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Research article

The structure of fuzzy extended b-metric spaces and some fixed-point theorems with applications

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Abstract: This paper establishes a novel concept of fuzzy extended b-metric space, which serves as an extension to both fuzzy metric spaces and extended b-metric spaces. The introduction of fuzzy extended b-metric spaces is motivated by the need to model uncertainty and vagueness in real-world problems, where classical metric spaces may fail to capture imprecise relationships. This framework extends traditional b-metric spaces, enhancing their applicability in fuzzy environments and providing a foundation for advanced fixed-point results and applications. By utilizing the extended b-comparison function, some contractive type fixed-point theorems are proved. As an application, we study the homeomorphism between the fuzzy extended b-metric space and the extended b-metric space. In addition, the existence and uniqueness of solutions to the Fredholm integral equation is presented.

Keywords: fuzzy extended b-metric space; fixed point theorem; homeomorphism
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1. Introduction

In 1965, Zadeh [1] proposed fuzzy set theory, which opened up new research directions. In 1972, Chang and Zadeh [2] defined fuzzy numbers in terms of fuzzy sets on $\mathbb{R}$ with special properties.

Subsequently, many researchers have been engaged with the operations of fuzzy numbers and fuzzy differential equations (see [3–5]). Especially, the fuzzy mapping theory formed by combining fuzzy numbers and set value mapping theory has many applications [6]. There are two fundamental approaches to constructing fuzzy spaces. One approach is to define a fuzzy mapping within a space that has a crisp (classical, not fuzzy) structure. The other approach is to consider a space with an inherent fuzzy structure while maintaining usual mappings. These two perspectives provide different ways to study fuzzy spaces, forming the foundation for their mathematical development and applications.

The concept of a metric is fundamental in measuring distances between elements in a space and plays a crucial role in fixed-point theorems. In the context of fuzzy spaces, Kaleva and Seikkala [7] introduced a new fuzzy metric in 1984 by defining the distance between two points in terms of fuzzy numbers. Subsequently, Kaleva-Seikkala's fuzzy metric space has been studied deeply and many results have been proved (see [8–10]). With the development of fuzzy set theory, some authors extended fixed-point theorems from classical metric spaces to fuzzy metric spaces. Heilpern [11] introduced fuzzy mappings and proved a fixed-point theorem for fuzzy mappings. Qiu et al. [12], Abbas and Turkoglu [13] proved some fixed-point theorems for fuzzy mappings under the ϕ-contraction condition and a generalized contractive condition, respectively. As an application of the fixed-point theorem for fuzzy mappings, Liu and Yu [14] derived an existence theorem for Nash equilibria in generalized fuzzy games by using the fuzzy Kakutani-Fan-Glicksberg fixed-point theorem. Especially, Xiao et al. [15] studied some kinds of fixed-point theorems under nonlinear contractions in Kaleva-Seikkala's fuzzy metric spaces. Irkin et al. [16] also proved that fuzzy soft sets are useful in optimizing fermentation processes, where understanding the complex interactions between variables is crucial.

In 1989, the concept of b-metric was introduced by Bakhtin [17] with a weaker condition replacing the usual triangle inequality. A few years later, Czerwik [18] formally defined the b-metric. Based on the b-metric, in 2017, Kamran et al. [19] proposed the definition of extended b-metric and proved a Banach type fixed-point theorem under this new metric. There have been several recent papers on b-metric spaces. For example, Latif et al. [20] proved some fixed-point theorems under a Suzuki-type contractive condition. Samreen et al. [21] used a new class of comparison functions to prove some fixed-point theorems in the extended b-metric space. Cobzaş and Czerwik [22] introduced the generalized b-metric and gave some fixed-point results in generalized b-metric spaces. Guran and Bota [23] considered the existence of solutions to nonlinear fractional differential equations in extended b-metric spaces.

Fixed-point theorems in extended b-metric space are mainly applied to a class of integral equations. The analysis of integral equations is an important branch of modern mathematics. Many problems in mathematics, natural science and engineering can be regarded as integral equation problems. Kamran et al. [19] considered the Fredholm integral equation in extended b-metric spaces. Mlaiki et al. [24] provided Volterra integral inclusions and Urysohn integral equations as applications. Shatanawi and Shatnawi [25] used an integral equation as an application to illustrate a new fixed-point theorem for new contraction conditions they constructed. Fixed-point theorems are fundamental tools in mathematics, used to prove the existence of solutions in various problems. While applicable to integral equations, their significance extends to diverse areas. For example, fixed-point theorems play a role in the study of algebraic curves [26], vector bundle moduli spaces [27], Higgs bundles and the Hitchin integrable system [28], and even in physics related to Langlands duality [29]. Meanwhile, Nadaban utilized fuzzy sets in b-metric spaces and introduced the notion of a fuzzy b-metric space in [30]. He also discussed the topological properties of this new space. Kamran et al. [19] further generalized the definition of [17] by introducing the idea of extended b-metric space, while in [31], Mehmood et al. applied fuzzy sets to the definition in [19] by introducing the notion of an extended fuzzy b-metric space based on continuous t-norms and proved a Banach-type contraction mapping principle on this space. Mehmood et al. [32] defined a fuzzy version of rectangular b-metric space, while in [33], Asim et al. introduced the concept of extended rectangular b-metric space and proved a related fixed-point theorem. Recently, Saleem et al. [34] introduced the notion of fuzzy double controlled metric space and proved a Banach-type contraction mapping principle on such spaces. Abdeljawad et al. [35] modified the definition of controlled metric type space defined in Mlaiki et al. [36] by giving the idea of a double controlled metric type space.

While extended b-metric spaces effectively generalize distance, they do not allow for fuzzy data. Conversely, fuzzy metrics model uncertainty well but lack the geometric flexibility for nonlinear systems. our fuzzy extended b-metric space, based on two functions that are more general than continuous t-norms, bridges this gap by integrating fuzzy and extended b-metric properties, enabling robust analysis of complex fuzzy systems. This advance, providing new fixed-point results and application examples, enhances the capabilities of metric space theory.

The motivation of this work is to bridge a specific gap in the existing structure of generalized metric spaces. While extended b-metric spaces have proven useful in relaxing the triangle inequality and fuzzy metric spaces are effective in handling vagueness and uncertainty, there is limited work combining these two frameworks to tackle problems involving both non-standard distance behavior and imprecision in data. Our proposed fuzzy extended b-metric space is designed to generalize and unify these concepts, allowing for the analysis of systems where both fuzziness and relaxed metric constraints coexist. This is particularly relevant in modeling real-world phenomena such as uncertain decision systems, fuzzy control systems, and integral equations with imprecise kernels. Unlike previous studies, such as [19], which solved the Fredholm integral equation using extended b-metric spaces, our framework enables solving such equations under a fuzzified structure, broadening the scope of solvable problems by incorporating vagueness directly into the space's definition.

Moreover, the fixed-point results developed within this new space are not straightforward generalizations of existing theorems. They introduce new contractive mappings and fuzzy comparison conditions that are essential in this hybrid setting. These results, in turn, support applications like the existence and uniqueness of solutions for Fredholm-type equations under fuzzified constraints. Thus, our work not only extends existing theory but also presents new theoretical tools that can be adapted in fuzzy analysis and operator theory.

The structure of this paper is as follows: We review some preliminaries in Section 2. In Section 3, we propose the concept of Kaleva-Seikkala type fuzzy extended b-metric space and present some interesting properties. In Section 4, we consider the existence and uniqueness of fixed points of operators under different contractive conditions, which generalize the results in the pre-existing literature. In Section 5, as applications, the existence and uniqueness of solutions to the Fredholm integral equation are proved. In Section 6, conclusions and future research directions are given.

2. Preliminaries

In this paper, suppose that $\mathbb{N}^+$ is the set of all positive integers, $\mathbb{R} = (-\infty, +\infty)$.

Definition 2.1. [2] A fuzzy number is a mapping $u : \mathbb{R} \to [0, 1]$ with the following properties:

(1) u is upper semicontinuous;

(2) u is fuzzy convex, that is, $u(\lambda p + (1 - \lambda)q) \geq \min\{u(p), u(q)\}$ for all $p, q \in \mathbb{R}$, $\lambda \in [0, 1]$;

(3) u is normal, that is, there exists $p_0 \in \mathbb{R}$ for which $u(p_0) = 1$;

(4) $\operatorname{supp} u = \{p \in \mathbb{R} \mid u(p) > 0\}$ is the support of u, and its closure $\operatorname{cl}(\operatorname{supp} u)$ is compact.

The set of all fuzzy numbers is denoted by E. $\bar{0} \in E$ is a special element which satisfies $\bar{0}(0) = 1$ and $\bar{0}(p) = 0$ for $p \neq 0$. $u \in E$ is called a nonnegative fuzzy number if u satisfies $u(p) = 0$ for all $p < 0$. The set of all nonnegative fuzzy numbers is denoted by E^+.

