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Research article

Finite-time blow-up in semilinear Kirchhoff-type plate equations with distributed delay and polynomial source term

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Abstract: This article investigates a Kirchhoff problem by introducing a delayed Kirchhoff plate equation with polynomial nonlinearity. For initial data with negative energy, we establish the blow-up of local solutions by employing energy methods together with appropriate functional inequalities.

Keywords: blow-up; distributed delay; Kirchhoff equation; polynomial nonlinearity; energy method; Lyapunov functional; nonlinear equation
Mathematics Subject Classification: 35B44, 35L05

1. Introduction

In this work, we study the Kirchhoff equation with a distributed delay and a polynomial source term:

$$\begin{cases} \varphi_{tt} + \mathcal{A}^2\varphi + \sigma(\int_\Gamma \left|\frac{\partial \varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x})(-\mathcal{A}\varphi) + \lambda_1\varphi_t + \int_{\tau_1}^{\tau_2} |\lambda_2(q)|\varphi_t(\mathrm{x}, t-q)dq = |\varphi|^{k-2}\varphi, & \mathrm{x} \in \Gamma, t > 0, \\ \varphi(\mathrm{x}, 0) = \varphi_0(\mathrm{x}), \quad \varphi_t(\mathrm{x}, 0) = \varphi_1(\mathrm{x}), \quad \mathrm{x} \in \Gamma, \\ \varphi(\mathrm{x}, t) = \frac{\partial \varphi}{\partial \nu} = 0, \qquad \mathrm{x} \in \partial\Gamma, \\ u_t(\mathrm{x}, -t) = g_0(\mathrm{x}, t), \qquad \mathrm{x} \in \Gamma, t \in (0, \tau_2), \end{cases}$$

where $\lambda_1 > 0$, Γ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Gamma$, τ_1, τ_2 are delayed times such that $0 < \tau_1 < \tau_2$, $k > 2$, λ_2 is an L^∞ function and $\sigma(s)$ a continuous function on $[0, +\infty)$ under the assumption (H1) and (H2). The operator $\mathcal{A}$ denotes the Laplacian operator ($\mathcal{A}\varphi = \Delta\varphi$).

The Kirchhoff plate equation describes the bending and vibration of a thin elastic plate under applied loads. It relates the plate's deflection to its bending stiffness, mass, and external forces. Physically, it models the flexural behavior of thin structures, predicting deformations, natural frequencies, and vibration modes. For further physical details, see [1–3]. For the mathematical analysis, we begin with the ppioneering work [4]. In 1985, A. Arosio [5] studied one of the earliest Kirchhoff equations in the following problem:

$$u_{tt} = m(\int_{\Omega} |u_x(x,t)|^2 dx)\Delta_x u, \text{ for } x \in \Omega, t > 0.$$

By incorporating the initial and boundary conditions, he analyzed the solvability of the problem originally proposed by R. Narasimha [6] and G. F. Carrier [7].

Subsequently, the same authors generalized the problem by introducing a source term [8]

$$u_{tt} - m(\int_{\Omega} |u_x(x,t)|^2 dx)\Delta_x u = f(x,t), \text{ for } x \in \Omega, t > 0.$$

They established the well-posedness of the problem in low-order Sobolev spaces.

One of the most recent contributions is that of Y. Han and Q. Li [9], who employed the refined potential well technique together with variational approach to investigate the long-term behavior of solutions to a Kirchhoff-type parabolic equation

$$u_{tt} - m(\int_{\Omega} |u_x(x,t)|^2 dx)\Delta_x u = |u|^{q-1}u.$$

To further complicate the problem, Pereira et al. [10] introduced a damping term and investigated the asymptotic behavior of a beam equation with a Kirchhoff term and a polynomial damping term, employing the Nakao method

$$u_{tt} - m(\int_{\Omega} |u_x(x,t)|^2 dx)\Delta_x u + u_t = |u|^{r-1}u.$$

We also refer the reader to additional citations that provide a mathematical overview of this type of problem, such as [11–13].

