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*Research article*

# Fixed point theorems for graphic Θ-contractions in $\mathcal{F}$-metric spaces with applications to image processing

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**Abstract:** The aim of this research article is to introduce the notion of graphic Θ-contractions in the setting of $\mathcal{F}$-metric spaces, and establish some novel fixed point results. To demonstrate the validity of our theoretical contributions, we include some illustrative examples. In addition, we extend our analysis by proving fixed point theorems for orbitally $G$ continuous graphic Θ-contractions. A significant application of our results is the formulation of image denoising as a fixed point problem. Specifically, we define a contractive operator that refines pixel intensities through iterative updates based on the local neighborhood of each pixel. Using the principles of fixed point theory, we prove the existence and uniqueness of a stable denoised image that corresponds to the fixed point of the constructed mapping. Furthermore, we investigate the convergence properties of the iterative scheme under the assumptions of graphic Θ-contractions, thereby confirming its reliability and effectiveness.

**Keywords:** fixed point; Θ-contraction; $\mathcal{F}$-metric space; graphs; self-mappings; image denoising
**Mathematics Subject Classification:** 46S40, 47H10, 54H25

## 1. Introduction

Fixed point theory (FPT) has emerged as one of the most influential and widely applicable areas of modern mathematics. The essence of this theory lies in identifying a point that remains invariant under the action of a given mapping, that is, a point $\hbar^*$ such that $\mathcal{V}\hbar^* = \hbar^*$. Although the concept appears simple, its implications are profound and far-reaching. FP results not only provide elegant tools for the study of nonlinear analysis but also serve as a unifying framework for solving problems in diverse branches of mathematics and applied sciences. In particular, they have played a central role in the development of differential and integral equations, optimization theory, game theory,

economics, computer science, control theory, and many other fields. The importance of FP results stems from their ability to guarantee the existence (and often uniqueness) of solutions to complex problems that cannot be tackled directly. For instance, fixed point techniques allow researchers to establish the existence of equilibria in dynamical systems, steady states in population models, and solutions to boundary value problems in partial differential equations. Furthermore, continuous generalizations of classical fixed point theorems have opened pathways to new structures in abstract spaces, including metric spaces, cone metric spaces, ordered metric spaces, and other generalized settings. Each extension not only broadens the scope of applications but also reveals deeper insights into the underlying geometry and analysis of the spaces under consideration.
In this theory, Banach contraction principle [1] is regarded as a pioneering theorem that laid the foundation of the subject and inspired countless extensions in different directions [2–5]. Jachymski [6] generalized this principle for single-valued mappings in the framework of complete metric spaces (CMSs) endowed with a graph structure, thereby enriching the classical approach. Jleli et al. [7] gave a fashionable assortment of contraction by employing a precise function is said to be Θ-contraction and furnished some examples to show the boldness of such generalizations. Jleli et al. [7] proved a fixed point result by utilizing the concept of Θ-contraction and generalized conventional Banach contraction principle (BCP). For a comprehensive overview, the reader may consult [8–10].

In other direction, the supreme part in FP theory is the underlying space. The idea of metric space (MS) was laid by the French mathematician Maurice Fréchet in 1905, laying the foundation for subsequent advancements in the field. The framework of MSs forms a fundamental part of mathematical analysis, serving as the foundation for various branches including real analysis, complex analysis, and multivariable calculus. Over the past several decades, numerous researchers have explored generalizations of MSs by modifying the traditional triangle inequality. The theory of generalized metric spaces has grown significantly since the classical framework was extended by Bakhtin [11] through the notion of a $b$-metric space ($b$-MS). In this setting, the triangle inequality is relaxed with a constant coefficient $s \geq 1$, which broadens the applicability of FP results while preserving a workable distance structure. Later, Czerwik [12] systematically investigated $b$-MSs, establishing their fundamental properties and emphasizing their importance in nonlinear analysis. Subsequently, Branciari [13] introduced the concept of a rectangular metric space (RMS), in which the usual triangle inequality is replaced by a four-point inequality. This new framework further generalized the metric structure by allowing distances to be controlled through three intermediate points rather than one, which opened new directions for FP theory and its applications in analysis and integral equations. Building upon these developments, Khojasteh et al. [14] proposed the framework of $\theta$-metric spaces ($\theta$-MSs), motivated by the idea of controlling the metric structure through an auxiliary function $\theta$. Unlike the $b$-MS and RMS, where the generalization arises from modifying the inequality directly, the $\theta$-MSs introduces a control function that governs how distances behave, thereby unifying and extending several previous notions of generalized metrics. Fagin et al. [15] introduced the notion of $s$-relaxed$_p$ metric space. Note that any $s$-relaxed$_p$ metric is a $b$-metric, but the converse is not true in general. Continuing in this direction, Jleli et al. [16] proposed a comprehensive extension known as $\mathcal{F}$-metric spaces ($\mathcal{F}$-MSs). In this setting, the classical distance function is replaced by a function $\tau : \Xi \times \Xi \to [0, +\infty)$ that is controlled through an auxiliary function $\xi$ together with additional conditions. This framework is broad enough to encompass many previously known generalizations such as MSs, $b$-mss, and RMS as special cases. Moreover, it provides new flexibility in dealing with convergence, continuity, and completeness, while still supporting the development of FP results parallel to those in the metric setting. Subsequently, Al-Mezel et al. [17] and Hussain et al. [18] further explored the framework of $\mathcal{F}$-MSs and established FP theorems for generalized contractions, such as ($\alpha\beta$-$\psi$)-contractions and and Ćirić-type ($\alpha$-$\psi$)-contractions respectively and then they applied their results to investigate the existence and uniqueness of solutions for neutral differential equations with unbounded delay. Alansari et al. [19] extended the framework of $\mathcal{F}$-MSs to the fuzzy setting and obtained fuzzy FP results. Their work highlighted how the structure of $\mathcal{F}$-MSs can be effectively utilized to establish the existence of fuzzy FPs, thereby enriching the theory and providing new tools for handling problems with inherent uncertainty. Recently, Faraji et al. [20] introduced and investigated the concept of $F$-$G$-contractions in the setting of $\mathcal{F}$-MSs endowed with a graph. By combining the structure of $\mathcal{F}$-MSs, originally proposed by Jleli et al. [16], with the additional layer of a graph structure, they obtained new FP results that generalize and extend several classical theorems. Their approach highlights how the interplay between graph theory and generalized metric spaces can provide a rich framework for analyzing the existence and uniqueness of FPs. The results not only unify earlier contributions in this theory but also demonstrate the versatility of graph-theoretic techniques in broadening the applicability of contraction principles. For additional developments along these lines, see [21–23].

FP theory has emerged as a powerful mathematical tool with significant applications in various scientific and engineering disciplines. One of its notable applications is in image processing, particularly in image restoration and enhancement. The use of FP methods in this domain has gained substantial attention due to their robustness, efficiency, and ability to handle complex optimization problems. Several studies have demonstrated the effectiveness of FP methods in image restoration. Hanjing et al. [24] introduced a fast image restoration algorithm that integrates this theory with optimization techniques, showcasing improved computational efficiency. Similarly, Kim et al. [25] employed a FP approach to solve a total variation $L_2(TVL2)$ regularization problem, which plays a crucial role in reducing noise while preserving important image features. FP methodologies have also been successfully implemented in hardware based image processing. Cabello et al. [26] designed a FP 2D Gaussian filter for image processing using field-programmable gate arrays (FPGAs), highlighting the applicability of FP arithmetic in real-time signal processing tasks. Furthermore, Mishra et al. [27] explored contraction mapping principles in digital image processing, demonstrating their relevance in improving image reconstruction techniques.

In this research article, we introduce the notion of graphic $\Theta$-contractions in the setting of $\mathcal{F}$-MS and obtain FP theorems for the self mappings. The scope is further broadened by extending these results to orbitally $G$-continuous graphic $\Theta$-contractions. To validate our theoretical developments, we present several illustrative examples. Our findings also encompass and generalize the principal results of Onsod et al. [9], Jleli et al. [16] and Mohanta et al. [23]. A notable application of the established results is provided in the context of image denoising, where the problem is formulated as a fixed point model. In particular, we construct a contractive operator that iteratively adjusts pixel intensities by taking into account their neighborhood relationships. Leveraging FP theory, we demonstrate both the existence and uniqueness of a stable denoised image corresponding to the FP of the operator. Additionally, we analyze the convergence behavior of the iterative scheme under the framework of graphic $\Theta$-contractions, thereby confirming its robustness and practical effectiveness.

## 2. Preliminaries

This section introduces the key definitions, notations, and essential results that underpin our main findings. As our primary results are are established for graphic Θ-contractions in $\mathcal{F}$-MSs, we begin by presenting FP theorems for contractions, graphic contractions, and Θ-contractions in metric spaces to establish a solid foundation. Furthermore, we discuss important properties related to completeness and convergence, providing a comprehensive framework for our FP results. These preliminary concepts will be crucial in formulating and proving the main theorems in the following sections. Throughout this article, we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of non-negative real numbers, and by $\mathbb{N}$ the set of natural numbers.

The Banach Contraction Principle asserts that any self-mapping $\mathcal{V}$ defined on a CMS $(\mathcal{P}, \tau)$ satisfying the condition

$$\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) \leq \lambda\tau(\hbar, \varsigma)$$

for all $\hbar, \varsigma \in \mathcal{P}$, where $\lambda \in [0, 1)$ has a unique FP. We will present graph-theoretic concepts based on the work of Jachymski [6]. Let $(\mathcal{P}, \tau)$ be a MS and let $\Delta$ denote the diagonal of $\mathcal{P} \times \mathcal{P}$. Consider a directed graph $G$ composed of a vertex set $V(G)$ identical to $\mathcal{P}$ and an edge set $E(G)$ encompassing all loops $(\Delta \subseteq E(G))$ . Importantly, $G$ contains no multiple edges.