Since each $x \in \mathbb{R}$ can be considered as a fuzzy number $\tilde{x}$ defined by

$$\bar{x}(t) = \begin{cases} 1, & \text{if } t = x, \\ 0, & \text{if } t \neq x, \end{cases} \tag{2.1}$$

the real numbers can be embedded in E.

The β-level set of a fuzzy number u is denoted by $[u]_\beta = \{p \in \mathbb{R} : u(p) \geq \beta, \beta \in (0, 1]\}$ and $[u]_0 = \overline{\{p \in \mathbb{R} : u(p) > 0\}}$ when $\beta = 0$. For convenience, it can be briefly written as $[u]_\beta = [\underline{u}_\beta, \overline{u}_\beta]$ with $\underline{u}_\beta, \overline{u}_\beta \in \mathbb{R}$.

The equality of fuzzy numbers u and v is defined by

$$u = v \text{ if and only if } u(p) = v(p) \text{ for all } p \in \mathbb{R}.$$

The arithmetic operations of fuzzy numbers u and v are defined (see [7]) by

$$\begin{cases} (u + v)(p) = \sup\limits_{q \in \mathbb{R}} \min(x(q), y(p - q)), \ p \in \mathbb{R}, \\ (u - v)(p) = \sup\limits_{q \in \mathbb{R}} \min(x(q), y(q - p)), \ p \in \mathbb{R}, \\ (u \cdot v)(p) = \sup\limits_{q \in \mathbb{R} \setminus \{0\}} \min(x(q), y(p/q)), \ p \in \mathbb{R}, \\ (u/v)(p) = \sup\limits_{q \in \mathbb{R}} \min(x(pq), y(q)), \ p \in \mathbb{R}. \end{cases} \tag{2.2}$$

Define a partial ordering $\preceq$ in E by

$$u \preceq v \text{ if and only if } \underline{u}_\beta \leq \underline{v}_\beta \text{ and } \overline{u}_\beta \leq \overline{v}_\beta, \forall \beta \in (0, 1],$$

where $[u]_\beta = [\underline{u}_\beta, \overline{u}_\beta]$ and $[v]_\beta = [\underline{v}_\beta, \overline{v}_\beta]$.

Definition 2.2. [18] Let Λ be a non-empty set, $b \geq 1$. A mapping $d_b : \Lambda \times \Lambda \to [0, \infty)$ is called a b-metric if for all $\xi, \chi, \zeta \in \Lambda$, it satisfies:

$(d1)$ $d_b(\xi, \chi) = 0$ iff $\xi = \chi$;

$(d2)$ $d_b(\xi,\chi) = d_b(\chi,\xi)$;
$(d3)$ $d_b(\xi,\zeta) \leq b[d_b(\xi,\chi) + d_b(\chi,\zeta)]$.

The pair (Λ, d_b) is called a b-metric space.

Definition 2.3. [19] Let Λ be a non-empty set, $\theta : \Lambda \times \Lambda \to [1, \infty)$. A mapping $d_\theta : \Lambda \times \Lambda \to [0, \infty)$ is called an extended b-metric if for all $\xi, \chi, \zeta \in \Lambda$, it satisfies:

$(d_\theta 1)$ $d_\theta(\xi,\chi) = 0$ iff $\xi = \chi$;
$(d_\theta 2)$ $d_\theta(\xi,\chi) = d_\theta(\chi,\xi)$;
$(d_\theta 3)$ $d_\theta(\xi,\zeta) \leq \theta(\xi,\zeta)[d_\theta(\xi,\chi) + d_\theta(\chi,\zeta)]$.

The pair (Λ, d_θ) is called an extended b-metric space.

Definition 2.4. [7] Let Λ be a non-empty set, $d : \Lambda \times \Lambda \to E^+$. Suppose that $L, R : [0, 1] \times [0, 1] \to [0, 1]$ are two symmetric and nondecreasing functions such that $L(0, 0) = 0$, $R(1, 1) = 1$. Denote

$$[d(\xi,\chi)]_\beta = [\lambda_\beta(\xi,\chi), \rho_\beta(\xi,\chi)], \quad \forall \xi,\chi \in \Lambda, \text{ and } \beta \in (0, 1],$$

where $\lambda_\beta, \rho_\beta : \Lambda \times \Lambda \to \mathbb{R}$. The quadruple (Λ, d, L, R) is called a fuzzy metric space, and d a fuzzy metric, if

(i) $d(\xi,\chi) = \bar{0}$ iff $\xi = \chi$;
(ii) $d(\xi,\chi) = d(\chi,\xi)$, $\forall \xi,\chi \in \Lambda$;
(iii) $d(\xi,\chi)(p + q) \geq L(d(\xi,\zeta)(p), d(\zeta,\chi)(q))$, whenever $p \leq \lambda_1(\xi,\zeta)$, $q \leq \lambda_1(\zeta,\chi)$ and $p + q \leq \lambda_1(\xi,\chi)$, $\xi,\chi,\zeta \in \Lambda$;
(iv) $d(\xi,\chi)(p + q) \leq R(d(\xi,\zeta)(p), d(\zeta,\chi)(q))$, whenever $p \geq \lambda_1(\xi,\zeta)$, $q \geq \lambda_1(\zeta,\chi)$ and $p + q \geq \lambda_1(\xi,\chi)$, $\xi,\chi,\zeta \in \Lambda$.

Lemma 2.5. *[7] The triangle inequality (iii) in Definition 2.4 with* $L(\xi,\chi) = \min(\xi,\chi)$ *on* $\Lambda \times \Lambda$ *is equivalent to the triangle inequality*

$$\lambda_\beta(\xi,\chi) \leq \lambda_\beta(\xi,\zeta) + \lambda_\beta(\zeta,\chi), \ \forall \xi,\chi,\zeta \in \Lambda, \ \textit{and}\, \beta \in (0, 1].$$

Lemma 2.6. *[7] The triangle inequality (iv) in Definition 2.4 with* $R(\xi,\chi) = \max(\xi,\chi)$ *on* $\Lambda \times \Lambda$ *is equivalent to the triangle inequality*

$$\rho_\beta(\xi,\chi) \leq \rho_\beta(\xi,\zeta) + \rho_\beta(\zeta,\chi), \ \forall \xi,\chi,\zeta \in \Lambda, \ \textit{and}\, \beta \in (0, 1].$$

3. A fuzzy extended b-metric

Based on Definitions 2.3 and 2.4, we show a new concept of the fuzzy extended b-metric.

Definition 3.1. Let Λ be a non-empty set, $d : \Lambda \times \Lambda \to E^+$, $\theta : \Lambda \times \Lambda \to [1, \infty)$. Suppose that $L, R : [0, 1] \times [0, 1] \to [0, 1]$ are two symmetric and nondecreasing functions such that $L(0, 0) = 0$, $R(1, 1) = 1$. For each $\beta \in (0, 1]$, $\xi, \chi \in \Lambda$, the β-level set of $d(\xi,\chi)$ is written as

$$[d(\xi,\chi)]_\beta = [\lambda_\beta(\xi,\chi), \rho_\beta(\xi,\chi)].$$

If for each $\xi,\chi,\zeta \in \Lambda$, $d(\xi,\chi)$ satisfies:

(M1) $d(\xi,\chi) = \bar{0}$ iff $\xi = \chi$;

(M2) $d(\xi,\chi) = d(\chi,\xi)$;

(M3) $d(\xi,\chi)(\theta(\xi,\chi)\cdot(p+q)) \geq L(d(\xi,\zeta)(p), d(\zeta,\chi)(q))$, whenever $p \leq \lambda_1(\xi,\zeta)$, $q \leq \lambda_1(\zeta,\chi)$ and $\theta(\xi,\chi)\cdot(p+q) \leq \lambda_1(\xi,\chi)$;

(M4) $d(\xi,\chi)(\theta(\xi,\chi)\cdot(p+q)) \leq R(d(\xi,\zeta)(p), d(\zeta,\chi)(q))$, whenever $p \geq \lambda_1(\xi,\zeta)$, $q \geq \lambda_1(\zeta,\chi)$ and $\theta(\xi,\chi)\cdot(p+q) \geq \lambda_1(\xi,\chi)$,

then d is called a fuzzy extended b-metric, the quintuple $(\Lambda, d, L, R, \theta)$ is called a fuzzy extended b-metric space.