Time delays represent one of the most active areas of study today, as they arise naturally in economic, biological, physical, and chemical systems. In mathematics, three principal types of delays are distiguished:

- Discrete delay: In this case, the time is retarded by a fixed rate

  $$y(t) = x(t - \tau).$$

  This form of delay was examined in [14], where the authors studied the existence and the exponential decay of solutions to a wave equation with a delay

  $$u_{tt}(x,t) + \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) + \mu_2 u(x,t-\tau) = 0.$$

  After, building on this earlier work, H.Yuksekkaya and collaborators [15–17], studied the asymptotic behavior of the same problem, this time with a logarithmic source term

  $$u_{tt}(x,t) + \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) + \mu_2 u(x,t-\tau) = uln|u|^{\gamma}.$$

- Variable delay: This problem is similar to the fixed delay, but here, τ is a function of t

$$y(t) = x(t - \tau(t)).$$

As an example of researchers who examined such terms, we cite [18, 19], where G. Liu and L. Diao studied this problem

$$u_{tt}(x,t) - \Delta u(x,t) + \alpha(t)\int_0^t g(t-s)\Delta u(x,s)ds + \mu u_t(x, t-\tau(t)) = 0,$$

to which they established the stability of the solution.
In 2021, A. Benguessoum [20] studied the energy decay of the following wave equation:

$$u_{tt}(x,t) - \Delta u(x,t) - \Delta_x u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) = 0.$$

- Distributed delay: this is the type we focus on in our work. In this case, the system depends on a continuum of past times

$$y(t) = \int_0^\infty K(\theta)\, x(t-\theta)\, d\theta.$$

H. Yüksekkaya and E. Pişkin [21] investigated the blow-up of solution under certain assumptions for a viscoelastic plate equation with distributed delay

$$u_{tt} + \Delta^2 u - \omega\Delta u_t - \int_0^t g(t-s)\Delta^2 u(s)dqs\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(q)| u_t(x, t_q)\, dq = b|u|^{p-2}u.$$

Additional literature on this field of research can be found in [22–24]. In our work, we investigate the Kirchhoff-type equation coupled with a distributed delay and a polynomial nonlinearity. We rely on recent research, such as in [25–27], which the authors proved the finite-time blow-up for the Kirchhoff-type equation with a viscoelastic term, logarithmic nonlinearity, and Balakrishnan-Taylor damping. For further details on the polynomial source term, see [28–30].

The work is divided as follows: a general introduction with relevant references, followed by two main sections. Section 2 establishes the preliminaries and conditions necessary for the development of our work. The main results appear in Section 3, where we prove the blow-up of the solution at a fixed time. Finally, we conclude with a brief summary.

2. Statement of the problem

In this problem, the notation $\|.\|_k$ denotes the norm in the L^k space, while $(.,.)$ represents the inner product in L^2. We usually write $\|.\|$ instead of $\|.\|_2$.
$(H1)$: $\lambda_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is an L^∞ function such that

$$\left(\frac{\delta+1}{2}\right)\int_{\tau_1}^{\tau_2} |\lambda_2(q)|dq \le \lambda_1, \quad \delta > 1. \tag{2.1}$$

$(H2)$:

$$\sigma : [0, +\infty) \to \mathbb{R}, \text{ such that } \sigma(\lambda) \le \delta_0. \tag{2.2}$$

Following the work of [31], we introduce a new variable

$$\chi(\mathrm{x}, p, q, t) = \varphi_t(\mathrm{x}, t - qp) \ \ 0 \le p \le 1,$$

we obtain

$$\begin{cases} q\chi_t(\mathrm{x}, p, q, t) + \chi_p(\mathrm{x}, p, q, t) = 0, \\ \chi(\mathrm{x}, 0, q, t) = \varphi_t(\mathrm{x}, t). \end{cases}$$

Then, the system becomes

$$\begin{cases} \varphi_{tt} + \mathcal{A}^2\varphi + \sigma(\int_\Gamma \left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x})(-\mathcal{A}u) + \lambda_1\varphi_t + \int_{\tau_1}^{\tau_2} |\lambda_2(q)|\chi(\mathrm{x}, 1, q, t)dq = |\varphi|^{k-2}\varphi, \\ q\chi_t(\mathrm{x}, p, q, t) + \chi_p(\mathrm{x}, p, q, t) = 0, \end{cases} \tag{2.3}$$

which satisfies the following initial and boundary conditions:

