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*Research article*

# Iterative approaches to Yosida variational inclusion problem involving averaged operator and logical operations

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**Abstract:** This paper investigated a Yosida variational inclusion problem (YVIP) in a real ordered Hilbert space, where logical operations are incorporated through an averaged-operator framework. By reformulating the YVIP and its associated resolvent equation as equivalent fixed-point problems, we designed an iterative scheme that systematically integrates these logical and averaged-operator components. Furthermore, we analyzed the convergence of the proposed algorithms. A comparative study with existing algorithms, supported by numerical experiments, demonstrated the improved computational behavior of the proposed method. To illustrate its practical relevance, a representative MATLAB-based numerical result was also presented.

**Keywords:** algorithms; averaged operator; logical operation; numerical result; resolvent operator; Yosida approximation operator
**Mathematics Subject Classification:** 47H05, 47J25, 49H10

## 1. Introduction

The concept of variational inequalities, first introduced by Hartman and Stampacchia in 1966 [1], has become a cornerstone of mathematical analysis in fields ranging from optimal control and economics to mechanics and data science. This framework was significantly expanded by Rockafellar [2] in 1976 with the introduction of variational inclusions, a more general formulation that accommodates multi-valued mappings. Hassouni and Moudafi [3] later formalized this concept in 1994, establishing variational inclusions as a powerful generalization of variational inequalities [4, 5]. The subsequent decades have seen extensive research and application of this framework to complex problems in areas such as financial modeling, image restoration, transportation systems, and network analysis [6, 7]. A prevalent and powerful technique for solving variational inclusions is to reformulate them as fixed-point problems [8, 9]. This is typically achieved using the resolvent operator

$[I + \lambda\mathcal{M}]^{-1}$, where $\lambda > 0$ is a regularization parameter and $\mathcal{M}$ is a multi-valued operator [10, 11]. This transformation is fundamental because it enables the construction of iterative algorithms to solve a broad spectrum of nonlinear problems [12, 13].

The Yosida operator [14], a known as the Yosida approximation was introduced by Kosaku Yosida. The Yosida approximation is a technique used to approximate solutions to variational inclusion problems, particularly those involving monotone operators. Specifically, Kosaku Yosida introduced the concept of approximating a set-valued maximal monotone operator with a sequence of single-valued and Lipschitz-continuous operators. This approximation is crucial for solving certain types of problems in functional analysis and related fields. Beyond the Yosida approximation, significant contributions encompass the advancement of functional analytic and operator-theoretic methods within differential equations and probability theory. These operators have been extensively used in the study of data science, fluid dynamics, heat transfer processes, and coupled linear acoustic equations.

Baillon, Bruck, and Reich first introduced the concept of the average operator [15] in their study of the asymptotic behavior of nonexpansive mappings and semigroups. This class of operators is pivotal for establishing convergence in iterative algorithms designed to solve fixed-point problems, optimization, and variational inclusions. A key connection exists between averaged operators and the resolvent operator, a cornerstone of monotone operator theory. Dao and Tam [16] further expanded this theory by analyzing the union of averaged operators and demonstrating their applicability to problems involving convex functions [17, 18]. For a complete overview of averaged operators and their diverse applications, see [19, 20]. Parallel to the advancement of variational theories, logical operations, most notably exclusive OR (XOR) and exclusive NOR (XNOR), have played a pivotal role in digital computation [21]. Specifically, the XOR operation returns a true output when the two Boolean inputs differ, while the XNOR operation yields a true output when the inputs coincide. These operations are fundamental in various computational domains, including fault-tolerant system design, parity verification, cryptographic protocols, and pseudorandom number generation [22]. Due to their associative and commutative properties, both XOR and XNOR are particularly advantageous in contexts that require linear separability and in the synthesis of logical circuits [23, 24].

In this work, we explore a Yosida-type variational inclusion that incorporates multi-valued operators together with logical operations, motivated by both their theoretical significance and their applications. We introduce an iterative method derived from an equivalent fixed-point reformulation and establish comprehensive results on its solvability and convergence behavior. The proposed framework is suitable for various physical models, including heat transfer, wave dynamics, and thermal flow processes. To demonstrate the effectiveness of the method, a numerical experiment is carried out in MATLAB R2024b, accompanied by computational tables and convergence plots.

## 2. Elementary tools

Let $\mathcal{E}$ be a real-ordered Hilbert space endowed with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. Let $C \subseteq \mathcal{E}$ be a closed convex cone, denoted by $C(\mathcal{E})$ which is the family of all nonempty compact subsets of $\mathcal{E}$, and by $2^{\mathcal{E}}$, the collection of all nonempty subsets of $C(\mathcal{E})$.

**Definition 2.1.** *[24] Let $\Psi_1, \Psi_2 \in \mathcal{E}$ and $C_{\mathcal{E}}$ be a cone that is denoted by $\lambda\Psi \in C_{\mathcal{E}}$. The cone $C_{\mathcal{E}}$ is normal if and only if there exists a constant $\lambda_{\Pi_{\mathcal{E}}} > 0$ such that*

$$\|\Psi_1\| \leq \lambda_{\Pi_{\mathcal{E}}} \leq \|\Psi_2\|,$$

*where $\lambda > 0$ and $0 \leq \Psi_1 \leq \Psi_2$. This cone $C_{\mathcal{E}}$ induces a partial order relation $\leq$ on $\mathcal{E}$ defined by*

$$\Psi_1 \leq \Psi_2 \Leftrightarrow \Psi_1 - \Psi_2 \in C(\mathcal{E}).$$

*Two elements $\Psi_1$ and $\Psi_2 \in \mathcal{E}$, are said to be comparable, denoted by $\Psi_1 \propto \Psi_2$, if either $\Psi_1 \leq \Psi_2$ or $\Psi_2 \leq \Psi_1$.*

**Definition 2.2.** *[24] A mapping $F : \mathcal{E} \to \mathcal{E}$ is called a $\delta_F$ order non-extended mapping if there exists a constant $\delta_F > 0$ such that*

$$F(\Psi_1) \oplus F(\Psi_2) \geq \delta_F(\Psi_1 \oplus \Psi_2),\ \forall\ \Psi_1, \Psi_2 \in \mathcal{E}.$$

**Definition 2.3.** *A single-valued mapping $F : \mathcal{E} \to \mathcal{E}$ is*

*(i) Lipschitz continuous, if there exists a constant $\lambda_F > 0$ such that*

$$\|F(\Psi_1) - F(\Psi_2)\| \leq \lambda_F \|\Psi_1 - \Psi_2\|,\ \forall\ \Psi_1, \Psi_2 \in \mathcal{E};$$

*(ii) relaxed Lipschitz continuous if there exists a constant $c > 0$ such that*

$$\left\langle F(\Psi_1) - F(\Psi_2), \Psi_1 - \Psi_2 \right\rangle \leq -c\|\Psi_1 - \Psi_2\|^2,\ \forall\ \Psi_1, \Psi_2 \in \mathcal{E}.$$

**Definition 2.4.** *[24, 25] Let $F : \mathcal{E} \to \mathcal{E}$ denote a single-valued mapping and $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ represent a multi-valued mapping. We define the following:*

*(i) The mapping $F$ is termed a strong comparison mapping if it qualifies as a comparison mapping and satisfies the condition*

$$F(\Psi_1) \propto F(\Psi_2) \quad \text{if and only if} \quad \Psi_1 \propto \Psi_2,\ \forall\ \Psi_1, \Psi_2 \in \mathcal{E}.$$

*(ii) The mapping $\mathcal{M}$ is referred to as a comparison mapping provided that for every $\Psi_1 \in \mathcal{E}$, there exists $\vartheta_{\Psi_1} \in \mathcal{M}(\Psi_1)$ such that $\Psi_1 \propto \vartheta_{\Psi_1}$, and for any pair $\Psi_1, \Psi_2 \in \mathcal{E}$ with $\Psi_1 \propto \Psi_2$, it holds that*

$$\vartheta_{\Psi_1} \propto \vartheta_{\Psi_2}, \quad \forall\ \vartheta_{\Psi_1} \in \mathcal{M}(\Psi_1),\ \vartheta_{\Psi_2} \in \mathcal{M}(\Psi_2).$$

*(iii) A comparison mapping $\mathcal{M}$ is said to be an $\alpha$-non-ordinary difference mapping if for any $\Psi_1, \Psi_2 \in \mathcal{E}$, and for all $\vartheta_{\Psi_1} \in \mathcal{M}(\Psi_1)$, $\vartheta_{\Psi_2} \in \mathcal{M}(\Psi_2)$, the following identity is satisfied:*

$$(\vartheta_{\Psi_1} \oplus \vartheta_{\Psi_2}) \oplus \alpha_F(\Psi_1 \oplus \Psi_2) = 0.$$

*(iv) The operator $\mathcal{M}$ is a $\lambda$-ordered rectangular comparison mapping if there exists a constant $\lambda > 0$ such that for all $\Psi_1, \Psi_2 \in \mathcal{E}$, there exist elements $\vartheta_{\Psi_1} \in \mathcal{M}(\Psi_1)$ and $\vartheta_{\Psi_2} \in \mathcal{M}(\Psi_2)$ fulfilling*

$$\left\langle \vartheta_{\Psi_1} \odot \vartheta_{\Psi_2} - (\Psi_1 \oplus \Psi_2) \right\rangle \geq \lambda \|\Psi_1 \oplus \Psi_2\|^2.$$

**Definition 2.5.** *Let $\xi : \mathcal{E} \to \mathcal{E}$ be called the $\gamma$-averaged operator for some $\gamma \in (0, 1)$ if it can be written as*

$$\xi = (1 - \gamma)I + \gamma H,$$

*where $I$ denotes the identity operator on $\mathcal{E}$, and $H : \mathcal{E} \to \mathcal{E}$ is a non-expansive operator.*