Jachymski [6] introduced the following definition of $G$-contraction:

**Definition 1.** *( [6]) Let $(\mathcal{P}, \tau)$ be a MS and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$. A mapping $\mathcal{V}$ is termed a Banach graphic contraction if*

*(a) $\forall\ \hbar, \varsigma \in \mathcal{P}$ with $(\hbar, \varsigma) \in E(G)$, we have $(\mathcal{V}(\hbar), \mathcal{V}(\varsigma)) \in E(G)$,*

*(b) there exists $\lambda \in (0, 1)$ such that, $\forall\ \hbar, \varsigma \in \mathcal{P}$ with $(\hbar, \varsigma) \in E(G)$, we have*

$$\tau(\mathcal{V}(\hbar), \mathcal{V}(\varsigma)) \leq \lambda\tau(\hbar, \varsigma). \tag{2.1}$$

$G^{-1}$ is the converse graph of $G$ that is the edge of $G^{-1}$ is established by reversing the direction of edges of $G$, that is

$$E\left(G^{-1}\right) = \{(\hbar, \varsigma) \in \mathcal{P} \times \mathcal{P} : (\varsigma, \hbar) \in E(G)\}.$$

Given two vertices, $\hbar$ and $\varsigma$ in a graph $G$, a path connecting $\hbar$ to $\varsigma$ of length $N$ (a natural number) is a finite sequence $\{\hbar_i\}_{i=0}^N$ of $N + 1$ vertices such that $\hbar_0 = \hbar$, $\hbar_N = \varsigma$ and $(\hbar_{i-1}, \hbar_i) \in E(G)$, $\forall\ i = 1, \cdots, N$. A graph $G$ is connected if any two distinct vertices within the graph can be joined by a path. If $\widetilde{G} = \left(\mathcal{P}, E(\widetilde{G})\right)$ represents symmetric graph established by putting the vertices of both $G$ and $G^{-1}$, that is

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}),$$

then the graph $G$ is said to be weakly connected if $\widetilde{G}$ is connected. For any vertex $\hbar_0 \in \Xi$, the notation $\widetilde{G}_{\hbar_0}$ refers to the connected component of $\widetilde{G}$ that contains $\hbar_0$. This component consists of all points in $\Xi$ that can be joined to $\hbar_0$ through a finite path in the symmetric graph $\widetilde{G}$. Let $G$ be a graph such that $E(G)$ is symmetric and let $\hbar \in V(G)$. The subgraph $G_\hbar$, formed by all vertices and edges that lie on some path of $G$ starting at $\hbar$, is called the component of $G$ containing $\hbar$. In this situation, we have $V(G_\hbar) = [\hbar]_G$, where $[\hbar]_G$ denotes the equivalence class with respect to the relation $R$ on $V(G)$ defined by $\varsigma R z$ whenever there exists a path in $G$ joining $\varsigma$ and $z$. It follows directly that $G_\hbar$ is connected for every $\hbar \in G$. Now, let $\Omega = \{G : G$ is a directed graph with $V(G) = \mathcal{P}$ and $\Delta \subseteq E(G)\}$ and define $\mathcal{P}_{\mathcal{V}} := \{\hbar \in \mathcal{P} : (\hbar, \mathcal{V}(\hbar)) \in E(G)\}$ for $\mathcal{V} : \mathcal{P} \to \mathcal{P}$.

In 2008, Jachymski [6] gave the following property which is also required in the proof of our result.

(P) for $\{\hbar_n\} \subseteq \mathcal{P}$, if $\hbar_n \to \hbar$ as $n \to \infty$ and $(\hbar_n, \hbar_{n+1}) \in E(G)$, then there exists a subsequence $\{\hbar_{j_n}\}$ such that $(\hbar_{j_n}, \hbar) \in E(G)$, $\forall\, n \in \mathbb{N}$.

Property (P) ensures that the graph structure is compatible with the convergence of sequences in $\Xi$. In particular, even if only consecutive terms $(\hbar_n, \hbar_{n+1})$ are connected in $E(G)$, the property guarantees the existence of a subsequence whose elements remain connected to the limit point $\hbar$. This connection is essential for validating the iterative process used in our main FP results, as it allows the graph based contractive conditions to pass to the limit. Without Property (P), the convergence of the sequence would not necessarily imply the required edge relation with its limit, and the FP theorems could fail.

**Definition 2.** *( [22]) Consider the MS $(\mathcal{P}, \tau)$ and a mapping $\mathcal{V} : \mathcal{P} \to \mathcal{P}$.The mapping $\mathcal{V}$ is termed a Picard operator if it possesses a unique FP denoted by $\hbar^*$ and $\mathcal{V}^n\hbar \to \hbar^*$, as $n \to \infty$, for all $\hbar \in \mathcal{P}$.*

**Definition 3.** *( [6]) Given a MS $(\mathcal{P}, \tau)$ and a mapping $\mathcal{V} : \mathcal{P} \to \mathcal{P}$. Then $\mathcal{V}$ is classified as a weakly Picard operator if for any $\hbar \in \mathcal{P}$, $\lim_{n\to\infty} \mathcal{V}^n\hbar$ exists and its limit is a FP of $\mathcal{V}$.*

Jleli et al. [7] initiated a novel contraction type and subsequently established new FP theorems within the context of generalized MS.

**Definition 4.** *Consider a function $\Theta : \mathbb{R}^+ \to (1, \infty)$ such that*

($\Theta_1$): $\Theta(t_1) \leq \Theta(t_2)$ for $t_1 \leq t_2$;
($\Theta_2$) for every sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \Theta(t_n) = 1$ if and only if $\lim_{n\to\infty}(t_n) = 0^+$;
($\Theta_3$) there exists $0 < r < 1$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\Theta(t)-1}{t^r} = l$.

A mapping $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is termed a $\Theta$-contraction if there exist the function $\Theta$ satisfying the conditions ($\Theta_1$)-($\Theta_3$) and a constant $\lambda \in (0, 1)$ such that

$$\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) \neq 0 \Longrightarrow \Theta(\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma)) \leq [\Theta(\tau(\hbar, \varsigma)]^\lambda \tag{2.2}$$

for all $\hbar, \varsigma \in \mathcal{P}$.

We represent $\Psi$, the set of the functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the conditions ($\Theta_1$)-($\Theta_3$).

Jleli et al. [16] initiated a novel extension of MSs, termed $\mathcal{F}$-MSs in this way.

Let $\mathcal{F}$ be the set of functions $\xi : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$): $\xi$ is non-decreasing, that is, $0 < \hbar_1 < \hbar_2 \Rightarrow \xi(\hbar_1) \leq \xi(\hbar_2)$.

($\mathcal{F}_2$): for every sequence $\{\hbar_n\} \subseteq \mathbb{R}^+$, we have

$$\lim_{n\to\infty} \hbar_n = 0 \iff \lim_{n\to\infty} \xi(\hbar_n) = -\infty.$$

**Definition 5.** *( [16] ) Let $\mathcal{P}$ be a nonempty set, and let $\tau : \mathcal{P} \times \mathcal{P} \to [0, +\infty)$. Suppose that there exists $(\xi, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that*

*($D_1$) $(\hbar, \varsigma) \in \mathcal{P} \times \mathcal{P}$, $\tau(\hbar, \varsigma) = 0$ if and only if $\hbar = \varsigma$,*
*($D_2$) $\tau(\hbar, \varsigma) = \tau(\varsigma, \hbar)$, for all $\hbar, \varsigma \in \mathcal{P}$,*

*$(D_3)$ for all $(\hbar, \varsigma) \in \Xi \times \Xi$, and for every finite sequence $(\hbar_i)_{i=1}^N \subset \Xi$, with*

$$\hbar_1 = \hbar, \quad \hbar_N = \varsigma,$$

*we have*

$$\tau(\hbar, \varsigma) > 0 \Rightarrow \xi(\tau(\hbar, \varsigma)) \leq \xi(\sum_{i=1}^{N-1} \tau(\hbar_i, \hbar_{i+1})) + \alpha,$$

*for all $N \geq 2$. Consequently, the pair $(\mathcal{P}, \tau)$ is said to be an $\mathcal{F}$-MS.*

**Example 1.** *( [16]) Let $\Xi$ be the set of natural numbers. Then $\tau : \Xi \times \Xi \to [0, +\infty)$ defined by*

$$\tau(\hbar, \varsigma) = \begin{cases} (\hbar - \varsigma)^2 & \text{if } \hbar, \varsigma \in \{0, 1, 2, 3\} \\ |\hbar - \varsigma|, & \text{otherwise,} \end{cases}$$

*with $\xi(t) = \ln(t)$ and $\alpha = \ln(3)$ is an $\mathcal{F}$-metric.*

**Definition 6.** *( [16]) Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS. A subset $A \subseteq \mathcal{P}$ is called an $\mathcal{F}$-open if, for each point $\hbar \in A$, there exists a radius $r > 0$ such that $\mathcal{F}$-ball $B(\hbar, r)$ is entirely contained in A, where*

$$B(\hbar, r) = \{\varsigma \in \mathcal{P} : \tau(\hbar, \varsigma) < r\}.$$

*A subset $C \subseteq \mathcal{P}$ is called $\mathcal{F}$-closed whenever its complement $\mathcal{P} \backslash C$ is $\mathcal{F}$-open. The collection of all $\mathcal{F}$-open subsets of $\mathcal{P}$ will be denoted by $\tau_{\mathcal{F}}$. The collection $\tau_{\mathcal{F}}$ is a topology on $\mathcal{P}$.*

**Definition 7.** *( [16]) Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS and A be a subset of $\mathcal{P}$. A point $\hbar \in \mathcal{P}$ is said to be an interior point of a set A if there exists some $r > 0$ such that $B(\hbar, r) \subseteq A$. A point $\hbar \in \mathcal{P}$ is said to be a limit point of A whenever for every $r > 0$, we have*

$$B(\hbar, r) \cap (A \backslash \{\hbar\}) \neq \emptyset.$$

*The closure of a set A with respect to the topology $\tau_{\mathcal{F}}$ is denoted by $\overline{A}$. It is defined as the intersection of all $\mathcal{F}$-closed subsets of $\mathcal{P}$ that contain A. Equivalently, $\overline{A}$ is the smallest $\mathcal{F}$-closed set containing A.*

**Definition 8.** *( [16]) Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS.*

(i) A sequence $\{\hbar_n\}$ in $\mathcal{P}$ is termed $\mathcal{F}$-convergent to $\hbar \in \mathcal{P}$ if $\{\hbar_n\}$ is convergent to $\hbar$ if it converges to $\hbar$ under the $\mathcal{F}$-metric $\tau$.

