Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 30828–30850.
DOI: 10.3934/math.20251353
Received: 16 June 2025
Revised: 20 December 2025
Accepted: 23 December 2025
Published: 30 December 2025

Research article

The ternary Goldbach problem with primes in arithmetic progressions modulus prime numbers

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Abstract: In this paper, assuming the Landau's conjecture, we obtained an explicit numerical upper bound for the prime modulus of arithmetic progressions, in which the ternary Goldbach problem is solvable. Our result on the quantitative upper bound improved the previous results.

Keywords: ternary Goldbach problem; prime solutions; arithmetic progressions; circle method; exponential sums
Mathematics Subject Classification: 11P32, 11P55

1. Introduction

The famous ternary Goldbach conjecture shows that every odd integer $N \geq 9$ can be represented as a sum of three odd primes. When N is sufficiently large, the equation

$$N = p_1 + p_2 + p_3 \tag{1.1}$$

has been shown to be solvable in prime variables p_1, p_2, and p_3 by Vinogradov [11] in 1937. Later, many authors [1, 14] began to study the ternary Goldbach problem concerned with satisfying specific congruence conditions. For example, for any number N satisfying $N \equiv l_1+l_2+l_3 \pmod k$, where $(l_j, k) = 1 (1 \leq j \leq 3)$, researchers try to determine the existence of a solution to the equation $N = p_1 + p_2 + p_3$ under the conditions $p_j \equiv l_j \pmod k$. Although the research in this area is still in the exploratory stage, some preliminary results and methods have been obtained. The papers [1, 14] proved that an equation with $N \equiv l_1 + l_2 + l_3 \pmod k$ is solvable in prime variables $p_j \equiv l_j \pmod k$, and they also give the results for any k with $1 \leq k \leq (logN)^C$ and C being any fixed positive number. Then these results were extended to a large modulus k with $1 \leq k \leq N^\theta$, where $0 \leq \theta \leq 1$ is an absolute computable constant by [9, 10]. Their results are as follows.

Theorem 0. *Let l_1, l_2, l_3, and $k \geq 1$ be integers satisfying $(l_j, k) = 1$ for $1 \leq j \leq 3$. Let N be a sufficiently large odd number satisfying $N \equiv l_1 + l_2 + l_3 \pmod{k}$. Then there exists an absolute computable constant θ with $0 \leq \theta \leq 1$ such that for any k with $1 \leq k \leq N^{\theta}$, the equation $N = p_1 + p_2 + p_3$ is solvable in prime variables p_1, p_2, p_3 with $p_j \equiv l_j \pmod{k} (1 \leq j \leq 3)$.*

For a quantitative result of θ, [6] obtained that 1/8 for "almost all" integer moduli and 3/20 for "almost all" prime moduli. For the value of θ with respect to all integer moduli, the paper [13] showed an explicit numerical upper bound for the modulus of arithmetic progressions, in which the ternary Goldbach problem is solvable. Their result implies a quantitative upper bound for the Linnik constant such that the θ in Theorem 0 can be taken as 1/42.

However, if we limit the range of the positive integer k so that k can only take prime numbers, the numerical value of θ corresponding to all integer moduli remains unspecified. Under the assumption that the Landau conjecture holds, this paper improves the upper bound of θ to $1/40$ by simplifying the analytical processes of integral interval extension and error estimation. Here we briefly explain why: Under the assumption that the Landau conjecture holds, there are no exceptional zeros $\widetilde{\beta}$ of Dirichlet L-functions—that is, exceptional zeros do not exist. Landau's conjecture typically refers to the conjecture regarding the lower bound of Dirichlet L-functions near $s = 1$. Its core statement can be expressed as follows: For any real, nontrivial primitive Dirichlet character χ, there exists an absolute constant $c > 0$ which is independent of q such that

$$L(1, \chi) > \frac{c}{\log q}.$$

If the above lower bound can be established, then the Dirichlet L-function will not have any zeros too close to $s = 1$, that is, Siegel zeros cannot exist. Thus this proves that the effective lower bound for $L(1, \chi)$ improves the zero-free region results, strengthening the conclusion from "at most one zero" to "there exists no zero abnormally close to 1". The primary objective of this work is to establish the proof of the following Theorem 1.

Theorem 1. *Let k be a prime number, and let l_1, l_2, l_3 be integers satisfying $(l_j, k) = 1$ for $1 \leq j \leq 3$. Let N be a sufficiently large odd integer satisfying $N \equiv l_1 + l_2 + l_3 \pmod{k}$. Assuming Landau's conjecture holds, then for any k with $1 \leq k \leq N^{1/40}$, the equation $N = p_1 + p_2 + p_3$ is solvable in prime variables p_1, p_2, p_3 with $p_j \equiv l_j \pmod{k}$ $(1 \leq j \leq 3)$.*

2. Preliminaries

Let l_1, l_2, l_3 be integers and $k \geq 1$ be a prime number satisfying $(l_j, k) = 1$ for $1 \leq j \leq 3$. Let N be a sufficiently large odd number satisfying $N \equiv l_1 + l_2 + l_3 \pmod{k}$. Let ε_0 be a fixed, sufficiently small positive number, and let

$$\mathcal{L} = logN, \; Q = k^{2+\varepsilon_0}\mathcal{L}^9, \; \tau = N^{-1}k^{2+2\varepsilon_0}\mathcal{L}^{9.5}, \; T = k^{4+10/3\varepsilon_0}\mathcal{L}^{20}, \; L = \log(kQ). \tag{2.1}$$

Let n and p, with or without subscripts, always denote a positive integer and a prime number respectively. Let $\chi \pmod{k}$ to denote a Dirichlet character modulo k and $L(\sigma + it, \chi)$ to denote a Dirichlet L-function. This section primarily provides the lemmas required for the subsequent proofs.

Lemma 2.1. *Assuming the Landau's conjecture, i.e., the non-existence of the exceptional zero $\widetilde{\beta}$, the function*

$$\prod_{q\le kQ}\prod_{\chi\,(\mathrm{mod}\,k)}{}^{*}L(\sigma+it,\chi)$$

has no zeros in the region

$$\sigma > 1-\frac{0.364}{L},\quad |t|\le C.$$

Proof. This follows Proposition 2.3 of [8]. □

Lemma 2.2. *For any $x\ge 2$ and $y\ge 1$, let*

$$N(\alpha,x,y)=\sum_{q\le x}\sum_{\chi\,(\mathrm{mod}\,q)}{}^{*}\sum_{\substack{\rho=\beta+i\gamma\\|\gamma|\le y\\ \beta\ge\alpha}}1.$$

The summation $\sum^{*}_{\chi\,(\mathrm{mod}\,q)}$ is over all primitive characters $\chi\,(\mathrm{mod}\,q)$ and $\rho=\beta+i\gamma$ is any nontrivial zero of $L(s,\chi)$. Then, we have

$$N(\alpha,\ x,\ y)\ll (x^2y)^{12/5(1-\alpha)+\varepsilon},\quad \textit{for}\ \frac{1}{2}\le\alpha\le\frac{4}{5}, \tag{2.2}$$

$$N(\alpha,\ x,\ y)\ll (x^2y)^{(2+\varepsilon)(1-\alpha)},\quad \textit{for}\ \frac{4}{5}\le\alpha\le 1. \tag{2.3}$$

Proof. Equation (2.2) follows from Eq (1.1) of [4], and Eq (2.3) follows from Theorem 1 of [5]. □

3. Simplification

Denote by $\Lambda(n)$ the von Mangoldt function, and $e(y)=\exp(2\pi iy)$ for any real number y. Set for $1\le j\le 3$,

$$S_j(x):=\sum_{\substack{N/4\le n\le N\\ n\equiv l_j(\mathrm{mod}\,k)}}\Lambda(n)e(xn), \tag{3.1}$$

and let

$$I(N):=\sum_{(n_1,n_2,n_3)}\Lambda(n_1)\Lambda(n_2)\Lambda(n_3),$$

where the sum $\sum_{(n_1,n_2,n_3)}$ is over all triplets (n_1,n_2,n_3) satisfying $N/4\le n_j\le N$, $n_j\equiv l_j(\mathrm{mod}\,k)$ for $1\le j\le 3$ and $n_1+n_2+n_3=N$.