**Definition 2.6.** *[25] Let $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ be a multi-valued mapping. The resolvent operator associated with $\mathcal{M}$, denoted by $\nabla^{\mathcal{M}}_{I,\lambda} : \mathcal{E} \to \mathcal{E}$ is defined for all $\Psi_1, \Psi_2 \in \mathcal{E}$ and $\lambda > 0$ by*

$$\nabla^{\mathcal{M}}_{I,\lambda}(\Psi_2) = [I + \lambda\mathcal{M}]^{-1}(\Psi_2), \forall\, \Psi_1, \Psi_2 \in \mathcal{E},$$

*where $I$ denotes the identity operator on $\mathcal{M}$. For any constant $\lambda > 0$*

**Definition 2.7.** *Let $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ be a multi-valued mapping. The Yosida approximation operator $\mathcal{G}^{\mathcal{M}}_{I,\lambda} : \mathcal{E} \to \mathcal{E}$ is defined by*

$$\mathcal{G}^{\mathcal{M}}_{I,\lambda}(\Psi_2) = \frac{1}{\lambda}\Big[I - \nabla^{\mathcal{M}}_{I,\lambda}\Big](\Psi_2), \forall\, \Psi_1, \Psi_2 \in \mathcal{E}, \lambda > 0.$$

**Proposition 2.1.** *[24, 25] Let $\oplus$ and $\odot$ be the XOR operation and XNOR operation, respectively. The operators $\vee$ and $\wedge$ are defined to represent the least upper bound (lub) and the greatest lower bound (glb), respectively. Let $\Psi_1, \Psi_2 \in \mathcal{E}$. Then, the following conditions hold:*

*(i)* $\Psi_1 \vee \Psi_2 = \inf\{\Psi_1, \Psi_2\}$;
*(ii)* $\Psi_1 \wedge \Psi_2 = \sup\{\Psi_1, \Psi_2\}$;
*(iii)* $\Psi_1 \oplus \Psi_2 = (\Psi_1 - \Psi_2) \vee (\Psi_2 - \Psi_1)$;
*(iv)* $\Psi_1 \odot \Psi_2 = (\Psi_1 - \Psi_2) \wedge (\Psi_2 - \Psi_1)$;
*(v)* $\Psi_1 \odot \Psi_2 = 0, (\Psi_1 \odot \Psi_2) = (\Psi_2 \odot \Psi_1), (\Psi_1 \oplus \Psi_2) = 0, (\Psi_1 \oplus \Psi_2) = (\Psi_2 \oplus \Psi_1), (\Psi_1 \odot \Psi_2) = -(\Psi_1 \oplus \Psi_2)$;
*(vi)* *if* $\Psi_1 \propto 0$ *then* $-\Psi_1 \oplus 0 \leq \Psi_1 \leq \Psi_1 \oplus 0$;
*(vii)* $(\lambda\Psi_1) \oplus (\lambda\Psi_2) = |\lambda|(\Psi_1 \oplus \Psi_2)$;
*(viii)* $0 \leq \Psi_1 \oplus \Psi_2$, *if* $\Psi_1 \propto \Psi_2$;
*(ix)* *if* $\Psi_1 \propto \Psi_2$ *then* $\Psi_1 \oplus \Psi_2 = 0$ *if and only if* $\Psi_1 = \Psi_2$.

**Proposition 2.2.** *[24, 25] Let $C_{\mathcal{E}}$ be a normal cone in $\mathcal{E}$ with normal constant $\lambda_{\Pi_{\mathcal{E}}} > 0$. Then, for every $\Psi_1, \Psi_2 \in \mathcal{E}$, the following postulates hold:*

*(i)* $\|0 \oplus 0\| = \|0\| = 0$;
*(ii)* $\|\Psi_1 \vee \Psi_2\| = \|\Psi_1\| \vee \|\Psi_2\| \leq \|\Psi_1\| + \|\Psi_2\|$;
*(iii)* $\|\Psi_1 \oplus \Psi_2\| \leq \|\Psi_1 - \Psi_2\| \leq \lambda_{\Pi}\|\Psi_1 \oplus \Psi_2\|$;
*(iv)* *if* $\Psi_1 \propto \Psi_2$, *then* $\|\Psi_1 \oplus \Psi_2\| = \|\Psi_1 - \Psi_2\|$.

**Lemma 2.1.** *[25] Let $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ be a maximal monotone operator. Then, the associated resolvent operator $\nabla^{\mathcal{M}}_{I,\lambda} : \mathcal{E} \to \mathcal{E}$ is Lipschitz continuous with Lipschitz constant $\nabla_\theta > 0$, where $\nabla_\theta = \frac{1}{\rho}$ for some $\rho > 0$. Specifically, the following inequality holds:*

$$\left\|\nabla^{\mathcal{M}}_{I,\lambda}(\Psi_1) \oplus \nabla^{\mathcal{M}}_{I,\lambda}(\Psi_2)\right\| \leq \nabla_\theta \|\Psi_1 \oplus \Psi_2\|, \ \forall\, \Psi_1, \Psi_2 \in \mathcal{E}.$$

*Hence, the resolvent operator $\nabla^{\mathcal{M}}_{I,\lambda}$ exhibits Lipschitz-type continuity.*

**Proposition 2.3.** *Let $\xi : \mathcal{E} \to \mathcal{E}$ be an average operator. Suppose that $H : \mathcal{E} \to \mathcal{E}$ is relaxed Lipschitz continuous with a constant $C > 0$. Then, the operator $\xi$ is also relaxed Lipschitz continuous with a constant $\theta_C > 0$, where*

$$\theta_C = \gamma(1 + C) - 1.$$

*Proof.* Since $H$ is relaxed ordered Lipschitz continuous, this implies that

$$\begin{aligned}
\Big\langle \xi(\Psi_1) \oplus \xi(\Psi_2), \Psi_1 \oplus \Psi_2 \Big\rangle &= \Big\langle \Big((1-\gamma)\Psi_1 + \gamma H(\Psi_1)\Big) \oplus \Big((1-\gamma)\Psi_2 + \gamma H(\Psi_2)\Big), \Psi_1 \oplus \Psi_2 \Big\rangle \\
&\leq \Big\langle (1-\gamma)(\Psi_1 \oplus \Psi_2) + \gamma(H(\Psi_1) \oplus H(\Psi_2)), \Psi_1 \oplus \Psi_2 \Big\rangle \\
&= (1-\gamma)\Big\langle \Psi_1 \oplus \Psi_2, \Psi_1 \oplus \Psi_2 \Big\rangle + \gamma\Big\langle H(\Psi_1) \oplus H(\Psi_2), \Psi_1 \oplus \Psi_2 \Big\rangle \\
&\leq (1-\gamma)||\Psi_1 \oplus \Psi_2||^2 - \gamma C||\Psi_1 \oplus \Psi_2||^2 \\
&\leq (1-\gamma(1+C))||\Psi_1 \oplus \Psi_2||^2 \\
&\leq -(\gamma(1+C)-1)||\Psi_1 \oplus \Psi_2||^2 \\
&\leq -\theta_C||\Psi_1 \oplus \Psi_2||^2.
\end{aligned}$$

Thus, the averaged operator $\xi$ is $\theta_C$-relaxed ordered Lipschitz continuous where $0 \leq \frac{1}{1+C} \leq \gamma < 1$. □

**Lemma 2.2.** *Let $\xi : \mathcal{E} \to \mathcal{E}$ be an averaged operator. Suppose that $H : \mathcal{E} \to \mathcal{E}$ is relaxed Lipschitz continuous with a constant $C > 0$. Then, the operator $\xi$ is $\xi_\theta$-relaxed Lipschitz continuous in the sense that*

$$||\xi(\Psi_1) - \xi(\Psi_2)|| \leq \xi_\theta||\Psi_1 - \Psi_2||, \quad \forall\, \Psi_1, \Psi_2 \in \mathcal{E},$$

*where*

$$\xi_\theta = \sqrt{2\gamma(1+\theta_C) - 2\theta_C - 1}, \quad \textit{and} \quad \theta_C = \gamma(1+C) - 1.$$

*Proof.* Since an averaged operator $\xi$ is $\theta_C$- relaxed Lipschitz continuous, it holds that

$$\begin{aligned}
||\xi(\Psi_1) - \xi(\Psi_2)||^2 &\leq ||\Psi_1 - \Psi_2||^2 - \left[\frac{1-\gamma}{\gamma}\right]||(I-\xi)(\Psi_1) - (I-\xi)(\Psi_2)||^2 \\
&\leq ||\Psi_1 - \Psi_2||^2 - \left[\frac{1-\gamma}{\gamma}\right]\Big\{||\Psi_1 - \Psi_2||^2 - 2\langle \Psi_1 - \Psi_2, \xi(\Psi_1) - \xi(\Psi_2)\rangle \\
&\quad + ||\xi(\Psi_1) - \xi(\Psi_2)||^2\Big\} \\
&\leq ||\Psi_1 - \Psi_2||^2 - \left[\frac{1-\gamma}{\gamma}\right]\Big\{||\Psi_1 - \Psi_2||^2 + 2\theta_C||\Psi_1 - \Psi_2||^2 + ||\xi(\Psi_1) - \xi(\Psi_2)||^2\Big\} \\
&\leq ||\Psi_1 - \Psi_2||^2 - (1+2\theta_C)\left[\frac{1-\gamma}{\gamma}\right]||\Psi_1 - \Psi_2||^2 - \left[\frac{1-\gamma}{\gamma}\right]||\xi(\Psi_1) - \xi(\Psi_2)||^2 \\
&\leq \gamma\Big\{1 - \frac{(1-\gamma)(1+2\theta_C)}{\gamma}\Big\}||\Psi_1 - \Psi_2||^2 \\
||\xi(\Psi_1) - \xi(\Psi_2)|| &\leq \sqrt{2\gamma(1+\theta_C) - 2\theta_C - 1}||\Psi_1 - \Psi_2|| \\
&= \xi_\theta||\Psi_1 - \Psi_2||,
\end{aligned}$$