(ii) A sequence $\{\hbar_n\}$ is considered $\mathcal{F}$-Cauchy, if

$$\lim_{n,m \to \infty} \tau(\hbar_n, \hbar_m) = 0.$$

(iii) $(\mathcal{P}, \tau)$ is said to be $\mathcal{F}$-complete, if each $\mathcal{F}$-Cauchy sequence in $\mathcal{P}$ is $\mathcal{F}$-convergent to a certain point in $\mathcal{P}$.

This article introduces graphic $\Theta$-contractions in $\mathcal{F}$-MS and establishes FP theorems, further extended to orbitally $G$-continuous mappings. The results generalize earlier works and are illustrated with examples. As an application, we model image denoising as a FP problem, proving the existence and uniqueness of a stable denoised image and analyzing the convergence of the iterative scheme.

## 3. Main results

In this section, we prove FP theorems for graphic Θ-contractions in the framework of $\mathcal{F}$-MSs. We begin by introducing this novel contraction and then state and prove new FP results. These theorems generalize and extend previously established results in $\mathcal{F}$-MSs. The outcomes contribute to the wider exploration of FP theory in generalized metric structures and lay the groundwork for potential applications, such as image denoising. In the whole section, we suppose that $\mathcal{P}$ is a $\mathcal{F}$-MS with a $\mathcal{F}$-metric $\tau$ and $G = \{G : G$ is a directed graph with $V(G) = \mathcal{P}$ and $\Delta \subseteq E(G)\}$. The set of all FPs of $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ will be denoted by $Fix(\mathcal{V})$. We proceed to define the concept of a graphic Θ-contraction as follows.

**Definition 9.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS equipped with a graph $G$. A mapping $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is termed a graphic Θ-contraction if*

(i)

$$(\hbar, \varsigma) \in E(G) \Longrightarrow (\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(G), \quad \forall \hbar, \varsigma \in \mathcal{P}, \tag{3.1}$$

(ii) there exists some $\lambda \in (0, 1)$ such that for all $\hbar, \varsigma \in \mathcal{P}, (\hbar, \varsigma) \in E(G)$ , we have

$$\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) \neq 0 \implies \Theta\left(\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma)\right) \leq \left[\Theta\left(\tau(\hbar, \varsigma)\right)\right]^{\lambda}. \tag{3.2}$$

**Example 2.** *Let $\mathcal{P} \neq \emptyset$ and $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS. Then for any $\Theta \in \Psi$ and $G \in \Omega$, a constant function $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is graphic Θ-contraction because in this way $G = (\mathcal{P}, E(G))$.*

**Example 3.** *Suppose $\Theta \in \Psi$ be an arbitrary. Then each Θ-contraction is a graphic Θ-contraction for the complete graph $G_0$ defined by $V(G_0) = \mathcal{P}$ and $E(G_0) = \mathcal{P} \times \mathcal{P}$.*

**Example 4.** *Let $G \in \Omega$. Then each graphic contraction is graphic Θ-contraction for $\Theta : (0, \infty) \to (1, \infty)$ given by $\Theta(t) = e^{\sqrt{t}}$, for $t > 0$.*

**Example 5.** *Define the sequence $\{\mu_n\}$ as follows:*
$\mu_1 = \ln(1)$
$\mu_2 = \ln(1 + 4)$
.
.
.
$\mu_n = \ln(1 + 4 + 7 + ... + (3n - 2)) = \ln\left(\frac{n(3n-1)}{2}\right)$, *for all $n \in \mathbb{N}$. Let $\mathcal{P} = \{\mu_n : n \in \mathbb{N}\}$ endowed with the $\mathcal{F}$-metric given by*

$$\tau(\hbar, \varsigma) = \begin{cases} e^{|\hbar - \varsigma|}, & if\ \hbar \neq \varsigma \\ 0, & if\ \hbar = \varsigma \end{cases}$$

*with $\xi(t) = \frac{-1}{t}$ and $\alpha = 1$. Then, $(\mathcal{P}, \tau)$ is an $\mathcal{F}$-complete $\mathcal{F}$-MS (see. [16]) equipped with the graph $G$ given by $V(G) = \mathcal{P}$ and*

$$E(G) = \{(\mu_n, \mu_n) : n \in \mathbb{N}\} \cup \{(\mu_1, \mu_n) : n \in \mathbb{N}\}.$$

*Define $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ by*

$$\mathcal{V}(\mu_n) = \begin{cases} \mu_1, & if\ n = 1, \\ \mu_{n-1}, & if\ n > 1 \end{cases}.$$

*Let $\hbar = \mu_i, \varsigma = \mu_j$ and suppose $(\hbar, \varsigma) \in E(G)$. Then by definition of $E(G)$, there are two possibilities.*
*(1) $\left(\mu_i, \mu_j\right) = (\mu_n, \mu_n)$ for some $n$. Then $\mathcal{V}(\mu_n)$ equals $\mu_1$ if $n = 1$ and $\mu_{n-1}$, if $n > 1$. In either case $\mathcal{V}(\mu_n) = \mathcal{V}(\mu_n)$, so $(\mathcal{V}(\mu_n), \mathcal{V}(\mu_n))$ is a loop and therefore belongs to $E(G)$.*
*(2) $\left(\mu_i, \mu_j\right) = (\mu_1, \mu_n)$ for some $n$. Then $\mathcal{V}(\mu_n) = \mu_1$ and $\mathcal{V}(\mu_n) = \mu_{n-1}$ for $n > 1$ (if $n = 1$ this reduces to case 1). Hence*

$$(\mathcal{V}(\mu_n), \mathcal{V}(\mu_n)) = (\mu_1, \mu_{n-1}),$$

*and this pair belongs to $E(G)$ because $E(G)$ contains $(\mu_1, \mu_k)$ for every $k \in \mathbb{N}$. Since every edge of $G$ falls into one of these two cases, we conclude*

$$(\hbar, \varsigma) \in E(G) \Longrightarrow (\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(G).$$

*Thus $\mathcal{V}$ preserves edges. We prove that $\mathcal{V}$ satisfies the condition (3.2). Clearly $(\hbar, \varsigma) \in E(G)$ with $\mathcal{V}\hbar \neq \mathcal{V}\varsigma$ if and only if $\hbar = \mu_1$ and $\varsigma = \mu_n$ for some $n > 2$. Let the mapping $\Theta : (0, \infty) \to (1, \infty)$ defined by*

$$\Theta(t) = e^{\sqrt{t}}, \ t > 0.$$

*It is simple to show that $\Theta \in \Psi$. As*

$$\lim_{n \longrightarrow \infty} \frac{\tau(\mathcal{V}(\mu_1), \mathcal{V}(\mu_n))}{\tau(\mu_1, \mu_n)} = 1,$$

*so $\mathcal{V}$ is not a graphic contraction. We now check the graphic $\Theta$-contraction condition. It is enough to check the nontrivial case when $(\hbar, \varsigma) = (\mu_1, \mu_n) \in E(G)$ with $n > 2$. For $n > 2$, we have $\mathcal{V}(\mu_1) = \mu_1$ and $\mathcal{V}(\mu_n) = \mu_{n-1}$, so*

$$\tau(\mathcal{V}(\mu_1), \mathcal{V}(\mu_n)) = \tau(\mu_1, \mu_{n-1}) = e^{\mu_{n-1} - \mu_1} > 0,$$

*and*

$$\tau(\mu_1, \mu_n) = e^{\mu_n - \mu_1} > 0.$$

*Hence, $\Theta$ is evaluated at positive arguments as required. Now compute the ratio that appears in the contraction inequality in this way*

$$\frac{\Theta\left(\tau(\mathcal{V}(\mu_1), \mathcal{V}(\mu_n))\right)}{\Theta\left(\tau(\mu_1, \mu_n)\right)} = e^{\left(\sqrt{\mu_{n-1} - \mu_1} - \sqrt{\mu_{n-1} - \mu_1}\right)} < e^{-1} = e^{-\lambda}.$$

*Since $e^{-\lambda} \in (0, 1)$, this verifies a graphic $\Theta$-contraction condition (3.2) in the nontrivial case.*

**Definition 10.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS. Then any two sequences $\{\hbar_n\}$ and $\{\varsigma_n\}$ are equivalent if $\tau(\hbar_n, \varsigma_n) \to 0$ as $n \to \infty$.*

**Proposition 1.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ be a graphic $\Theta$-contraction. Then $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is graphic $\Theta$-contraction for both $G^{-1}$ and $\widetilde{G}$, it means (3.1) and (3.2) holds for $G^{-1}$ and $\widetilde{G}$.*

*Proof.* As an $\mathcal{F}$-metric is symmetric, so $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is also graphic $\Theta$-contraction for both $G^{-1}$ and $\widetilde{G}$. □

**Theorem 1.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS. Then following conditions are equivalent:*

(i) $G$ is weakly connected,

(ii) for any graphic $\Theta$-contraction $\mathcal{V}:\mathcal{P}\to\mathcal{P}$ and $\hbar,\varsigma\in\mathcal{P}$, $\{\hbar_n\}$ and $\{\varsigma_n\}$ are equivalent and Cauchy,

(iii) for any graphic $\Theta$-contraction $\mathcal{V}:\mathcal{P}\to\mathcal{P}$, Card($Fix\mathcal{V}$)$\leq 1$.