Then by the integral theory of complex variable functions, we have

$$I(N)=\int_{\tau}^{1+\tau}e(-Nx)\prod_{j=1}^{3}S_j(x)dx.$$

By Dirichlet's lemma on rational approximations, each $x\in[\tau,1+\tau]$ may be written in the form:

$$x=hq^{-1}+z,\ (h,q)=1,\ 1\le q\le\tau^{-1},\ |z|\le\tau/q.$$

Let $\mathcal{M}_1 := \{x: \ x \in [\tau, 1+\tau], \ x = hq^{-1} + z, \ (h, q) = 1, 1 \le q \le Q, |z| \le \tau/q\}$ and $\mathcal{M}_2 := [\tau, 1+\tau] \setminus \mathcal{M}_1$. Then, in view of $k \le N^{1/40}$ and (2.1), $\mathcal{M}_1$ is a union of mutually disjoint intervals of the form $m(h, q) = [\frac{h-\tau}{q}, \ \frac{h+\tau}{q}]$.

Then we divide $I(N)$ into two parts $I_1(N)$ and $I_2(N)$ as

$$\begin{aligned} I(N) &= \int_{\mathcal{M}_1} e(-Nx) \prod_{j=1}^{3} S_j(x) dx + \int_{\mathcal{M}_2} e(-Nx) \prod_{j=1}^{3} S_j(x) dx \\ &=: I_1(N) + I_2(N). \end{aligned} \tag{3.2}$$

Let $d := (k, q)$ and $D := [k, q]$, denote the greatest common divisors and the least common multiples of k and q. Since k is prime, we have

$$d = (k, q) = \begin{cases} k, & \text{if } q = k^m, m \ge 1, \\ 1, & \text{otherwise.} \end{cases}; \quad D = [k, q] = \begin{cases} q, & \text{if } q = k^m, m \ge 1, \\ kq, & \text{otherwise.} \end{cases} \tag{3.3}$$

For any l with $(l, q) = 1$ we see that the pair of congruent equations $n \equiv l_j (\bmod\, k)$ and $n \equiv l (\bmod\, k)$ is solvable if and only if $l \equiv l_j (\bmod\, k)$, and then the solution s_j is unique modulo D, and $(s_j, D) = 1$.

Note that by $(l_j, k) = 1$ and $q \le Q$, we have

$$\sum_{\substack{n \le N \\ n \equiv l_j (\bmod\, k) \\ (n, D) \ne 1}} \Lambda(n) \le \sum_{p | q} \log N \ll \log N \sum_{p | N} 1 \ll \mathcal{L}^2.$$

Then by (3.1) and the definition of $S_j(x)$, we have

$$S_j(x) = \sum_{\substack{N/4 \le n \le N \\ n \equiv l_j (\bmod\, k) \\ (n, D) = 1}} \Lambda(n) e(xn) + O(\mathcal{L}^2).$$

By writing $n \equiv l\, (\bmod\, q)$ with $(l, q) = 1$ and letting $x = h/q + z$, we can rewrite $S_j(x)$ as

$$S_j(x) = \sum_{\substack{l=1 \\ (l, q) = 1 \\ l \equiv l_j\, (\bmod\, k)}}^{q} e\left(\frac{hl}{q}\right) \sum_{\substack{N/4 \le n \le N \\ n \equiv s_j\, (\bmod\, D)}} \Lambda(n) e(zn). \tag{3.4}$$

Considering the orthogonality relation of Dirichlet characters,

$$\sum_{\chi\, (\bmod\, D)} \overline{\chi}(s_j)(n) = \begin{cases} \varphi(D), \ \text{if } n \equiv s_j\, (\bmod\, D) \\ 0, \ \text{other wises,} \end{cases}$$

we get

$$\sum_{\substack{n \le t \\ n \equiv s_j (\bmod\, D)}} \Lambda(n) = \varphi(D)^{-1} \sum_{\chi (\bmod\, D)} \psi(t, \chi) \overline{\chi}(s_j), \tag{3.5}$$

where φ denotes the Euler function and $\psi(t, \chi) = \sum_{n \le t} \Lambda(n) \chi(n)$.

If we express the second sum in $S_j(x)$ by the integral

$$\sum_{\substack{N/4\le n\le N\\ n\equiv s_j\,(\mathrm{mod}\,D)}}\Lambda(n)e(nz)=\int_{N/4}^{N}e(zt)d\left(\sum_{\substack{n\le t\\ n\equiv s_j\,(\mathrm{mod}\,D)}}\Lambda(n)\right),\tag{3.6}$$

then, by combining (3.4)–(3.6) we get

$$S_j(x)=\varphi(D)^{-1}\sum_{\chi\,(\mathrm{mod}\,D)}G_j(\overline{\chi},h,q)\int_{N/4}^{N}e(zt)d\psi(t,\chi)+O(\mathcal{L}^2),\tag{3.7}$$

where

$$G_j(\chi,h,q):=\sum_{\substack{l=1\\ (l,q)=1\\ l\equiv l_j\,(\mathrm{mod}\,d)}}^{q}e(hl/q)\chi(s_j).$$

Denote by χ_{0q} the principal character modulo q; for abbreviation, we write $\chi_0=\chi_{0D}$. Let

$$\delta_\chi:=\begin{cases}1, & \text{if }\chi=\chi_0,\\ 0, & \text{otherwise,}\end{cases}$$

then by [3], since $\tilde{\beta}$ does not exist, we have

$$\psi(t,\chi)=\delta_\chi t-\sum_{|\gamma|\le T}{}'(t^\rho/\rho)+R(t,D,T),\tag{3.8}$$

where $R(t,D,T)\ll N\mathcal{L}^2T^{-1}$, and we use $\sum'_{|\gamma|\le T}$ to denote the summation over all nontrivial zeros $\rho=\beta+\mathrm{i}\gamma$ of $L(s,\chi)$ satisfying $\beta\ge 1/2$, $|\gamma|\le T$.

For $j=1,2,3$ denote

$$H_j(h,q,z):=G_j(\chi_0,h,q)I(z)-\sum_{\chi\,(\mathrm{mod}\,D)}G_j(\tilde{\chi},h,q)I(\chi,z),\tag{3.9}$$

where

$$I(z):=\int_{N/4}^{N}e(zt)dt;\,I(\chi,z):=\int_{N/4}^{N}e(zt)\sum_{|\gamma|\le T}{}'t^{\rho-1}dt.$$

Then, combining (3.7)–(3.9), we have

$$S_j(x)=\varphi(D)^{-1}H_j(h,q,z)+O((1+|z|N)\varphi(q)N\mathcal{L}^2T^{-1}):=A_j+O(R),\ 1\le j\le 3.\tag{3.10}$$

4. Upper-bound estimate for the integral on minor arcs

According to the results of exponential sums over primes in an arithmetic progression in [2,12], we have, for $x\in\mathcal{M}_2$,

$$S_1(x)\ll(Nq^{-1/2}+N^{1/2}q^{1/2}+N^{4/5}k^{3/5})\mathcal{L}^{5/2}$$

$$\ll (NQ^{-1/2} + N^{1/2}\tau^{-1/2} + N^{4/5}k^{3/5})\mathcal{L}^{5/2}$$
$$\ll Nk^{-1-\varepsilon_0/2}\mathcal{L}^{-2},$$

therefore, by the defintion of $I_2(N)$ in Eq (3.2), we have

$$\begin{aligned} I_2(N) &\ll Nk^{-1-\varepsilon_0/2}\mathcal{L}^{-2}\int_{\mathcal{M}_2}\prod_{j=2}^{3}|S_j(x)|dx \\ &\ll Nk^{-1-\varepsilon_0/2}\mathcal{L}^{-2}\prod_{j=2}^{3}(\int_{\tau}^{1+\tau}|S_j(x)|^2dx)^{1/2} \\ &\ll Nk^{-1-\varepsilon_0/2}\mathcal{L}^{-2}\cdot Nk^{-1}\mathcal{L}^{1+\varepsilon} \\ &\ll N^2k^{-2-\varepsilon_0/2}\mathcal{L}^{-1+\varepsilon}. \end{aligned} \tag{4.1}$$

Here we apply the trivial estimate for $\int_{\tau}^{1+\tau}|S_1(x)|^2dx$, that is,

$$\begin{aligned} &\int_{\tau}^{1+\tau}|S_1(x)|^2dx \\ &= \int_{\tau}^{1+\tau}|\sum_{\substack{N/4\le n\le N \\ n\equiv l_1 \ (\mathrm{mod}\, k)}}\Lambda(n)e(xn)|^2dx \\ &= \sum_{\substack{N/4\le n\le N \\ n\equiv l_1 \ (\mathrm{mod}\, k)}}\Lambda^2(n) \\ &\ll \log N\sum_{\substack{N/4\le n\le N \\ n\equiv l_1 \ (\mathrm{mod}\, k)}}\Lambda(n) \\ &\ll Nk^{-1}\mathcal{L}^{1+\varepsilon}. \end{aligned}$$