where

$$\xi_\theta = \sqrt{2\gamma(1+\theta_C) - 2\theta_C - 1}.$$

□

**Proposition 2.4.** *Let $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ be a maximal monotone operator, $\nabla^{\mathcal{M}}_{I,\lambda} : \mathcal{E} \to \mathcal{E}$ be the resolvent operator, and the Yosida approximation operator $\mathcal{G}^{\mathcal{M}}_{I,\lambda}$ be $\lambda_{\mathcal{G}}$- Lipschitz continuous, which provided $\Psi_1 \propto \Psi_2, F(\Psi_1) \propto F(\Psi_2)$, $\nabla^{\mathcal{M}}_{I,\lambda}(\Psi_1) \propto \nabla^{\mathcal{M}}_{I,\lambda}(\Psi_2)$ and $\mathcal{G}^{\mathcal{M}}_{I,\lambda}(\Psi_1) \propto \mathcal{G}^{\mathcal{M}}_{I,\lambda}(\Psi_2)$ such that*

$$\left\|\mathcal{G}^{\mathcal{M}}_{I,\lambda}(\Psi_1) \oplus \mathcal{G}^{\mathcal{M}}_{I,\lambda}(\Psi_2)\right\| \leq \lambda_{\mathcal{G}}\|\Psi_1 \oplus \Psi_2\|, \forall\, \Psi_1, \Psi_2 \in \mathcal{E},$$

*where*

$$\lambda_{\mathcal{G}} = \frac{(1 + \nabla_\theta)}{\lambda}.$$

*Proof.* Using the Lipschitz continuity of $\nabla^{\mathcal{M}}_{I,\lambda}$, we evaluate

$$\begin{aligned}
\left\|\mathcal{G}^{\mathcal{M}}_{I,\lambda}(\Psi_1) \oplus \mathcal{G}^{\mathcal{M}}_{I,\lambda}(\Psi_2)\right\| &= \left\|\frac{1}{\lambda}\Big[I - \nabla^{\mathcal{M}}_{I,\lambda}\Big](\Psi_1) \oplus \frac{1}{\lambda}\Big[I - \nabla^{\mathcal{M}}_{I,\lambda}\Big](\Psi_2)\right\| \\
&= \frac{1}{\lambda}\left\|\Big[I - \nabla^{\mathcal{M}}_{I,\lambda}\Big](\Psi_1) \oplus \Big[I - \nabla^{\mathcal{M}}_{I,\lambda}\Big](\Psi_2)\right\| \\
&\leq \frac{1}{\lambda}\|\Psi_1 \oplus \Psi_2\| + \frac{1}{\lambda}\left\|\nabla^{\mathcal{M}}_{I,\lambda}(\Psi_1) \oplus \nabla^{\mathcal{M}}_{I,\lambda}(\Psi_2)\right\| \\
&\leq \frac{1}{\lambda}\|\Psi_1 \oplus \Psi_2\| + \frac{\nabla_\theta}{\lambda}\|\Psi_1 \oplus \Psi_2\| \\
&\leq \frac{(1 + \nabla_\theta)}{\lambda}\|\Psi_1 \oplus \Psi_2\| \\
&\leq \lambda_{\mathcal{G}}\|\Psi_1 \oplus \Psi_2\|,
\end{aligned}$$

where

$$\lambda_{\mathcal{G}} = \frac{(1 + \nabla_\theta)}{\lambda}.$$

□

## 3. Fixed-point method and main results for Yosida inclusions with logical operators

Let $f, F : \mathcal{E} \to \mathcal{E}$ be single-valued mappings and $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ be a maximal monotone multi-valued mapping. Let $H : \mathcal{E} \to \mathcal{E}$ be a nonexpansive, $C$-relaxed Lipschitz continuous single-valued mapping. Assume that $\xi : \mathcal{E} \to \mathcal{E}$ is a $\gamma$-average operator that is also $\theta_C$-relaxed Lipschitz continuous. Let $\mathcal{G}^{\mathcal{M}}_{I,\lambda}$ denote the Yosida approximation operator corresponding to $\mathcal{M}$ for any $\lambda > 0$. The problem is to find $\Psi \in \mathcal{E}$ such that

$$0 \in \mathcal{G}^{\mathcal{M}}_{I,\lambda}(F(\Psi)) + \nabla^{\mathcal{M}}_{I,\lambda}(\xi(\Psi)) \oplus \mathcal{M}(f(\Psi)). \tag{3.1}$$

If we have $\mathcal{G}^{\mathcal{M}}_{I,\lambda}(F(\Psi)) = 0$, $\nabla^{\mathcal{M}}_{I,\lambda}(\xi(\Psi)) = 0$, and $\mathcal{M}(f(\Psi)) = \mathcal{M}(\Psi)$, then problem (3.1) reduces to the classical form:

$$0 \in \mathcal{M}(\Psi).$$

This is a fundamental problem involving the Yosida operator, the averaged operator, and the logical operation studied by many researchers in the literature [25].

Now, we establish the equivalence between the problem (3.1) and the corresponding fixed-point formulation.

**Lemma 3.1.** *Let $\Psi \in \mathcal{E}$ be the solution of the Yosida variational inclusion problem (3.1), including an averaged operator and logical operations if and only if the following equation holds:*

$$f(\Psi) = \nabla_{I,\lambda}^{\mathcal{M}}\left[f(\Psi) + \lambda\left\{\nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi)\Big)\right\}\right]. \tag{3.2}$$

*Proof.* Suppose $\Psi \in \mathcal{E}$ satisfies Eq (3.2). Then, we have

$$\begin{aligned}
f(\Psi) &= \nabla_{I,\lambda}^{\mathcal{M}}\left[f(\Psi) + \lambda\left\{\nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi)\Big)\right\}\right],\\
&= (I + \lambda\mathcal{M})^{-1}\left[f(\Psi) + \lambda\left\{\nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi)\Big)\right\}\right],\\
(I + \lambda\mathcal{M})f(\Psi) &= f(\Psi) + \lambda\left\{\nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi)\Big)\right\},\\
\mathcal{M}(f(\Psi)) &= \nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi)\Big),\\
\nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \odot \mathcal{M}(f(\Psi)) &= \nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \odot \nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi)\Big),\\
&= \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi)\Big),\\
0 &\in \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi)\Big) + \nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi)\Big) \oplus \mathcal{M}(f(\Psi)).
\end{aligned}$$

This represents the required YVIP involving logical operations and the averaged operator.

We now formulate the corresponding iterative algorithm, based on Lemma 3.1, to solve the problem involving logical operations and the averaged operator. □

**Algorithm 3.1.** *For every $\Psi_0 \in \mathcal{E}$, enumerate the sequence $\{\Psi_n\}$ by taking after the iterative algorithm*

$$f(\Psi_{n+1}) = (1-\alpha)f(\Psi_n) + \alpha\nabla_{I,\lambda}^{\mathcal{M}}\left[f(\Psi_n) + \lambda\left\{\nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi_n)\Big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi_n)\Big)\right\}\right], \tag{3.3}$$

*where $0 \leq \alpha \leq 1, \gamma \in (0,1)$ and $\lambda > 0$ are constants and $n = 0, 1, 2, 3, \cdots$*

We develop an existence and convergence result for the YVIP using an averaged-operator framework enhanced with logical operations.

**Theorem 3.1.** *Let $\mathcal{E}$ be a real ordered Hilbert space, and let $C_{\mathcal{E}} \subset \mathcal{E}$ be a normal cone. Consider the mappings $f, F : \mathcal{E} \to \mathcal{E}$, both Lipschitz continuous with respective constants $\lambda_f > 0$ and $\lambda_F > 0$. Let $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ be a multi-valued mapping. Define $\mathcal{G}_{I,\lambda}^{\mathcal{M}}$ as the Yosida approximation operator, assumed to be Lipschitz continuous with constant $\lambda_{\mathcal{G}}$, and $\nabla_{I,\lambda}^{\mathcal{M}}$ as the associated resolvent operator, which is $\nabla_\theta$-Lipschitz continuous. Suppose that for all $\Psi_1, \Psi_2 \in \mathcal{E}$, the following relations hold:*

$$\Psi_{n+1} \propto \Psi_n, \nabla_{I,\lambda}^{\mathcal{M}}(\Psi_{n+1}) \propto \nabla_{I,\lambda}^{\mathcal{M}}(\Psi_n), \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi_{n+1})\Big) \propto \mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi_n)\Big), f(\Psi_{n+1}) \propto f(\Psi_n), F(\Psi_{n+1}) \propto F(\Psi_n),$$

*where $\lambda > 0$ is a constant. Assume further that the following condition is satisfied:*

$$0 < \frac{\lambda_{\Pi_{\mathcal{E}}}^2}{\delta_f}\left\{(1-\alpha)\lambda_f + \alpha\nabla_\theta\lambda_f + \alpha\lambda\nabla_\theta\lambda_{\mathcal{G}}\lambda_F + \alpha\lambda\xi_\theta\nabla_\theta^2\right\} < 1, \tag{3.4}$$