*Proof.* (i)$\Longrightarrow$(ii)

Suppose $G$ is weakly connected $\mathcal{V}:\mathcal{P}\to\mathcal{P}$ is graphic $\Theta$-contraction and $\hbar,\varsigma\in\mathcal{P}$. Then $\mathcal{P}=[\hbar]_{\widetilde{G}}$. Take $\varsigma=\mathcal{V}\hbar\in[\hbar]_{\widetilde{G}}$, so there is a path $\{\hbar_i\}_{i=0}^{N}$ in $\widetilde{G}$ from $\hbar$ to $\varsigma$ with $\hbar_0=\hbar$ and $\hbar_N=\varsigma$ and $(\hbar_{i-1},\hbar_i)\in E(\widetilde{G})$ for all $i=1,2,...,N$. If $\mathcal{V}^{j+1}\hbar=\mathcal{V}^{j}\hbar$ for some $j\in\mathbb{N}$, then the sequence $\{\mathcal{V}^n\hbar\}$ becomes constant sequence and hence it is Cauchy, where $n$ denotes the iteration index of the sequence $\{\mathcal{V}^n\hbar\}$. So assume that $\mathcal{V}^{n+1}\hbar\neq\mathcal{V}^n\hbar$, for all $n\in\mathbb{N}$, then $\tau_n=\tau\left(\mathcal{V}^n\hbar,\mathcal{V}^{n+1}\hbar\right)>0$, for all $n\in\mathbb{N}$. From Proposition 1, $\mathcal{V}:\mathcal{P}\to\mathcal{P}$ is graphic $\Theta$-contraction for $\widetilde{G}$, then we have

$$(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i)\in E(\widetilde{G})$$

consequently

$$\Theta\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)\leq\left[\Theta\left(\tau\left(\mathcal{V}^{n-1}\hbar_{i-1},\mathcal{V}^{n-1}\hbar_i\right)\right)\right]^{\lambda}.$$

Thus continuing in this way, we get

$$\Theta\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)\leq\left[\Theta\left(\tau\left(\hbar_{i-1},\hbar_i\right)\right)\right]^{\lambda^n} \tag{3.3}$$

$\forall\ n\in\mathbb{N}$ with $i=1,2,...,N$. Thus

$$\lim_{n\to\infty}\Theta\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)=1$$

which implies that

$$\lim_{n\to\infty}\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)=0.$$

From the condition $(\Theta_3)$, there exist $0<r<1$ and $l\in(0,\infty]$ such that

$$\lim_{n\to\infty}\frac{\Theta\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)-1}{\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)^r}=l. \tag{3.4}$$

Suppose that $l<\infty$. In this case, let $B=\frac{l}{2}>0$. From the definition of the limit, there exists $n_0\in\mathbb{N}$ such that

$$\left|\frac{\Theta\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)-1}{\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)^r}-l\right|\leq B$$

for all $n>n_0$. It implies

$$\frac{\Theta\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)-1}{\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)^r}\geq l-B=\frac{l}{2}=B$$

for all $n>n_0$. Then

$$n(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right))^r\leq An\left[\Theta\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)-1\right]$$

for all $n>n_0$, where $A=\frac{1}{B}$. Now we assume that $l=\infty$. Let $B>0$ be an arbitrary positive number. From the definition of the limit, there exists $n_1\in\mathbb{N}$ such that

$$B\leq\frac{\Theta\left(\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right)-1}{\left[\tau\left(\mathcal{V}^n\hbar_{i-1},\mathcal{V}^n\hbar_i\right)\right]^r}$$

for all $n > n_1$. It implies that

$$n[\tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)]^r \leq An\left[\Theta(\tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)) - 1\right],$$

for all $n > n_1$, where $A = \frac{1}{B}$. Thus in all cases, there exist a positive integer $A$ and a natural number $n_2 = \max\{n_0, n_1\}$ such that

$$n[\tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)]^r \leq An\left[\Theta(\tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)) - 1\right], \tag{3.5}$$

for all $n > n_2$. From (3.3) and (3.5), we have

$$n[\tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)]^r \leq An\left[(\Theta(\tau(\hbar_{i-1}, \hbar_i)))^{\lambda^n} - 1\right],$$

for all $n > n_2$. Taking $n \to \infty$, we have

$$\lim_{n\to\infty} n\tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)^r = 0,$$

for each $i = 1, 2, ..., N$. Thus for every $i$, there exist a natural number $n_3$ such that

$$n\tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)^r < 1$$

for all $n > n_3$. which is equivalent to

$$\tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i) < \frac{1}{n^{\frac{1}{r}}}. \tag{3.6}$$

Summing those $N$ inequalities over $i$ (path edges) with the same fixed $n$ gives

$$\sum_{i=1}^{N} \tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i) < \sum_{i=1}^{N} \frac{1}{n^{\frac{1}{r}}} = \frac{N}{n^{\frac{1}{r}}}$$

for all $n > n_3$. Since $N$ is fixed and $1/r > 0$, we have

$$\sum_{i=1}^{N} \tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i) < \sum_{i=1}^{N} \frac{1}{n^{\frac{1}{r}}} = \frac{N}{n^{\frac{1}{r}}} \to 0 \text{ as } n \to \infty.$$

Hence, there exists a sufficiently large natural number $n_4 \in \mathbb{N}$ such that for all $n > n_4 > n_3$, we have

$$\sum_{i=1}^{N} \tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i) < \delta. \tag{3.7}$$

Now let $(\xi, \alpha) \in \mathcal{F} \times [0, +\infty)$ be such that $(D_3)$ is satisfied and $\epsilon > 0$ be fixed. From $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \text{ implies } \xi(t) < \xi(\epsilon) - \alpha. \tag{3.8}$$

Then by (3.7), (3.8) and $(\mathcal{F}_1)$, we get

$$\xi\left(\sum_{i=1}^{N} \tau(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i)\right) < \xi(\epsilon) - \alpha, \tag{3.9}$$

for $n > n_4$. Using $(D_3)$ and (3.9), we get

$$\tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma) > 0,\ n > n_3 \Longrightarrow \xi\left(\tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma)\right) \leq \xi\left(\sum_{i=1}^{N} \tau\left(\mathcal{V}^n\hbar_{i-1}, \mathcal{V}^n\hbar_i\right)\right) + \alpha < \xi(\epsilon)$$

which, from $(\mathcal{F}_1)$, gives that

$$\tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma) < \epsilon$$

$n > n_4$. Thus $\tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma) \to 0$ as $n \to \infty$ and hence $\{\mathcal{V}^n\hbar\}$ is Cauchy Sequence. Now, we show that (ii) $\Longrightarrow$ (iii). Let $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ be a graphic $\Theta$-contraction and $\hbar, \varsigma \in Fix\mathcal{V}$. By (ii), $\{\hbar_n\}$ and $\{\varsigma_n\}$ are equivalent. Then we obtain

$$\tau(\hbar, \varsigma) = \tau(\mathcal{V}^n\hbar, \mathcal{V}^n\varsigma) \to 0$$

as $n \to \infty$, i.e, $\hbar = \varsigma$. In the end, we show that (iii) $\Longrightarrow$ (i). We suppose on the contrary that $G$ is not weakly connected, i.e, $\widetilde{G}$ is disconnected. Assume that $\exists\ \hbar_0 \in \mathcal{P}$ such that both sets $[\hbar_0]_{\widetilde{G}} \neq \emptyset$ and $\mathcal{P}-$ $[\hbar_0]_{\widetilde{G}} \neq \emptyset$. Suppose $\varsigma_0 \in \mathcal{P}-\ [\hbar_0]_{\widetilde{G}}$ and define

$$\mathcal{V}\hbar = \hbar_0 \text{ if } \hbar \in [\hbar_0]_{\widetilde{G}}\,,\ \ \mathcal{V}\hbar = \varsigma_0 \text{ if } \hbar \in \mathcal{P}-\ [\hbar_0]_{\widetilde{G}}\,.$$

Therefore, $Fix(\mathcal{V}) = \{\hbar_0, \varsigma_0\}$ . Now, we prove that $\mathcal{V}$ is graphic $\Theta$-contraction. Assume $(\hbar, \varsigma) \in E(G)$, so $[\hbar]_{\widetilde{G}} = [\varsigma]_{\widetilde{G}}$, i.e., $\hbar, \varsigma \in [\hbar_0]_{\widetilde{G}}$ or $\hbar, \varsigma \in \mathcal{P}-\ [\hbar_0]_{\widetilde{G}}$. Then, we have $\mathcal{V}\hbar = \mathcal{V}\varsigma$, so $(\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(G)$ as $\triangle \subset E(G)$. Thus Equation (3.1) holds. Also as there is no $(\hbar, \varsigma) \in E(G)$ with $\mathcal{V}\hbar \neq \mathcal{V}\varsigma$, therefore, inequality (3.2) is satisfied. Thus $\mathcal{V}$ is graphic $\Theta$-contraction having two FPs that contravenes (iii). Thus $G$ is necessarily weakly connected. □