5. Simplification for the integral on the major arcs

Recalling the definition of $I_1(N)$ in (3.2), we have

$$I_1(N) = \sum_{q\le Q}\sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(-\frac{h}{q}N\right)\int_{-\tau/q}^{\tau/q}e(-Nz)\prod_{j=1}^{3}S_j\left(\frac{h}{q}+z\right)dz. \tag{5.1}$$

In view of (3.10), the following elementary estimate will be applied:

$$S_1S_2S_3 = A_1A_2A_3 + O(|S_1S_2|R + |S_1|^2R + R^3). \tag{5.2}$$

Denote by E_j ($j = 1, 2, 3$) the total contribution to (5.1) from the first to third terms in the $O-$terms on the right side of (5.1), respectively. Noting

$$R = (1 + N|z|)N\mathcal{L}^2T^{-1}\varphi(q)$$

we have

$$E_1 \ll \sum_{q\le Q} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \int_{-\tau/q}^{\tau/q} (1+N|z|) B\mathcal{L}^2 T^{-1}\varphi(q) \prod_{j=1}^{2} \left| S_j\left(\frac{h}{q}+z\right)\right| dz$$

$$\ll N^2\tau\mathcal{L}T^{-1}\int_{\tau}^{1+\tau} \prod_{j=1}^{2} |S_j(x)|dx.$$

Using Cauchy's inequality and $\Lambda(n) \ll \mathcal{L}$ for $n \le N$, the above can be estimated further as

$$\begin{aligned}
E_1 &\ll N^2\tau\mathcal{L}^2T^{-1}\int_{\tau}^{1+\tau} \mathcal{L}^2 \sum_{\substack{N/4\le n\le N \\ n\le l_j (\mathrm{mod}\, k)}} 1 \\
&\ll N^3\tau\mathcal{L}^4T^{-1}k^{-1} \\
&= N^3N^{-1}k^{2+2\varepsilon_0}\mathcal{L}^{9.5}\mathcal{L}^4k^{-4-10/3\varepsilon_0}\mathcal{L}^{-20}k^{-1} \\
&= N^2k^{-3-4/3\epsilon_0}\mathcal{L}^{-6.5} \\
&\ll N^2k^{-2-\epsilon_0}\mathcal{L}^{-1} := \Omega.
\end{aligned}$$

Similarly we can derive

$$\begin{aligned}
E_3 &\ll \sum_{q\le Q} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \int_{-\tau/q}^{\tau/q} (1+N|z|)^3 (N\mathcal{L}^2T^{-1}\varphi(q))^3 dz \\
&\ll N^3\mathcal{L}^6T^{-3}\sum_{q\le Q}\varphi(q)^3 \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \left\{\int_0^{N^{-1}} dz + \int_{N^{-1}}^{\tau/q} N^3z^3dz\right\} \\
&\ll N^6\mathcal{L}^6T^{-3}\tau^4Q \\
&= N^6\mathcal{L}^6k^{-12-10\varepsilon_0}\mathcal{L}^{-60}N^{-4}k^{8+8\varepsilon_0}\mathcal{L}^{38}k^{2+\varepsilon_0}\mathcal{L}^9 \\
&= N^2k^{-2-\varepsilon_0}\mathcal{L}^{-7} \\
&\ll \Omega.
\end{aligned}$$

For E_2, we need to note that $|S_1|R^2 \le |S_1|^2R + R^3$. Therefore by (5.1), (5.2), and (3.10), we have

$$I_1(N) = \sum_{q\le Q}\varphi(D)^{-3} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(-\frac{h}{q}N\right)\int_{-\tau/q}^{\tau/q} e(-Nz)\prod_{j=1}^{3} H_j(h,q,z)dz + O(\Omega), \tag{5.3}$$

where

$$\Omega := N^2k^{-2-\epsilon_0}\mathcal{L}^{-1}.$$

Since $\widetilde{\beta}$ does not exist, now multiplying out the product $\prod_{j=1}^{3} H_j(h,q,z)$ by (3.9) we get 8 terms. They are grouped into the following two categories:

($\mathcal{T}_1$) the term $\prod_{j=1}^{3}(G_j(\chi_0,h,q)I(z)$;

($\mathcal{T}_2$) 7 terms, each has at least one $\sum_{\chi (\mathrm{mod}\, D)} G_j(\overline{\chi},h,q)I(\chi,z)$ as a factor.

For $i = 1, 2$, define

$$\mathcal{M}_i := \sum_{q\le Q} \varphi(D)^{-3} \sum_{\substack{h=1\\(h,q)=1}}^{q} e\left(-\frac{h}{q}N\right) \int_{-\tau/q}^{\tau/q} e(-Nz)\{\text{sum of the terms in } (\mathcal{T}_i)\} dz. \tag{5.4}$$

In view of (5.3) and (5.4), we get

$$I_1(N) = \mathcal{M}_1 + \mathcal{M}_2 + O(\Omega). \tag{5.5}$$

6. An asymptotic formula for $\mathcal{M}_1$

For any positive integer q, define

$$A(q) := \left(\frac{\varphi(d)}{\varphi(q)}\right)^3 \sum_{\substack{h=1\\(h,q)=1}}^{q} e\left(-\frac{h}{q}N\right) \prod_{j=1}^{3} G_j(\chi_0, h, q). \tag{6.1}$$

It can be proved that for any q_1, q_2, and $(q_1, q_2) = 1$, we have $A(q_1q_2) = A(q_1)A(q_2)$; for any h with $(h, q) = 1$, we have

$$G_j(\chi, h, q) = \sum_{\substack{l=1\\(l,q)=1\\l\equiv l_j \,(\mathrm{mod}\, d)}}^{q} e(hl/q)\chi(s_j),$$

and $(s_j, D) = 1, \chi_0 = \chi \,(\mathrm{mod}\, D)$, so $\chi_0(s_j) = 1$.

$$G_j(\chi_0, h, q) = \sum_{\substack{l=1\\(l,q)=1\\l\equiv l_j \,(\mathrm{mod}\, q)}} e\left(\frac{h}{q}l\right) = \begin{cases}\mu(q/d)e(hul_j/d), & \text{if } (d, q/d) = 1,\\ 0, & \text{otherwise,}\end{cases} \tag{6.2}$$

where $0 \le u < d$ is the unique solution of the congruence $qu/d \equiv 1 \,(\mathrm{mod}\, d)$.

For the estimation of $\mathcal{M}_1$, we need to consider several cases where $(d, q/d) = 1$ will occur when $q = p^m$, where p is prime and m is any positive integer. Now, we prove this assertion.

In short, $G_j(\chi_0, h, p^m)$ is not equal to 0 only when $(k, p) = 1$ and $p^m = k$.

Lemma 6.1. *By the definition of $A(q)$, for prime p and any positive integer m, when k is prime, we have*

$$A(p^m) = \begin{cases} p-1, & \text{if } p = k, m = 1,\\ -(1-p)^{-3}, & \text{if } p \nmid N, (p,k) = 1, m = 1,\\ -(1-p)^{-2}, & \text{if } p \mid N, (p,k) = 1, m = 1,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. For any h with $(h, q) = 1$, Eq (6.2) gives

$$G_j(\chi_0, h, p^m) = 0,$$

and hence

$$A(p^m) = 0 \quad \text{unless} \quad m = 1. \tag{6.3}$$

We now show that if

$$G_j(\chi_0, h, p^m) \neq 0 \quad \Longrightarrow \quad m = 1.$$

Since in (6.2), $q = p^m$ is a prime power, any pair $(d,\ q)$ satisfying $(d, q/d) = 1$ must be

$$d = 1, \quad q/d = 1, \quad \text{or} \quad d,\ q/d > 1,$$

but the third case cannot occur (as then $(d, q/d) > 1$). We consider the remaining two cases:

(i) *Case* $d = 1$. Here $(k, p^m) = 1$, and since k is prime this implies $(p, k) = 1$ and $m = 1$.
(ii) *Case* $q/d = 1$. Here $d = q = p^m$, and because $d = (k, p^m)$ with k prime, we must have $p^m = k$. Primehood of k then forces $m = 1$. So we have $p = k$ and $m = 1$.

These shows that when $m = 1$, $G_j(\chi_0, h, p^m) \neq 0$.

The third case $d, q/d > 1$ is impossible. Suppose, for contradiction, that $(d, p^m/d) = 1$ while $d > 1$ and $p^m/d > 1$. Since $d = (k,\ p^m) > 1$ and $d \mid p^m$, we have

$$d = p^{\alpha},\ 1 \le \alpha < m,$$

and

$$\frac{p^m}{d} = p^{m-\alpha} > 1,$$

then

$$\gcd(d, p^m/d) = \gcd(p^{\alpha}, p^{m-\alpha}) = p^{\min\{\alpha, m-\alpha\}} > 1,$$

contradicting the assumption $\gcd(d, p^m/d) = 1$. Hence, this case cannot occur.

