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Research article

Closed-form solutions of stochastic solitary waves for certain type of nonlinear Schrödinger equation

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Abstract: This study investigates the stochastic nonlinear Schrödinger equation with a delta potential (δ-NLSE), a model capturing the combined effects of nonlinear dispersion, localized defects, and environmental randomness. Analytical solutions are constructed using unified solver techniques to evaluate the influence of noise intensity and potential strength on wave propagation and solitary-wave formation. The solitary waves in the stochastic δ-NLSE reveal how randomness and defect-induced localization interact, resulting in novel phenomena such as stochastic modification of transmission and reflection coefficients, noise-stabilized limit states, and random soliton diffusion. The issue is significant because real-world media are rarely homogeneous or noise-free, and understanding how randomness interacts with localized singularities remains a challenge in both theory and practice. The proposed stochastic solutions are ground-breaking and highly relevant for modeling complex physical processes in nonlinear wave theory, quantum mechanics, water waves, nonlinear optics, and Bose-Einstein condensates, where localized impurities or interfaces strongly affect wave propagation.

Keywords: Brownian motion; nonlinear Schrödinger equation; solitary waves; effects of noise; physical applications
Mathematics Subject Classification: 35C07, 60H15, 35R60, 35Q55, 35Q62, 68W30

Abbreviations

NLSE Nonlinear Schrödinger equation
PDE Partial differential equation
NPDEs Nonlinear partial differential equations
ODEs Ordinary differential equations
ZNNs Zeroing neural networks
KdV Korteweg–de Vries
SPDEs Stochastic partial differential equations
SNLSE Stochastic nonlinear Schrödinger equation
2D Two-dimensional
3D Three-dimensional

1. Introduction

Nonlinear partial differential equations (NPDEs) are fundamental to mathematical modeling in modern science [1,2]. They arise naturally across diverse fields, including superfluid, nonlinear optics, quantum field theory, plasma physics, biological systems, traffic flow, and financial mathematics, where they govern the evolution of multivariate physical quantities subject to spatial and temporal variations. The nonlinearity inherent in these systems often produces complex solution structures, such as solitary waves, breathers, shock waves, and blow-up phenomena, which are absent in linear models. A central objective in the study of NPDEs is to obtain exact or approximate solutions that reveal the physical properties of the modeled system [3,4]. Although monotone iterative techniques and quasilinearization procedures are typically developed for nonlinear ordinary or impulsive differential equations with integral terms, they share deep conceptual and methodological connections with the broader theory of NPDEs [5]. Moreover, the theoretical foundations of a noise-tolerant fuzzy-type zeroing neural network are closely related to the analysis and solution of NPDEs, even though ZNNs are usually presented as computational schemes for real-time control or synchronization of nonlinear systems [6]. Each of these approaches embodies distinct principles and advantages, offering valuable tools for deriving analytical solutions.

One of the most remarkable types of nonlinear wave phenomena is the solitary waves, which can propagate over long distances without losing speed or shape [7]. Solitary waves are central to understanding nonlinear dynamics in diverse physical systems because of their stability and persistence, arise from a delicate balance between nonlinear steepening and dispersive spreading. Over the past century, they have become a fundamental concept in applied mathematics, physics, and engineering. Nonlinear partial differential equations, such as the Korteweg–de Vries (KdV) equation, the first analytical foundation of soliton theory, rigorously describe their behavior [8]. Since then, solitary waves have been identified in numerous fields, including Bose–Einstein condensates, nonlinear optics, shallow-water hydrodynamics, plasma physics, and biological systems such as nerve-pulse transmission. They have significant theoretical and practical ramifications due to their elastic interaction, which allows them to emerge from collisions with their form and speed intact. Solitary waves underpin robust communication channels in optical fibers, aid in modeling tsunamis in geophysics, and inspire mathematical techniques for studying integrable systems in modern physics and engineering. Thus, solitary waves represent both a fundamental component of nonlinear scientific research and a fascinating natural phenomenon [9].