**Corollary 1.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS equipped with a weakly connected graph $G$. Then for any graphic $\Theta$-contraction $\mathcal{V} : \mathcal{P} \to \mathcal{P}$, there exists $\hbar^* \in \mathcal{P}$ such that* $\lim_{n\to\infty} \mathcal{V}^n\hbar = \hbar^*$ *for all $\hbar \in \mathcal{P}$.*

*Proof.* Fix $\hbar \in \Xi$. By Theorem 1, $\{\mathcal{V}^n\hbar\}$ is $\mathcal{F}$-Cauchy sequence in $\Xi$. Since $(\Xi, \tau)$ is an $\mathcal{F}$-complete $\mathcal{F}$-MS, so there exists a point $\hbar^* \in \Xi$ such that $\lim_{n\to\infty} \mathcal{V}^n\hbar = \hbar^*$. Now, for any $\varsigma \in \Xi$, Theorem 1 implies that sequences $\{\mathcal{V}^n\varsigma\}$ and $\{\mathcal{V}^n\hbar\}$ are equivalent. Therefore $\{\mathcal{V}^n\varsigma\}$ also converges to $\hbar^*$. Thus $\lim_{n\to\infty} \mathcal{V}^n\varsigma = \hbar^*$ for all $\varsigma \in \Xi$. □

**Theorem 2.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS equipped with a graph $G$ and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ be a graphic $\Theta$-contraction. Then $[\hbar_0]_{\widetilde{G}}$ is $\mathcal{V}$-invariant and $\mathcal{V}|_{[\hbar_0]_{\widetilde{G}}}$ is graphic $\Theta$-contraction for $\widetilde{G}_{\hbar_0}$, where $\hbar_0 \in \Xi$ and $\mathcal{V}(\hbar_0) \in [\hbar_0]_{\widetilde{G}}$. Furthermore, if $\hbar, \varsigma \in [\hbar_0]_{\widetilde{G}}$, then $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$ are equivalent and Cauchy.*

*Proof.* Let $\hbar \in [\hbar_0]_{\widetilde{G}}$, so there is a path $\{\hbar_i\}_{i=0}^N$ in $\widetilde{G}$ from $\hbar_0$ to $\hbar$, that is $\hbar = \hbar_N$ and $(\hbar_{i-1}, \hbar_i) \in E(\widetilde{G})$. By proposition 1, every $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is graphic $\Theta$-contraction for graph $\widetilde{G}$ too. So we get $(\mathcal{V}\hbar_{i-1}, \mathcal{V}\hbar_i) \in E(\widetilde{G})$ for all $i = 1, 2, ...N$. Therefore $\{\mathcal{V}\hbar_i\}_{i=0}^N$ is a path in $\widetilde{G}$ from $\mathcal{V}\hbar_0$ to $\mathcal{V}\hbar$. Then $\mathcal{V}\hbar \in [\mathcal{V}\hbar_0]_{\widetilde{G}}$. But $\mathcal{V}\hbar_0 \in [\hbar_0]_{\widetilde{G}}$. So $[\mathcal{V}\hbar_0]_{\widetilde{G}} = [\hbar_0]_{\widetilde{G}}$. Hence $\mathcal{V}\hbar \in [\hbar_0]_{\widetilde{G}}$. It follows that $[\hbar_0]_{\widetilde{G}}$ is $\mathcal{V}$-invariant. Now, let $(\hbar, \varsigma) \in E(\widetilde{G}_{\hbar_0})$. Accordingly, there exists a path $\{\hbar_i\}_{i=0}^N$ in $\widetilde{G}$ from $\hbar_0$ to $\varsigma$ such that $\hbar_{N-1} = \hbar$. Following the approach from the beginning of the proof, it follows that $\{\mathcal{V}\hbar_i\}_{i=0}^N$ is a path in $\widetilde{G}$ from $\mathcal{V}\hbar_0$ to $\mathcal{V}\varsigma$. Since $\mathcal{V}\hbar_0 \in [\hbar_0]_{\widetilde{G}}$. Thus, we obtain a path $\{\varsigma_i\}_{i=0}^M$ in $\widetilde{G}$ from $\hbar_0$ to $\mathcal{V}\hbar_0$. It follows that $(\varsigma_0, \varsigma_1, ..., \varsigma_M, \mathcal{V}\hbar_1, \mathcal{V}\hbar_2, ..., \mathcal{V}\hbar_N)$ is a path in $\widetilde{G}$ from $\hbar_0$ to $\mathcal{V}\varsigma$. Notably $(\mathcal{V}\hbar_{N-1}, \mathcal{V}\hbar_N) \in E(\widetilde{G_{\hbar_0}})$. It means $(\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(\widetilde{G_{\hbar_0}})$. Since $E(\widetilde{G_{\hbar_0}}) \subseteq E(\widetilde{G})$, that is $\widetilde{G}_{\hbar_0}$ is a subgraph of $G$ and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is graphic $\Theta$-contraction, so $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is graphic $\Theta$-contraction for the graph $\widetilde{G}_{\hbar_0}$ as well.Thus $\mathcal{V}|_{[\hbar_0]_{\widetilde{G}}}$ is graphic $\Theta$-contraction for $\widetilde{G}_{\hbar_0}$. Eventually, from the connectedness of $\widetilde{G}_{\hbar_0}$ and Theorem 1, $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$ are equivalent and Cauchy. □

**Theorem 3.** *Let $(\mathcal{P},\tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS equipped with a graph G satisfying the property (P), $\mathcal{V}:\mathcal{P}\to\mathcal{P}$ is a graphic $\Theta$-contraction and the set*

$$\mathcal{P}_{\mathcal{V}}=\{\hbar\in\mathcal{P}\colon(\hbar,\mathcal{V}\hbar)\in E(G)\}$$

*is nonempty. Then*

(i) Card($Fix\mathcal{V}$)= $Card\left\{[\hbar]_{\widetilde{G}}:\hbar\in\mathcal{P}_{\mathcal{V}}\right\}$,
(ii) $Fix\mathcal{V}\neq\emptyset\Longleftrightarrow\mathcal{P}_{\mathcal{V}}\neq\emptyset$,
(iii) $\mathcal{V}$ possesses a unique FP if and only if there exists $\hbar_0\in\mathcal{P}_{\mathcal{V}}$ such that $\mathcal{P}_{\mathcal{V}}\subseteq[\hbar_0]_{\widetilde{G}}$,
(iv) $\mathcal{V}|_{[\hbar]_{\widetilde{G}}}$ is Picard Operator, for any $\hbar\in\mathcal{P}_{\mathcal{V}}$,
(v) if $G$ is weakly connected and $\mathcal{P}_{\mathcal{V}}\neq\emptyset$, then $\mathcal{V}$ is a Picard Operator,
(vi) if $\mathcal{P}'=\cup\left\{[\hbar]_{\widetilde{G}}:\hbar\in\mathcal{P}_{\mathcal{V}}\right\}$, then the restriction $\mathcal{V}|_{\mathcal{P}'}$ is weakly Picard Operator,
(vii) if $\mathcal{V}\subseteq E(G)$, consequently, $\mathcal{V}$ is classified as a weakly Picard Operator.

*Proof.* We commence by proving (iv) that is $\mathcal{V}|_{[\hbar_0]_{\widetilde{G}}}$ is Picard Operator, for any $\hbar\in\mathcal{P}_{\mathcal{V}}$. Let $\hbar\in\mathcal{P}_{\mathcal{V}}$, then $(\hbar,\mathcal{V}\hbar)\in E(G)$ which implies that $\mathcal{V}\hbar\in[\hbar]_{\widetilde{G}}$. Then the sequence $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$, for $\varsigma\in\mathcal{P}$ are equivalent and Cauchy by Theorem 2. Now as $(\mathcal{P},\tau)$ be an $\mathcal{F}$-complete, so there exists $\hbar^*\in\mathcal{P}$ such that $\mathcal{V}^n\hbar\to\hbar^*\longleftarrow\mathcal{V}^n\varsigma$ as $n\to\infty$. As $(\hbar,\mathcal{V}\hbar)\in E(G)$, so from (3.1), we have

$$(\mathcal{V}^n\hbar,\mathcal{V}^{n+1}\hbar)\in E(G), \tag{3.10}$$

$\forall n\in\mathbb{N}$. As $G$ satisfies the property (P), so there exists a subsequence $\{\mathcal{V}^{j_n}\hbar\}$ of $\{\mathcal{V}^n\hbar\}$ such that

$$(\mathcal{V}^{j_n}\hbar,\hbar^*)\in E(G),$$

$\forall n\in\mathbb{N}$. Now from (3.10), there exists a path $\left(\hbar,\mathcal{V}\hbar,\mathcal{V}^2\hbar,...,\mathcal{V}^{j_1}\hbar,\hbar^*\right)$ in $G$ from $\hbar$ to $\hbar^*$. So $\hbar^*\in[\hbar]_{\widetilde{G}}$. Now since the mapping $\mathcal{V}:\Xi\to\Xi$ is a graphic $\Theta$-contraction, so for all $n\in\mathbb{N}$, we have

$$\begin{aligned}\Theta\left(\tau\left(\mathcal{V}^{j_n+1}\hbar,\mathcal{V}\hbar^*\right)\right) &\leq \left[\Theta\left(\tau\left(\mathcal{V}^{j_n}\hbar,\hbar^*\right)\right)\right]^{\lambda}\\ &< \Theta\left(\tau\left(\mathcal{V}^{j_n}\hbar,\hbar^*\right)\right),\end{aligned}$$

because $\lambda<1$. By $(\mathcal{F}_1)$, we have

$$\tau\left(\mathcal{V}^{j_n+1}\hbar,\mathcal{V}\hbar^*\right)<\tau\left(\mathcal{V}^{j_n}\hbar,\hbar^*\right),$$

for all $n\in\mathbb{N}$. Taking the limit as $n\to\infty$, we have

$$\tau\left(\hbar^*,\mathcal{V}\hbar^*\right)=0,$$

which implies $\hbar^*=\mathcal{V}\hbar^*$. Thus $\mathcal{V}|_{[\hbar]_{\widetilde{G}}}$ is Picard Operator. □

Now we prove the condition (v). Let $\mathcal{P}_{\mathcal{V}}\neq\emptyset$ and $\hbar\in\mathcal{P}_{\mathcal{V}}$. Also suppose that $G$ is weakly connected. Then $\mathcal{P}=[\hbar]_{\widetilde{G}}$ and $\mathcal{V}$ is Picard Operator. Condition (vi) is direct consequence of (iv).