In fact, in Definition 3.1, $L, R : [0,1]\times[0,1] \to [0,1]$ are two symmetric and nondecreasing functions including continuous t-norms. For example, they can be given by

$$L(t,s) = \min\{t,s\}, \quad R(t,s) = \max\{t,s\}, \quad \forall t,s \in [0,1], \tag{3.1}$$

or

$$L(t,s) \equiv 0, \quad R(t,s) = \begin{cases} 0, & t = s = 0, \\ 1, & \text{otherwise}. \end{cases} \tag{3.2}$$

Example 3.2. *Let $\Lambda = E$ with $\theta(\xi,\chi) = 1$ for all $\xi,\chi \in \Lambda$, and let $L(t,s) \equiv 0$, $R(t,s) = \max\{t,s\}$ for all $t,s \in [0,1]$. If for all $\xi,\chi \in \Lambda$, $p \in \mathbb{R}$, $d(\xi,\chi)(p)$ is defined by*

$$d(\xi,\chi)(p) = \begin{cases} \max\{(\xi-\chi)(p), (\chi-\xi)(p)\}, & \textit{if } \xi \neq \chi, \\ \bar{0}(p), & \textit{if } \xi = \chi, \end{cases}$$

then, by Lemma 2.1 in [7], for $\xi \neq \chi$, we have

$$[d(\xi,\chi)]_\beta = \left[\max\left\{0, \underline{\xi}_\beta - \overline{\chi}_\beta, \overline{\xi}_\beta - \underline{\chi}_\beta\right\}, \max\left\{\left|\underline{\xi}_\beta - \overline{\chi}_\beta\right|, \left|\overline{\xi}_\beta - \underline{\chi}_\beta\right|\right\}\right] := [\lambda_\beta(\xi,\chi), \rho_\beta(\xi,\chi)],$$

where $[\xi]_\beta = \left[\underline{\xi}_\beta, \overline{\xi}_\beta\right]$ and $[\chi]_\beta = \left[\underline{\chi}_\beta, \overline{\chi}_\beta\right]$.

Moreover, it is easy to see that (M1)–(M3) in Definition 3.1 hold. For any $\xi,\chi,\zeta \in E$, since

$$\rho_\beta(\xi,\chi) = \max\left\{\left|\underline{\xi}_\beta - \overline{\chi}_\beta\right|, \left|\overline{\xi}_\beta - \underline{\chi}_\beta\right|\right\},$$

one has

$$\rho_\beta(\xi,\chi) \leq \rho_\beta(\xi,\zeta) + \rho_\beta(\zeta,\chi).$$

Hence, by Lemma 2.6, (M4) holds for any $p \geq \lambda_1(\xi,\zeta)$, $q \geq \lambda_1(\zeta,\chi)$, and $p+q \geq \lambda_1(\xi,\chi)$. Therefore, the quintuple $(\Lambda, d, L, R, \theta)$ is a fuzzy extended b-metric space.

Similarly to Example 3.2, one has

Example 3.3. *Let* $\Lambda = C([a,b],E)$ *with* $\theta(\xi,\chi) = 1$ *for all* $\xi,\chi \in \Lambda$*, and let* $L(t,s) = \min\{t,s\}$, $R(t,s) = \max\{t,s\}$ *for all* $t,s \in [0,1]$*. If for all* $\xi,\chi \in \Lambda$*,* $p \in \mathbb{R}$*,* $d(\xi,\chi)(p)$ *is defined by*

$$d(\xi,\chi)(p) = \begin{cases} \sup\limits_{\tau\in[a,b]} \max\{(\xi(\tau)-\chi(\tau))(p),(\chi(\tau)-\xi(\tau))(p)\}, & \text{if } \xi \neq \chi, \\ \bar{0}(p), & \text{if } \xi = \chi, \end{cases}$$

then $(\Lambda, d, \min, \max, \theta)$ *is a fuzzy extended b-metric space.*

Example 3.4. *Let* (Λ, m) *be a metric space,* $d : \Lambda\times\Lambda \to E^+$ *is defined by*

$$d(\xi,\chi)(p) = \begin{cases} 1, & p = m(\xi,\chi), \\ 0, & \text{elsewhere}, \end{cases}$$

where $m(\xi,\chi) = \lambda_\beta(\xi,\chi) = \rho_\beta(\xi,\chi)$*, for all* $\xi,\chi \in \Lambda$ *and* $\beta \in (0,1]$.

Let $\theta : \Lambda\times\Lambda \to [1,\infty)$*, since*

$$m(\xi,\chi) \leq m(\xi,\zeta) + m(\zeta,\chi) \leq \theta(\xi,\chi)\left[m(\xi,\zeta) + m(\zeta,\chi)\right],$$

so from Definition 2.3, we obtain that (Λ, m) *is an extended b-metric space, then we have*

$$\lambda_\beta(\xi,\chi) \leq \theta(\xi,\chi)\left[\lambda_\beta(\xi,\zeta) + \lambda_\beta(\zeta,\chi)\right],$$

and

$$\rho_\beta(\xi,\chi) \leq \theta(\xi,\chi)\left[\rho_\beta(\xi,\zeta) + \rho_\beta(\zeta,\chi)\right],\ \forall \xi,\chi,\zeta \in \Lambda \text{ and } \beta \in (0,1].$$

If $L(0,1) = 1$ *and* R *is right continuous, then we can obtain that* $(\Lambda, d, L, R, \theta)$ *is a fuzzy extended b-metric space. However, if* $\lambda_1(\xi,\chi) = 0$ *for all* $\xi,\chi \in \Lambda$*, we can obtain that* $(\Lambda, d, L, R, \theta)$ *is not a fuzzy metric space and thus not a fuzzy extended b-metric space.*

From Example 3.4, we can obtain the following remark.

Remark 3.5. *Both fuzzy b-metric spaces [30] and our fuzzy extended b-metric spaces generalize the concept of a fuzzy metric space by relaxing the triangular inequality. While standard fuzzy metric spaces require a strict triangular inequality, fuzzy b-metric spaces introduce a constant* $b \geq 1$*, permitting a weaker form. Our extended fuzzy b-metric space further generalizes this relaxation. The key distinction is that, instead of a constant b, we use a function* $\theta(\xi,\chi) : \Lambda\times\Lambda \to [1,\infty)$*. This allows the degree of triangular inequality relaxation to vary with the specific points* ξ *and* χ*, offering greater modeling flexibility. Consequently, a fuzzy b-metric space is a special case of our framework where* $\theta(\xi,\chi)$ *is a constant function equal to b.*

From the properties of fuzzy numbers and Definition 3.1, we directly obtain the next lemma, so we omit the proof.

Lemma 3.6. *Let* $(\Lambda, d, L, R, \theta)$ *be a fuzzy extended b-metric space. Then for each* $\beta \in (0,1]$, $\xi,\chi \in \Lambda$,

(1) $\lim\limits_{q\to-\infty} d(\xi,\chi)(q) = 0$, $\lim\limits_{q\to+\infty} d(\xi,\chi)(q) = 0$;

(2) $\rho_\beta(\xi,\chi)$ *is a non-increasing and left continuous function w.r.t.* $\beta \in (0,1]$;

(3) $d(\xi,\chi)(q)$ *is a non-increasing and left continuous function w.r.t.* $q \in (\lambda_1(\xi,\chi), +\infty)$.

Lemma 3.7. *Let* $(\Lambda, d, L, R, \theta)$ *be a fuzzy extended b-metric space, and consider the statements*

(R-1) $R(\xi,\chi) \leq \max\{\xi,\chi\}$;

(R-2) for each $\beta \in (0,1]$, *there exists* $s \in (0,\beta]$ *such that* $R(s,r) < \beta$ *for all* $r \in (0,\beta)$;

(R-3) $\lim_{\eta\to 0^+} R(\eta,\eta) = 0$.

Then, (R-1)⇒(R-2)⇒(R-3).

Proof. Suppose that (R-1) holds. Then for each $\beta \in (0,1]$, there exists $s = \frac{1}{2}\beta \in (0,\beta]$ such that

$$R(s,r) \leq \max\left\{\frac{1}{2}\beta, r\right\} < \beta,$$

for all $r \in (0,\beta)$. Therefore, (R-1)⇒(R-2).

Suppose that (R-2) holds. Then for each $\varepsilon \in (0,1]$, there exists $s \in (0,\varepsilon]$ and $r = \frac{1}{2}s$ such that $R\left(s, \frac{1}{2}s\right) < \varepsilon$. From Definition 3.1, we obtain

$$0 \leq R(\eta,\eta) \leq R\left(s, \frac{1}{2}s\right) < \varepsilon,$$

for all $\eta \in \left(0, \frac{1}{2}s\right]$, that is, $\lim_{\eta\to 0^+} R(\eta,\eta) = 0$. Therefore, (R-2)⇒(R-3). □

Lemma 3.8. *Let* $(\Lambda, d, L, R, \theta)$ *be a fuzzy extended b-metric space. Then*

(1) *(R-1)* $\Rightarrow$ *for each* $\beta \in (0,1]$, $\rho_\beta(\xi,\chi) \leq \theta(\xi,\chi)[\rho_\beta(\xi,\zeta) + \rho_\beta(\zeta,\chi)], \forall \xi,\chi,\zeta \in \Lambda$;

(2) *(R-2)* $\Rightarrow$ *for each* $\beta \in (0,1]$, *there exists* $s = s(\beta) \in (0,\beta]$ *such that* $\forall \xi,\chi,\zeta \in \Lambda$,
$\rho_\beta(\xi,\chi) \leq \theta(\xi,\chi)[\rho_s(\xi,\zeta) + \rho_\beta(\zeta,\chi)]$;

(3) *(R-3)* $\Rightarrow$ *for each* $\beta \in (0,1]$, *there exists* $s = s(\beta) \in (0,\beta]$ *such that* $\forall \xi,\chi,\zeta \in \Lambda$,
$\rho_\beta(\xi,\chi) \leq \theta(\xi,\chi)[\rho_s(\xi,\zeta) + \rho_s(\zeta,\chi)]$.