In view of (6.1), we obtain, for $m = 1$ and $(p, k) = 1$,

$$A(p) = \varphi(p)^{-3} \sum_{h=1}^{p-1} e\left(-\frac{h}{p}N\right) \prod_{j=1}^{3} \sum_{l=1}^{p-1} e\left(\frac{hl}{p}\right) = \begin{cases} -(1-p)^{-3}, & \text{if } p \nmid N, \\ -(1-p)^{-2}, & \text{if } p \mid N. \end{cases} \tag{6.4}$$

For $m = 1$ and $p = k$, we have, by (6.1) and the condition $l_1 + l_2 + l_3 \equiv N (\operatorname{mod} k)$,

$$A(p) = \sum_{\substack{h=1 \\ (h,p)=1}}^{p} e\left(-\frac{h}{p}N\right) \prod_{j=1}^{3} e\left(\frac{h}{p} l_j\right) = \sum_{\substack{h=1 \\ (h,p)=1}}^{p} e\left(\frac{l_1 + l_2 + l_3 - N}{p} h\right) = \varphi(k) = k - 1. \tag{6.5}$$

This, together with (6.3)–(6.5), proves Lemma 6.1. □

We use $\operatorname{ord}_p(n)$ to denote the largest integer α such that $p^{\alpha} \mid n$. Consider k to be prime. By Lemma 6.1, we have, for any positive number y,

$$\begin{aligned} \sum_{q \le y} |A(q)| \le & \prod_{\substack{p \le y \\ (p,k)=1}} (1 + |A(p)|) \prod_{\substack{p \le y \\ p \mid k}} (1 + |A(p)| + \cdots + |A(p^{\operatorname{ord}_p(k)})|) \\ = & (1 + |A(k)|) \prod_{\substack{p \le y \\ (p,k)=1}} (1 + |A(p)|) = \prod_{p \le y} (1 + |A(p)|) \end{aligned}$$

$$\leq \prod_{p\leq y}\left(1+\frac{1}{(p-1)^2}\right) \ll 1. \tag{6.6}$$

Hence the series $\sum A(q)$ is absolutely convergent. For any p with $(p,k)=1$, define

$$s(p) := 1 + A(p). \tag{6.7}$$

It is easy to see that

$$\sum A(q) = (1 + A(k)) \prod_{(p,k)=1} (1 + A(p)) = k \prod_{(p,k)=1} s(p). \tag{6.8}$$

We also need the following.

Lemma 6.2. *For any complex number p_j with $0 < \mathrm{Re}\rho_{\mathrm{j}} \leq 1$, $j = 1, 2, 3$, we have*

$$\int_{-\infty}^{+\infty} e(-Nz)\left(\prod_{j=1}^{3}\int_{N/4}^{N} t^{\rho_j-1}e(zt)dt\right)dz = N^2\int_{\mathcal{D}}\prod_{j=1}^{3}(Nx_j)^{\rho_j-1}dx_1dx_2,$$

where $x_3 = 1 - x_1 - x_2$, and $\mathcal{D} := \{(x_1, x_2) :\ 1/4 \leq x_1,\ x_2,\ x_3 \leq 1\}$.

Proof. This follows Lemma 4.7 of [7]. □

Then we have the following lemma.

Lemma 6.3. *Let $\mathcal{M}_1$ be defined as in (5.4) and define*

$$\mathcal{M}_0 := N^2k(k-1)^{-3}\prod_{(p,k)=1} s(p)\int_{\mathcal{D}} dx_1dx_2.$$

We have

$$(i)\ \mathcal{M}_1 = \mathcal{M}_0 + O(\Omega), \quad (ii)\ \mathcal{M}_0 \gg N^2k^{-2}.$$

Proof. By the definitons of $\mathcal{M}_1$ and $A(q)$ we can derive the following

$$\mathcal{M}_1 = (k-1)^{-3}\sum_{q\leq Q} A(q)\int_{-\tau/q}^{\tau/q} e(-Nz)I^3(z)dz. \tag{6.9}$$

We first extend the range of the integration in the above equation to $(-\infty, +\infty)$. By the estimation $I(z) \ll \min(N, |z|^{-1})$ from (2.1), (5.3), and (6.6), the total error induced from this extension is:

$$\begin{aligned}
&\ll (k-1)^{-3}\sum_{q\leq Q}|A(q)|\int_{\tau/q}^{+\infty} z^{-3}dz\\
&\ll (k-1)^{-3}(\tau/Q)^{-2}\sum_{q\leq Q}|A(q)|\\
&\ll k^{-3+\varepsilon_0}\tau^{-2}Q^2\\
&= k^{-3+\varepsilon_0}N^2k^{-4-4\varepsilon_0}\mathcal{L}^{-19}k^{4+2\varepsilon_0}\mathcal{L}^{18}
\end{aligned}$$

$$= N^2 k^{-3-\varepsilon_0} \mathcal{L}^{-1}$$
$$\ll \Omega. \tag{6.10}$$

Again, we have

$$\begin{aligned}
\sum_{q>Q} |A(q)| &\le Q^{-1} \sum_{q=1}^{+\infty} q|A(q)| \\
&= Q^{-1} \prod_{(p,k)=1} (1 + p|A(p)|) \prod_{p|k} (1 + pA(p) + \cdots + p^{\mathrm{ord}_p(k)} A(p^{\mathrm{ord}_p(k)})) \\
&= Q^{-1}(1 + kA(k)) \prod_{(p,k)=1} (1 + p|A(p)|) \\
&\ll Q^{-1} k^{2-\varepsilon_0} \prod_{\substack{(p,k)=1 \\ p|N}} \left(1 + \frac{p}{(p-1)^2}\right) \prod_{\substack{(p,k)=1 \\ p\nmid N}} \left(1 + \frac{p}{(p-1)^3}\right) \\
&\ll Q^{-1} k^{2-\varepsilon_0} \mathcal{L}.
\end{aligned} \tag{6.11}$$

Thus by (6.7) to (6.11) and Lemma 6.2, we get

$$\mathcal{M}_1 = N^2 (k-1)^{-3} \int_{\mathcal{D}} dx_1 dx_2 \left(\sum_{q=1}^{+\infty} A(q) + O\left(\sum_{q>Q} |A(q)| \right) \right) + O(\Omega) = \mathcal{M}_0 + O(\Omega),$$

with $\rho_1 = \rho_2 = \rho_3 = 1$, and $\int_{\mathcal{D}} dx_1 dx_2 = 1/32$ by the definition of $\mathcal{D}$ in Lemma 6.2. This proves (*i*). For (*ii*) we only need to note that

$$1 \ll \prod_{p\ge 3} \left(1 - \frac{1}{(p-1)^2}\right) \le \prod_{(p,k)=1} s(p) \le \prod_{(p,k)=1} \left(1 + \frac{1}{(p-1)^3}\right) \ll 1.$$

So we have $\mathcal{M}_0 \gg N^2 k (k-1)^{-3} \gg N^2 k^{-2}$. The proof of Lemma 6.3 is complete. □

7. An expression for $Z(q; \chi_1, \chi_2, \chi_3)$

Let $r_1,\ r_2,\ r_3$ be any positive integers and denote by $r = [r_1,\ r_2,\ r_3]$ their common multiple. For any primitive characters $\chi_j (\mathrm{mod}\, r_j)$ $(1 \le j \le 3)$ and $r \mid D$, define

$$Z_1(q) := Z_1(q; \chi_1, \chi_2, \chi_3) := \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(-\frac{h}{q} N\right) \prod_{j=1}^{3} G_j(\chi_j \chi_0,\ h,\ q), \tag{7.1}$$

and

$$\sum\nolimits_2 := \varphi(k)^{-3} \sum_{\substack{q \le Q \\ r|D}} \left(\frac{\varphi(d)}{\varphi(q)} \right)^{-3} Z_1(q). \tag{7.2}$$

Let $v := \mathrm{ord}_p(k)$, $v = \begin{cases} 1 & , p = k \\ 0 & , \text{otherwise} \end{cases}$, $\alpha := \mathrm{ord}_p(r), \alpha_j := \mathrm{ord}_p(r_j)$, $j = 1,\ 2,\ 3$, and put

$$r' = \prod_{\substack{p|r \\ \alpha > v}} p^{\alpha}, \quad r'_j = \prod_{\substack{p|r_j \\ \alpha_j \ge v}} p^{\alpha_j}, \quad r'' = \frac{r}{r'}, \quad r''_j = \frac{r_j}{r'_j}. \tag{7.3}$$