*where $C > 0$, $0 \leq \alpha \leq 1$, $\gamma \in (0,1)$, $n = 0, 1, 2, \ldots$, and the constants are given by*

$$\xi_\theta = \sqrt{2\gamma(1+\theta_C) - 2\theta_C - 1}, \quad \theta_C = \gamma(1+C) - 1, \quad \lambda_{\mathcal{G}} = \frac{1+\nabla_\theta}{\lambda}.$$

*Then, the sequence $\{\Psi_n\}$ generated by Algorithm 3.1 converges strongly to a solution $\Psi$ of the YVIP involving the averaged operator and logical operations.*

*Proof.* We have

$$\begin{aligned}
0 &\le f(\Psi_{n+1}) \oplus f(\Psi_n)\\
&= \Big\{(1-\alpha) f(\Psi_n) + \alpha \nabla^{\mathcal{M}}_{I,\lambda}\Big[f(\Psi_n) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)\Big\}\Big]\Big\}\\
&\quad \oplus \Big\{(1-\alpha) f(\Psi_{n-1}) + \alpha \nabla^{\mathcal{M}}_{I,\lambda}\Big[f(\Psi_{n-1}) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\}\Big]\Big\}\\
&\le (1-\alpha)\Big(f(\Psi_n) \oplus f(\Psi_{n-1})\Big) + \alpha\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big[f(\Psi_n) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big)\\
&\quad \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)\Big\}\Big] \oplus \nabla^{\mathcal{M}}_{I,\lambda}\Big[f(\Psi_{n-1}) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\}\Big]\Big\}. \qquad (3.5)
\end{aligned}$$

Using the definition of normal cone and (3.5), we obtain

$$\begin{aligned}
\|f(\Psi_{n+1}) \oplus f(\Psi_n)\| &\le (1-\alpha)\lambda_{\Pi_\mathcal{E}}\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \alpha\lambda_{\Pi_\mathcal{E}}\Big\|\nabla^{\mathcal{M}}_{I,\lambda}\Big[f(\Psi_n) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big)\\
&\quad \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)\Big\}\Big] \oplus \nabla^{\mathcal{M}}_{I,\lambda}\Big[f(\Psi_{n-1}) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\}\Big]\Big\|\\
&\le \lambda_{\Pi_\mathcal{E}}\Big\{(1-\alpha)\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \alpha\nabla_\theta\Big\|\Big[f(\Psi_n) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big)\\
&\quad \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)\Big\}\Big] \oplus \Big[f(\Psi_{n-1}) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\}\Big]\Big\|\Big\}\\
&\le \lambda_{\Pi_\mathcal{E}}\Big\{(1-\alpha)\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \alpha\nabla_\theta\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \alpha\lambda\nabla_\theta\|\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big)\\
&\quad \oplus \nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big)\| + \alpha\lambda\nabla_\theta\Big\|\mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big) \oplus \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\|\Big\}\\
&\le \lambda_{\Pi_\mathcal{E}}\Big\{(1-\alpha)\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \alpha\nabla_\theta\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \alpha\lambda\nabla^2_\theta\|\xi(\Psi_n)\\
&\quad \oplus \xi(\Psi_{n-1})\| + \alpha\lambda\nabla_\theta\lambda_\mathcal{G}\Big\|F(\Psi_n) \oplus F(\Psi_{n-1})\Big\|\Big\}. \qquad (3.6)
\end{aligned}$$

Since $f$ is a $\delta_f$ order non-extended mapping, we have

$$f(\Psi_{n+1}) \oplus f(\Psi_n) \ge \delta_f(\Psi_{n+1} \oplus \Psi_n),$$

which implies that

$$(\Psi_{n+1} \oplus \Psi_n) \le \frac{1}{\delta_f}(f(\Psi_{n+1}) \oplus f(\Psi_n)).$$

Using the definition of the normal cone, we have

$$\|\Psi_{n+1} \oplus \Psi_n\| \le \frac{\lambda_{\Pi_\mathcal{E}}}{\delta_f}\|f(\Psi_{n+1}) \oplus f(\Psi_n)\|. \qquad (3.7)$$

Combining (3.6) and (3.7), we obtain

$$\begin{aligned}
\|\Psi_{n+1} \oplus \Psi_n\| &\le \frac{\lambda^2_{\Pi_\mathcal{E}}}{\delta_f}\Big\{(1-\alpha)\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \alpha\nabla_\theta\|f(\Psi_n) \oplus f(\Psi_{n-1})\|\\
&\quad + \alpha\lambda\nabla^2_\theta\|\xi(\Psi_n) \oplus \xi(\Psi_{n-1})\| + \alpha\lambda\nabla_\theta\lambda_\mathcal{G}\Big\|F(\Psi_n) \oplus F(\Psi_{n-1})\Big\|\Big\}. \qquad (3.8)
\end{aligned}$$

Since $\Psi_{n+1} \propto \Psi_n$, $\nabla^{\mathcal{M}}_{I,\lambda}(\Psi_{n+1}) \propto \nabla^{\mathcal{M}}_{I,\lambda}(\Psi_n)$, $\mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n+1})\Big) \propto \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)$, $f(\Psi_{n+1}) \propto f(\Psi_n)$, $F(\Psi_{n+1}) \propto F(\Psi_n)$ and also using (iv) of Proposition 2.2 in (3.8), we have

$$\|\Psi_{n+1} - \Psi_n\| = \|\Psi_{n+1} \oplus \Psi_n\|$$

$$\leq \frac{\lambda_{\Pi_{\mathcal{E}}}^2}{\delta_f}\Big\{(1-\alpha)\lambda_f\|\Psi_n - \Psi_{n-1}\| + \alpha\nabla_\theta\lambda_f\|\Psi_n - \Psi_{n-1}\| + \alpha\lambda\nabla_\theta\lambda_{\mathcal{G}}\lambda_F\|\Psi_n - \Psi_{n-1}\| + \alpha\lambda\nabla_\theta^2\|\xi(\Psi_n) - \xi(\Psi_{n-1})\|\Big\},$$

which implies that

$$\begin{aligned}\|\Psi_{n+1} - \Psi_n\| &\leq \frac{\lambda_{\Pi_{\mathcal{E}}}^2}{\delta_f}\Big\{(1-\alpha)\lambda_f + \alpha\nabla_\theta\lambda_f + \alpha\lambda\nabla_\theta\lambda_{\mathcal{G}}\lambda_F + \alpha\lambda\xi_\theta\nabla_\theta^2\Big\}\|\Psi_n - \Psi_{n-1}\| \\ &= \Omega(\theta)\|\Psi_n - \Psi_{n-1}\|. \end{aligned} \tag{3.9}$$

where

$$\Omega(\theta) = \frac{\lambda_{\Pi_{\mathcal{E}}}^2}{\delta_f}\Big\{(1-\alpha)\lambda_f + \alpha\nabla_\theta\lambda_f + \alpha\lambda\nabla_\theta\lambda_{\mathcal{G}}\lambda_F + \alpha\lambda\xi_\theta\nabla_\theta^2\Big\}.$$

From condition (3.4), it follows that $\Omega(\theta) < 1$, where

$$\Omega(\theta) = \frac{\lambda_{\Pi_{\mathcal{E}}}^2}{\delta_f}\Big\{(1-\alpha)\lambda_f + \alpha\nabla_\theta\lambda_f + \alpha\lambda\nabla_\theta\lambda_{\mathcal{G}}\lambda_F + \alpha\lambda\xi_\theta\nabla_\theta^2\Big\}.$$

As a consequence, relation (3.9) ensures that the sequence $\{\Psi_n\}$ is Cauchy in $\mathcal{E}$. Therefore, there exists an element $\Psi \in \mathcal{E}$ such that

$$\Psi_n \to \Psi \text{ as } n \to \infty.$$

Now we apply the continuity of $f, F, \xi, \nabla_{I,\lambda}^{\mathcal{M}}$, and $\mathcal{G}_{I,\lambda}^{\mathcal{M}}$, which implies that

$$f(\Psi) = \nabla_{I,\lambda}^{\mathcal{M}}\Big[f(\Psi) + \lambda\Big\{\nabla_{I,\lambda}^{\mathcal{M}}\big(\xi(\Psi)\big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\big(F(\Psi)\big)\Big\}\Big].$$

Using Lemma 3.1, $\Psi \in \mathcal{E}$ is the solution of YVIP involving the averaged operator and logical operations. □

## 4. Yosida resolvent equation involving logical operation

Within the framework of the YVIP, which involves an averaged operator and logical operations, we introduce the following associated resolvent equation. Find $\Psi, S \in \mathcal{E}$ such that

$$\mathcal{G}_{I,\lambda}^{\mathcal{M}}\big(F(\Psi)\big) \odot \nabla_{I,\lambda}^{\mathcal{M}}\big(\xi(\Psi)\big) - \lambda^{-1} Q_{I,\lambda}^{\mathcal{M}}(S) = 0, \tag{4.1}$$

where, $Q_{I,\lambda}^{\mathcal{M}}(S) = \big[I - \nabla_{I,\lambda}^{\mathcal{M}}\big](S)$, and $I$ is an identity mapping and a constant.