Stochastic partial differential equations (SPDEs) extend conventional partial differential equation (PDE) theory by incorporating randomness, thereby modeling systems influenced by noise and uncertainty. Unlike deterministic PDEs, which describe smooth and predictable dynamics, SPDEs capture the random fluctuations that naturally arise in physics, biology, engineering, and finance. The inclusion of stochastic factors enables SPDEs to represent real-world phenomena such as turbulent fluid flows, heat transfer in random media, population dynamics under environmental noise, and pricing of financial derivatives under stochastic volatility [10]. The SPDE applications have increased dramatically during the last few decades. In fluid dynamics, SPDEs are used to simulate turbulent transport and random forcing in Navier-Stokes systems [11]. Temperature fluctuations, molecular noise, material imperfections, and stochastic optical forcing significantly affect nematic systems, necessitating SPDE based models [12]. In climate research, SPDE explain large-scale variability and uncertainty in atmospheric and oceanic models [13]. Zhang investigated exact solutions for Wick-type stochastic KdV equations using the Hermite transform method and white noise theory [14]. In financial mathematics, SPDEs underpin models of stochastic volatility surfaces and uncertainty-based option pricing [15]. Thus, SPDEs have become a cornerstone of modern applied mathematics: they bridge the gap between deterministic modeling and probabilistic uncertainty, integrating the analytical depth of PDE theory with the complexity of stochastic processes [16, 17]. Their study continues to shape both theoretical mathematics and practical modeling across complex systems.

Brownian motion refers to the erratic and unpredictable movement of tiny particles suspended in a fluid, caused by countless collisions with surrounding environment [17]. This phenomenon is one of the most fundamental examples of a stochastic process, laying the groundwork for contemporary probability theory and stochastic calculus. Mathematically, Brownian motion is defined as a continuous-time process with stationary and independent increments, commonly known as the Wiener process. It plays a pivotal in modeling of diffusion phenomena in physics, effectively capturing the random behavior of particles at microscopic scales. Beyond physics, Brownian motion serves as a foundational principle across multiple disciplines: in finance, it models fluctuations in stock prices; in biology, it explains the random movements of molecules and cells; and in mathematics, it represents a universal scaling limit for random walks. Its simplicity, universality, and profound connection between randomness and structure make Brownian motion one of the most influential and extensively widely applied concepts in the scientific community [18–20].

The complex cubic nonlinear Schrödinger equation (NLSE) featuring a δ-potential serves as a significant model in mathematical physics for examining localized wave interactions with point-like impurities or defects within nonlinear media [21]. The cubic NLSE, in its conventional form, outlines the progression of a complex-valued wave function influenced by both dispersion and cubic nonlinearity, which may manifest as either focusing or defocusing. The introduction of a δ-potential signifies an idealized impurity or trapping site located at a singular spatial point, resulting in pronounced localized interactions. This adjustment is especially pertinent in scenarios such as nonlinear optics, where it simulates the impact of a thin defect or impurity within an optical fiber, and in Bose–Einstein condensates, where it denotes an atomic cloud subjected to a sharply localized external potential. By modifying the scattering characteristics of solitons and permitting the presence of bound states, the δ-potential mathematically enhances the dynamics. For example, depending on their energy and contact with the δ-site, light solitons may be trapped or reflected in the focusing scenario, whereas dark solitons may exhibit phase shifts and changed propagation in the defocusing regime. Analysis of stability, bifurcations, and long-time asymptotics of solutions is made possible by the interaction of nonlinearity, dispersion, and singular potential. The cubic NLSE with δ-potential is therefore a typical model for investigating the interaction of nonlinear waves with localized defects, providing insights into fundamental theory and applications in quantum transport, waveguides, and condensed matter systems. The deterministic complex cubic nonlinear Schrödinger equation (NLSE) with a δ-potential (δ-NLSE) is defined as follows [21]:

$$i\Phi_t + \frac{1}{2}\Phi_{xx} - \eta \mid \Phi \mid^2 \Phi - \alpha\delta \, \Phi = 0, \quad i = \sqrt{-1}, \tag{1.1}$$

$\alpha, \eta \in \mathbb{R} - \{0\}$, whereas δ represents the Dirac measure at the origin [22] and $\Phi(x, t)$ denotes the complex-valued wave function. The delta potential is either repulsive or attractive depending upon $\alpha > 0$ or $\alpha < 0$, respectively [23]. The δ-potential acts as a localized source of scattering and energy leakage, linking the nonlinear dynamics of the soliton with the generation of radiation. This interplay between nonlinearity, localization, and radiation is fundamental to understanding energy transfer and stability in systems governed by the NLSE with localized defects. Goodman et al. [22] discussed the stability of solitary wave solutions. Small-amplitude linear waves that travel away from the defect location are the manifestation of this radiation. In this research, we analyze model (1.1) in the context of a Brownian motion process in the Itô sense [24], which is outlined as follows:

$$i\Phi_t + \frac{1}{2}\Phi_{xx} - \eta \mid \Phi \mid^2 \Phi - \alpha\delta\,\Phi + \sigma\Xi_t\Phi = 0, \quad i = \sqrt{-1}, \tag{1.2}$$

where σ denotes the strength of noise and Ξ_t characterizes the noise effect, which represents the time derivative of the Brownian motion process $\{\Xi(t)\}_{t\geq 0}$. The primary characteristics of the Brownian motion process are [16]:

(a) $\Xi(t)$, $t \geq 0$, represents a continuous function of time t.
(b) For $s < t < u < k$, $\Xi(t) - \Xi(s)$ and $\Xi(k) - \Xi(u)$ are independent.
(c) $\Xi(t) - \Xi(s)$ follows a normal distribution with zero mean and variance $t - s$.