Since $\chi_j (\bmod r_j)$ is primitive and $r_j = r'_j r''_j$, we can write $\chi_j (\bmod r_j) = \chi'_j (\bmod r'_j)\chi''_j (\bmod r''_j)$ $(1 \le j \le 3)$, where both χ'_j and χ''_j are primitive. Next, define

$$Z_2(q) := Z_2(q;\chi'_1,\chi'_2,\chi'_3) = \sum_{\substack{h=1\\(h,q)=1}}^{q} e\left(-\frac{h}{q}N\right)\prod_{j=1}^{3}\sum_{\substack{l=1\\(l,q)=1\\l\equiv l_j (\bmod d)}}^{q} e\left(\frac{h}{q}l\right)\chi'_j(l). \tag{7.4}$$

Lemma 7.1. *Under the notations of* $\sum_2$ *and* $s(p)$ *defined as* (7.2) *and* (6.7)*, respectively, we have*

$$\sum\nolimits_2 \le 2.140782k^{-2+\varepsilon_0}\prod_{(p,k)=1} s(p).$$

Proof. By Lemma 6.1 and (6.7), we have

$$\begin{aligned}
\left|\sum_{\substack{q\le Q/r'\\(q,r')=1}} A(q)\right| &\le \prod_{\substack{p\nmid r'\\p|k}}(1+\varphi(p)+\cdots+\varphi(p^{\nu}))\prod_{\substack{p\nmid kr'\\p|N}}(1+A(p))\prod_{\substack{p\nmid kr'\\p\nmid N}}(1+A(p))\\
&= ((1+\varphi(k))\prod_{\substack{p\nmid kr'\\p\nmid N}}(1+A(p))\prod_{\substack{p|kr'\\p\nmid N}}(1+A(p))\\
&\le k\prod_{\substack{p\nmid kr'\\p|N}}\left(1+A(p)+\frac{2}{(p-1)^2}\right)\prod_{\substack{p\nmid kr'\\p\nmid N}}(1+A(p))\\
&\le k\prod_{p\nmid kr'}(1+A(p))\prod_{\substack{p\ge3\\p|N}}\frac{1+A(p)+\frac{2}{(p-1)^2}}{1+A(p)}\\
&\le k\prod_{p\nmid kr'}s(p)\prod_{p\ge3}\frac{(p-1)^2+1}{(p-1)^2-1}\\
&\le 2.140782k\prod_{p\nmid kr'}s(p).
\end{aligned} \tag{7.5}$$

By Lemma 5.3 in [9], suppose $q = q_1q_2$, $r' \mid q$, $(r', q_2) = 1$, and q_1 and r' have the same prime factors. Then

$$Z_2(q) = Z_2(q_1)(\varphi(q_2)/\varphi(d_2))^3 A(q_2), \tag{7.6}$$

where $d_2 = (k, q_2)$ and $Z_2(q_1) = 0$ if $q_1 \ne r'$. Thus, we have

$$\begin{aligned}
\sum\nolimits'_2 &:= \sum_{\substack{q\le Q\\r'|q}}(\varphi(d)/\varphi(q))^3 Z_2(q)\\
&=(\varphi((k,r'))/\varphi(r'))^3 Z_2(r')\sum_{\substack{q_2\le Q/r'\\(q_2,r')=1}} A(q_2).
\end{aligned} \tag{7.7}$$

Decompose $r' = r^{(1)}r^{(2)}$ where

$$r^{(i)} = [r'_{1i},\ r'_{2i},\ r'_{3i}],\quad i = 1,\ 2, \tag{7.8}$$

and write $\chi'_{j1}(\bmod r'_{j1}) = \prod_{p\mid r'_{j1}} \chi'_{j1p}(\bmod r'_{j1p})$ for $j = 1, 2, 3$. Then by (7.2), since $Z_2(q)$ is a multiplicative function of q, we have

$$Z_2(r') = Z_3 \prod_{p\mid r^{(1)}} Z_4(p), \tag{7.9}$$

where by $d = (r^{(2)}, k) = 1$ we have

$$Z_3 := \sum_{\substack{h=1\\(h,r^{(2)})=1}}^{r^{(2)}} e(-Nh/r^{(2)}) \prod_{j=1}^{3} \sum_{\substack{l=1\\(l,r^{(2)})=1}}^{r^{(2)}} e(a_j hl/r^{(2)})\chi'_{j2}(l),$$

and by

$$\alpha = \mathrm{ord}_p(r^{(1)}) \text{ and } d = (p^\alpha, k) = p^v,$$

we have

$$Z_4(p) := \sum_{\substack{h=1\\p\nmid h}}^{p^\alpha} e(-Nh/p^\alpha) \prod_{j=1}^{3} \prod_{\substack{l=1\\l\equiv l_j\,(\bmod p^v)}}^{p^\alpha} e(a_j hl/p^\alpha)\chi'_{j1p}(l).$$

For Z_3, we can derive that

$$|Z_3| \le \varphi(r^{(2)})^3 \prod_{p\mid r^{(2)}} s(p). \tag{7.10}$$

For $Z_4(p)$ and $p \mid r^{(1)}$, we have

$$Z_4(p) = p^\alpha \sum_{(l)} \prod_{j=1}^{3} \chi'_{j1p}(l^{(j)}), \tag{7.11}$$

where $\sum_{(l)}$ is taken over all $l^{(j)} = 1, \cdots, p^\alpha$, $j = 1, 2, 3$ satisfying $l^{(j)} \equiv l_j\,(\bmod p^v)$ and $\sum_{j=1}^{3} l^{(j)} \equiv N\,(\bmod p^\alpha)$. So, by (7.11),

$$\begin{aligned}
|Z_4(p)| &\le p^\alpha \sum_{(l)} 1\\
&= \sum_{h=1}^{p^\alpha} e(-Nh/p^\alpha) \prod_{j=1}^{3} \sum_{\substack{l=1\\l\equiv l_j\,(\bmod p^v)}}^{p^\alpha} e(hl/p^\alpha)\\
&= \sum_{h=1}^{p^\alpha} e(-Nh/p^\alpha) \prod_{j=1}^{3} e(hl_j/p^\alpha) \sum_{t=0}^{p^{\alpha-v}-1} e(ht/p^{\alpha-v})\\
&= \sum_{\substack{h=1\\p^{\alpha-v}\mid h}}^{p^\alpha} e(h(-N + \sum_{j=1}^{3} l_j)/p^v)p^{3(\alpha-v)}\\
&= (\sum_{\substack{h=1\\p^{\alpha-v}\mid h}}^{p^\alpha} 1)p^{3(\alpha-v)}\\
&= p^{3\alpha-2v}.
\end{aligned} \tag{7.12}$$

Therefor, by (7.5), (7.7), (7.9), (7.10), and (7.12), we have

$$\sum\nolimits_2' \ll (\varphi((k,r'))/\varphi(r'))^3\varphi(r^{(2)})^3\prod_{p|r^{(2)}} s(p)\prod_{p|r^{(1)}} p^{3\alpha-2\nu}k\prod_{p\nmid r'k} s(p). \tag{7.13}$$

Here

$$(\varphi((k,r'))/\varphi(r'))\varphi(r^{(2)}) = \varphi((k,r^{(1)})/\varphi(r^{(1)}) = \prod_{p|r^{(1)}} p^{\nu-\alpha}, \tag{7.14}$$

and

$$\prod_{p|r^{(2)}} s(p) = \prod_{\substack{p|r'\\ p\nmid k}} s(p). \tag{7.15}$$

Then, by (7.2), (7.7) and (7.13)–(7.15),

$$\begin{aligned}\sum\nolimits_2 &= \varphi(k)^{-3}\sum_{\substack{q\le Q\\ r|D}}(\varphi(d)/\varphi(q))^3 Z_1(q)\\ &= \varphi(k)^{-3}\left(\prod_{j=1}^{3}\chi_j''(l_j)\right)\sum\nolimits_2'\\ &\le 2.140782k(k-1)^{-3}\prod_{p\nmid k} s(p)\\ &\le 2.140782k^{-2+\varepsilon_0}\prod_{p\nmid k} s(p). \end{aligned}\tag{7.16}$$