Now we prove the condition (vii). Let $\mathcal{V}\subseteq E(G)$. This implies $\mathcal{P}=\mathcal{P}_{\mathcal{V}}$ which gives $\mathcal{P}'=\mathcal{P}$. Thus $\mathcal{V}$ is weakly Picard Operator directly from the condition (iv).

Now we prove the condition (i). For this, we define $\wp : Fix\mathcal{V} \to \Phi$ by $\wp(\hbar) = [\hbar]_{\widetilde{G}}$, $\forall \hbar \in Fix\mathcal{V}$, where

$$\Phi = \left\{ [\hbar]_{\widetilde{G}} : \hbar \in \Xi_{\mathcal{V}} \right\}.$$

Then we just have to prove that $\wp$ is bijective mapping. Now let $\hbar \in \mathcal{P}_{\mathcal{V}}$. Then from (iv), $\mathcal{V}|_{[\hbar]_{\widetilde{G}}}$ is Picard Operator. Now let

$$\hbar^* = \lim_{n\to\infty} \mathcal{V}^n \hbar.$$

Then

$$\hbar^* \in Fix\mathcal{V} \cap [\hbar]_{\widetilde{G}}$$

and $\wp(\hbar^*) = [\hbar^*]_{\widetilde{G}} = [\hbar]_{\widetilde{G}}$. Thus $\wp$ is onto. Also suppose $\hbar_1, \hbar_2 \in Fix\mathcal{V}$ with $[\hbar_1]_{\widetilde{G}} = [\hbar_2]_{\widetilde{G}}$. Then $\hbar_2 \in [\hbar_1]_{\widetilde{G}}$. Now from (iv),

$$\lim_{n\to\infty} \mathcal{V}^n \hbar_2 \in Fix\mathcal{V} \cap [\hbar_1]_{\widetilde{G}} = \{\hbar_1\}.$$

But $\mathcal{V}^n \hbar_2 = \hbar_2$, $\forall n \in \mathbb{N}$. Hence we have $\hbar_1 = \hbar_2$ that is $\wp$ is one-one. And finally $\wp$ is bijective. Lastly, (i) directly follows from (ii) and (iii).

**Corollary 2.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS equipped with a graph $G$ satisfying the property (P). Then these conditions are equivalent*

(i) $G$ is weakly connected,

(ii) $\mathcal{V}$ is Picard Operator for every graphic $\Theta$-contraction $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ such that $(\hbar_0, \mathcal{V}\hbar_0) \in E(G)$ for some $\hbar_0 \in \mathcal{P}$

(iii) for any graphic $\Theta$-contraction $\mathcal{V} : \mathcal{P} \to \mathcal{P}$, Card$(Fix\mathcal{V}) \leq 1$.

*Proof.* From condition (v) of Theorem (3), (i)$\Longrightarrow$ (ii). Now we prove (ii)$\Longrightarrow$ (iii). Suppose $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is graphic $\Theta$-contraction. If $\mathcal{P}_{\mathcal{V}} = \emptyset$, then it is obvious that Card$(Fix\mathcal{V}) \leq 1$. Now if $\mathcal{P}_{\mathcal{V}} \neq \emptyset$, then by (ii) $Fix\mathcal{V}$ is singleton. Thus Card$(Fix\mathcal{V}) \leq 1$. Lastly, (iii)$\Longrightarrow$(i) follows directly from Theorem (3). □

**Definition 11.** *( [20]) Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS equipped with a graph $G$ and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$. Then $\mathcal{V}$ is said to be orbitally continuous if $\forall \hbar, \varsigma \in \mathcal{P}$ and any sequence $\{j_n\}$ of positive numbers, then $\mathcal{V}^{j_n}\hbar \to \varsigma$ implies $\mathcal{V}(\mathcal{V}^{j_n}\hbar) \to \mathcal{V}\varsigma$ as $n \to \infty$.*

**Definition 12.** *( [20]) Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS equipped with a graph $G$ and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$. Then $\mathcal{V}$ is said to be G-continuous if for $\hbar \in \mathcal{P}$ and a sequence $\{\hbar_n\}$ with $\hbar_n \to \hbar$ as $n \to \infty$ and $(\hbar_n, \hbar_{n+1}) \in E(G)$, then $\mathcal{V}\hbar_n \to \mathcal{V}\hbar$ as $n \to \infty$.*

**Definition 13.** *( [20]) Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS equipped with a graph $G$ and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$. Then $\mathcal{V}$ is said to be orbitally G- continuous if $\forall \hbar, \varsigma \in \mathcal{P}$ and any sequence $\{j_n\}$ of positive numbers $\mathcal{V}^{j_n}\hbar \to \varsigma$ and $(\mathcal{V}^{j_n}\hbar, \mathcal{V}^{j_n+1}\hbar) \in E(G)$ implies $\mathcal{V}(\mathcal{V}^{j_n}\hbar) \to \mathcal{V}\varsigma$ as $n \to \infty$.*

**Remark 1.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-MS endowed with a graph $G$ and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$. Then Continuity of $\mathcal{V}$ implies G-Continuity of $\mathcal{V}$ implies orbital G-continuity of $\mathcal{V}$. Also Continuity of $\mathcal{V}$ implies orbital Continuity of $\mathcal{V}$ implies orbital G-continuity of $\mathcal{V}$.*

**Theorem 4.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS equipped with a graph $G$ and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is orbitally G-continuous graphic $\Theta$-contraction. Then these conditions hold:*

(i) $\mathcal{P}_{\mathcal{V}} \neq \emptyset \Longleftrightarrow Fix(\mathcal{V}) \neq \emptyset$,

(ii) for any $\hbar \in \mathcal{P}_{\mathcal{V}}$ and $\varsigma \in [\hbar]_{\widetilde{G}}$, the sequence $\{\mathcal{V}^n\varsigma\}$ converges to the FP of $\mathcal{V}$ and $\lim_{n\to\infty} \mathcal{V}^n\varsigma$ does not depend on $\varsigma$,

(iii) if $G$ is weakly connected and $\mathcal{P}_{\mathcal{V}} \neq \emptyset$, then $\mathcal{V}$ is Picard Operator,

(iv) if $\mathcal{V} \subseteq E(G)$, then $\mathcal{V}$ is weakly Picard Operator.

*Proof.* We prove (i)$\Longrightarrow$ (ii). Let $\hbar \in \mathcal{P}$ with $(\hbar, \mathcal{V}\hbar) \in E(G)$ and $\varsigma \in [\hbar]_{\widetilde{G}}$. From Theorem (2), $\{\mathcal{V}^n\hbar\}$ and $\{\mathcal{V}^n\varsigma\}$ converges to a point $\hbar^*$. Also $\left(\mathcal{V}^n\hbar, \mathcal{V}^{n+1}\hbar\right) \in E(G)$, $\forall n \in \mathbb{N}$. Now using the orbitally $G$-continuity of $\mathcal{V}$ we have

$$\mathcal{V}(\mathcal{V}^n\hbar) \to \mathcal{V}(\hbar^*).$$

Also since $\mathcal{V}(\mathcal{V}^n\hbar) = \mathcal{V}^{n+1}\hbar \to \hbar^*$, thus we obtain $\mathcal{V}(\hbar^*) = \hbar^*$. Thus (i) implies (ii). Now since $\Delta \subseteq E(G)$, so (ii) implies (i). Also since $\mathcal{V} \subseteq E(G)$ so $\mathcal{P}_{\mathcal{V}} = \mathcal{P}$, so (iv) follows directly from (ii). Now if $\mathcal{P}_{\mathcal{V}} \neq \emptyset$ and $\hbar_0 \in \mathcal{P}_{\mathcal{V}}$, then $[\hbar_0]_{\widetilde{G}} = \mathcal{P}$, so $\mathcal{V}$ is a Picard Operator by (ii) which proved (iii). □

**Corollary 3.** *Let $(\mathcal{P}, \tau)$ be an $\mathcal{F}$-complete $\mathcal{F}$-MS equipped with a graph $G$ and $\mathcal{V} : \mathcal{P} \to \mathcal{P}$. Then these conditions are equivalent.*

(i) $G$ is weakly connected,

(ii) every orbitally continuous graphic $\Theta$-contraction $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ is Picard Operator,

(iii) for every orbitally continuous graphic $\Theta$-contraction $\mathcal{V} : \mathcal{P} \to \mathcal{P}$,Card($Fix\mathcal{V}$)$\leq 1$.

## 4. Derivation of known results

In this section, we demonstrate how several well-known fixed-point results from the literature emerge as special cases of our main theorem. This not only validates the generality of our approach but also highlights the unifying framework we have introduced. More importantly, we emphasize how our results go beyond mere combination or slight modification of existing work, offering genuine and nontrivial extensions.