Adopting the Itô interpretation directly influences on the analytical treatment of the stochastic model. In the Itô framework, stochastic integrals are martingales, and no drift-correction terms appear, which simplifies expectation calculations and the derivation of moment equations. The authors introduced the stochastic solutions of model (1.2) via the Sardar sub-equation method [25]. The authors investigated the complex cubic NLSE with δ-potential, under Brownian forcing via the modified $exp(-\Psi(\xi))$-expansion method [26]. The stochastic δ-NLSE effectively integrates nonlinearity, localized defects (δ potentials), and stochastic perturbations. This model enables researchers to predict scattering, trapping, and destabilization of solitons or coherent wave packets in noisy, defect-containing media—phenomena directly relevant to robust wave control, signal integrity, and device design. The authors selected this model because the cubic nonlinearity preserves the basic physical structure of Kerr-type systems, while the δ-defect imposes clear matching conditions and an analytically tractable linear operator, allowing for rigorous analysis and precise numerical simulation. The proposed approach, typically combines energy estimates with, Itô calculus, providing a coherent framework that can simultaneously accommodates the singular potential and the stochastic forcing. The stochastic δ-NLSE is a perfect benchmark for comprehending actual nonlinear wave dynamics under uncertainty thus serves as an ideal both theoretical clarity and useful forecasting capability.

In this research, we employ unified solver approaches [27, 28] to construct a novel stochastic solitary-wave solution for the nonlinear Schrödinger equation (NLSE) with a δ-potential under the influence of Brownian motion process. We also demonstrate how solution behavior of the solutions is influenced by physical parameters and noise term. Compared to earlier, more intricate methods, the proposed solvers offer several advantages, they eliminate cumbersome computations, yield accurate results through direct use of physical parameters. They can be implemented as pre-built functions or box solvers for diverse equations and systems that arise in applied science. These solvers are unpretentious, dependable, and durable. In fact, these approaches enable mathematicians, physicists, and engineers to capture and analyze fascinating nonlinear phenomena in practical contexts.

The arrangement of the present article is as follows. Section 2 discusses solid analytical approaches for solving the Duffing equation $L\phi''(\zeta) + M\phi^3(\zeta) + N\phi(\zeta) = 0$. The Duffing equation, a well-known nonlinear differential equation, that models the dynamics of a damped and driven oscillator with a nonlinear stiffness component. Section 3 presents critical stochastic solutions for the nonlinear Schrödinger equation (NLSE) with a δ-potential. Section 4 describes the potential solutions obtained, including several 2D and 3D graphical representations generated for appropriate parameter choices. Section 5 provides the conclusion based on the study's findings and offers suggestions for future research.

2. The solvers' approaches

The Duffing equation serves as a key nonlinear oscillator model that bridges theoretical analysis with practical applications. Its mathematical simplicity conceals a dynamic richness that is essential for understanding bifurcations, chaos, and nonlinear resonance. Moreover, broad applications in mechanics, electronics, climate science, and quantum systems ensure its continued relevance in both classical and modern research. Here we report the form of solutions for the following Duffing equation.

Consider the NPDE for $\Phi(x,t)$ in the form

$$\Lambda(\Phi, \Phi_x, \Phi_t, \Phi_{xx}, \Phi_{xt}, \Phi_{tt}, ...) = 0. \tag{2.1}$$

Using the wave transformation

$$\Phi(x,t) = \phi(\xi), \qquad \xi = x - vt, \tag{2.2}$$

Eq (2.1) transformed into the following ODE:

$$\Gamma(\phi, \phi', \phi'', \phi''', ...) = 0. \tag{2.3}$$

Many nonlinear partial differential equations (NPDEs) governing wave propagation in dispersive and nonlinear media can be transformed into a diffusion-type equation with cubic nonlinearity

$$L\,\phi''(\xi) + M\,\phi^3(\xi) + N\,\phi(\xi) = 0. \tag{2.4}$$

By include a wave variable, complex evolution equations can be reduced to an ordinary differential form. This reduction captures the essential physical balance between dispersion (or diffusion) and nonlinearity, while eliminating explicit time dependence. The resulting equation serves as a classical model for investigating stationary wave profiles, solitary structures, and soliton solutions. Thus, examining this reduced equation not only simplifies the mathematical analysis of NLPDEs, but also provides detailed insight into the qualitative behavior, amplitude, and stability of nonlinear waves across diverse physical systems, including fluid dynamics, plasma, and optics. Equation (2.4) admits the following solutions when solved using the methods described in [27, 28]:

$$\phi_{1,2}(\xi) = \pm\sqrt{\frac{-2L}{M}}\,tanh\,(\xi)\,,\ \frac{L}{M} < 0. \tag{2.5}$$

$$\phi_{3,4}(\xi) = \pm\sqrt{\frac{-2L}{M}}\,coth\,(\xi)\,,\ \frac{L}{M} < 0, \tag{2.6}$$

$N = 2L$.

$$\phi_{5,6}(\xi) = \pm\sqrt{\frac{-2N}{M}}\,sech(\pm\sqrt{\frac{-N}{L}}\xi)\ ,\ \frac{N}{L} < 0,\ \frac{N}{M} < 0. \tag{2.7}$$

3. The stochastic solutions

Using the transformation

$$\Phi(x,t)=\phi(\xi)e^{i(p\,x+ct+\sigma\Xi(t))},\ \ \xi=\beta(x-v\,t),\tag{3.1}$$

where p, c, and v denote the wave number, frequency, and wave speed, respectively, we have

$$\begin{aligned} i\Phi_t &= (-i\beta\,v\phi'-(c+\sigma\Xi_t)\phi)e^{i(p\,x+ct+\sigma\Xi(t))},\\ \Phi_{xx} &= (\beta^2\,\phi''+2i\beta\,p\phi'-p^2\,\phi)e^{i(p\,x+ct+\sigma\Xi(t))},\\ \mid\Phi\mid^2\ \Phi &= \phi^3\,e^{i(p\,x+ct+\sigma\Xi(t))}. \end{aligned}\tag{3.2}$$

Upon substituting Eqs (3.2) and (3.1) into Eq (1.2) produces

$$L\,\phi''(\xi)+M\,\phi^3(\xi)+N\,\phi(\xi)=0,\tag{3.3}$$

from real part, where $L=\beta^2, M=-2\eta$ & $N=-(2\alpha\delta+2\,c+p^2)$. On the other hand, $p=v$ from the imaginary part. Using the suggested unified solvers, the solutions for Eq (3.3) are

$$\phi_{1,2}(\xi)=\pm\sqrt{\frac{\beta^2}{\eta}}\,tanh\,(\xi)\,,\eta>0.\tag{3.4}$$

$$\phi_{3,4}(\xi)=\pm\sqrt{\frac{\beta^2}{\eta}}\,coth\,(\xi)\,,\eta>0,\tag{3.5}$$

$c=-\alpha\delta-\beta^2-\frac{1}{2}p^2$.

$$\phi_{5,6}(\xi)=\pm\sqrt{\frac{2\alpha\delta+2\,c+p^2}{-\eta}}sech(\pm\sqrt{\frac{2\alpha\delta+2\,c+p^2}{\beta^2}}\xi),\tag{3.6}$$

$\frac{2\alpha\delta+2\,c+p^2}{\eta}<0,\ 2\alpha\delta+2\,c+p^2>0$.

Thus, the solutions for Eq (1.2) are

$$\Phi_{1,2}(x,t)=\pm\sqrt{\frac{\beta^2}{\eta}}\,tanh\,(\beta(x-v\,t))\,e^{i(p\,x+ct+\sigma\Xi(t))},\eta>0,\tag{3.7}$$

$$\Phi_{3,4}(x,t)=\pm\sqrt{\frac{\beta^2}{\eta}}\,coth\,(\beta(x-v\,t))\,e^{i(p\,x+ct+\sigma\Xi(t))},\eta>0,\tag{3.8}$$

$c=-\alpha\delta-\beta^2-\frac{1}{2}p^2$.

$$\Phi_{5,6}(x,t)=\pm\sqrt{\frac{2\alpha\delta+2\,c+p^2}{-\eta}}sech(\pm\sqrt{\frac{2\alpha\delta+2\,c+p^2}{\beta^2}}(\beta(x-v\,t)))e^{i(p\,x+ct+\sigma\Xi(t))},\tag{3.9}$$

$\frac{2\alpha\delta+2\,c+p^2}{\eta}<0,\ 2\alpha\delta+2\,c+p^2>0$.