Proof. (1) On the contrary, for some $\beta_0 \in (0,1]$, $\xi_0,\chi_0,\zeta_0 \in \Lambda$,

$$\rho_{\beta_0}(\xi_0,\chi_0) > \theta(\xi_0,\chi_0)[\rho_{\beta_0}(\xi_0,\zeta_0) + \rho_{\beta_0}(\zeta_0,\chi_0)].$$

We can find $p, q \in \mathbb{R}$ such that

$$\begin{cases} p > \rho_{\beta_0}(\xi_0,\zeta_0) \geq \lambda_1(\xi_0,\zeta_0), \\ q > \rho_{\beta_0}(\zeta_0,\chi_0) \geq \lambda_1(\zeta_0,\chi_0), \\ \theta(\xi_0,\chi_0)(p+q) = \rho_{\beta_0}(\xi_0,\chi_0) \geq \lambda_1(\xi_0,\chi_0). \end{cases}$$

Lemma 3.6 (3) implies that $d(\xi_0,\zeta_0)(p) < \beta_0$, $d(\zeta_0,\chi_0)(q) < \beta_0$. Then by (M4) and (R-1), we obtain that

$$\begin{aligned} \beta_0 &\leq d(\xi_0,\chi_0)(\rho_{\beta_0}(\xi_0,\chi_0)) \\ &= d(\xi_0,\chi_0)(\theta(\xi_0,\chi_0)\cdot(p+q)) \\ &\leq R(d(\xi_0,\zeta_0)(p), d(\zeta_0,\chi_0)(q)) \\ &\leq \max\{d(\xi_0,\zeta_0)(p), d(\zeta_0,\chi_0)(q)\} \\ &< \beta_0, \end{aligned}$$

which is a contradiction.

(2) Suppose that (R-2) holds, that is, for each $\beta \in (0,1]$, there exists $s \in (0,\beta]$ such that $R(s,r) < \beta$ for all $r \in (0,\beta)$. It is sufficient to prove that

$$\rho_\beta(\xi,\chi) \leq \theta(\xi,\chi)[\rho_s(\xi,\zeta) + \rho_r(\zeta,\chi)], \text{ for all } r \in (0,t).$$

Suppose for the purposes of contradiction, that for some $\beta_0 \in (0,1]$, $\xi_0, \chi_0, \zeta_0 \in \Lambda$,

$$\rho_{\beta_0}(\xi_0,\chi_0) > \theta(\xi_0,\chi_0)[\rho_s(\xi_0,\zeta_0) + \rho_r(\zeta_0,\chi_0)].$$

We can find $p, q \in \mathbb{R}$ such that

$$\begin{cases} p > \rho_s(\xi_0,\zeta_0) \geq \lambda_1(\xi_0,\zeta_0), \\ q > \rho_r(\zeta_0,\chi_0) \geq \lambda_1(\zeta_0,\chi_0), \\ \theta(\xi_0,\chi_0)\cdot(p+q) = \rho_{\beta_0}(\xi_0,\chi_0) \geq \lambda_1(\xi_0,\chi_0). \end{cases}$$

Lemma 3.6 (3) implies that $d(\xi_0,\zeta_0)(p) < s$, $d(\zeta_0,\chi_0)(q) < r$. Then by (M4) and (R-2), we obtain that

$$\begin{aligned} \beta_0 &\leq d(\xi_0,\chi_0)\left(\rho_{\beta_0}(\xi_0,\chi_0)\right) \\ &= d(\xi_0,\chi_0)(\theta(\xi_0,\chi_0)\cdot(p+q)) \\ &\leq R(d(\xi_0,\zeta_0)(p), d(\zeta_0,\chi_0)(q)) \\ &\leq R(s,r) \\ &< \beta_0, \end{aligned}$$

which is a contradiction.

(3) Suppose that (R-3) holds, that is, $\lim\limits_{\beta\to 0^+} R(\beta,\beta) = 0$. Then, for each $\beta \in (0,1]$, there exists $\delta \in (0,\beta]$ such that $R(\xi,\xi) < \beta$ for $\xi \in (0,\delta)$. Suppose for the purposes of contradiction, that for some $\beta_0 \in (0,1]$, $\xi_0, \chi_0, \zeta_0 \in \Lambda$,

$$\rho_{\beta_0}(\xi_0,\chi_0) > \theta(\xi_0,\chi_0)[\rho_s(\xi_0,\zeta_0) + \rho_s(\zeta_0,\chi_0)].$$

We can find $p, q \in \mathbb{R}$ such that

$$\begin{cases} p > \rho_s(\xi_0,\zeta_0) \geq \lambda_1(\xi_0,\zeta_0), \\ q > \rho_s(\zeta_0,\chi_0) \geq \lambda_1(\zeta_0,\chi_0), \\ \theta(\xi_0,\chi_0)\cdot(p+q) = \rho_{\beta_0}(\xi_0,\chi_0) \geq \lambda_1(\xi_0,\chi_0). \end{cases}$$

From Lemma 3.6, it implies that $d(\xi_0,\zeta_0)(p) < s$, $d(\zeta_0,\chi_0)(q) < s$. Then, by (M4) and (R-3), we obtain that

$$\begin{aligned} \beta_0 &\leq d(\xi_0,\chi_0)\left(\rho_{\beta_0}(\xi_0,\chi_0)\right) \\ &= d(\xi_0,\chi_0)(\theta(\xi_0,\chi_0)\cdot(p+q)) \\ &\leq R(d(\xi_0,\zeta_0)(p), d(\zeta_0,\chi_0)(q)) \\ &\leq R(\xi,\xi) \\ &< \beta_0, \end{aligned}$$

which leads to a contradiction. □

Definition 3.9. Let $(\Lambda, d, L, R, \theta)$ be a fuzzy extended b-metric space and $\{\xi_n\}$ a sequence in Λ.

(1) $\{\xi_n\}$ is said to converge to $\xi \in \Lambda$, if $\lim_{n\to\infty} d(\xi_n, \xi) = \bar{0}$, that is, $\lim_{n\to\infty} \rho_\beta(\xi_n, \xi) = 0$ for each $\beta \in (0, 1]$.

(2) $\{\xi_n\}$ is said to be a Cauchy sequence, if $\lim_{n,m\to\infty} d(\xi_n, \xi_m) = \bar{0}$, that is, for any $\varepsilon > 0, \beta \in (0, 1]$, $s \in (0, \beta]$, there exists $N = N(s, \beta) \in N^+$ such that $\rho_\beta(\xi_n, \xi_m)(s) < \varepsilon$, whenever $n, m \geq N$.

(3) $(\Lambda, d, L, R, \theta)$ is said to be complete, if every Cauchy sequence in Λ converges to some point in Λ.

Let the symbols '$\sum$' and '$\prod$' denote the continued summation and the continued product, respectively.

Definition 3.10. [21] Let $(\Lambda, d, L, R, \theta)$ be a fuzzy extended b-metric space. A function $\varphi : [0, \infty) \to [0, \infty)$ is called an extended b-comparison function if it is non-decreasing and there exists a mapping $\Upsilon : D \subset \Lambda \to \Lambda$ such that for some $\xi_0 \in D$, $\sum_{n=1}^{\infty} \varphi^n(\zeta) \prod_{i=1}^{n} \theta(\xi_i, \xi_m)$ converges for each $\zeta \in [0, \infty)$, $m \in \mathbb{N}^+$. Here, $\xi_n = \Upsilon^n \xi_0$, $n \in \mathbb{N}^+$.

We denote by ϕ_θ the collection of all extended b-comparison functions. If we take $\theta(\xi, \chi) = b \geq 1$ in Definition 3.10, then φ is a b-comparison function. Let ψ_b be the collection of all b-comparison functions.