**Lemma 4.1.** *The Yosida variational inclusion problem (YVIP) admits a solution $\Psi \in \mathcal{E}$ if and only if the corresponding Yosida resolvent equation problem (YREP) possesses solutions $\Psi, S \in \mathcal{E}$ satisfying*

$$f(\Psi) = \nabla_{I,\lambda}^{\mathcal{M}}(S), \tag{4.2}$$

$$\textit{where} \quad S = f(\Psi) + \lambda\left\{\nabla_{I,\lambda}^{\mathcal{M}}\big(\xi(\Psi)\big) \odot \mathcal{G}_{I,\lambda}^{\mathcal{M}}\big(F(\Psi)\big)\right\}. \tag{4.3}$$

*Proof.* Let $\Psi \in \mathcal{E}$ be a solution of YVIP involving an averaged operator and logical operations. Then, by definition, the following equation must hold:

$$f(\Psi) = \nabla^{\mathcal{M}}_{I,\lambda}\left[f(\Psi) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\}\right]. \tag{4.4}$$

Define

$$f(\Psi) = \nabla^{\mathcal{M}}_{I,\lambda}(S), \tag{4.5}$$

$$\text{where} \quad S = f(\Psi) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\}. \tag{4.6}$$

This establishes the equivalence between the solutions of the YVIP and the YREP. We obtain from (4.5) and (4.6)

$$\begin{aligned}
S &= \nabla^{\mathcal{M}}_{I,\lambda}(S) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
S - \nabla^{\mathcal{M}}_{I,\lambda}(S) &= \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
(I - \nabla^{\mathcal{M}}_{I,\lambda})(S) &= \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
\mathcal{Q}^{\mathcal{M}}_{I,\lambda}(S) &= \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
\mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big) &\odot \nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) - \lambda^{-1}\mathcal{Q}^{\mathcal{M}}_{I,\lambda}(S) = 0.
\end{aligned}$$

Conversely, let $\Psi, S \in \mathcal{E}$ be the solutions of YREP. Then, we have

$$\begin{aligned}
\mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big) &\odot \nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) - \lambda^{-1}\mathcal{Q}^{\mathcal{M}}_{I,\lambda}(S) = 0,\\
\mathcal{Q}^{\mathcal{M}}_{I,\lambda}(S) &= \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
(I - \nabla^{\mathcal{M}}_{I,\lambda})(S) &= \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
S - \nabla^{\mathcal{M}}_{I,\lambda}(S) &= \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
S &= \nabla^{\mathcal{M}}_{I,\lambda}(S) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
f(\Psi) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\} &= \nabla^{\mathcal{M}}_{I,\lambda}\Big[f(\Psi) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\}\Big],\\
&\quad +\lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\},\\
f(\Psi) &= \nabla^{\mathcal{M}}_{I,\lambda}\Big[f(\Psi) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi)\big)\right\}\Big].
\end{aligned}$$

Which is the required YVIP, including logical operations and an averaged operator. In addition, the solution $\Psi$ satisfies Lemma 3.1.

We now proceed to develop the following algorithm based on Lemma 4.1. □

**Algorithm 4.1.** *For every $\Psi_0, S_0 \in \mathcal{E}$. Compute the sequence $\{\Psi_n\}, \{S_n\}$ step by step*

$$f(\Psi_n) = \nabla^{\mathcal{M}}_{I,\lambda}(S_n) \tag{4.7}$$

$$\textit{where}, S_{n+1} = f(\Psi_n) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\big(\xi(\Psi_n)\big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\big(F(\Psi_n)\big)\right\}, \tag{4.8}$$

*where* $n = 0, 1, 2, \cdots$

**Theorem 4.1.** *Let $\mathcal{E}$ be a real ordered Hilbert space, and let $C_{\mathcal{E}} \subset \mathcal{E}$ be a normal cone. Consider the mappings $f, F : \mathcal{E} \to \mathcal{E}$, both Lipschitz continuous with respective constants $\lambda_f > 0$ and $\lambda_F > 0$. Let $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ be a multi-valued mapping. Define $\mathcal{G}^{\mathcal{M}}_{I,\lambda}$ and $\nabla^{\mathcal{M}}_{I,\lambda}$ as the generalized Yosida approximation and resolvent operators, respectively, assumed to be Lipschitz continuous with constants $\lambda_{\mathcal{G}}$ and $\nabla_\theta$. Assume that for all $\Psi_1, \Psi_2 \in \mathcal{E}$, the following relations hold:*

$$\Psi_{n+1} \propto \Psi_n, \quad \nabla^{\mathcal{M}}_{I,\lambda}(\Psi_{n+1}) \propto \nabla^{\mathcal{M}}_{I,\lambda}(\Psi_n), \quad \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n+1})\Big) \propto \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big),$$

$$f(\Psi_{n+1}) \propto f(\Psi_n), \quad S_{n+1} \propto S_n, \quad F(\Psi_{n+1}) \propto F(\Psi_n),$$

*where $\lambda > 0,$ and $C > 0$ are fixed constants. Suppose further that the following condition is satisfied:*

$$0 \le \lambda_{\Pi_{\mathcal{E}}} \left(\lambda_f + \lambda\lambda_{\mathcal{G}}\lambda_F + \lambda\nabla_\theta\xi_\theta\right) < 1, \tag{4.9}$$

*where the parameters are given by:*

$$\lambda_{\mathcal{G}} = \frac{1+\nabla_\theta}{\lambda}, \quad \xi_\theta = \sqrt{2\gamma(1+\theta_c) - 2\theta_c - 1}, \quad \theta_c = \gamma(1+C) - 1,$$

*with $0 \le \alpha \le 1$, $\gamma \in (0,1)$, and $n = 0, 1, 2, \ldots$.*

*Then, the pair $(\Psi, S)$ is the solution to the YREP involving an averaged operator and logical operations. The sequence $\{\Psi_n\}$ and $\{S_n\}$ generated by Algorithm 4.1 converges strongly to $\Psi$ and $S$, respectively.*

*Proof.* We have

$$\begin{aligned}
0 &\le S_{n+1} \oplus S_n \\
&= \Big[f(\Psi_n) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)\Big\}\Big] \oplus \Big[f(\Psi_{n-1})) + \lambda\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\}\Big] \\
&\le \Big\{f(\Psi_n) \oplus f(\Psi_{n-1})\Big\} + \lambda\Big[\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)\Big\} \oplus \Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\}\Big].
\end{aligned}$$

Using the definition of the normal cone, we obtain

$$\begin{aligned}
\|S_{n+1} \oplus S_n\| &\le \lambda_{\Pi_{\mathcal{E}}}\Big\{\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \lambda\Big\|\Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)\Big\} \oplus \Big\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big) \\
&\quad \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\}\Big\|\Big\} \\
&\le \lambda_{\Pi_{\mathcal{E}}}\Big\{\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \lambda\Big\|\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_n)\Big) \oplus \nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi_{n-1})\Big)\Big\| + \lambda\Big\|\mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big) \\
&\quad \oplus \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n-1})\Big)\Big\|\Big\} \\
&\le \lambda_{\Pi_{\mathcal{E}}}\Big\{\|f(\Psi_n) \oplus f(\Psi_{n-1})\| + \lambda\nabla_\theta\|\xi(\Psi_n) \oplus \xi(\Psi_{n-1})\| \\
&\quad + \lambda\lambda_{\mathcal{G}}\|F(\Psi_n) \oplus F(\Psi_{n-1})\|\Big\}.
\end{aligned} \tag{4.10}$$

Since $\Psi_{n+1} \propto \Psi_n$, $\nabla^{\mathcal{M}}_{I,\lambda}(\Psi_{n+1}) \propto \nabla^{\mathcal{M}}_{I,\lambda}(\Psi_n)$, $\mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_{n+1})\Big) \propto \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi_n)\Big)$, $f(\Psi_{n+1}) \propto f(\Psi_n)$, $S_{n+1} \propto S_n$, $F(\Psi_{n+1}) \propto F(\Psi_n)$, and using (iv) of Proposition 2.2 in (4.10), we have

$$\begin{aligned}
\|S_{n+1} - S_n\| &= \|S_{n+1} \oplus S_n\| \\
&\le \lambda_{\Pi_{\mathcal{E}}}\Big\{\lambda_f\|\Psi_n - \Psi_{n-1}\| + \lambda\nabla_\theta\|\xi(\Psi_n) - \xi(\Psi_{n-1})\| + \lambda\lambda_{\mathcal{G}}\lambda_F\|\Psi_n - \Psi_{n-1}\|\Big\}
\end{aligned}$$

$$= \lambda_{\Pi_{\mathcal{E}}}\Big\{(\lambda_f + \lambda\lambda_{\mathcal{G}}\lambda_F)\|\Psi_n - \Psi_{n-1}\| + \lambda\nabla_\theta\|\xi(\Psi_n) - \xi(\Psi_{n-1})\|\Big\}.$$

Now, using Lemma 2.2, we obtain

$$\begin{aligned}\|S_{n+1} - S_n\| &\le \lambda_{\Pi_{\mathcal{E}}}\Big\{(\lambda_f + \lambda\lambda_{\mathcal{G}}\lambda_F)\|\Psi_n - \Psi_{n-1}\| + \lambda\nabla_\theta\xi_\theta\|\Psi_n - \Psi_{n-1}\|\Big\} \\ &= \lambda_{\Pi_{\mathcal{E}}}(\lambda_f + \lambda\lambda_{\mathcal{G}}\lambda_F + \lambda\nabla_\theta\xi_\theta)\|\Psi_n - \Psi_{n-1}\| \\ &= \omega(\theta)\|\Psi_n - \Psi_{n-1}\|, \end{aligned} \tag{4.11}$$

where

$$\omega(\theta) = \lambda_{\Pi_{\mathcal{E}}}\left(\lambda_f + \lambda\lambda_{\mathcal{G}}\lambda_F + \lambda\nabla_\theta\xi_\theta\right).$$

From condition (4.9), it follows that $\omega(\theta) < 1$. Consequently, the relation (4.11) implies that the sequences $\{\Psi_n\}$ and $\{S_n\}$ are Cauchy in $\mathcal{E}$. Therefore, there exist elements $\Psi, S \in \mathcal{E}$ such that

$$\Psi_n \to \Psi \quad \text{and} \quad S_n \to S \quad \text{as} \quad n \to \infty.$$

Applying the continuity of the mappings $f$, $F$, $\xi$, $\nabla^{\mathcal{M}}_{I,\lambda}$, and $\mathcal{G}^{\mathcal{M}}_{I,\lambda}$, we obtain

$$f(\Psi) = \nabla^{\mathcal{M}}_{I,\lambda}\left[f(\Psi) + \lambda\left\{\nabla^{\mathcal{M}}_{I,\lambda}\Big(\xi(\Psi)\Big) \odot \mathcal{G}^{\mathcal{M}}_{I,\lambda}\Big(F(\Psi)\Big)\right\}\right].$$

Hence, according to Lemma 3.1, $\Psi \in \mathcal{E}$ is a solution of the YVIP involving an averaged operator and logical operations. □

## 5. Numerical result

To demonstrate the applicability of Theorems 3.1 and 4.1, we present the following numerical experiment carried out using MATLAB R2024b. The findings are demonstrated using five computational tables along with the corresponding convergence graphs.