$$\begin{cases} \varphi(\mathrm{x}, 0) = \varphi_0(\mathrm{x}), \quad \varphi_t(\mathrm{x}, 0) = \varphi_1(\mathrm{x}), \\ \varphi(\mathrm{x}, t) = \frac{\partial\varphi}{\partial\nu} = 0, \qquad \mathrm{x} \in \partial\Gamma, \\ \chi(\mathrm{x}, 0, p, t) = \varphi_t(\mathrm{x}, t), \qquad \chi(\mathrm{x}, q, p, 0) = g_0(\mathrm{x}, qp), \end{cases} \tag{2.4}$$

where

$$(\mathrm{x}, p, q, t) \in \Gamma \times (0, 1) \times [\tau_1, \tau_2] \times [0, +\infty).$$

Theorem 1. *Suppose that* (2.1) *hold. Let*

$$\begin{cases} k \ge 2, n = 1, 2, 3, 4, \\ 2 < k < \frac{2(n-2)}{n-4}, n \ge 5. \end{cases} \tag{2.5}$$

Thus, for any initial data

$$(\varphi_0, \varphi_1, g_0) \in H_0^2(\Gamma) \times H_0^2(\Gamma) \times L^2\left(\Gamma \times (0, 1) \times (\tau_1, \tau_2)\right),$$

the problem (2.3)-(2.4) *has a unique solution*

$$\varphi \in C\left([0, T]; H_0^2(\Gamma)\right),$$

for some $T > 0$.

Lemma 1. *Assume that our conditions are satisfied, then*

$$\mathcal{E}(t) = \frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\mathcal{A}\varphi\|^2 + \frac{1}{2}\sigma(\int_\Gamma \left|\frac{\partial\varphi}{\partial x}\right|^2 dx)\int_\Gamma \left|\frac{\partial\varphi}{\partial x}\right|^2 dx + \frac{1}{2}\int_\Gamma\int_0^1\int_{\tau_1}^{\tau_2} q|\lambda_2(q)||\chi(x,1,q,t)|dqdpdx - \frac{1}{k}\|\varphi\|_k^k, \tag{2.6}$$

and

$$\mathcal{E}'(t) \le -c_1\left(\|\varphi_t\|^2 + \int_\Gamma\int_{\tau_1}^{\tau_2} |\lambda_2(q)| \left|\chi^2(x, 1, q, t)\right| dqdx\right). \tag{2.7}$$

Proof. By multiplying the first equation by φ_t and integrating on Γ, we get

$$\frac{d}{dt}\left\{\frac{1}{2}\|\varphi_t\|^2+\frac{1}{2}\|\mathcal{A}\varphi\|^2+\frac{1}{2}\sigma(\int_\Gamma\left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x})\int_\Gamma\left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}+\frac{1}{k}\|\varphi\|_k^k\right\}=-\lambda_1\|\varphi_t\|^2+\int_\Gamma\varphi_t\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\chi(\mathrm{x},1,q,t)dqd\mathrm{x},$$

the second equation of (2.3), holds:

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Gamma\int_0^1\int_{\tau_1}^{\tau_2} q\,|\lambda_2(q)|\left|\chi^2(\mathrm{x},\rho,q,t)\right| dqd\rho d\mathrm{x}\\
=\;& -\frac{1}{2}\int_\Gamma\int_0^1\int_{\tau_1}^{\tau_2} 2\,|\lambda_2(q)|\,y\chi_\rho dqd\rho d\mathrm{x}\\
=\;& \frac{1}{2}\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\left|\chi^2(\mathrm{x},0,q,t)\right| dqd\mathrm{x}-\frac{1}{2}\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\left|\chi^2(\mathrm{x},1,q,t)\right| dqd\mathrm{x}\\
=\;& \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\,dq\right)\|\varphi_t\|^2-\frac{1}{2}\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\left|\chi^2(\mathrm{x},1,q,t)\right| dqd\mathrm{x}.
\end{aligned}\tag{2.8}$$

So,

$$\mathcal{E}'(t)=-\lambda_1\|\varphi_t\|^2+\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\varphi_t\chi(\mathrm{x},1,q,t)dqd\mathrm{x}+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\lambda_2(q)|dq\right)\|\varphi_t\|^2-\frac{1}{2}\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\left|\chi^2(\mathrm{x},1,q,t)\right|dqd\mathrm{x}.$$