Example 3.11. *Let* $\Lambda = [0, \infty)$, $\varphi(\zeta) = \frac{\zeta}{2}$, $\zeta \in [0, \infty)$, $\xi_0 = 0$, $\xi_n = n$, $\theta(\xi_n, \xi_m) = \mathrm{e}^{\frac{1}{2^{m+n}}}$, $m, n \in \mathbb{N}^+$. *Hence,*

$$\sum_{n=1}^{\infty} \varphi^n(\zeta) \prod_{i=1}^{n} \theta(\xi_i, \xi_m) = \sum_{n=1}^{\infty} \frac{\zeta}{2^n} \prod_{i=1}^{n} \mathrm{e}^{\frac{1}{2^{m+i}}} = 0, \quad \zeta \in [0, \infty), \quad m \in \mathbb{N}^+.$$

Therefore, the function $\varphi : [0, \infty) \to [0, \infty)$ *is an extended b-comparison function.*

Remark 3.12. *If* $\varphi : [0, \infty) \to [0, \infty)$ *is an extended b-comparison function, then* $\varphi(\zeta) < \zeta$ *for each* $\zeta > 0$ *and* $\varphi(0) = 0$. *Hence, it is easy to see that* $\lim_{n\to\infty} \varphi^n(\zeta) = 0$ *for each* $\zeta \geq 0$.

The uniqueness of limit leads to the following lemma, so we omit the proof.

Lemma 3.13. *Let* $(\Lambda, d, L, R, \theta)$ *be a complete fuzzy extended b-metric space with (R-2) and* $\{\xi_n\}$ *be a sequence in* Λ. *If the sequence* $\{\xi_n\}$ *converges to both* $\xi \in \Lambda$ *and* $\chi \in \Lambda$, *then* $\xi = \chi$.

4. Fixed-point theorems

Section 3 introduced the Kaleva-Seikkala's type fuzzy extended b-metric space. In this section, we will explore various contractive conditions and establish the corresponding fixed-point theorems within this space, thereby building the mathematical machinery needed for the later application. This section provides critical context to the applications in Section 5 and extends the body of knowledge in the field of metric spaces.

Lemma 4.1. *Let* $(\Lambda, d, L, R, \theta)$ *be a complete fuzzy extended b-metric space with (R-2) and* $\{\xi_n\}$ *be a sequence in* Λ. *Then, for any* $m, n \in \mathbb{N}^+$ *with* $m > n$, *and for each* $\beta \in (0, 1]$, *there exists* $s = s(\beta) \in (0, \beta]$ *such that*

$$\rho_\beta(\xi_n, \xi_m) \leq \sum_{j=n}^{m-1} \rho_s(\xi_j, \xi_{j+1}) \prod_{i=n}^{j} \theta(\xi_i, \xi_m). \tag{4.1}$$

Proof. Since $(\Lambda, d, L, R, \theta)$ satisfies (R-2), from Lemma 3.8, for each $\beta \in (0, 1]$, there exists $s = s(\beta) \in (0, \beta]$ such that

$$\rho_\beta(\xi, \chi) \leq \theta(\xi, \chi)[\rho_s(\xi, \zeta) + \rho_\beta(\zeta, \chi)], \forall \xi, \chi, \zeta \in \Lambda. \tag{4.2}$$

For $m, n \in \mathbb{N}^+$ with $m > n$, by (4.2), we obtain

$$\begin{aligned}
\rho_\beta(\xi_n, \xi_m) &\leq \theta(\xi_n, \xi_m)[\rho_s(\xi_n, \xi_{n+1}) + \rho_\beta(\xi_{n+1}, \xi_m)] \\
&\leq \sum_{j=n}^{n+1} \rho_s(\xi_j, \xi_{j+1}) \prod_{i=n}^{j} \theta(\xi_i, \xi_m) + \rho_\beta(\xi_{n+2}, \xi_m) \prod_{i=n}^{n+1} \theta(\xi_i, \xi_m) \\
&\leq \cdots \\
&\leq \sum_{j=n}^{m-2} \rho_s(\xi_j, \xi_{j+1}) \prod_{i=n}^{j} \theta(\xi_i, \xi_m) + \rho_\beta(\xi_{m-1}, \xi_m) \prod_{i=n}^{m-2} \theta(\xi_i, \xi_m) \\
&\leq \sum_{j=n}^{m-1} \rho_s(\xi_j, \xi_{j+1}) \prod_{i=n}^{j} \theta(\xi_i, \xi_m).
\end{aligned}$$

This ends the proof. □

Theorem 4.2. *Let $(\Lambda, d, L, R, \theta)$ be a complete fuzzy extended b-metric space with (R-2). Let $\Upsilon : \Lambda \to \Lambda$ be a mapping such that*

$$\rho_\beta(\Upsilon\xi, \Upsilon\chi) \leq k\rho_\beta(\xi, \chi), \ \forall \xi, \chi \in \Lambda, \ \beta \in (0, 1], \tag{4.3}$$

where $k \in (0, 1)$ is such that for each $\xi_0 \in \Lambda$, $\lim_{n,m\to\infty} \theta(\xi_n, \xi_m) < \frac{1}{k}$ with $\xi_n = \Upsilon^n \xi_0, \ n \in \mathbb{N}^+$. Then Υ has a unique fixed point u in Λ.

Proof. From (4.3), we deduce that

$$\rho_\beta(\xi_n, \xi_{n+1}) = \rho_\beta(\Upsilon\xi_{n-1}, \Upsilon\xi_n) \leq k\rho_\beta(\xi_{n-1}, \xi_n) \leq \ldots \leq k^n \rho_\beta(\xi_0, \xi_1). \tag{4.4}$$

For $m, n \in \mathbb{N}^+$ with $m > n$, from Lemma 4.1 and (4.4), we obtain

$$\rho_\beta(\xi_n, \xi_m) \leq \sum_{j=n}^{m-1} \rho_s(\xi_j, \xi_{j+1}) \prod_{i=n}^{j} \theta(\xi_i, \xi_m) \leq \rho_s(\xi_0, \xi_1) \sum_{j=n}^{m-1} k^j \prod_{i=n}^{j} \theta(\xi_i, \xi_m).$$

Since $\theta(\xi, \chi) \geq 1$, we deduce

$$\rho_\beta(\xi_n, \xi_m) \leq \rho_s(\xi_0, \xi_1) \sum_{j=n}^{m-1} k^j \prod_{i=1}^{j} \theta(\xi_i, \xi_m). \tag{4.5}$$

Let $S_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j} \theta(\xi_i, \xi_m)$, we can deduce that the series $\sum_{j=1}^{\infty} k^j \prod_{i=1}^{j} \theta(\xi_i, \xi_m)$ converges by ratio test for each $m \in \mathbb{N}^+$. Denote $S = \lim_{n\to\infty} S_n$, for $m > n$, from (4.5) we have

$$\rho_\beta(\xi_n, \xi_m) \leq \rho_s(\xi_0, \xi_1)[S_{m-1} - S_n].$$

Then, we conclude that $\{\xi_n\}$ is a Cauchy sequence. Since Λ is complete, let $\xi_n \to u \in \Lambda$ as $n \to \infty$, that is, $\lim_{n\to\infty} \rho_\beta(\xi_n, u) = 0$ for each $\beta \in (0, 1]$. By virtue of (4.3), we deduce that

$$\rho_\beta(\xi_{n+1}, \Upsilon u) = \rho_\beta(\Upsilon\xi_n, \Upsilon u) \leq k\rho_\beta(\xi_n, u).$$

Thus, $\lim_{n\to\infty} \rho_\beta(\xi_{n+1}, \Upsilon u) = 0$, that is, $\xi_{n+1} \to \Upsilon u$ as $n \to \infty$. From Lemma 3.13, we deduce that $\Upsilon u = u$. Therefore, u is a fixed point of Υ.

If Υ has another fixed point $v \in \Lambda$, that is, $\Upsilon v = v,\ v \neq u$, then

$$\rho_\beta(u, v) = \rho_\beta(\Upsilon u, \Upsilon v) \leq k\rho_\beta(u, v) < \rho_\beta(u, v),$$

which is a contradiction. Therefore, Υ has a unique fixed point. □

Theorem 4.3. *Let $(\Lambda, d, L, R, \theta)$ be a complete fuzzy extended b-metric space with (R-2). Let $\Upsilon : \Lambda \to \Lambda$ be a mapping, $\varphi \in \phi_\theta$ such that*

$$\rho_\beta(\Upsilon\xi, \Upsilon\chi) \leq \varphi(\rho_\beta(\xi, \chi)), \tag{4.6}$$

for all $\xi, \chi \in \Lambda,\ \beta \in (0, 1]$. Then Υ has a unique fixed point u in Λ.