□

Lemma 7.2. *Let* $\sum_2$ *be defined as in* (7.2). *If at least one of the* χ_j'*s* $(1\le j\le 3)$ *is* χ_{01}, *then we have*

$$\sum\nolimits_2 \ll k^{-1+\varepsilon_0}r^{-1}(\log\log r')^2.$$

Proof. See the proof of Lemma 5.2 of [13] and (7.16). □

8. An upper bound for $\mathcal{M}_2$

Recall that

$$\mathcal{M}_2 = \sum_{q\le Q}\varphi(D)^{-3}\sum_{\substack{h=1\\ (h,q)=1}}^{q} e\left(-\frac{h}{q}N\right)\int_{-\tau/q}^{\tau/q} e(-Nz)\{\text{sum of the terms in } (\mathcal{T}_2)\}\mathrm{d}z,$$

where

$$\mathcal{T}_2 : 7 \text{ terms, and each has at least one } \sum_{\chi\,(\mathrm{mod}\,D)} G_j(\chi,h,q)I(\chi,z) \text{ as a factor.}$$

Then they are of the following three types:

($\mathcal{T}_{21}$) 3 terms of the form $\prod_{j=1}^{2}(G_j(\chi_0,h,q)I(z))\sum_{\chi\,(\mathrm{mod}\,D)} G_3(\overline{\chi},h,q)I(\chi,z)$;

($\mathcal{T}_{22}$) 3 terms of the form $G_1(\chi_0,\ h,\ q)I(z)\prod_{j=2}^{3}\sum_{\chi\,(\mathrm{mod}\,D)}G_j(\overline{\chi},\ h,\ q)I(\chi,z)$;

($\mathcal{T}_{23}$) The remaining one term $\prod_{j=1}^{3}\sum_{\chi\,(\mathrm{mod}\,D)}G_j(\overline{\chi},\ h,\ q)I(\chi,z)$.

Before the treatments for these three types, we provide the following lemmas.

To estimate $\mathcal{M}_2$, we also need the explicit estimates for two triple sums, that is

$$\sum\nolimits_4 := \sum_{q\le kQ}\sum_{\chi\,(\mathrm{mod}\,q)}{}^{*}\sum_{|\gamma|\le C}{}'(N/4)^{\beta-1},$$

and

$$\sum\nolimits_5 := \sum_{q\le kQ^{\varepsilon_0}}\sum_{\chi\,(\mathrm{mod}\,q)}{}^{*}\sum_{|\gamma|\le C}{}'(N/4)^{\beta-1},$$

where ε_0 and ε_1 are two fixed, sufficiently small positive constants.

Let $N(\chi,\alpha,C)$ denote the number of zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$ lying inside the rectangle $\alpha\le\beta\le 1-0.365/L$, $|\gamma|\le C$. Put $N^*(\alpha,\ kQ,\ C) := \sum_{q\le kQ}\sum^{*}_{\chi\,(\mathrm{mod}\,q)}N(\chi,\ \alpha,\ C)$.

Lemma 8.1. *Since the exceptional zero $\widetilde{\beta}$ does not exist, then for $k\le N^{1/40}$ we have*

$$\sum\nolimits_4 = \sum_{q\le kQ}\sum_{\chi\,(\mathrm{mod}\,q)}{}^{*}\sum_{|\gamma|\le C}{}'(N/4)^{\beta-1} \le 0.8369.$$

Proof. By Lemma 2.1 and in view of the bounds for λ in Lemma 6.2 of [8], we can write

$$\begin{aligned}\sum\nolimits_4 \le&(N/4)^{-1/2}N^*(1/2,kQ,C)+\{\int_{1/2}^{4/5}+\int_{4/5}^{1-L^{-1}\log\log L}\\ &+\int_{1-L^{-1}\log\log L}^{1-6/L}+\int_{1-6/L}^{1-2/L}+\int_{1-2/L}^{1-1/L}\\ &+\int_{1-1/L}^{1-0.696/L}+\int_{1-0.696/L}^{1-0.504/L}+\int_{1-0.504/L}^{1-0.364/L}(N/4)^{\alpha-1}N^*(\alpha,kQ,C)\mathcal{L}d\alpha\\ =:&\sum_{j=1}^{9}D_j,\ \text{say.}\end{aligned}$$

For D_1,

$$\begin{aligned}D_1 &= (N/4)^{-1/2}N^*(1/2,kQ,C)\\ &\ll N^{-1/2}((kQ)^2C)^{12/5(1-1/2)+\varepsilon}\\ &\ll N^{-1/2}(kQ)^{12/5}\\ &\ll (kQ)^{-4.2}\ll(\log L)^{-9}.\end{aligned}$$

For D_2,

$$D_2 = \int_{1/2}^{4/5}(N/4)^{\alpha-1}N^*(\alpha,kQ,C)\mathcal{L}\,\mathrm{d}\alpha$$

$$
\begin{aligned}
&= \int_{1/2}^{4/5} (N/4)^{\alpha-1}((kQ)^2C)^{\frac{12}{5}(1-\alpha)}\mathcal{L}\,\mathrm{d}\alpha \\
&\ll \int_{1/2}^{4/5} (N(kQ)^{-24/5})^{\alpha-1}\mathcal{L}\,\mathrm{d}\alpha \\
&\ll (kQ)^{-1.7} \ll (\log L)^{-9}.
\end{aligned}
$$

For D_3,

$$
\begin{aligned}
D_3 &= \int_{4/5}^{1-L^{-1}\log\log L} (N/4)^{\alpha-1}N^*(\alpha, kQ, C)\mathcal{L}\,\mathrm{d}\alpha \\
&= \int_{4/5}^{1-L^{-1}\log\log L} (N/4)^{\alpha-1}((kQ)^2C)^{(2+\varepsilon)(1-\alpha)}\mathcal{L}\,\mathrm{d}\alpha \\
&\ll \int_{4/5}^{1-L^{-1}\log\log L} (N(kQ)^{-4})^{\alpha-1}\mathcal{L}\,\mathrm{d}\alpha \\
&\ll (\log L)^{-28/3} \ll (\log L)^{-9}.
\end{aligned}
$$

So, we have

$$
D_1 + D_2 + D_3 \ll (\log L)^{-9}.
$$

For D_4, by Lemma 3.1 in [8], if $1 - L^{-1}\log\log L \le \alpha < 1 - 6/L$, we have

$$
\begin{aligned}
N_4^*(\alpha, Q, C) &\le N_4^* \\
&\le 42.54\left(1 + \frac{35.385}{6}\right)\left(e^{2.87538\log\log L} - \frac{e^{2.07176*6} - e^{1.92136*\log\log L}}{0.1504*6}\right) \\
&\le 293.42\left((\log L)^{2.87538} + 1.108156(\log L)^{1.92136} - 250336.1766\right),
\end{aligned}
$$

then

$$
\begin{aligned}
D_4 &= \int_{1-L^{-1}\log\log L}^{1-6/L} (N/4)^{\alpha-1}N^*\log(N)d\alpha \\
&\le N^*\frac{\log N}{\log N - \log 4}((N/4)^{-6/L} - (N/4)^{-\log\log L/L}) \\
&\le N^*\frac{40/3L}{40/3L - \log 4}(e^{-6*40/3} - e^{-40/3\log\log L}) \\
&\le \frac{40/3L}{40/3L - \log 4}293.42((\log L)^{2.87538} + 1.108156(\log L)^{1.92136} - 250336.1766) \\
&\quad \times (e^{-80} - (\log L)^{-40/3}) \\
&\le \frac{40/3L}{40/3L - \log 4}293.42 \times 3.1 \times 10^7 \times e^{-80} \le 1.64 \times 10^{-25}.
\end{aligned}
$$

For D_5, if $1 - 6/L \le \alpha < 1 - 2/L$, we have

$$
N_5^*(\alpha, Q, C) \le N_5^*
$$

$$\leq 167.67/2\left(e^{3.116796\times 6}-\frac{(e^{2.223794\times 2}-e^{1.869794\times 6})}{0.354\times 2}\right)$$
$$\leq 1.1\times 10^{10},$$

then

$$\begin{aligned} D_5 &= \int_{1-6/L}^{1-2/L}(N/4)^{\alpha-1}N_5^*\log N\,d\alpha \\ &\leq N_5^*\frac{\log N}{\log N-\log 4}((N/4)^{-2/L}-(N/4)^{-6/L}) \\ &\leq N_5^*\frac{40/3L}{40/3L-\log 4}(e^{-2*40/3}-e^{-6*40/3}) \\ &\leq \frac{40/3L}{40/3L-\log 4}1.1\times 10^{10}e^{-2*40/3} \\ &\leq 0.0291. \end{aligned}$$

