(\int_{\mathbb{R}^2\setminus S} \left|\hat{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\left\{\hat{h}_o\right\}(\omega)\right|^2 \mathrm{d}\omega\right),$$

then

$$\begin{aligned}&\int_{\mathbb{R}^2} \left|\tilde{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} \left|\tilde{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} \left|\hat{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} \left|\hat{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\\ &\leq C\left(\int_{\mathbb{R}^2\setminus S} \left|\tilde{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus S} \left|\tilde{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus S} \left|\hat{h}_e(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2\setminus S} \left|\hat{h}_o(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right.\\ &+ \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\left\{\tilde{h}_e\right\}(\omega)\right|^2 \mathrm{d}\omega + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\left\{\tilde{h}_o\right\}(\omega)\right|^2 \mathrm{d}\omega\\ &\left.+ \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\left\{\hat{h}_e\right\}(\omega)\right|^2 \mathrm{d}\omega + \int_{\mathbb{R}^2\setminus V} \left|O^Q_{M_1,M_2}\left\{\hat{h}_o\right\}(\omega)\right|^2 \mathrm{d}\omega\right).\end{aligned}$$

Sort out the above formula to get a conclusion.

$$\int_{\mathbb{R}^3} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \le C\left(\int_{\mathbb{R}^3\setminus V} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + 2\pi |b_3| \int_{\mathbb{R}^3\setminus V} \left|O^{O}_{M_1,M_2,M_3}\{h\}(\boldsymbol{\omega})\right|^2 \mathrm{d}\boldsymbol{\omega}\right).$$

We have successfully completed the proof of the theorem.

Corollary 4.5. If $M_j = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \end{array}\right]$, $j = 1, 2, 3$, then Theorem 4.4 will be reduced to

$$\int_{\mathbb{R}^3} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \le C\left(\int_{\mathbb{R}^3\setminus V} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + 2\pi \int_{\mathbb{R}^3\setminus V} |\mathcal{F}_O\{h\}(\boldsymbol{\omega})|^2 \mathrm{d}\boldsymbol{\omega}\right).$$

Corollary 4.6. If $M_j = \left[\begin{array}{cc|c} \cos\theta_j & \sin\theta_j & 0 \\ -\sin\theta_j & \cos\theta_j & 0 \end{array}\right]$, $j = 1, 2, 3$, then Theorem 4.4 will be reduced to

$$\int_{\mathbb{R}^3} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \le C\left(\int\limits_{\mathbb{R}^3\setminus V} |h(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + 2\pi |\sin\theta_3| \int\limits_{\mathbb{R}^3\setminus V} |\mathcal{F}_O\{h\}(\boldsymbol{\omega})|^2 \mathrm{d}\boldsymbol{\omega}\right)^{\frac{1}{r}}.$$

5. Conclusions

This paper studies the modulatory and differential properties of the OOCLCT. Through rigorous mathematical derivation, the uncertainty principle under this transformation is established, and the practical value of the theoretical results is clarified: the differential properties of the OOCLCT can be applied to 3D medical image reconstruction scenarios. By leveraging its precise capture capability of signal gradient changes, the method enhances the accuracy of image edge computation and enables efficient detection of lesion areas or tissue boundaries. The uncertainty principle of the OOCLCT provides theoretical constraints for high-dimensional signal sampling. In complex signal processing applications such as radar and sonar, this principle can guide the optimization of sampling rate design, effectively reducing data redundancy while preserving the integrity of signal transmission and storage. This paper not only provides a novel perspective for investigating the mathematical properties of offset octonions, but also bridges the gap between the theoretical framework and practical applications of the OOCLCT, thereby contributing to the further development of its theoretical system.

Based on the theoretical results presented in this paper, future research can proceed in multiple directions, particularly focusing on the development of efficient algorithms for the OOCLCT. In light of the current challenge of high computational complexity in transformation processes, accelerated strategies based on matrix decomposition and the fast Fourier transform may be explored to improve its operational efficiency in large-scale data processing. At the same time, cross-disciplinary research can be advanced through the integration of deep learning technologies. The OOCLCT can be incorporated as a feature extraction module into convolutional neural networks and other machine learning models, thereby further exploring its potential to enhance performance in tasks such as high-dimensional image recognition and remote sensing data interpretation.

Author contributions

Yixuan Lv: Writing-original draft, Investigation, Validation. Qiang Feng: Writing-review & editing, Investigation. Bo Li, Jinyi Ji and Ziyuan Guo: Writing-review & editing.

All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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