(i) Considering the function $\Theta(t) = e^{\sqrt{t}}$ and and taking $E(G) = \Xi\times\Xi$ in Theorem (3), the graphical structure becomes trivial, and the contraction condition reduces to the one used by Jleli et al. [16]. Their result is thus recovered as a particular instance of our theorem. However, our framework is strictly more general because we work in a graph controlled $\mathcal{F}$-MS, which allows for a finer analysis of distances only along admissible paths, and because the use of a wider class of functions $\xi$ in the $\mathcal{F}$-metric context permits handling a broader family of nonlinear contractions. Hence, our theorem provides a graph-aware generalization of their result, applicable even when the underlying space is not complete in the usual sense but only complete along graph paths.

(ii) If we take $\xi(t) = \ln t$ and $\alpha = 0$ in Definition (5), then the notion of $\mathcal{F}$-MS reduces to classical MS and from our Theorem (3), we get the chief result of Onsod et al. [9]. The novelty of our work lies in the fact that we introduce and employ the $\mathcal{F}$-metric with a variable scaling parameter $\alpha$, which can model phenomena where distances behave sub or super additively. Even when specialized to $\alpha = 0$, the proof technique via $\Theta$-contractions in a graph structured environment is new and offers a different perspective on FP problems. Thus, our contribution is not a trivial re-proof of known results, but a unified approach that recovers classical theorems as limiting cases.

(iii) By selecting $\xi(t) = \ln t$ and $\alpha \geq 1$ in Definition (5), then the notion of $\mathcal{F}$-MS reduces to $b$-MS. Then Theorem (3) reduces to a result of Mohanta et al. [23]. The significant generalization here is twofold. First, functional flexibility: our use of an arbitrary continuous, strictly increasing function $\xi$ allows us to encode various nonlinear distortions of distances, while the $b$-metric case corresponds only to the logarithmic distortion. Second, graph based control: the incorporation of a directed graph $G$ enables us to impose constraints on the contraction condition, which need only hold along edges of $G$, a stricter and more structured requirement than in the usual $b$-metric setting.

## 5. Application to image denoising using graph-based filtering

Image denoising can be viewed as a problem of finding a "clean" image from a noisy one. Here's how our main Theorem 1 can be applied.

Let the image be represented as a graph $G = (\mathcal{P}, E)$, where $\mathcal{P}$ is the set of pixels and $E$ represents edges connecting neighboring pixels (e.g., based on spatial proximity or intensity similarity). Since the set $P$ of pixels is finite and there is a path between any two pixels in the graph $G = (\mathcal{P}, E)$, so the first two conditions of Theorem 1 are satisfied. Let $\mathcal{P}$ be the set of all pixels in the image. For $\hbar, \varsigma \in \mathcal{P}$, define

$$\tau(\hbar, \varsigma) = |I(\hbar) - I(\varsigma)|$$

where $I(\hbar)$ is the intensity value of pixel $\hbar \in \mathcal{P}$. To introduce the $\mathcal{F}$-metric structure, we choose the admissible pair $\xi(t) = \ln t$ and $\alpha = 0$. With this choice, any metric $\tau$ on a set automatically becomes an $\mathcal{F}$-metric (Jleli et al. [16]). Therefore $(\mathcal{P}, \tau)$ is an $\mathcal{F}$-MS. Define a mapping $\mathcal{V} : \mathcal{P} \to \mathcal{P}$ that updates the intensity of each pixel based on its neighbors. For example

$$\mathcal{V}(\hbar) = \frac{2I(\hbar) + \sum\limits_{\varsigma \in N(\hbar)} I(\varsigma)}{2 + |N(\hbar)|}$$

where $N(\hbar)$ is the set of neighbors of $\hbar$ in the graph $G$. The mapping $\mathcal{V}$ can be interpreted as a graph-based filter that smooths the image by averaging neighboring pixel intensities. To apply our main corollary, we need to show that $\mathcal{V}$ is a contraction mapping.

- Graph Preservation: If $(\hbar, \varsigma) \in E(G) \implies (\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(G)$. This holds if $\mathcal{V}$ preserves the graph structure (e.g., by only averaging over connected pixels).
- For all $\hbar, \varsigma \in \mathcal{P}$ with $(\hbar, \varsigma) \in E(G)$, we have

$$\Theta(\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma)) \leq [\Theta(\tau(\hbar, \varsigma))]^{\lambda}$$

where $\lambda \in (0, 1)$ and $\Theta : \mathbb{R}^+ \to (1, \infty)$. This ensures that $\mathcal{V}$ reduces the distance between connected pixels. Starting from the noisy image $I_0$, the sequence $I_{n+1} = \mathcal{V}(I_n)$ converges to $I^*$. In this way, the condition (iii) of Theorem 1 is satisfied and $\mathcal{V}$ has a unique FP $I^*$, which represents the denoised image.

**Example 6.** *Consider a* $2 \times 2$ *noisy image:*

$$\begin{pmatrix} 100 & 130 \\ 140 & 120 \end{pmatrix}_{2\times 2}.$$

*Let*

$$\mathcal{P} = \{p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}\}$$

*denote the set of pixels, where each pixel is assigned its corresponding intensity*

$$\begin{pmatrix} p_{1,1} = 100 & p_{1,2} = 130 \\ p_{2,1} = 140 & p_{2,2} = 120 \end{pmatrix}_{2\times 2}.$$

*Edges E connect neighboring pixels. Since this is a $2 \times 2$ grid, we define edges based on spatial proximity (spatial proximity refers to the measure of how close two or more objects, points, or sets are within a given space).*

Horizontal Edges (Row-wise neighbors)

- $(p_{1,1}, p_{1,2})$
- $(p_{2,1}, p_{2,2})$

Vertical Edges (Column-wise neighbors)

- $(p_{1,1}, p_{2,1})$
- $(p_{1,2}, p_{2,2})$.

Thus, $E(G)$=$(p_{1,1}, p_{1,2})$,$(p_{2,1}, p_{2,2})$,$(p_{1,1}, p_{2,1})$,$(p_{1,2}, p_{2,2})$ are the only edges of the graph $G = (\mathcal{P}, E(G))$. Since there is a path between any two pixels in the graph $G$, so $G$ is weakly connected. Hence the condition (i) of the Theorem 1 is satisfied. Moreover, since $\mathcal{P}$ is finite, so the condition (ii) of Theorem 1 is fulfilled.

Define the distance function in this way

$$\tau(\hbar, \varsigma) = |I(\hbar) - I(\varsigma)|$$

which is the absolute difference in intensity, that is,

$$\tau(p_{1,1}, p_{1,2}) = |100 - 130| = 30$$

$$\tau(p_{2,1}, p_{2,2}) = |140 - 120| = 20$$

$$\tau(p_{1,1}, p_{2,1}) = |100 - 140| = 40$$

$$\tau(p_{2,1}, p_{2,2}) = |130 - 120| = 10.$$

Then $(\mathcal{P}, \tau)$ is an $\mathcal{F}$-complete $\mathcal{F}$-MS with $\alpha = \ln 1$. Define the mapping $\mathcal{V}$ by using the averaging rule, that is,

$$\mathcal{V}(\hbar) = \frac{2I(\hbar) + \sum\limits_{\varsigma \in N(\hbar)} I(\varsigma)}{2 + |N(\hbar)|}.$$

Applying this formula:

(1) For $p_{1,1}$, the neighbors are $p_{1,2}$ and $p_{2,1}$. So

$$\mathcal{V}(p_{1,1}) = \frac{2(100) + 130 + 140}{2 + 2} = 92.5.$$

(2) For $p_{1,2}$, the neighbors are $p_{1,1}$ and $p_{2,2}$. So

$$\mathcal{V}(p_{1,2}) = \frac{2(130) + 100 + 120}{2 + 2} = 120.$$

(3) For $p_{2,1}$, the neighbors are $p_{1,1}$ and $p_{2,2}$. So

$$\mathcal{V}(p_{2,1}) = \frac{2(140) + 100 + 120}{2 + 2} = 125.$$

(4) For $p_{2,2}$, the neighbors are $p_{1,2}$ and $p_{2,1}$. So

$$\mathcal{V}(p_{2,2}) = \frac{2(120) + 130 + 140}{2 + 2} = 127.5.$$

Updated intensities:

$$\mathcal{V}(I) = \begin{pmatrix} 92.5 & 120 \\ 125 & 127.5 \end{pmatrix}_{2\times 2}.$$

Since $\mathcal{V}$ modifies intensities but not pixel positions, the graph structure is preserved

$$(\hbar, \varsigma) \in E(G) \Longrightarrow (\mathcal{V}\hbar, \mathcal{V}\varsigma) \in E(G).$$

Thus, the edge-preservation condition is satisfied. Now computing the new distances

$$\tau(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{1,2})) = |92.5 - 120| = 27.5$$

$$\tau(\mathcal{V}(p_{2,1}), \mathcal{V}(p_{2,2})) = |125 - 127.5| = 2.5$$

$$\tau(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{2,1})) = |92.5 - 125| = 32.5$$

$$\tau(\mathcal{V}(p_{1,2}), \mathcal{V}(p_{2,2})) = |130 - 127| = 7.5.$$

Now define $\Theta : \mathbb{R}^+ \to (1, \infty)$ by $\Theta(t) = e^{\sqrt{t}}$, for $t > 0$. Then $\Theta$ belongs to $\Psi$ and satisfies conditions $(\Theta_1)$-$(\Theta_3)$. Now we discuss the following four distinct cases.