Using the hypothesis in (2.1) and Young's inequality, we obtain

$$\mathcal{E}'(t)\leq -C\left(\|\varphi_t\|^2+\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\left|\chi^2(\mathrm{x},1,q,t)\right| dqd\mathrm{x}\right).$$

□

Lemma 2. *([32]) There is a constant $\kappa > 0$, depending solely on Γ, for which*

$$\left(\int_\Gamma|\varphi|^k dx\right)^{s/k}\leq\kappa\left[\|\int_\Gamma\left|\frac{\partial\varphi}{\partial x}\right|^2\|^2+\|\varphi\|_k^k\right],$$

for all $\varphi\in L^{k+1}(\Gamma)$ and $2\leq s\leq k$.

By applying the preceding lemma together with the Sobolev embedding theorem, we derive the following corollary.

Corollary 1. *It exists $\kappa > 0$, depending on Γ only, for which*

$$\left(\int_\Gamma|\varphi|^k dx\right)^{s/k}\leq\kappa\left[\|\mathcal{A}\varphi\|^2+\|\varphi\|_k^k\right],\tag{2.9}$$

for all $\varphi\in L^{k+1}(\Gamma)$ and $2\leq s\leq k$.

Using $\|\varphi\|_2^2\leq\kappa\|\varphi\|_k^2\leq\kappa\left(\|\varphi\|_k^k\right)^{2/k}$, we have the following corollary.

Corollary 2. *There is a constant $\kappa > 0$, depending solely on Γ, for which*

$$\|\varphi\|_2^2\leq\kappa\left[\|\mathcal{A}\varphi\|_2^{4/k}+\left(\|\varphi\|_k^k\right)^{2/k}\right].\tag{2.10}$$

3. Blow-up analysis

In this section, we establish the blow-up result for the problem (2.3)-(2.4).

Theorem 2. *Under conditions* (2.1) *and* (2.2)*, the solution of system* (2.6)-(2.7) *blows up in finite time.*

Proof. By (2.6) and (2.7), we have

$$\mathcal{E}(t) \le \mathcal{E}(0) \le 0. \tag{3.1}$$

Now we define the following functional:

$$\begin{aligned}&\mathcal{H}(t) = -\mathcal{E}(t)\\ =&\frac{1}{k}\|\varphi\|_k^k - \frac{1}{2}\|\varphi_t\|^2 - \frac{1}{2}\|\mathcal{A}\varphi\|^2 - \frac{1}{2}\sigma\Big(\int_\Gamma \left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}\Big)\int_\Gamma \left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x} - \frac{1}{2}\int_\Gamma\int_0^1\int_{\tau_1}^{\tau_2} q|\lambda_2(q)||\chi(\mathrm{x},p,q,t)|dqdpd\mathrm{x}.\end{aligned} \tag{3.2}$$

Thus,

$$\begin{aligned}\mathcal{H}'(t) &= -\mathcal{E}'(t)\\ &\ge C\left(\|\varphi_t\|^2 + \int_\Gamma\int_{\tau_1}^{\tau_2} |\lambda_2(q)|\left|\chi^2(\mathrm{x},1,q,t)\right| dqd\mathrm{x}\right)\\ &\ge C\int_\Gamma\int_{\tau_1}^{\tau_2} |\lambda_2(q)|\left|\chi^2(\mathrm{x},1,q,t)\right| dqd\mathrm{x},\end{aligned} \tag{3.3}$$

and

$$0 \le \mathcal{H}(0) \le \mathcal{H}(t) \le \frac{1}{k}\|\varphi\|_k^k. \tag{3.4}$$