Proof. Taking $\xi_0 \in \Lambda$, we obtain the iterative sequence $\{\xi_n\}$ by $\xi_n = \Upsilon^n\xi_0,\ n \in \mathbb{N}^+$. Suppose that $a_n(\beta) = \rho_\beta(\xi_{n-1}, \xi_n)$ for $\beta \in (0, 1]$. From (4.6), we deduce that

$$a_{n+1}(\beta) = \rho_\beta(\xi_n, \xi_{n+1}) = \rho_\beta(\Upsilon\xi_{n-1}, \Upsilon\xi_n) \leq \varphi(\rho_\beta(\xi_{n-1}, \xi_n)) = \varphi(a_n(\beta)) \leq a_n(\beta). \tag{4.7}$$

From $a_n(\beta) \geq 0$ and (4.7), it follows that $\{a_n(\beta)\}$ is a non-increasing sequence, that is, $\{a_n(\beta)\}$ converges. Let $\lim_{n\to\infty} a_n(\beta) = a(\beta) \geq 0$, by (4.7), we obtain

$$a_{n+1}(\beta) \leq \varphi(a_n(\beta)) \leq \varphi^2(a_{n-1}(\beta)) \leq \cdots \leq \varphi^n(a_1(\beta)). \tag{4.8}$$

Let $n \to \infty$ in (4.8), according to Remark 3.12, we obtain that $a(\beta) = 0$. According to Lemma 4.1, for $m, n \in \mathbb{N}^+$ with $m > n$, we deduce that

$$\rho_\beta(\xi_n, \xi_m) \leq \sum_{j=n}^{m-1} a_{j+1}(s) \prod_{i=n}^{j} \theta(\xi_i, \xi_m) \leq \sum_{j=n}^{m-1} \varphi^j(a_1(s)) \prod_{i=n}^{j} \theta(\xi_i, \xi_m).$$

From $\theta(\xi, \chi) \geq 1$, it follows that

$$\rho_\beta(\xi_n, \xi_m) \leq \sum_{j=n}^{m-1} \varphi^j(a_1(s)) \prod_{i=1}^{j} \theta(\xi_i, \xi_m). \tag{4.9}$$

Let $S_n = \sum_{j=1}^{n} \varphi^j(a_1(s)) \prod_{i=1}^{j} \theta(\xi_i, \xi_m)$. We know $\sum_{j=1}^{\infty} \varphi^j(a_1(s)) \prod_{i=1}^{j} \theta(\xi_i, \xi_m)$ converges because $\varphi \in \phi_\theta$ with $a_1(s) \in [0, \infty)$. Denote $S = \lim_{n\to\infty} S_n$, thus for $m > n$, due to (4.9), we have

$$\rho_\beta(\xi_n, \xi_m) \leq S_{m-1} - S_n.$$

Then, we conclude that $\{\xi_n\}$ is a Cauchy sequence. Since Λ is complete, let $\xi_n \to u \in \Lambda$ as $n \to \infty$, that is, $\lim_{n\to\infty} \rho_\beta(\xi_n, u) = 0$ for each $\beta \in (0, 1]$. By virtue of (4.6), we obtain that

$$\rho_\beta(\xi_{n+1}, \Upsilon u) = \rho_\beta(\Upsilon\xi_n, \Upsilon u) \leq \varphi(\rho_\beta(\xi_n, u)) \leq \rho_\beta(\xi_n, u).$$

Thus, $\lim_{n\to\infty} \rho_\beta(\xi_{n+1}, \Upsilon u) = 0$, that is, $\xi_{n+1} \to \Upsilon u$ as $n \to \infty$. From Lemma 3.13, we deduce that $\Upsilon u = u$. Therefore, u is a fixed point of Υ.

If Υ has another fixed point $v \in \Lambda$, that is, $\Upsilon v = v,\ v \neq u$, then

$$\rho_\beta(u, v) = \rho_\beta(\Upsilon u, \Upsilon v) \leq \varphi(\rho_\beta(u, v)) < \rho_\beta(u, v),$$

which is a contradiction. Therefore, Υ has a unique fixed point. □

Theorem 4.4. *Let $(\Lambda, d, L, R, \theta)$ be a complete fuzzy extended b-metric space with (R-1). Let $\Upsilon : \Lambda \to \Lambda$ be a mapping, $\varphi \in \phi_\theta$ such that*

$$\rho_\beta(\Upsilon\xi, \Upsilon\chi) \leq \varphi(M_\beta(\xi, \chi)), \tag{4.10}$$

where

$$M_\beta(\xi, \chi) = \max\left\{\rho_\beta(\xi, \chi), \rho_\beta(\xi, \Upsilon\xi), \rho_\beta(\chi, \Upsilon\chi), \frac{\rho_\beta(\xi, \Upsilon\chi) + \rho_\beta(\chi, \Upsilon\xi)}{2\theta(\xi, \Upsilon\chi)}\right\},$$

for each $\xi, \chi \in \Lambda,\ \beta \in (0, 1]$. Then Υ has a unique fixed point u.

Proof. Taking $\xi_0 \in \Lambda$, we obtain the iterative sequence $\{\xi_n\}$ by $\xi_n = \Upsilon^n\xi_0,\ n \in \mathbb{N}^+$. Suppose that $a_n(\beta) = \rho_\beta(\xi_{n-1}, \xi_n)$ for $\beta \in (0, 1]$. From (4.10), we deduce that

$$a_{n+1}(\beta) = \rho_\beta(\xi_n, \xi_{n+1}) = \rho_\beta(\Upsilon\xi_{n-1}, \Upsilon\xi_n) \leq \varphi(M_\beta(\xi_{n-1}, \xi_n)), \tag{4.11}$$

and

$$\begin{aligned} M_\beta(\xi_{n-1}, \xi_n) &= \max\left\{\rho_\beta(\xi_{n-1}, \xi_n), \rho_\beta(\xi_{n-1}, \Upsilon\xi_{n-1}), \rho_\beta(\xi_n, \Upsilon\xi_n), \frac{\rho_\beta(\xi_{n-1}, \Upsilon\xi_n) + \rho_\beta(\xi_n, \Upsilon\xi_{n-1})}{2\theta(\xi_{n-1}, \Upsilon\xi_n)}\right\} \\ &= \max\left\{\rho_\beta(\xi_{n-1}, \xi_n), \rho_\beta(\xi_n, \xi_{n+1}), \frac{\rho_\beta(\xi_{n-1}, \xi_{n+1})}{2\theta(\xi_{n-1}, \xi_{n+1})}\right\}, \end{aligned} \tag{4.12}$$

because $\xi_n = \Upsilon\xi_{n-1},\ n \in \mathbb{N}^+$.

Since $(\Lambda, d, L, R, \theta)$ satisfies (R-1), according to Lemma 3.8, for each $\beta \in (0, 1]$, we have

$$\rho_\beta(\xi, \chi) \leq \theta(\xi, \chi)[\rho_\beta(\xi, \zeta) + \rho_\beta(\zeta, \chi)], \text{for all } \xi, \chi, \zeta \in \Lambda.$$

Then, we obtain

$$\rho_\beta(\xi_{n-1}, \xi_{n+1}) \leq \theta(\xi_{n-1}, \xi_{n+1})[\rho_\beta(\xi_{n-1}, \xi_n) + \rho_\beta(\xi_n, \xi_{n+1})]. \tag{4.13}$$

By (4.12) and (4.13), we deduce that

$$\begin{aligned} M_\beta(\xi_{n-1}, \xi_n) &\leq \max\left\{\rho_\beta(\xi_{n-1}, \xi_n), \rho_\beta(\xi_n, \xi_{n+1}), \frac{\rho_\beta(\xi_{n-1}, \xi_n) + \rho_\beta(\xi_n, \xi_{n+1})}{2}\right\}. \\ &= \max\{\rho_\beta(\xi_{n-1}, \xi_n), \rho_\beta(\xi_n, \xi_{n+1})\} \\ &= \max\{a_n(\beta), a_{n+1}(\beta)\}. \end{aligned} \tag{4.14}$$

Case 1: If $\max\{a_n(\beta), a_{n+1}(\beta)\} = a_{n+1}(\beta)$, by (4.11), we have

$$a_{n+1}(\beta) \leq \varphi(M_\beta(\xi_{n-1}, \xi_n)) \leq \varphi(a_{n+1}(\beta)) < a_{n+1}(\beta),$$

which is a contradiction.
Case 2: If $\max\{a_n(\beta), a_{n+1}(\beta)\} = a_n(\beta)$, then we have

$$a_{n+1}(\beta) \leq \varphi(M_\beta(\xi_{n-1}, \xi_n)) \leq \varphi(a_n(\beta)) < a_n(\beta). \tag{4.15}$$

These two cases cover all possibilities, so only Case 2 holds after eliminating contradictions.