4. Results and discussion

We employed unified solver approaches to construct robust stochastic solutions for the stochastic δ-NLSE. The investigation of the stochastic nonlinear Schrödinger equation (SNLSE) with a δ-potential is motivated by its dual significance in physical applications and mathematical complexity. Physically, the NLSE serves as a universal framework for wave propagation in nonlinear media, such as optical fibers and Bose–Einstein condensates, where δ-potentials represent sharply localized defects or impurities. In realistic scenarios, these systems inevitably encounter random perturbations arising from thermal noise, environmental fluctuations, or quantum effects, thereby necessitating a stochastic formulation. Mathematically, the interaction between singular perturbations induced by δ-potentials and stochastic influences raises fundamental challenges concerning well-posedness, stability, and scattering theory. Although deterministic NLSEs with δ-defects and SNLSEs without singularities have been studied independently, their integration remains largely unexplored and promises to reveal new dynamics at the intersection of localization and randomness. Understanding how solitons scatter, localize, or destabilize in the presence of both δ-defects and noise not only advances nonlinear dispersive theory but also provides valuable insights for robust quantum transport, optical communication, and impurity-driven condensate dynamics.

In the stochastic δ-NLSE, solitary waves represent a new class of randomly disturbed nonlinear bound states in which noise-induced delocalization and defect-induced localization compete. In addition to posing challenging analytical and physical problems, their study is essential for understanding how solitons behave in realistic noisy, defect-laden environments. These issues include stability, probabilistic scattering rules, and noise-driven transitions between trapped and radiated states. To illustrate the propagation of solitary waves in the stochastic δ-NLSE, 2D and 3D charts of chosen solutions are created using Matlab. Figures 1 and 2 illustrate the behavior of the developed solution $\Phi_1(x,t)$. The waveform has a periodic structure, but randomness results in uneven peaks, expanded oscillations, and local amplitude disturbances. The width of the wave does not increase significantly. The reliability of optical communication systems is enhanced by these graphical tools, which support, loss reduction, and improve data transmission efficiency. These representations are valuable for examining soliton interactions, as they highlight regions of high and low density, thereby offering insights into modulation, merging, energy exchange, and soliton interaction. Furthermore, the inclusion of a stochastic component in nonlinear wave equations introduces unpredictability that can substantially alter the solitary-wave behavior in hydrodynamic or optical contexts. The delicate balance between nonlinearity and dispersion that sustains solitons is disrupted by stochastic perturbations arising from thermal noise, impurities, or environmental fluctuations, leading to waveform distortion, positional jitter, and eventual extinction. Transmission errors may increase due to soliton pulses broadening caused by amplifier noise and stochastic dispersion variations in optical fibers.

We demonstrate how the dynamic behavior of solutions is influenced by the nonlinear coefficient parameter η. The generated solution varies with x for different η values in both deterministic and stochastic cases as shown in Figures 3 and 4. These graphics illustrate that the amplitude of the periodic solution decreases as γ increases. Moreover, the profile does not exhibit reversals or changes in direction. Figure 5 shows the variations of soliton solution $\Phi_1(x,t)$ with α. Figure 6 shows the variations of soliton solution $\Phi_1(x,t)$ with δ. Figures 5 and 6 indicate that α and δ affects the wave's spatial alignment or phase rather than its amplitude. Finally, Figure 7 presents the variations of the soliton solution $\Phi_5(x, t)$ with respects to c. It was found that the increasing of c raises the amplitudes of the soliton form without introducing a phase shift.

In summary, the results demonstrate the effectiveness of the proposed approach and its ability to generate a variety of isolated solutions for the generalized nonlinear Schrödinger model. In the context of applications of soliton theory, these solutions are particularly advantageous.