Put

$$\mathcal{L}(t) = \mathcal{H}^{1-\gamma} + \epsilon\int_\Gamma \varphi_t\varphi d\mathrm{x} + \frac{\epsilon\lambda_1}{2}\int_\Gamma \varphi^2 d\mathrm{x}, \tag{3.5}$$

where $\epsilon > 0$, and

$$\frac{2(k-2)}{k^2} < \gamma < \frac{k-2}{2k} < 1. \tag{3.6}$$

By deriving (3.5) and exploiting the first equation of (2.3), we obtain

$$\begin{aligned}\mathcal{L}'(t) =&(1-\gamma)\mathcal{H}^{-\gamma}\mathcal{H}'(t) + \epsilon\|\varphi_t\|^2 - \epsilon\|\mathcal{A}\varphi\|^2 - \epsilon\sigma\Big(\int_\Gamma \left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}\Big)\int_\Gamma \left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}\\ &+ \epsilon\|\varphi\|_k^k - \epsilon\int_\Gamma\int_{\tau_1}^{\tau_2} |\lambda_2(q)||\varphi\chi(\mathrm{x},1,q,t)|dqd\mathrm{x}.\end{aligned} \tag{3.7}$$

Utilizing

$$\epsilon\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)||\varphi\chi(\mathrm{x},1,q,t)|dqd\mathrm{x} \le \epsilon\left\{\varepsilon_1\left(\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\,dq\right)\|\varphi\|^2 + \frac{1}{4\varepsilon_1}\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\left|\chi^2(\mathrm{x},1,q,t)\right| dqd\mathrm{x}\right\},$$

and employing (2.2), we get

$$\begin{aligned}\mathcal{L}'(t) \ge\ & (1-\gamma)\mathcal{H}^{-\gamma}\mathcal{H}'(t) + \epsilon\|\varphi_t\|^2 - \epsilon\|\mathcal{A}\varphi\|^2 - \epsilon\delta_0\int_\Gamma\left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x} + \epsilon\|\varphi\|_k^k\\ &-\epsilon\left\{\varepsilon_1\left(\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\,dq\right)\|\varphi\|^2 + \frac{1}{4\varepsilon_1}\int_\Gamma\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\left|\chi^2(\mathrm{x},1,q,t)\right| dqd\mathrm{x}\right\}.\end{aligned} \tag{3.8}$$

Taking ε_1 sucht that $\frac{1}{4\varepsilon_1 c} = \rho\mathcal{H}^{-\gamma}$ and exploiting (3.3), we will get

$$\mathcal{L}'(t) \geq [(1-\gamma)-\epsilon\rho]\mathcal{H}^{-a}\mathcal{H}'(t)+\epsilon\|\varphi_t\|^2-\epsilon\|\mathcal{A}\varphi\|^2-\epsilon\delta_0\int_\Gamma\left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}+\epsilon\|\varphi\|_k^k-\frac{\epsilon\mathcal{H}^\gamma}{4c\rho}\left(\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\,dq\right)\|\varphi\|^2. \quad (3.9)$$

For $(0 < b < 1)$

$$\begin{aligned}\epsilon\|\varphi\|_k^k =&\epsilon k(1-\gamma)\mathcal{H}(t)+\frac{\epsilon k(1-b)}{2}\|\varphi_t\|^2+\frac{\epsilon k(1-b)}{2}\|\mathcal{A}\varphi\|^2+\frac{\epsilon k(1-b)}{2}\sigma(\int_\Gamma\left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x})\int_\Gamma\left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}\\ &+\frac{\epsilon k(1-b)}{2}\int_\Gamma\int_0^1\int_{\tau_1}^{\tau_2} q|\lambda_2(q)||\chi^2(\mathrm{x},p,q,t)|dqdpd\mathrm{x}+\epsilon b\|\varphi\|_k^k,\end{aligned} \quad (3.10)$$

with (3.9) implies

$$\begin{aligned}&\mathcal{L}'(t)\\ \geq&[(1-\gamma)-\epsilon\rho]\mathcal{H}^{-a}\mathcal{H}'(t)+\left[\epsilon+\frac{\epsilon k(1-b)}{2}\right]\|\varphi_t\|^2+\left[\frac{\epsilon k(1-b)}{2}-\epsilon\right]\|\mathcal{A}\varphi\|^2\delta_0\left[\frac{\epsilon k(1-b)}{2}-\epsilon\right]\int_\Gamma\left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}\\ &+\epsilon k(1-\gamma)\mathcal{H}(t)+\epsilon b\|\varphi\|_k^k+\frac{\epsilon k(1-b)}{2}\int_\Gamma\int_0^1\int_{\tau_1}^{\tau_2}q|\lambda_2(q)||\chi^2(\mathrm{x},p,q,t)|dqdpd\mathrm{x}-\frac{\epsilon\mathcal{H}^\gamma}{4c\rho}\left(\int_{\tau_1}^{\tau_2}|\lambda_2(q)|\,dq\right)\|\varphi\|^2.\end{aligned} \quad (3.11)$$