The facts that $a_n(\beta) \geq 0$ and (4.15) show that $\{a_n(\beta)\}$ is a non-increasing sequence, that is, $\{a_n(\beta)\}$ converges. Let $\lim\limits_{n\to\infty} a_n(\beta) = a(\beta) \geq 0$, from (4.15), we obtain

$$a_{n+1}(\beta) \leq \varphi(a_n(\beta)) \leq \varphi^2(a_{n-1}(\beta)) \leq \cdots \leq \varphi^n(a_1(\beta)). \tag{4.16}$$

Let $n \to \infty$ in (4.16), we can obtain $a(\beta) = 0$. For $m, n \in \mathbb{N}^+$ with $m > n$, by (4.13) and (4.16), we have

$$\begin{aligned}
\rho_\beta(\xi_n, \xi_m) &\leq \sum_{j=n}^{m-1} \rho_\beta(\xi_j, \xi_{j+1}) \prod_{i=n}^{j} \theta(\xi_i, \xi_m) \\
&\leq \sum_{j=n}^{m-1} a_{j+1}(\beta) \prod_{i=n}^{j} \theta(\xi_i, \xi_m) \\
&\leq \sum_{j=n}^{m-1} \varphi^j(a_1(\beta)) \prod_{i=n}^{j} \theta(\xi_i, \xi_m).
\end{aligned}$$

By $\theta(\xi, \chi) \geq 1$, we obtain

$$\rho_\beta(\xi_n, \xi_m) \leq \sum_{j=n}^{m-1} \varphi^j(a_1(\beta)) \prod_{i=1}^{j} \theta(\xi_i, \xi_m). \tag{4.17}$$

Let $S_n = \sum\limits_{j=1}^{n} \varphi^j(a_1(\beta)) \prod\limits_{i=1}^{j} \theta(\xi_i, \xi_m)$, we can know $\sum\limits_{j=1}^{\infty} \varphi^j(a_1(\beta)) \prod\limits_{i=1}^{j} \theta(\xi_i, \xi_m)$ converges by $\varphi \in \phi_\theta$ with $a_1(s) \in [0, \infty)$. Denote $S = \lim\limits_{n\to\infty} S_n$, thus for $m > n$, by (4.17), we have

$$\rho_\beta(\xi_n, \xi_m) \leq S_{m-1} - S_n.$$

Then, we conclude that $\{\xi_n\}$ is a Cauchy sequence. Since Λ is complete, let $\xi_n \to u \in \Lambda$ as $n \to \infty$, that is, $\lim\limits_{n\to\infty} \rho_\beta(\xi_n, u) = 0$ for each $\beta \in (0, 1]$. By virtue of (4.10), we obtain that

$$\rho_\beta(\xi_{n+1}, \Upsilon u) = \rho_\beta(\Upsilon\xi_n, \Upsilon u) \leq \varphi(\rho_\beta(\xi_n, u)) \leq \rho_\beta(\xi_n, u).$$

Thus, $\lim\limits_{n\to\infty} \rho_\beta(\xi_{n+1}, \Upsilon u) = 0$, that is, $\xi_{n+1} \to \Upsilon u$ as $n \to \infty$. From Lemma 3.13, we deduce that $\Upsilon u = u$. Therefore, u is a fixed point of Υ.

If Υ has another fixed point $v \in \Lambda$, that is, $\Upsilon v = v,\ v \neq u$, then

$$\rho_\beta(u, v) = \rho_\beta(\Upsilon u, \Upsilon v) \leq \varphi(\rho_\beta(u, v)) < \rho_\beta(u, v),$$

which is a contradiction. Therefore, Υ has a unique fixed point. □

Considering that the fuzzy extended b-metric space is the generalization of fuzzy b-metric space. Taking $\theta(\xi,\chi) = b \geq 1$ in Theorems 4.2–4.4, we obtain the following consequences.

Corollary 4.5. *Let (Λ, d, L, R, b) be a complete fuzzy b-metric space with (R-2). Let $\Upsilon : \Lambda \to \Lambda$ be a mapping such that*

$$\rho_\beta(\Upsilon\xi, \Upsilon\chi) \leq k\rho_\beta(\xi,\chi),\ \forall \xi,\chi \in \Lambda,\ \beta \in (0, 1]$$

where $k \in (0, 1)$ be such that for each $\xi_0 \in \Lambda$, $bk < 1$, and $\xi_n = \Upsilon^n\xi_0$, $n \in \mathbb{N}^+$. Then Υ has a unique fixed point u in Λ.

Corollary 4.6. *Let (Λ, d, L, R, b) be a complete fuzzy b-metric space with (R-2). Let $\Upsilon : \Lambda \to \Lambda$ be a mapping, $\varphi \in \psi_b$ such that*

$$\rho_\beta(\Upsilon\xi, \Upsilon\chi) \leq \varphi(\rho_\beta(\xi,\chi)),$$

for all $\xi,\chi \in \Lambda$, $\beta \in (0, 1]$. Then Υ has a unique fixed point u in Λ.

Corollary 4.7. *Let (Λ, d, L, R, b) be a complete fuzzy b-metric space with (R-1). Let $\Upsilon : \Lambda \to \Lambda$ be a mapping, $\varphi \in \psi_b$ such that*

$$\rho_\beta(\Upsilon\xi, \Upsilon\chi) \leq \varphi(M_\beta(\xi,\chi)),$$

where

$$M_\beta(\xi,\chi) = \max\left\{\rho_\beta(\xi,\chi), \rho_\beta(\xi,\Upsilon\xi), \rho_\beta(\chi,\Upsilon\chi), \frac{\rho_\beta(\xi,\Upsilon\chi) + \rho_\beta(\chi,\Upsilon\xi)}{2\theta(\xi,\Upsilon\chi)}\right\},$$

for all $\xi,\chi \in \Lambda$, $\beta \in (0, 1]$. Then Υ has a unique fixed point u in Λ.

5. Applications

This section is devoted to the applications of the new results of this paper.

Lemma 5.1. *Let (Λ, d_θ) be an extended b-metric space, where $d_\theta : \Lambda \times \Lambda \to [0, \infty)$ is a continuous function, and θ is a mapping from $\Lambda \times \Lambda$ into $[1, \infty)$. Taking $L = \min, R = \max$ as given in* (3.1)*, if $d(\xi,\chi)$ is defined by*

$$d(\xi,\chi)(q) = \bar{0}(q - d_\theta(\xi,\chi)),\ q \in \mathbb{R}, \tag{5.1}$$

then $(\Lambda, d, \min, \max, \theta)$ is a fuzzy extended b-metric space.

Proof. According to (5.1), it's easy to verify (M1) and (M2).

For each $\xi,\chi,\zeta \in \Lambda$, let

$$\begin{cases} p \leq \lambda_1(\xi,\zeta), \\ q \leq \lambda_1(\zeta,\chi), \\ \theta(\xi,\chi)(p + q) \leq \lambda_1(\xi,\chi). \end{cases}$$

Suppose that $p < \lambda_1(\xi,\zeta)$ or $q < \lambda_1(\zeta,\chi)$, we have

$$L(d(\xi,\zeta)(p), d(\zeta,\chi)(q)) = \min\{d(\xi,\zeta)(p), d(\zeta,\chi)(q)\} = 0 \leq d(\xi,\chi)(\theta(\xi,\chi)(p+q)).$$

Suppose that $p = \lambda_1(\xi,\zeta)$ and $q = \lambda_1(\zeta,\chi)$, we have

$$\theta(\xi,\chi)(p+q) = \theta(\xi,\chi)[d_\theta(\xi,\zeta) + d_\theta(\zeta,\chi)] \geq d_\theta(\xi,\chi) = \lambda_1(\xi,\chi).$$

So, $\theta(\xi,\chi)(p+q) = d_\theta(\xi,\chi) = \lambda_1(\xi,\chi)$. Then,

$$d(\xi,\chi)(\theta(\xi,\chi)(p+q)) = d(\xi,\chi)(d_\theta(\xi,\chi)) = 1 = L(d(\xi,\zeta)(p), d(\zeta,\chi)(q)).$$

Taking

$$\begin{cases} p \geq \lambda_1(\xi,\zeta), \\ q \geq \lambda_1(\zeta,\chi), \\ \theta(\xi,\chi)(p+q) \geq \lambda_1(\xi,\chi). \end{cases}$$

Suppose that $p = \lambda_1(\xi,\zeta)$ or $q = \lambda_1(\zeta,\chi)$, we have

$$\max\{d(\xi,\zeta)(p), d(\zeta,\chi)(q)\} = 1,$$

so,

$$d(\xi,\chi)(\theta(\xi,\chi)(p+q)) \leq R(d(\xi,\zeta)(p), d(\zeta,\chi)(q)) = 1.$$

Suppose that $p > \lambda_1(\xi,\zeta)$ and $q > \lambda_1(\zeta,\chi)$, we have

$$\theta(\xi,\chi)(p+q) > \theta(\xi,\chi)[d_\theta(\xi,\zeta) + d_\theta(\zeta,\chi)] \geq d_\theta(\xi,\chi) = \lambda_1(\xi,\chi).$$

It implies that

$$d(\xi,\chi)(\theta(\xi,\chi)(p+q)) = 0 = R(d(\xi,\zeta)(p), d(\zeta,\chi)(q)).$$

Therefore, $d(\xi,\chi)$ also satisfies (M3) and (M4). According to Definition 3.1, $(\Lambda, d, \min, \max, \theta)$ is a fuzzy extended b-metric space. □