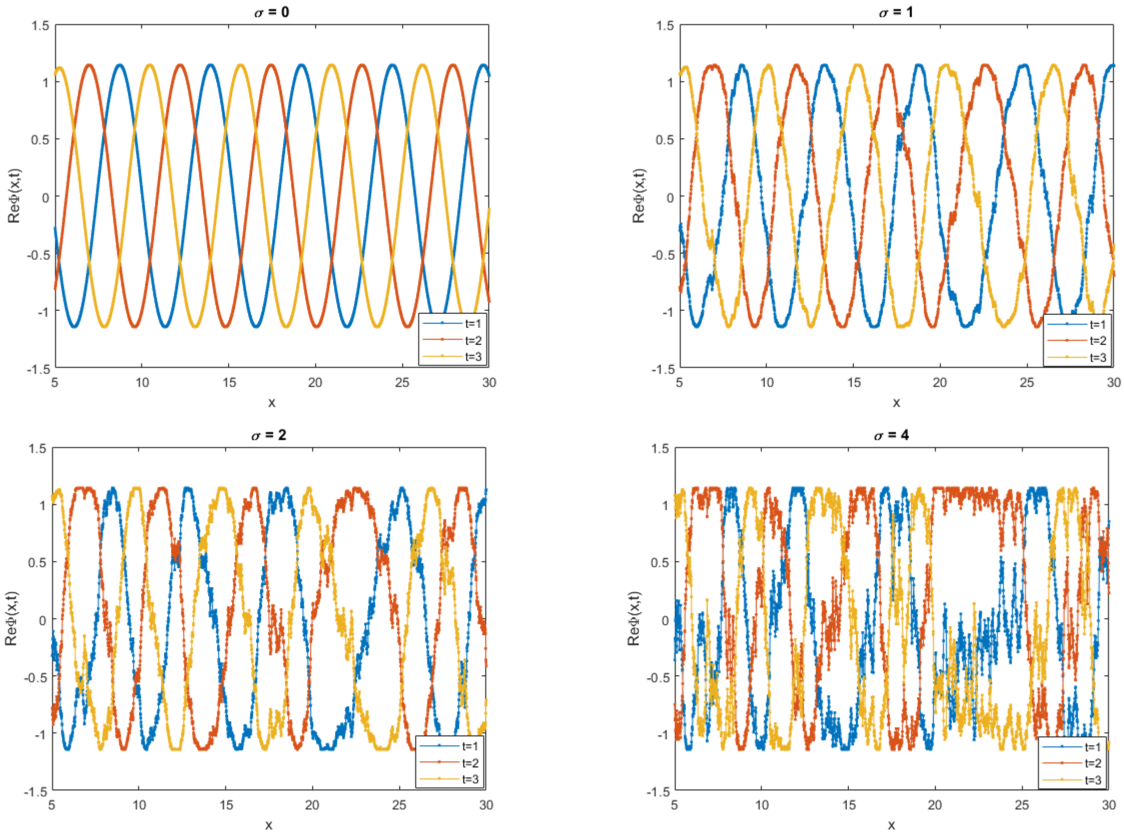

Figure 1. 2D plots of solution $\Phi_1(x, t)$ with different values of σ for $\beta = 1.4$, $\eta = 1.5$ $v = 1.2$, $p = 1.2$, $\alpha = 1.5$ and $\delta = 1$.