Using Young's inequality, Eqs (2.9), (2.10), and (3.4), we obtain

$$\begin{aligned}\mathcal{H}^\gamma\|\varphi\|^2 &\leq \left(\int_\Gamma|\varphi|^k d\mathrm{x}\right)^\gamma\|\varphi\|^2\\ &\leq \kappa\left\{\left(\int_\Gamma|\varphi|^k d\mathrm{x}\right)^{\frac{\gamma+2}{k}}+\left(\int_\Gamma|\varphi|^k d\mathrm{x}\right)^\gamma\|\mathcal{A}\varphi\|^{\frac{4}{k}}\right\}\\ &\leq \kappa\left\{\left(\int_\Gamma|\varphi|^k d\mathrm{x}\right)^{(k\gamma+2)/k}+\|\mathcal{A}\varphi\|^2+\left(\int_\Gamma|\varphi|^k d\mathrm{x}\right)^{k\gamma/(k-2)}\right\}.\end{aligned}$$

So, by Lemma 2 we obtain

$$\mathcal{H}^\gamma\|\varphi\|^2 \leq \kappa(\|\varphi\|_k^k+\|\mathcal{A}\varphi\|^2). \quad (3.12)$$

Together with (3.11), we get

$$\begin{aligned}\mathcal{L}'(t) \geq& [(1-\gamma)-\epsilon\rho]\mathcal{H}^{-\gamma}\mathcal{H}'(t)+\epsilon\left[1+\frac{k(1-b)}{2}\right]\|\varphi_t\|^2+\epsilon\left[\frac{k(1-b)}{2}-1-\frac{\kappa}{4cp}\int_{\tau_1}^{\tau_2}|\lambda_2(q)|dq\right]\|\mathcal{A}\varphi\|^2\\ &+\delta_0\epsilon\left[\frac{k(1-b)}{2}-1\right]\int_\Gamma\left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}+\epsilon k(1-\gamma)\mathcal{H}(t)+\epsilon\left[b-\frac{\kappa}{4c\rho}\int_{\tau_1}^{\tau_2}|\lambda_2(q)|dq\right]\|\varphi\|_k^k\\ &+\frac{\epsilon k(1-b)}{2}\int_\Gamma\int_0^1\int_{\tau_1}^{\tau_2}q|\lambda_2(q)||\chi^2(\mathrm{x},p,q,t)|dqdpd\mathrm{x}.\end{aligned} \quad (3.13)$$

We have, for $0 < b < 1$ small enough, we have

$$\beta_1 = 1+\frac{k(1-b)}{2} > 0,$$

$$\beta_2 = \frac{k(1-b)}{2} - 1 > 0,$$

and

$$\beta_3 = \frac{k(1-b)}{2} > 0.$$

Fixing κ and ρ and choosing ϵ small enough, such that

$$\beta_4 = \frac{k(1-b)}{2} - 1 - \frac{\kappa}{4c\rho}\int_{\tau_1}^{\tau_2} |\lambda_2(q)|dq > 0,$$

$$\beta_5 = b - \frac{\kappa}{4cp}\int_{\tau_1}^{\tau_2} |\lambda_2(q)|dq > 0,$$

and

$$\beta_6 = (1-\gamma) - \epsilon\rho > 0.$$

Therefore, for some $B > 0$,

$$\mathcal{L}'(t) \geq B\left\{\mathcal{H}(t) + \|\varphi_t\|^2 + \|\mathcal{A}\varphi\|^2 + \int_\Gamma \left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x} + \|\varphi\|_k^k + \int_\Gamma\int_0^1\int_{\tau_1}^{\tau_2} q|\lambda_2(q)||\chi^2(\mathrm{x},p,q,t)|dqdpd\mathrm{x}\right\}. \quad (3.14)$$