(i) Let $\mathcal{E} = \mathbb{R}$, endowed with the canonical inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Consider the multi-valued mapping $\mathcal{M} : \mathcal{E} \to 2^{\mathcal{E}}$ defined as

$$\mathcal{M}(\Psi) = \left\{\frac{7\Psi}{6}\right\}.$$

Additionally, let $f, F, H : \mathcal{E} \to \mathcal{E}$ be single-valued mappings defined by

$$H(\Psi) = -\frac{3\Psi}{5}, \quad F(\Psi) = \frac{\Psi}{3}, \quad \text{and} \quad f(\Psi) = \frac{3\Psi}{7}.$$

For any $\Psi_1, \Psi_2 \in \mathcal{E}$, we have

$$\Big\|f(\Psi_1) - f(\Psi_2)\Big\| = \left\|\frac{3\Psi_1}{7} - \frac{3\Psi_2}{7}\right\| = \frac{3}{7}\|\Psi_1 - \Psi_2\| \le \frac{1}{2}\|\Psi_1 - \Psi_2\|.$$

Thus, $f$ is Lipschitz continuous with constant $\lambda_f = \frac{1}{2}$ and

$$\Big\|f(\Psi_1) \oplus f(\Psi_2)\Big\| = \left\|\frac{3\Psi_1}{7} \oplus \frac{3\Psi_2}{7}\right\|$$

$$
\begin{aligned}
&= \frac{3}{7}\|\Psi_1 \oplus \Psi_2\| \\
&\geq \frac{1}{3}\|\Psi_1 \oplus \Psi_2\|,
\end{aligned}
$$

and, $f$ is the non extended mapping with constant $\delta_f = \frac{1}{3}$.
Once more

$$
\left\|F(\Psi_1) - F(\Psi_2)\right\| = \left\|\frac{\Psi_1}{3} - \frac{\Psi_2}{3}\right\| = \frac{1}{3}\|\Psi_1 - \Psi_2\| \leq \frac{1}{2}\|\Psi_1 - \Psi_2\|.
$$

Thus, $F$ is Lipschitz continuous with constant $\lambda_F = \frac{1}{2}$.

(ii) Let $H$ be a nonexpansive mapping. It follows that

$$
\begin{aligned}
\left\|H(\Psi_1) - H(\Psi_2)\right\| &= \left\|\left(-\frac{3\Psi_1}{5}\right) - \left(-\frac{3\Psi_2}{5}\right)\right\| \\
&= \left|-\frac{3}{5}\right| \|\ \Psi_1 - \Psi_2\| \\
&\leq \|\ \Psi_1 - \Psi_2\|\ .
\end{aligned}
$$

Hence, $H$ is a nonexpansive mapping.

(iii) Let $H$ be $C$-relaxed Lipschitz continuous(where $C = \frac{2}{5}$). It follows that

$$
\begin{aligned}
\left\langle H(\Psi_1) - H(\Psi_2), \Psi_1 - \Psi_2\right\rangle &= \left\langle\left(-\frac{3\Psi_1}{5}\right) - \left(-\frac{3\Psi_2}{5}\right), \Psi_1 - \Psi_2\right\rangle \\
&= -\frac{3}{5}\left\langle\Psi_1 - \Psi_2, \Psi_1 - \Psi_2\right\rangle \\
&\leq -\frac{2}{5}\|\Psi_1 - \Psi_2\|^2.
\end{aligned}
$$

Hence, $H$ is $C$-relaxed Lipschitz continuous with constant $C = \frac{2}{5}$.

(iv) We calculate the averaged operator $\xi$ as

$$
\xi(\Psi) = [(1-\gamma)I + \gamma H](\Psi) = (1-\gamma)\Psi - \frac{3\gamma\Psi}{5} = \left(1 - \gamma - \frac{3\gamma}{5}\right)(\Psi) = \left(1 - \frac{8\gamma}{5}\right)(\Psi),
$$

where, $\gamma \in (0, 1)$.

(v) $\xi$ is $\theta_C$-relaxed Lipschitz continous, where $C = \frac{2}{5}$ and $\theta_C = \left(\frac{7\gamma}{5} - 1\right)$

$$
\begin{aligned}
\left\langle\xi(\Psi_1) - \xi(\Psi_2), \Psi_1 - \Psi_2\right\rangle &= \left\langle\left(1 - \frac{8\gamma}{5}\right)(\Psi_1) - \left(1 - \frac{8\gamma}{5}\right)(\Psi_2), \Psi_1 - \Psi_2\right\rangle \\
&= \left(1 - \frac{8\gamma}{5}\right)\left\langle\Psi_1 - \Psi_2, \Psi_1 - \Psi_2\right\rangle \\
&= \left(1 - \frac{8\gamma}{5}\right)\|\Psi_1 - \Psi_2\|^2 \\
&\leq -\left(\frac{7\gamma}{5} - 1\right)\|\Psi_1 - \Psi_2\|^2,\ \forall\ \Psi_1, \Psi_2 \in \mathbb{R}^2.
\end{aligned}
$$

Thus $\xi$ is $\theta_C$-relaxed Lipschitz continous with constant $\theta_C = \left(\frac{7\gamma}{5} - 1\right)$ where $\gamma \in (0, 1)$.

(vi) Let $\lambda$=1,Then we evaluate the resolvent operators $\nabla_{I,\lambda}^{\mathcal{M}}$ such that

$$\nabla_{I,\lambda}^{\mathcal{M}}(\Psi) = (I+\lambda\mathcal{M})^{-1}(\Psi) = \frac{6\Psi}{13}.$$

Also, we have

$$\begin{aligned} \left\|\nabla_{I,\lambda}^{\mathcal{M}}(\Psi_1)\oplus\nabla_{I,\lambda}^{\mathcal{M}}(\Psi_2)\right\| &= \left\|\frac{6\Psi_1}{13}\oplus\frac{6\Psi_2}{13}\right\| \\ &= \frac{6}{13}\|\Psi_1\oplus\Psi_2\| \\ &\leq \frac{1}{2}\|\Psi_1\oplus\Psi_2\|. \end{aligned}$$

Thus, $\nabla_{I,\lambda}^{\mathcal{M}}$ is Lipschitz continuous with constant $\nabla_\theta = \frac{1}{2}$.

(vii) Using the values of $\nabla_{I,\lambda}^{\mathcal{M}}$ , we obtain the Yosida approximation operator as

$$\mathcal{G}_{I,\lambda}^{\mathcal{M}}(\Psi) = \frac{1}{\lambda}\Big[I-\nabla_{I,\lambda}^{\mathcal{M}}\Big](\Psi) = \Big[1-\frac{6}{13}\Big](\Psi) = \frac{7\Psi}{13}.$$

Now, we have

$$\begin{aligned} \|\mathcal{G}_{I,\lambda}^{\mathcal{M}}(\Psi_1)\oplus\mathcal{G}_{I,\lambda}^{\mathcal{M}}(\Psi_2)\| &= \left\|\frac{7\Psi_1}{13}\oplus\frac{7\Psi_2}{13}\right\| \\ &= \frac{7}{13}\|\Psi_1\oplus\Psi_2\| \\ &\leq \frac{3}{2}\|\Psi_1\oplus\Psi_2\|. \end{aligned}$$

Thus, $\mathcal{G}_{I,\lambda}^{\mathcal{M}}$ is Lipschitz continuous with constant $\lambda_{\mathcal{G}} = \frac{3}{2}$ where $\lambda_{\mathcal{G}} = \frac{(1+\nabla_\theta)}{\lambda}$.

(viii) Now, we consider the interval $0 \leq \frac{1}{50} \leq \Psi_1 \leq \Psi_2 \leq 1$ and $\lambda_{\Pi_\varepsilon} = \frac{1}{8}$.

(ix) In light of the constants evaluated above, it is evident that the conditions specified in (3.4) and (4.9) of Theorems 3.1 and 4.1, respectively, are satisfied.

(x) Substituting all the computed values into Eq (3.3), we obtain

$$\begin{aligned} f(\Psi_{n+1}) &= (1-\alpha)f(\Psi_n)+\alpha\nabla_{I,\lambda}^{\mathcal{M}}\left[f(\Psi_n)+\lambda\left\{\nabla_{I,\lambda}^{\mathcal{M}}\Big(\xi(\Psi_n)\Big)\odot\mathcal{G}_{I,\lambda}^{\mathcal{M}}\Big(F(\Psi_n)\Big)\right\}\right], \\ \Psi_{n+1} &= (1+0.2071\alpha-0.9278\alpha\gamma)\Psi_n. \end{aligned}$$

In this numerical analysis, four cases are considered to construct the estimation tables and corresponding convergence graphs. The simulations are performed using MATLAB-R2024b with different initial values $\Psi_0 = -1.50,\ -0.5,\ 0.50,\ 1.50$, while keeping the parameters $\alpha$ and $\gamma$ fixed, where $0 \leq \alpha \leq 1$ and $\gamma \in (0,1)$. The iterative sequence $\{\Psi_{n+1}\}$ exhibits convergence for each chosen initial value $\Psi_0$. The convergence behavior is systematically analyzed and discussed in three cases, as well as in the comparison section.