For D_6, if $1-2/L\leq\alpha<1-1/L$, we have

$$\begin{aligned} N_6^*(\alpha,Q,C) &\leq N_6^* \\ &\leq 50.36/1\left(e^{3.753506\times 2}-\frac{(e^{2.747904\times 1}-e^{2.160104\times 2})}{0.58\times 1}\right) \\ &\leq 96868, \end{aligned}$$

then

$$\begin{aligned} D_6 &= \int_{1-2/L}^{1-1/L}(N/4)^{\alpha-1}N_6^*\log N\,d\alpha \\ &\leq N_6^*\frac{\log N}{\log N-\log 4}((N/4)^{-1/L}-(N/4)^{-2/L}) \\ &\leq N_6^*\frac{40/3L}{40/3L-\log 4}(e^{-1*40/3}-e^{-2*40/3}) \\ &\leq \frac{40/3L}{40/3L-\log 4}96868e^{-1*40/3} \\ &\leq 0.1569. \end{aligned}$$

For D_7, if $1-1/L\leq\alpha<1-0.696/L$, we have

$$\begin{aligned} N_7^*(\alpha,Q,C) &\leq N_7^* \\ &\leq 26.93/0.696\left(e^{4.28374\times 1}-\frac{(e^{3.19253\times 0.696}-e^{2.42653\times 1})}{0.766\times 0.696}\right) \\ &\leq 2957.6, \end{aligned}$$

then

$$D_7 = \int_{1-1/L}^{1-0.696/L}(N/4)^{\alpha-1}N_7^*\log N\,d\alpha$$

$$
\begin{aligned}
&\le N_7^* \frac{\log N}{\log N - \log 4}((N/4)^{-0.696/L} - (N/4)^{-1/L}) \\
&\le N_7^* \frac{40/3L}{40/3L - \log 4}(e^{-0.696*40/3} - e^{-1*40/3}) \\
&\le \frac{40/3L}{40/3L - \log 4} 2957.6(e^{-0.696*40/3} - e^{-1*40/3}) \\
&\le 0.2711.
\end{aligned}
$$

For D_8, if $1 - 0.696/L \le \alpha < 1 - 0.504/L$, we have

$$
\begin{aligned}
N_8^*(\alpha, Q, C) \le N_8^* & \\
&\le 8.86706/0.504 \left(e^{4.31403\times 0.696} - \frac{(e^{3.15402\times 0.504} - e^{2.32002\times 0.696})}{0.834 \times 0.504} \right) \\
&\le 359.5,
\end{aligned}
$$

then

$$
\begin{aligned}
D_8 &= \int_{1-0.696/L}^{1-0.504/L} (N/4)^{\alpha-1} N_8^* \log N \, d\alpha \\
&\le N_8^* \frac{\log N}{\log N - \log 4}((N/4)^{-0.504/L} - (N/4)^{-696/L}) \\
&\le N_8^* \frac{40/3L}{40/3L - \log 4}(e^{-0.504*40/3} - e^{-0.696*40/3}) \\
&\le \frac{40/3L}{40/3L - \log 4} 359.5(e^{-0.504*40/3} - e^{-0.696*40/3}) \\
&\le 0.4002.
\end{aligned}
$$

By Lemma 2.5 of [8], we get

$$
\begin{aligned}
D_9 &= \int_{1-0.504/L}^{1-0.364/L} (N/4)^{\alpha-1} N^*(\alpha, kQ, C) \mathcal{L} d\alpha \\
&\le \frac{((N/4)^{-0.364/L} - (N/4)^{-0.504/L})\mathcal{L}}{\mathcal{L} - \log 4} d\alpha \\
&\le (e^{-40/3\times 0.364} - e^{-40/3\times 0.504})(1 + \frac{\log 4}{\mathcal{L} - \log 4}) \\
&\le 0.0066.
\end{aligned}
$$

Summing up these estimates, we get the desired numerical estimate 0.8639 for $\sum_4$. □

Lemma 8.2. *Since the exceptional zero $\widetilde{\beta}$ does not exist, then for $k \le N^{1/40}$ we have*

$$
\sum\nolimits_5 = \sum_{q \le kQ^{\varepsilon_0}} \sum_{\chi \,(\mathrm{mod}\, q)}{}^{*} \sum_{|\gamma| \le C}{}' (N/4)^{\beta-1} \le 0.0078.
$$

Proof. By Lemma 2.1, the function $\Pi(kQ^{\varepsilon_0})$ has at most one zero $\widetilde{\beta}_1$ satisfying

$$1 - 0.364\log(kQ^{\varepsilon_0}) < \widetilde{\beta}_1 \le 1 - 0.364\log(kQ).$$

If $\widetilde{\beta}_1$ does not exist, then, completely similarly to case $\sum_4$, we have

$$kQ^{\varepsilon_0} = k^{1+2\varepsilon_0} = N^{\frac{1+2\varepsilon_0}{40}},$$

so we take θ with $\frac{1+2\varepsilon_0}{40}$, and we can obtain $\sum_5 \le 1.11\times 10^{-6}$.

When $\widetilde{\beta}_1$ exists, we can derive

$$\sum\nolimits_5 \le 5.167\times 10^{-7} + N^{\widetilde{\beta}_1-1} \le 5.167\times 10^{-7} + \exp(-0.364\log N/\log(kQ)) \le 0.0078.$$

□

At the end of this section, we give a crude upper bound for the triple sum. For $k \le N^{1/40}$, we have

$$\sum_{q\le kQ}\sum_{\chi\,(\mathrm{mod}\,q)}{}^{*}\sum_{|\gamma|\le T}{}^{\prime}(N/4)^{\beta-1} \le 1, \tag{8.1}$$

which can be obtained in precisely the same way as that of Lemma 4.5 from [7] with Q and N_j' replaced by kQ and $N/4$, respectively.

We now estimate $\mathcal{M}_2$. First we state a lemma which will be used frequently in this section.

Lemma 8.3. *Let $\rho = \beta + i\gamma$ and $1/2 \le \beta \le 1$. Then, for any real z, we have*

$$\int_{N/4}^{N} e(zt)t^{\rho-1}dt \ll \begin{cases} \min\left(N^{\beta}, |z|^{-\beta}\right), & \text{if } \gamma = 0\\ N^{\beta}|\gamma|^{-1}, & \text{if } |z| \le \frac{|\gamma|}{4\pi N},\\ N^{\beta}|\gamma|^{-1/2}, & \text{if } \frac{|\gamma|}{4\pi N} < |z| \le \frac{4|\gamma|}{\pi N},\\ N^{\beta-1}|z|^{-1}, & \text{if } \frac{4|\gamma|}{\pi N} < |z|. \end{cases}$$

Proof. This is Lemma 3.2 of [7]. □

Recall from (5.4) that there are 7 terms in the integrand of $\mathcal{M}_2$, and they are of the following three types:

- ($\mathcal{T}_{21}$): Three terms of the form $\prod_{j=1}^{2}(G_j(\chi_0,h,q)I(z))\sum_{\chi\,(\mathrm{mod}\,D)} G_3(\bar{\chi},h,q)I(\chi,z)$;
- ($\mathcal{T}_{22}$): Three terms of the form $G_1(\chi_0,h,q)I(z)\prod_{j=2}^{3}\sum_{\chi\,(\mathrm{mod}\,D)} G_j(\bar{\chi},h,q)I(\chi,z)$;
- ($\mathcal{T}_{23}$): The remaining one term $\prod_{j=1}^{3}\sum_{\chi\,(\mathrm{mod}\,D)} G_j(\bar{\chi},h,q)I(\chi,z)$.

The treatments for these three types are quite similar. We begin by illustrating the details with a term from the first type $\mathcal{M}_{21}$:

$$\mathcal{M}_{21} =: \sum_{q\le Q}\varphi(D)^{-3}\sum_{\substack{h=1\\(h,q)=1}}^{q} e\left(-\frac{h}{q}N\right)\int_{-\tau/q}^{\tau/q} e(-Nz)\prod_{j=1}^{2}(G_j(\chi_0,h,q)I(z))$$
$$\times \sum_{\chi\,(\mathrm{mod}\,D)} G_3(\bar{\chi},h,q)I(\chi,z)dz.$$

Noting the definition of $Z_1(q)$ in (7.1), we have

$$\mathcal{M}_{21} = \sum_{r_3 \le kQ} \sum_{\chi_3 \,(\mathrm{mod}\, r_3)}{}^{*} \sum_{\substack{q \le Q \\ [r_3] | D}} \frac{Z_1(q; \chi_{01}, \chi_{02}, \tilde{\chi}_3)}{\varphi(D)^3} \times \int_{-\tau/q}^{\tau/q} e(-Nz) I_1(z) I_2(z) I_3(\chi_3, z)\, dz. \tag{8.2}$$

We divide the sum over the nontrivial zeros into the two ranges $C < |\gamma_3| \le T$ and $|\gamma_3| \le C$. By the definition of $I(\chi, z)$, we rewrite the integral with respect to z in the above formula as

$$\begin{aligned} &= \sum_{|\gamma_3| \le T}{}' \int_{-\tau/q}^{\tau/q} e(-Nz) I_1(z) I_2(z) \left(\int_{N/4}^{N} e(zt) t^{\rho_3 - 1} dt \right) dz \\ &= \left\{ \sum_{|\gamma_3| \le C}{}' + \sum_{C < |\gamma_3| \le T}{}' \right\}, \end{aligned} \tag{8.3}$$

where C is a positive number with $4\pi \le C \le (kQ)^{\varepsilon_0/2}$, and its value being chosen later.