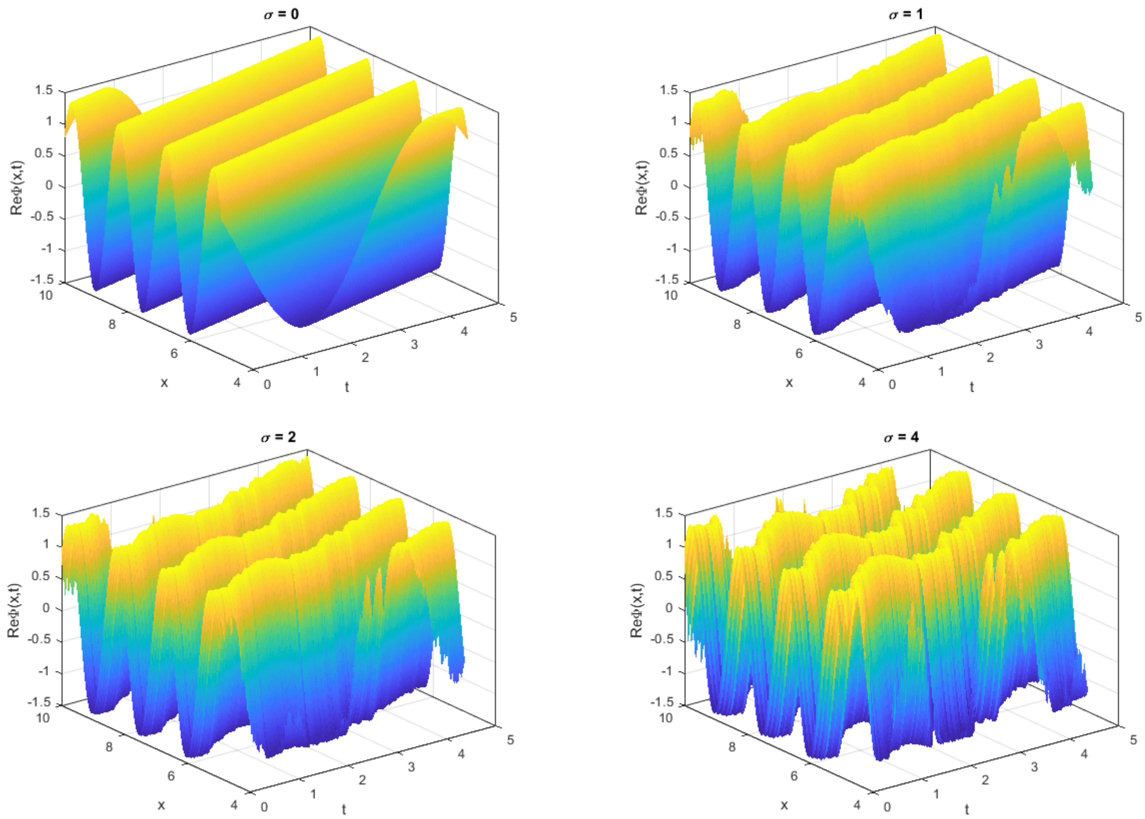

Figure 2. 3D plots of solution $\Phi_1(x, t)$ with different values of σ for $\beta = 1.4$, $\eta = 1.5$ $v = 1.2$, $p = 1.2$, $\alpha = 1.5$ and $\delta = 1$.

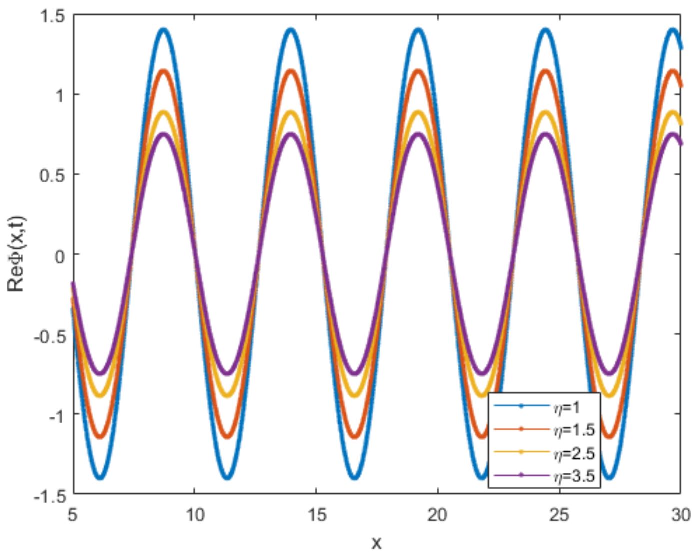

Figure 3. 2D plot of solution $\Phi_1(x, t)$ with different values of η for $\sigma = 0$, $\beta = 1.4$, $v = 1.2$, $p = 1.2$, $\alpha = 1.5$ and $\delta = 1$.

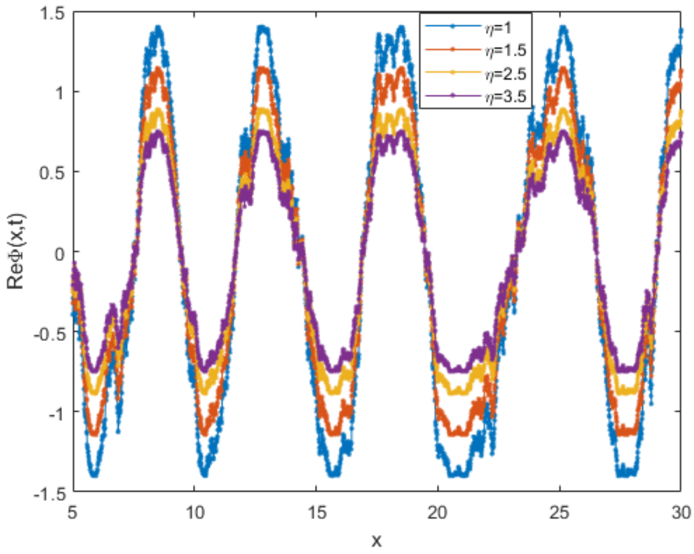

Figure 4. 2D plot of solution $\Phi_1(x,t)$ with different values of η for $\sigma = 2$, $\beta = 1.4$, $v = 1.2$, $p = 1.2$, $\alpha = 1.5$ and $\delta = 1$.