Consequently,

$$L(t) \geq L(0) > 0, \quad t > 0. \quad (3.15)$$

Now, by applying Young's and Holder's inequalities, we obtain:

$$\|\varphi\|_2 = \left(\int_\Gamma \varphi^2 d\mathrm{x}\right)^{\frac{1}{2}} \leq \left[\left(\int_\Gamma \left(|\varphi|^2\right)^{k/2} d\mathrm{x}\right)^{\frac{2}{k}} \left(\int_\Gamma 1 d\mathrm{x}\right)^{1-\frac{2}{k}}\right]^{\frac{1}{2}} \leq C\|\varphi\|_k, \quad (3.16)$$

and

$$\left|\int_\Gamma \varphi\varphi_t d\mathrm{x}\right| \leq C\|\varphi_t\| \, \|\varphi\|_k.$$

Hence,

$$\left|\int_\Gamma \varphi\varphi_t d\mathrm{x}\right|^{\frac{1}{1-\gamma}} \leq C\left(\|\varphi_t\|^{\frac{\xi}{1-\gamma}} + \|\varphi\|_k^{\frac{\mu}{1-\gamma}}\right), \quad (3.17)$$

where $\frac{1}{\xi} + \frac{1}{\mu} = 1$. Taking $\xi = 2(1-\gamma)$

$$\frac{\mu}{1-\gamma} = \frac{2}{1-2\gamma} \leq k.$$

Exploiting Corollary 1, we obtain

$$\left|\int_\Gamma \varphi\varphi_t d\mathrm{x}\right|^{\frac{1}{1-\gamma}} \leq C\left[\|\varphi_t\|^2 + \|\varphi\|_k^k + \|\mathcal{A}\varphi\|^2 + \int_\Gamma \left|\frac{\partial\varphi}{\partial\mathrm{x}}\right|^2 d\mathrm{x}\right]. \quad (3.18)$$

In contrast,

$$\begin{aligned}\mathcal{L}^{\frac{1}{1-\gamma}}(t) &= \left(\mathcal{H}(t)^{1-\gamma} + \epsilon\int_\Gamma \varphi_t\varphi d\mathrm{x} + \frac{\epsilon\lambda_1}{2}\int_\Gamma \varphi^2 d\mathrm{x}\right)^{\frac{1}{1-\gamma}} \\ &\leq C\left(\mathcal{H}(t) + \left|\int_\Gamma \varphi\varphi_t d\mathrm{x}\right|^{\frac{1}{1-\gamma}} + \|\varphi\|^{\frac{2}{1-\gamma}} + \|\mathcal{A}\varphi\|^{\frac{2}{1-\gamma}}\right).\end{aligned}$$

Combining it with (3.18)

$$\mathcal{L}^{\frac{1}{1-\gamma}}(t) \le A\left[\mathcal{H}(t) + \|\varphi_t\|^2 + \|\varphi\|_k^k + \|\mathcal{A}\varphi\|^2 + \int_\Gamma \left|\frac{\partial\varphi}{\partial \mathrm{x}}\right|^2 d\mathrm{x}\right]. \tag{3.19}$$

By (3.14) and (3.19),

$$\mathcal{L}'(t) \ge \Lambda\mathcal{L}^{\frac{1}{1-\gamma}}(t), \tag{3.20}$$

with $\Lambda > 0$.

Solving the corresponding ordinary differential equation, we obtain

$$\mathcal{L}^{\frac{\gamma}{1-\gamma}}(t) \ge \frac{1}{\mathcal{L}^{\frac{-\gamma}{1-\gamma}}(0) - \Lambda\dfrac{\gamma}{1-\gamma}t}.$$

From the positivity of the functional $\mathcal{L}(t)$

$$\mathcal{L}^{\frac{-\gamma}{1-\gamma}}(0) \ge \frac{\gamma\Lambda t}{1-\gamma}.$$

Consequently, we conclude that the solution blows up in finite time

$$T \le T^* = \frac{1-\gamma}{\Lambda\gamma\mathcal{L}^{\frac{\gamma}{1-\gamma}}(0)}. \tag{3.21}$$

The proof is now complete. □

4. Conclusions

Kirchhoff-type equations with time delays haveemerged as significant topics in physics and engineering in recent years. In our work, we established conditions under which the solution of a semilinear Kirchhoff-type equation with distributed delay and a polynomial source term, subject to initial and boundary conditions, blows up in finite time, with the blow-up time explicitly determined.

Author contributions

Every author did the same amount of work on this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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