Case 1. When $\hbar = p_{1,1}$ and $\varsigma = p_{1,2}$. Then $(p_{1,1}, p_{1,2}) = (100, 130) \in E(G)$, implies

$$(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{1,2})) = (92.5, 120) \in E(G).$$

Now $\tau\left(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{1,2})\right) = 27.5 \neq 0$ implies

$$\begin{aligned}
\Theta(\tau(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{1,2}))) &= e^{\sqrt{\tau(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{1,2}))}} \\
&\leq e^{\sqrt{27.5}} = e^{5.2440} \\
&\leq e^{0.98(5.4772)} = e^{0.98(\sqrt{30})} \\
&= \left[e^{\sqrt{\tau(p_{1,1}, p_{1,2})}}\right]^{\lambda} \\
&= \left[\Theta(\tau(p_{1,1}, p_{1,2}))\right]^{\lambda}.
\end{aligned}$$

Case 2. When $\hbar = p_{2,1}$ and $\varsigma = p_{2,2}$. Then $(p_{2,1}, p_{2,2}) = (140, 120) \in E(G)$, implies $(\mathcal{V}(p_{2,1}), \mathcal{V}(p_{2,2})) = (125, 127.5) \in E(G)$. Now $\tau\left(\mathcal{V}(p_{2,1}), \mathcal{V}(p_{2,2})\right) = 2.5 \neq 0$ implies

$$\Theta(\tau(\mathcal{V}(p_{2,1}), \mathcal{V}(p_{2,2}))) = e^{\sqrt{\tau(\mathcal{V}(p_{2,1}), \mathcal{V}(p_{2,2}))}}$$

$$
\begin{aligned}
&\leq e^{\sqrt{2.5}} = e^{1.5811} \\
&< e^{0.98(4.4721)} = e^{0.98(\sqrt{20})} \\
&= \left[e^{\sqrt{\tau(p_{2,1},p_{2,2})}}\right]^{\lambda} \\
&= \left[\Theta(\tau(p_{2,1},p_{2,2}))\right]^{\lambda}.
\end{aligned}
$$

Case 3. When $\hbar = p_{1,1}$ and $\varsigma = p_{2,1}$. Then $(p_{1,1}, p_{2,1}) = (100, 140) \in E(G)$, implies $(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{2,1})) = (92.5, 125) \in E(G)$. Now $\tau(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{2,1})) = 32.5 \neq 0$ implies

$$
\begin{aligned}
\Theta(\tau(\mathcal{V}(p_{1,1}), \mathcal{V}(p_{2,1}))) &= e^{\sqrt{\tau(\mathcal{V}(p_{2,1}),\mathcal{V}(p_{2,2}))}} \\
&\leq e^{\sqrt{32.5}} = e^{5.7009} \\
&< e^{0.98(6.3246)} = e^{0.98(\sqrt{40})} \\
&= \left[e^{\sqrt{\tau(p_{1,1},p_{2,1})}}\right]^{\lambda} \\
&= \left[\Theta(\tau(p_{1,1},p_{2,1}))\right]^{\lambda}.
\end{aligned}
$$

Case 4. When $\hbar = p_{1,2}$ and $\varsigma = p_{2,2}$. Then $(p_{1,2}, p_{2,2}) = (130, 120) \in E(G)$, implies $(\mathcal{V}(p_{1,2}), \mathcal{V}(p_{2,2})) = (120, 127.5) \in E(G)$. Now $\tau(\mathcal{V}(p_{1,2}), \mathcal{V}(p_{2,2})) = 7.5 \neq 0$ implies

$$
\begin{aligned}
\Theta(\tau(\mathcal{V}(p_{1,2}), \mathcal{V}(p_{2,2}))) &= e^{\sqrt{\tau(\mathcal{V}(p_{1,2}),\mathcal{V}(p_{2,2}))}} \\
&\leq e^{\sqrt{7.5}} = e^{2.7386} \\
&< e^{0.98(3.1622)} = e^{0.98(\sqrt{10})} \\
&= \left[e^{\sqrt{\tau(p_{1,2},p_{2,2})}}\right]^{\lambda} \\
&= \left[\Theta(\tau(p_{1,2},p_{2,2}))\right]^{\lambda}.
\end{aligned}
$$

Hence the condition (iii) of Theorem 1 are satisfied for $\lambda = 0.98 \in (0, 1)$. Hence, the mapping $\mathcal{V}$ is a graphic Θ-contraction and has a unique FP.

$$
I^* = \begin{pmatrix} 122.5 & 122.5 \\ 122.5 & 122.5 \end{pmatrix}_{2\times 2}.
$$

In this Example, the value $\lambda = 0.98$ is verified a posteriori for the given image. In general, the averaging operator

$$
\mathcal{V}(\hbar) = \frac{2I(\hbar) + \sum_{\varsigma\in N(\hbar)} I(\varsigma)}{2 + |N(\hbar)|}
$$

is non-expansive in the usual metric, i.e., $\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma) \leq \tau(\hbar, \varsigma)$. However, under the transformation $\Theta(t) = e^{\sqrt{t}}$, it becomes a strict Θ-contraction for connected graphs and non-constant images, because the nonlinear scaling of Θ amplifies even small reductions in intensity differences. An a priori estimate of $\lambda$ can be obtained from

$$
\lambda \approx \max_{(\hbar,\varsigma)\in E(G)} \frac{\ln\Theta(\tau(\mathcal{V}\hbar, \mathcal{V}\varsigma))}{\ln\Theta(\tau(\hbar, \varsigma))},
$$

which is typically $< 1$ for real images and for any nontrivial averaging window. This ensures that the Θ-contractive condition of Theorem 1 is satisfied without relying solely on numerical verification.

**Remark 2.** *The* $2\times 2$ *image in Example 6 is intentionally chosen for clarity so that every computational step can be shown explicitly. The proposed method is not restricted to small images; it applies in the same way to larger images, and its performance on standard datasets will be considered in future work.*

**Graph-Based Filtering Algorithm**
Here's a step-by-step algorithm for image denoising using graph-based filtering.
**Input:** Noisy image $I_0$, graph $G = (P, E)$, and contraction parameter $\eth$,
**Initialize:** Set $I = I_0$,
**Iterate:** For each pixel $\hbar \in P$, compute:

$$\mathcal{V}(\hbar) = \frac{2I(\hbar) + \sum_{\varsigma \in N(\hbar)} I(\varsigma)}{2 + |N(\hbar)|}.$$

**Update the image**: $I = \mathcal{V}(I)$,
**Check for convergence:** If $\tau(I_{n+1}, I_n) < \epsilon$ (for some small $\epsilon$), stop.
**Output:** Denoised image $I^*$.

## 6. Conclusions

Image denoising has been recognized as a crucial challenge in image processing, aiming to reconstruct a clear image from a noisy input. In this study, FP theory has been utilized to develop effective denoising techniques. The concept of graphic $\Theta$-contractions in the framework of $\mathcal{F}$-MSs has been introduced, leading to the establishment of new FP results, supported by illustrative examples. These findings have been further extended to orbitally $G$-continuous graphic $\Theta$-contractions. As a significant application, image denoising has been formulated as a FP problem, where a contractive mapping iteratively adjusts pixel intensities based on their neighborhood structure. The existence and uniqueness of a stable denoised image as the FP of the transformation have been demonstrated. Moreover, the convergence behavior of the iterative denoising process under graphic $\Theta$-contractions has been analyzed, ensuring both its effectiveness and stability.

## 7. Future work and open problems

This research introduces novel FP theorems in $\mathcal{F}$-MSs, laying the groundwork for further studies in both theoretical and practical contexts. However, several unresolved questions persist, presenting opportunities for future investigation.

**Extensions to Different Mathematical Frameworks**

- Extending the current results to more generalized spaces, such as interpolative metric spaces, perturbed metric spaces, and ($\alpha$-$\nu$)-relaxed polygonal metric spaces
- Exploring the existence of FPs under relaxed contraction conditions.
- Exploring cyclic, asymptotic, and quasi-contractive mappings for multivalued mappings.

Establishing FP results in these spaces can broaden applications in physics, differential equations, and optimization. Additionally, relaxing conditions improves real-world relevance, particularly in scenarios where strict contractions are not applicable.

**Applications to Fractional Differential Equations**

Although this study utilizes FP results for integral equations, a promising avenue for further research is their application to fractional differential equations. Specifically:

- Analyzing the existence and stability of solutions to fractional differential equations using FP theory.
- Applying FP methods to extend the results to systems with delay, especially in models incorporating memory effects or time-dependent delays.

Applying FP theory to fractional differential equations could result in notable progress both theoretically and computationally.

**Numerical Techniques and Algorithmic Approaches**

While the current study is theoretical, computational aspects are equally important. Future work could involve:

- Developing iterative numerical algorithms for approximating FPs.
- Utilizing machine learning techniques to enhance FP computations and their application in image denoising.
- Evaluating the efficiency and accuracy of various denoising techniques, including variational methods, wavelet transforms, and deep learning approaches, within a FP framework.

Such computational approaches would help validate theoretical findings and provide insights into their real-world applicability.

**Empirical Validation and Performance Analysis in Image Processing**

As a natural continuation of the application presented in Section 5, future research will focus on experimental validation of the proposed graph-based denoising framework. Specifically, we plan to:

- Test the framework on standard image datasets (e.g., BSD68, Set12, or COCO) corrupted with various noise types (Gaussian, salt-and-pepper, speckle).
- Compare the performance of the proposed $\Theta$-contractive averaging filter with classical filters such as Gaussian, median, and bilateral filters, as well as with modern variational and deep learning-based methods.
- Use quantitative metrics such as Peak Signal-to-Noise Ratio (PSNR), Structural Similarity Index (SSIM), and Mean Squared Error (MSE) to objectively assess denoising quality.
- Investigate the effect of graph construction (e.g., $k$-nearest neighbors, fully connected with weights based on intensity similarity) on denoising performance and convergence speed.
- Explore adaptive strategies for choosing the contraction parameter $\lambda$ and the function $\Theta$ based on image content and noise characteristics.

## Author contributions

Maliha Rashid: Formal analysis, Investigation; Haseeba Bibi: Validation, Methodology; Hadeel Z. Al-Zumi: Resources, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare that they have no conflicts of interest.

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