By Lemma 8.3, the second term in (8.3) can be estimated as

$$\begin{aligned} &\ll \sum_{C \le |\gamma_3| \le T}{}' \int_{-\tau/q}^{\tau/q} |I_1(z)| |I_2(z)| \left| \int_{N/4}^{N} e(zt) t^{\rho_3 - 1} dt \right| dz \\ &\ll \sum_{C \le |\gamma_3| \le T}{}' \left\{ \int_0^{N^{-1}} N^2 N^{\beta_3} |\gamma_3|^{-1}\, dz + \int_{J_1} z^{-2} N^{\beta_3} |\gamma_3|^{-1}\, dz \right. \\ &\qquad \left. + \int_{J_2} z^{-2} N^{\beta_3} |\gamma_3|^{-1/2}\, dz + \int_{J_3} z^{-3} N^{\beta_3 - 1}\, dz \right\} \\ &\ll \sum_{C \le |\gamma_3| \le T}{}' N^{\beta_3 + 1} |\gamma_3|^{-1} \ll C^{-1} \sum_{|\gamma_3| \le T}{}' N^{\beta_3 + 1}, \end{aligned} \tag{8.4}$$

where J_1, J_2, and J_3 are the intervals of $z \in [N^{-1},\ \tau/q]$ satisfying$z \le |\gamma_3|/(4\pi N),\ \ |\gamma_3|/(4\pi N) < z \le |\gamma_3|/(\pi N)$, and $z > 4|\gamma_3|/(\pi N)$, respectively.

By (8.1), (8.4), and Lemma 7.1, the total error induced in (8.2) by the second terms of (8.3) is

$$\ll N^2 k^{-2} C^{-1} \sum_{r_j \le kQ} \sum_{\chi_3 \,(\mathrm{mod}\, r_3)}{}^{*} \sum_{|\gamma_3| \le T}{}' N^{\beta_3 - 1} \ll N^2 k^{-2+\varepsilon_0} C^{-1}. \tag{8.5}$$

For the first term on the right-hand side of (8.3), we first extend the range of the integration over z to $(-\infty, +\infty)$, and let R_{21} be the total error induced in (8.2). In view of (2.1) and the bound $|\gamma_j| \le C \le (kQ)^{\varepsilon_0/2}$, we have $\tau/q \ge 4|\gamma_j/\pi N$.

Thus, by Lemma 8.3, we obtain

$$R_{21} \ll \sum_{|\gamma_3| \le T}{}' \int_{\tau/q}^{+\infty} z^{-3} N^{\beta - 1} dz \ll \tau^{-2} q^2 \sum_{|\gamma_3| \le T}{}' N^{\beta - 1},$$

and then the total error induced by R_{21} in (8.2) is

$$\ll k^{-2+\varepsilon_0}\tau^{-2}Q^2 \ll N^2k^{-2+\varepsilon_0}C^{-1}. \tag{8.6}$$

From (8.2), (8.3), (8.5), and (8.6), we get

$$\begin{aligned}
|\mathcal{M}_{21}| &\le \sum_{\chi_3 \,(\mathrm{mod}\, r_3)} {}^{*} \sum_{\substack{q\le Q\\ r_3|D}} \frac{Z(q;\chi_{01},\chi_{02},\tilde{\chi}_3)}{\varphi(D)^3}\\
&\times \sum_{|\gamma_3|\le T}{}^{'} \int_{-\infty}^{+\infty} e(-Nz)I_1(z)I_2(z)\left(\int_{N/4}^{N} e(zt)t^{\rho_3-1}dt\right)dz + O\left(N^2k^{-2+\varepsilon_0}C^{-1}\right)\\
&\le N^2\left\{\sum_{r_3\le kQ^{\varepsilon_0}} + \sum_{kQ^{\varepsilon_0}<r_3\le kQ}\right\} \sum_{\chi_3 \,(\mathrm{mod}\, r_3)}{}^{*} \sum_{|\gamma_3|\le C}{}^{'} \sum_{\substack{q\le Q\\ r_3|D}} \frac{1}{\varphi(D)^3}|Z(q;\chi_{01},\chi_{02},\tilde{\chi}_3)|\\
&\times \int_D (Nx_3)^{\rho_3-1}dx_3 + O\left(N^2k^{-2+\varepsilon_0}C^{-1}\right).
\end{aligned} \tag{8.7}$$

For the second multi-sum inside the curly brackets in (8.7), by Lemma 7.2 and (8.1), it can be estimated as

$$\begin{aligned}
&\ll N^2k^{-2+\varepsilon_0} \sum_{kQ^{\varepsilon_0}<r_3\le kQ} \frac{k}{r_2}\log^2\mathcal{L} \sum_{\chi_3 \,(\mathrm{mod}\, r_3)}{}^{*} \sum_{|\gamma_3|\le C}{}^{'} (N/4)^{\beta_3-1}\\
&\ll N^2k^{-2+\varepsilon_0}Q^{-\varepsilon_0}\log^2\mathcal{L} \ll N^2k^{-2+\varepsilon_0}C^{-1}.
\end{aligned}$$

Next, we use Lemma 7.1 to determine the absolute value of (8.7)

$$\begin{aligned}
&\le 2.140782\,\mathcal{M}_0 \sum_{r_3\le kQ^{\varepsilon_0}} \sum_{\chi_3 \,(\,\mathrm{mod}\ r_3)}{}^{*} \sum_{|\gamma_3|\le C}{}^{'} \left(\frac{N}{4}\right)^{\beta_3-1}\\
&\quad + O(N^2\,k^{-2+\varepsilon_0}\,C^{-1}).
\end{aligned}$$

Now, there are two other similar terms of the same type, corresponding to sums over r_1 and r_2. Hence, we can combine the sums over r_1, r_2, and r_3 into a single one. So:

$$\mathcal{M}_{21} \le (2.140782 + \varepsilon_2)\mathcal{M}_0\Sigma_5. \tag{8.8}$$

In precisely the same way we can derive

$$\mathcal{M}_{22} \le (2.140782 + \varepsilon_2)\mathcal{M}_0{\Sigma_5}^2;\ \mathcal{M}_{23} \le (2.140782 + \varepsilon_2)\mathcal{M}_0{\Sigma_4}^3. \tag{8.9}$$

Hence, by (8.8), (8.9), Lemma 8.1, and Lemma 8.2, we get

$$|\mathcal{M}_2| \le (2.140782)\mathcal{M}_0(3\times 0.0078 + 3\times 0.0078^2 + 0.8369^3) \le 0.6097\mathcal{M}_0.$$

9. Proof of Theorem 1

Then, we get

$$I_1(N) = \mathcal{M}_1 + \mathcal{M}_2 + O(\Omega) \geq (1 - 0.6106)\mathcal{M}_0 + O(N^2 k^{-2-\varepsilon_0}\mathcal{L}) \gg \mathcal{M}_0. \tag{9.1}$$

So by (4.1), (9.1), and Lemma 6.3, we find that the sum is

$$I(N) = I_1(N) + I_2(N) \gg \mathcal{M}_0 \gg N^2 k^{-2}.$$

This completes the proof of Theorem 1.

10. Conclusions

In this paper, we consider the equation $N = p_1 + p_2 + p_3$ with the restriction $p_j \equiv l_j \pmod{k}$, p being a prime number, and k being a prime number as large as possible. Our main result shows that the above equation is solvable when $k \leq N^{\frac{1}{40}}$ by assuming the truth of Landau's conjecture. Our methods for the proof of Theorem 1 in this paper is the circle method and follows the techniques on handling contributions of the major arcs from Liu and Wang's [9] and Zhang and Wang's [13]. By assuming the truth of Landau's conjecture, our application of the circle method becomes more concise. Moreover, by the restriction k being a prime number, we improve the main result $k \leq N^{\frac{1}{42}}$ of [13] on the solvability of the same equation. While, this improvement comes from Lemma 6.1 and Lemma 8.1 in this paper.

Author contributions

These authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No. 12301005) and Joint Training Base Construction Project for Graduate Students in Chongqing (Grant No. JDLHPYJD2021016). The authors would like to express their sincere thanks to the reviewers and editors for many useful suggestions and comments on the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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