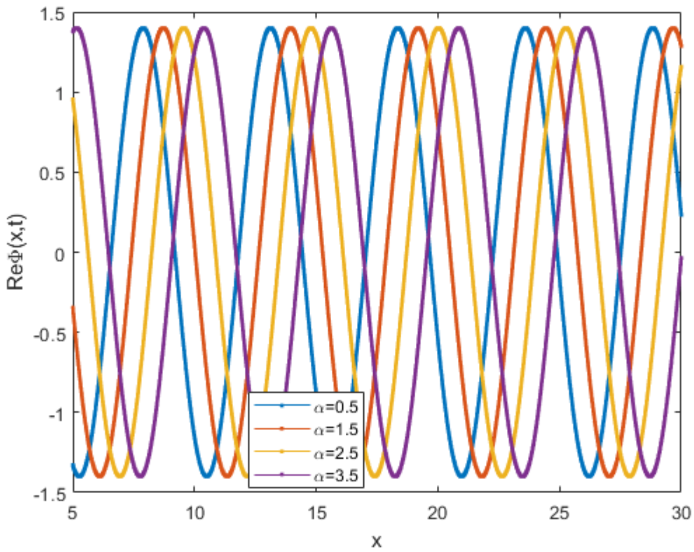

Figure 5. 2D plot of solution $\Phi_1(x,t)$ with different values of α for $\beta = 1.4$, $\eta = 1$, $v = 1.2$, $p = 1.2$ and $\delta = 1$.

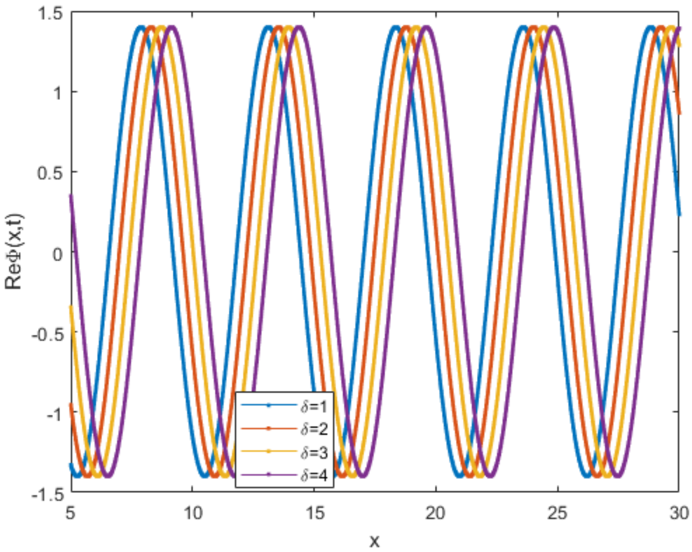

Figure 6. 2D plot of solution $\Phi_1(x,t)$ with different values of δ for $\beta = 1.4$, $\eta = 1$, $v = 1.2$, $p = 1.2$ and $\alpha = 0.5$.

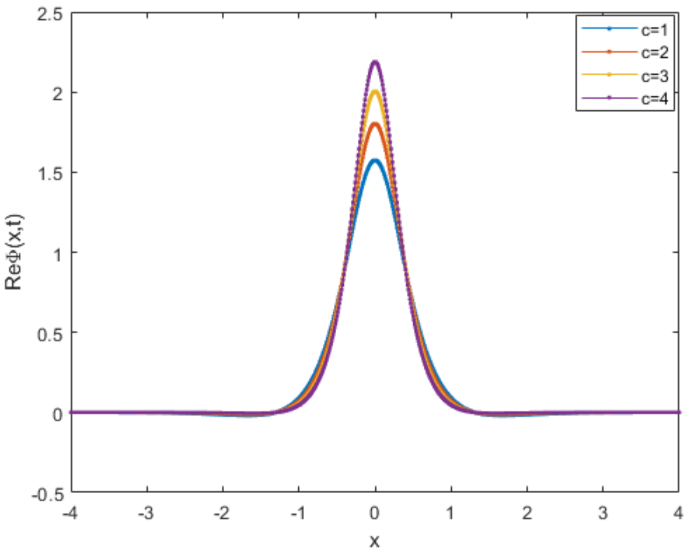

Figure 7. 2D plot of solution $\Phi_5(x,t)$ with different values of c for $\beta = 2.2$, $\eta = -2.6$, $v = 1.2$, $p = 1.2$, $\delta = 1$ and $\alpha = 1.5$.

5. Conclusions

The stochastic δ-NLSE represents a natural and compelling synthesis of nonlinear wave dynamics, singular perturbations, and stochastic forcing. By combining the localized effects of δ-potentials with the randomness introduced by Brownian noise, it captures a broad spectrum of realistic physical scenarios—from impurity-pinned solitons in optical fibers and Bose–Einstein condensates to quantum transport in noisy environments. The study of this equation presents not only mathematical challenges, requiring new strategies for well-posedness, stability, and stochastic scattering theory, but also yields significant physical insights, by clarifying how randomness interacts with localization to shape long-time wave behavior. In this paper, solitary waves provide a rich foundation for uncovering mechanisms of noise-induced stabilization, decoherence, and probabilistic scattering. Enhancing the understanding of the stochastic δ-NLSE is therefore promising both for advancing mathematical theory and for informing experimental design in nonlinear optics, condensed matter, and disordered quantum systems.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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