Case I: We consider the pair of parameters $(\alpha,\gamma) = (0.75, 0.75)$. The convergence sequence $\{\Psi_{n+1}\}$ demonstrates a strong convergence toward $\Psi = 0$ (after 30 iterations), as evidenced by the results in Table 1 and the corresponding convergence graph shown in Figure 1.

**Table 1.** The values of $\Psi_n$ with initial values $\Psi_0 = -1.50, \Psi_0 = -0.50, \Psi_0 = 0.50$ and $\Psi_0 = 1.50$.

| Number of Iteration n | $\Psi_0 = -1.50$ $\Psi_n$ | $\Psi_0 = -0.50$ $\Psi_n$ | $\Psi_0 = 0.50$ $\Psi_n$ | $\Psi_0 = 1.50$ $\Psi_n$ |
|---|---|---|---|---|
| 1 | -0.950173 | -0.316724 | 0.316724 | 0.950173 |
| 2 | -0.601886 | -0.200629 | 0.200629 | 0.601886 |
| 3 | -0.381264 | -0.127088 | 0.127088 | 0.381264 |
| 4 | -0.241511 | -0.080504 | 0.080504 | 0.241511 |
| 5 | -0.152985 | -0.050995 | 0.050995 | 0.152985 |
| 10 | -0.015603 | -0.005201 | 0.005201 | 0.015603 |
| 15 | -0.001591 | -0.000530 | 0.000530 | 0.001591 |
| 20 | -0.000162 | -0.000054 | 0.000054 | 0.000162 |
| 25 | -0.000017 | -0.000006 | 0.000006 | 0.000017 |
| 30 | -0.000002 | -0.000001 | 0.000001 | 0.000002 |
| 33 | -0.000000 | -0.000000 | 0.000000 | 0.000000 |
| 35 | -0.000000 | -0.000000 | 0.000000 | 0.000000 |

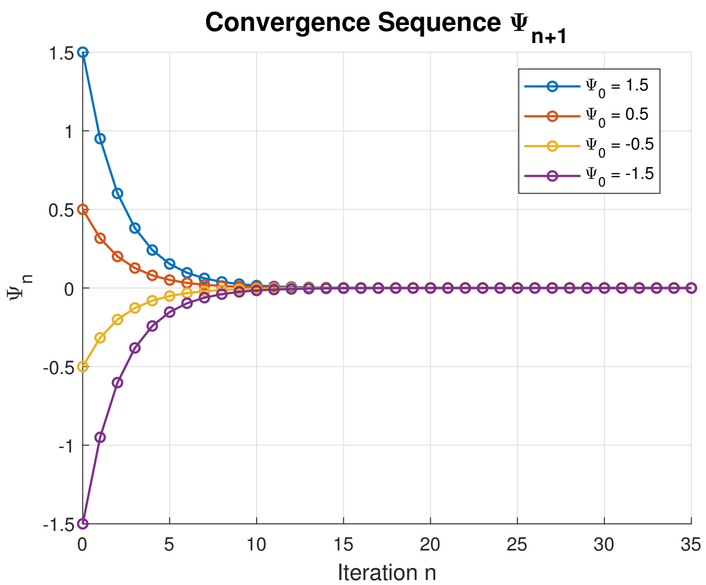



**Figure 1.** The convergence behavior of the sequence $\{\Psi_{n+1}\}$ is illustrated for variant parameter pairs $(\alpha, \gamma) = (0.75, 0.75)$.

Case II: We assume the parameter pair $(\alpha, \gamma) = (0.88, 0.88)$. The results findings are reported in Table 2, while the convergence pattern of the sequence $\{\Psi_{n+1}\}$ is depicted in Figure 2. As shown, the sequence converges to $\Psi = 0$ in 19 iterations.

**Table 2.** The values of $\Psi_n$ with initial values $\Psi_0 = -1.50, \Psi_0 = -0.50, \Psi_0 = 0.50$ and $\Psi_0 = 1.50$.

| Number of Iteration n | $\Psi_0 = -1.50$ $\Psi_n$ | $\Psi_0 = -0.50$ $\Psi_n$ | $\Psi_0 = 0.50$ $\Psi_n$ | $\Psi_0 = 1.50$ $\Psi_n$ |
|---|---|---|---|---|
| 1 | -0.695656 | -0.231885 | 0.231885 | 0.695656 |
| 2 | -0.322625 | -0.107542 | 0.107542 | 0.322625 |
| 3 | -0.149624 | -0.049875 | 0.049875 | 0.149624 |
| 4 | -0.069391 | -0.023130 | 0.023130 | 0.069391 |
| 5 | -0.032182 | -0.010727 | 0.010727 | 0.032182 |
| 7 | -0.006922 | -0.002307 | 0.002307 | 0.006922 |
| 10 | -0.000690 | -0.000230 | 0.000230 | 0.000690 |
| 14 | -0.000032 | -0.000011 | 0.000011 | 0.000032 |
| 17 | -0.000003 | -0.000001 | 0.000001 | 0.000003 |
| 18 | -0.000001 | -0.000000 | 0.000000 | 0.000001 |
| 19 | -0.000001 | -0.000000 | 0.000000 | 0.000001 |
| 20 | -0.000000 | -0.000000 | 0.000000 | 0.000000 |

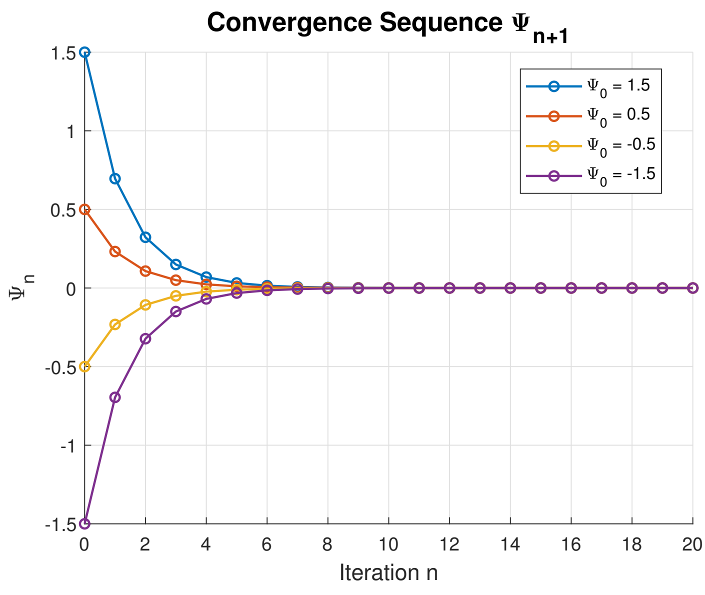



**Figure 2.** The convergence behavior of the sequence $\{\Psi_{n+1}\}$ is illustrated for variant parameter pairs $(\alpha, \gamma) = (0.88, 0.88)$.

Case III: We consider the parameter pair $(\alpha, \gamma) = (1.00, 0.99)$. In this setting, the convergence sequence $\{\Psi_{n+1}\}$ shows excellent performance, rapidly converging to the fixed point $\Psi = 0$ after 12 iterations. The results are presented in Table 3, and the convergence behavior is illustrated in Figure 3.

**Table 3.** The values of $\Psi_n$ with initial values $\Psi_0 = -1.50, \Psi_0 = -0.50, \Psi_0 = 0.50$ and $\Psi_0 = 1.50$.

| Number of Iteration n | $1\Psi_0 = -1.50$ $\Psi_n$ | $\Psi_0 = -0.50$ $\Psi_n$ | $\Psi_0 = 0.50$ $\Psi_n$ | $\Psi_0 = 1.50$ $\Psi_n$ |
|---|---|---|---|---|
| 1 | -0.432867 | -0.144289 | 0.144289 | 0.432867 |
| 2 | -0.124916 | -0.041639 | 0.041639 | 0.124916 |
| 3 | -0.036048 | -0.012016 | 0.012016 | 0.036048 |
| 4 | -0.010403 | -0.003468 | 0.003468 | 0.010403 |
| 5 | -0.003002 | -0.001001 | 0.001001 | 0.003002 |
| 6 | -0.000866 | -0.000289 | 0.000289 | 0.000866 |
| 7 | -0.000250 | -0.000083 | 0.000083 | 0.000250 |
| 8 | -0.000072 | -0.000024 | 0.000024 | 0.000072 |
| 9 | -0.000021 | -0.000007 | 0.000007 | 0.000021 |
| 10 | -0.000006 | -0.000002 | 0.000002 | 0.000006 |
| 11 | -0.000002 | -0.000001 | 0.000001 | 0.000002 |
| 12 | -0.000001 | -0.000000 | 0.000000 | 0.000001 |
| 15 | -0.000000 | -0.000000 | 0.000000 | 0.000000 |