Example 5.2. *Let* $\Lambda = C([a,b],\mathbb{R})$, $d_\theta(\xi,\chi) = \sup_{\tau\in[a,b]} |\xi(\tau) - \chi(\tau)|^2$ *with* $\theta(\xi,\chi) = |\xi(\tau)| + |\chi(\tau)| + 2$. *Example 3 in [19] indicates that the pair* (Λ, d_θ) *is a complete extended b-metric space. Let* $L = \min$, $R = \max$. *According to Lemma 5.1, if* $d(\xi,\chi)$ *is defined by*

$$d(\xi,\chi)(q) = \bar{0}(q - d_\theta(\xi,\chi)),\ q \in \mathbb{R},$$

then $(\Lambda, d, \min, \max, \theta)$ *is a fuzzy extended b-metric space.*

Theorem 5.3. *Let* (Λ, d_θ) *be a complete extended b-metric space, where* d_θ *is a continuous function. Let* $\Upsilon : \Lambda \to \Lambda$ *be a mapping,* $\varphi \in \phi_\theta$ *such that*

$$d_\theta(\Upsilon\xi, \Upsilon\chi) \leq \varphi(M_\theta(\xi,\chi)),$$

where

$$M_\theta(\xi,\chi) = \max\left\{d_\theta(\xi,\chi), d_\theta(\xi,\Upsilon\xi), d_\theta(\chi,\Upsilon\chi), \frac{d_\theta(\xi,\Upsilon\chi) + d_\theta(\chi,\Upsilon\xi)}{2\theta(\xi,\Upsilon\chi)}\right\}, \forall \xi,\chi \in \Lambda.$$

Then Υ *has a unique fixed point u in* Λ.

Proof. According to Lemma 5.1, we can see that $(\Lambda, d_\theta, \min, \max, \theta)$ is a fuzzy extended b-metric space. Then by Theorem 4.4 we obtain that Υ has a unique fixed point u . □

If we take $M_\theta(\xi,\chi) = d_\theta(\xi,\chi)$, $\varphi(\zeta) = k\zeta$ in Theorem 5.3, where $k \in (0,1)$ satisfies $\lim_{n,m\to\infty} \theta(\xi_n, \xi_m) < \frac{1}{k}$ for each $\xi_0 \in \Lambda$, $\xi_n = \Upsilon^n \xi_0$, $n \in \mathbb{N}^+$, then we can get the same conclusion in [19].

Example 5.4. *Let* $\Lambda = [0,\infty]$, $d_\theta(\xi,\chi) = |\xi - \chi|^3$ *with* $\theta(\xi,\chi) = \xi + \chi + 1$. *It's easy to prove that the pair* (Λ, d_θ) *is a complete extended b-metric space.*

Let $\Upsilon : \Lambda \to \Lambda$ *be defined by*

$$\Upsilon\xi = \frac{\xi}{3}.$$

For any $\xi,\chi \in \Lambda$, *we have*

$$d_\theta(\Upsilon\xi, \Upsilon\chi) = \left|\frac{\xi}{3} - \frac{\chi}{3}\right|^3 = \frac{1}{27}|\xi - \chi|^3,$$

$$M_\theta(\xi,\chi) = \max\left\{|\xi - \chi|^3, \frac{8\xi^3}{27}, \frac{8\chi^3}{27}, \frac{4\xi^3 + 4\chi^3}{27(\xi + \frac{\chi}{3} + 1)}\right\} = \max\left\{|\xi - \chi|^3, \frac{8\xi^3}{27}, \frac{8\chi^3}{27}\right\}.$$

Considering that

$$\xi_n = \Upsilon^n\xi = \frac{\xi}{3^n}, n = 1, 2, \ldots,$$

we have

$$\lim_{n,m\to\infty} \theta(\xi_n, \xi_m) = \lim_{n,m\to\infty} \left(\frac{\xi}{3^n} + \frac{\xi}{3^m} + 1\right) = 1 < 8.$$

Let $\varphi(\zeta) = \frac{\zeta}{8} \in \phi_\theta$, *we get*

$$d_\theta(\Upsilon\xi, \Upsilon\chi) \leq \frac{1}{8} M_\theta(\xi,\chi) = \varphi(M_\theta(\xi,\chi)).$$

The above results satisfy the conditions of Theorem 5.3. Then we can obtain that 0 is the unique fixed point of Υ.

Finally, we give an application of Theorem 4.2 to a Fredholm integral equation.

Example 5.5. *Consider the following Fredholm integral equation:*

$$\xi(\tau)(p) = \int_a^b G(\tau, s, \xi(s)(p))ds + g(\tau)(p), \quad \tau \in [a,b], \quad p \in \mathbb{R}, \tag{5.2}$$

where $g : [a,b] \to E$ *and* $G : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *are two continuous functions. If the following condition holds:*

$$d(G(\tau, s, \xi(s)(p)), G(\tau, s, \chi(s)(p))) \leq \frac{1}{4(b-a)} d(\xi(s)(p), \chi(s)(p)), \quad \xi,\chi \in \Lambda, \quad \tau, s \in [a,b], \quad p \in \mathbb{R},$$

where $(\Lambda, d, \min, \max, \theta)$ *is defined in Example 3.2, then* (5.2) *has a unique solution in* Λ.

Proof. Let $\Upsilon : \Lambda \to \Lambda$ be defined by

$$\Upsilon\xi(\tau)(p) = \int_a^b G(\tau, s, \xi(s)(p))ds + g(\tau)(p), \quad \tau \in [a, b], \quad p \in \mathbb{R}.$$

For any $\xi, \chi \in \Lambda$, we have

$$\begin{aligned} d(\Upsilon\xi(\tau)(p), \Upsilon\chi(\tau)(p)) &= d\left(\int_a^b G(\tau, s, \xi(s)(p))ds + g(\tau)(p), \int_a^b G(\tau, s, \chi(s)(p))ds + g(\tau)(p)\right) \\ &\leq \int_a^b d(G(\tau, s, \xi(s)(p)), G(\tau, s, \chi(s)(p)))ds \\ &\leq \frac{1}{4} \sup_{s\in[a,b]} d(\xi(s)(p), \chi(s)(p)), \end{aligned}$$

that is,

$$\sup_{s\in[a,b]} d(\Upsilon\xi(s)(p), \Upsilon\chi(s)(p)) \leq \frac{1}{4} \sup_{s\in[a,b]} d(\xi(s)(p), \chi(s)(p)).$$

Consider that $\rho_\beta(\xi, \chi) = \sup_{s\in[a,b]} d(\xi(s)(p), \chi(s)(p))$, for each $\beta \in (0, 1]$, $p \in \mathbb{R}$. Therefore, the Fredholm integral equation (5.2) has a unique solution in $C([a, b], E)$ according to Theorem 4.2. □

Remark 5.6. *Since each $x \in \mathbb{R}$ can be considered as a fuzzy number $\bar{x}$ (see* (2.1)*), Example 5.5 is the generalization of the corresponding result in [19].*

6. Conclusions

In this paper, we generalize existing structures to obtain a fuzzy metric space called fuzzy extended b-metric space. The concepts of Cauchy sequence, convergence of sequences and completeness are introduced in this space, and some properties concerning the quasi-metric $\rho_\beta(\xi, \chi)$ are also proved. By virtue of the extended b-comparison function and the quasi-metric $\rho_\beta(\xi, \chi)$, we obtain some fixed-point theorems as a generalization of the Banach contraction principle. Especially, Theorems 4.2 and 4.3 are proved under the weak condition (R-2) while Theorem 4.4 is proved under condition (R-1). As an application, the homeomorphism between this space and extended b-metric space is analyzed and some fixed-point theorems are given.

The fuzzy extended b-metric space introduced here offers numerous avenues for future research. We plan to investigate its topological properties, focusing on continuity and compactness. In particular, we want to determine conditions under which the fuzzy extended b-metric space is sequentially compact or totally bounded. Furthermore, we intend to investigate fixed-point theorems for cyclic contraction mappings within this space. This includes establishing conditions under which a cyclic contraction has a unique fixed point, and investigating the convergence of iterative sequences to this fixed point. We also plan to investigate the relationship between this fuzzy extended b-metric space and probabilistic metric spaces. A key aim is to establish links between the fuzzy metric and the probabilistic metric, potentially leading to new results in both areas. For example, we will work to see if this new space can be used in new machine learning algorithms, such as fuzzy C-means, Fuzzy Support Vector Machine (SVM) and so on. By pursuing these specific research directions, we aim to further elucidate the properties and applications of the fuzzy extended b-metric space.

Author contributions

Yanrong An: Writing original draft, Conceptualization and Methodology, Investigation and Validation. Muhammad Aamir Ali: Writing original draft, Conceptualization and Methodology, Investigation and Validation. Jarunee Sirisin: Writing original draft, Conceptualization and Methodology, Investigation and Validation. Thanin Sitthiwirattham: Writing original draft, Conceptualization and Methodology, Investigation and Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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