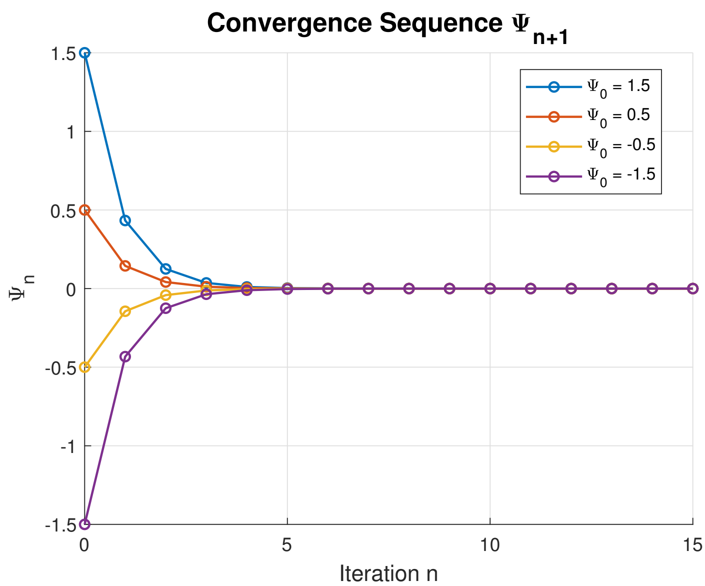


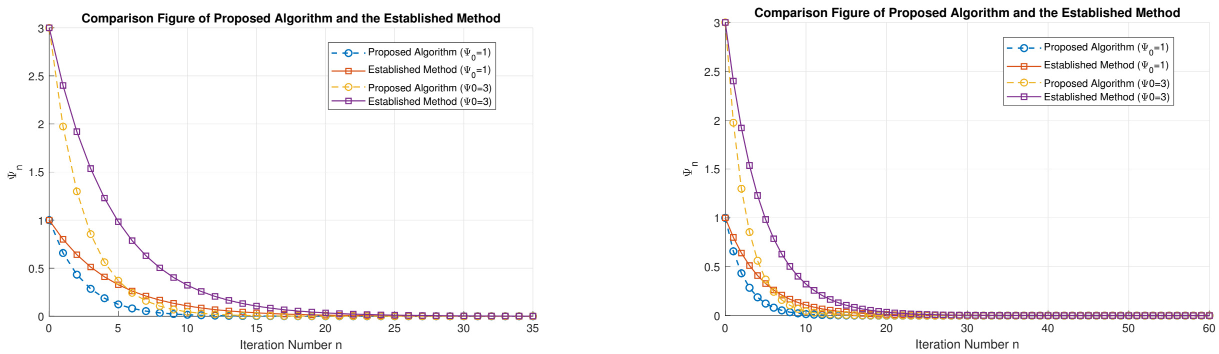

**Figure 3.** The convergence behavior of the sequence $\{\Psi_{n+1}\}$ is illustrated for variant parameter pairs $(\alpha, \gamma) = (1.00, 0.99)$.

## 6. Comparative analysis of the proposed algorithm and the established method

In this section, we compare the performance of the proposed Algorithm 3.1 with Algorithm 3.2 presented in [20] using a numerical example. The comparison focuses on the rate of convergence under the stopping criterion $\|\Psi_{n+1} - \Psi_n\| \leq 10^{-6}$. For the parameter pair $(\alpha, \gamma) = (0.70, 0.75)$, the corresponding numerical results are summarized in Table 4. As shown in Figure 4(a), Algorithm 3.1 (Proposed Algorithm) achieves convergence in approximately 30 iterations, while Algorithm 3.2 (Established Method) requires nearly 57 iterations, as illustrated in Figure 4(b), to attain the same level of accuracy. These findings clearly underscore the superior convergence performance of Algorithm 3.1. All computations were carried out using the initial values $\Psi_0 = 1$ and $\Psi_0 = 3$.

**Table 4.** Comparison table: For initial values $\Psi_0 = 1$ and $\Psi_0 = 3$.

| Number of Iteration n | Proposed Algorithm $\Psi_0 = 1$ $\Psi_n$ | Proposed Algorithm $\Psi_0 = 3$ $\Psi_n$ | Established Method in [20] $\Psi_0 = 1$ $\Psi_n$ | Established Method in [20] $\Psi_0 = 3$ $\Psi_n$ |
|---|---|---|---|---|
| 1 | 0.657880 | 1.973600 | 0.800000 | 2.400000 |
| 2 | 0.432800 | 1.298400 | 0.640000 | 1.920000 |
| 3 | 0.284730 | 0.854180 | 0.512000 | 1.536000 |
| 5 | 0.123230 | 0.369690 | 0.327680 | 0.983040 |
| 10 | 0.015186 | 0.045557 | 0.107370 | 0.322120 |
| 15 | 0.001871 | 0.005614 | 0.035184 | 0.105550 |
| 20 | 0.000230 | 0.000691 | 0.011529 | 0.034588 |
| 25 | 0.000002 | 0.000008 | 0.003777 | 0.011334 |
| 31 | 0.000000 | 0.000000 | 0.000990 | 0.002971 |
| 40 | 0.000000 | 0.000000 | 0.000132 | 0.000398 |
| 50 | 0.000000 | 0.000000 | 0.000001 | 0.000004 |
| 57 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |



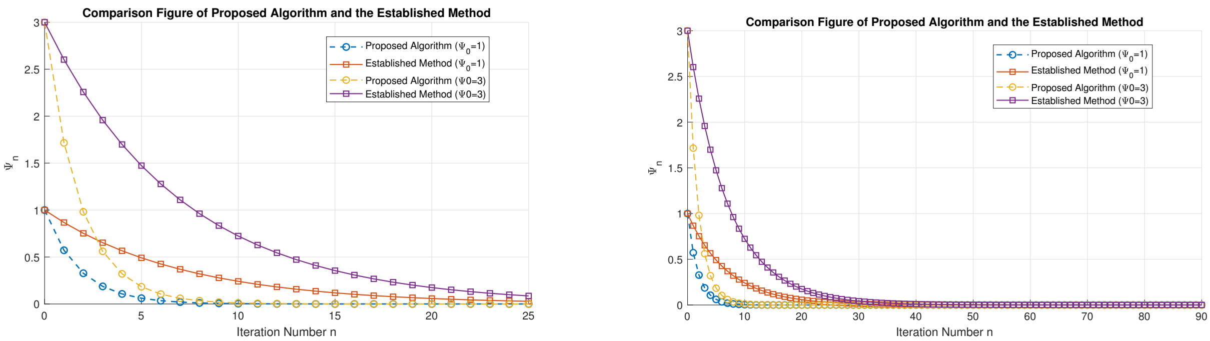

**(a)** Convergence performance comparison. **(b)** Convergence performance comparison.

**Figure 4.** The convergence behavior of the sequence $\{\Psi_{n+1}\}$ is illustrated for variant parameter pairs $(\alpha, \gamma) = (0.70, 0.75)$.

Again, consider the other parameter pair $(\alpha, \gamma) = (0.80, 0.80)$, the numerical results are reported in Table 5, and the corresponding convergence behavior of the sequence $\{\Psi_{n+1}\}$ is depicted in Figure 5. As observed, the sequence converges to $\Psi = 0$. Algorithm 3.1 (Proposed Method) achieves convergence in approximately 23 iterations, as shown in Figure 5(a), while Algorithm 3.2 (Established Method in [20]) requires about 89 iterations, as shown in Figure 5(b). These results clearly indicate that Algorithm 3.1 exhibits a significantly faster convergence rate.

**Table 5.** Comparison table: For initial values $\Psi_0 = 1$ and $\Psi_0 = 3$.

| Number of Iteration n | Proposed Algorithm $\Psi_0 = 1$ $\Psi_n$ | Proposed Algorithm $\Psi_0 = 3$ $\Psi_n$ | Established Method in [20] $\Psi_0 = 1$ $\Psi_n$ | Established Method in [20] $\Psi_0 = 3$ $\Psi_n$ |
|---|---|---|---|---|
| 1 | 0.571890 | 1.715700 | 0.867430 | 2.602300 |
| 2 | 0.327060 | 0.981170 | 0.752430 | 2.257300 |
| 3 | 0.187040 | 0.561120 | 0.652680 | 1.958000 |
| 4 | 0.106970 | 0.320900 | 0.566150 | 1.698500 |
| 5 | 0.061172 | 0.183520 | 0.491100 | 1.473300 |
| 10 | 0.003742 | 0.011226 | 0.241180 | 0.723530 |
| 20 | 0.000001 | 0.000004 | 0.058167 | 0.174500 |
| 30 | 0.000000 | 0.000000 | 0.014029 | 0.042086 |
| 40 | 0.000000 | 0.000000 | 0.003383 | 0.010150 |
| 50 | 0.000000 | 0.000000 | 0.000815 | 0.002448 |
| 60 | 0.000000 | 0.000000 | 0.000196 | 0.000590 |
| 70 | 0.000000 | 0.000000 | 0.000004 | 0.000142 |
| 80 | 0.000000 | 0.000000 | 0.000001 | 0.000003 |
| 90 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |




**(a)** Convergence performance comparison. **(b)** Convergence performance comparison.

**Figure 5.** The convergence behavior of the sequence $\{\Psi_{n+1}\}$ is illustrated for variant parameter pairs $(\alpha, \gamma) = (0.80, 0.80)$.

## 7. Conclusions

This work addressed a Yosida variational inclusion problem and its corresponding resolvent formulation in a real ordered Hilbert space, incorporating logical operations through an averaged-operator framework. The proposed iterative algorithm demonstrates notable improvements over existing methods [20] for such inclusions involving logically enriched Yosida and averaged operators. A numerical result carried out in MATLAB R2024b supported the theoretical findings, demonstrating rapid convergence and improved computational efficiency. These results confirm the practical effectiveness of the proposed algorithms in relation to established techniques.

## Author contributions

Arifuzzaman participated in the writing, editing and methodology of the manuscript. Syed Shakaib Irfan did computational and graphical work with appropriate software and also participated in the review and editing process. Iqbal Ahmad participated in the review of the main text of the manuscript. All authors read and approved the final version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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