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Research article

A multi-criteria approach to decision-making using a hybrid $\mathfrak{M}$-polar 3-spherical $\mathfrak{Q}$-fuzzy soft sets model

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Abstract: In this paper, we proposed a hybrid $\mathfrak{M}$-polar 3-spherical $\mathfrak{Q}$-fuzzy soft set model by integrating $\mathfrak{M}$-polar, 3-spherical, and $\mathfrak{Q}$-fuzzy soft sets. The model handles three-dimensional, $\mathfrak{M}$-polar parametrized data. Fundamental operations such as union, intersection, complement, maximum, minimum, direct sum, direct product, and a weighted geometric aggregation operator were defined within the proposed $\mathfrak{M}$-polar 3-spherical $\mathfrak{Q}$-fuzzy soft set framework. The new model enhanced flexibility and effectiveness in managing vagueness and uncertainty in complex decision-making problems. An application to artificial intelligence (AI) model selection and multi-criteria decision-making (MCDM) demonstrated the advantages of the proposed framework.

Keywords: $\mathfrak{M}$-polar 3-spherical $\mathfrak{Q}$-fuzzy soft set; decision-making; pattern recognition; cluster analysis; correlation coefficients
Mathematics Subject Classification: MSC2010, 03B52, 03E72

1. Introduction

Fuzzy set theory, initiated by Zadeh [1] as a generalization of crisp sets, brought Membership degree (MD), which then emerged across scientific spheres to establish models for uncertainty management and vagueness handling. According to the presentation, these mathematical ideas involve the concepts of inclusion, union, intersection, complement, relation, and convexity. The linguistic variables derived from natural or artificial languages contain a quintuple and compatibility function, meaning to approximate complex phenomena by Zadeh [2]. Fuzzy MADM methods composed of fuzzy AHP, fuzzy TOPSIS, fuzzy ELECTRE, and fuzzy VIKOR provide a comprehensive solution for making decisions under uncertainties in supply chain management, environmental assessment, and project selection by Chen et al. [3]. Molodtsov [4] pioneered the idea of soft set presentations of basic concepts and initial results while discussing upcoming challenges. Roy et al. [5] introduced a novel and innovative technique to recognize objects from imprecise multiobserver data through the creation of fuzzy soft set-based decision making models. Intuitionistic fuzzy sets (IFSs) represented a contemporary method to manage fuzzy sets (FSs) that improved the handling of uncertain and hesitant decision making and real-world scenarios by Atanassov [6]. IFS represented an element by two belonging measurement functions, Membership degree ($\mathcal{MD}$) and non-membership degree ($\mathcal{NMD}$), that fulfill the sum of $\mathcal{MD}$ and $\mathcal{NMD}$ in [0,1]. Maji [7] presented novel operational methods for intuitionistic fuzzy soft sets (IFSSs) and described their individual characteristics.Peng and Yang [8] established Pythagorean fuzzy sets (PFSs) as a new methodology for handling ambiguous $\mathcal{MD}$ and $\mathcal{NMD}$ while utilizing division and subtraction operations to solve uncertainty problems in multiple group decision-making (MGDM) processes. The scope of $\mathcal{MD}$ and $\mathcal{NMD}$ in PFSs extends beyond IFSs under specific conditions that the sum of the squares of $\mathcal{MD}$ and $\mathcal{NMD}$ in [0,1].These mathematical frameworks can calculate and model complicated uncertainty with greater accuracy. Building on these foundations, Broumi et al. [9] presented fermatean neutrosophic graphs (FNGs), which introduced innovative concepts to the fields of fuzzy and neutrosophic graph theory. Combining polarity and fuzziness for bipolar clustering, conflict resolution, and multi-agent coordination, the bipolar fuzzy set theory was introduced by [10] Zhang. Abdullah et al. [11] investigated bipolar fuzzy soft sets (BFSSs) together with their fundamental properties and two fundamental processes and demonstrated methods for set intersections and extended unions. Alghamdi et al. [12] introduced the BF-TOPSIS and BF-ELECTRE I methodologies, which implemented BFS for solving MCDM problems.Adam et al. [13] established a definition of $\mathfrak{Q}$-fuzzy soft sets ($\mathfrak{Q}$FSSs) and explained their role as an analytical tool to handle uncertain situations while introducing essential characteristics. The application of De Morgan's Law enabled the development of $\mathfrak{Q}$FSSs containing operations for union and intersection, along with logical functions AND and OR by Adam et al. [14]. Adam et al. [15] presented definitions alongside properties of multi $\mathfrak{Q}$-fuzzy soft (m$\mathfrak{Q}$-fs) matrices and demonstrated their representation through null, absolute, equal, submatrix, quality, symmetric, addition, subtraction, union, and intersection. Adam et al. [16] defined multi $\mathfrak{Q}$-Fuzzy sets (M$\mathfrak{Q}$FSs) as well as multi $\mathfrak{Q}$-Fuzzy parameterized soft sets (M$\mathfrak{Q}$FPSSs) alongside their operational methods and presented a decision-making framework based on the Mk$\mathfrak{Q}$FP-soft set theory. Adam et al. [17] developed the M$\mathfrak{Q}$FPSS while demonstrating its basic operations, including intersection, complement, union, AND, and OR operations.

Mahmood [18] examined T-bipolar soft sets (TBSSs) by defining BSSs while studying their relationship to bipolarity, exploring their operational laws, properties, algebraic structures, and decision-making applications. Chen et al. [19] demonstrated BFSSs and [0,1]2-sets as mathematically equivalent concepts while introducing $\mathfrak{M}$-Polar fuzzy sets ($\mathfrak{M}$PFSs) as a generalization of BFSSs. MPFSSs contribute to better decision-making in uncertain scenarios through their advanced approach to represent membership and possible event levels by Khalil et al. [20]. Zahedi et al. [21] introduced a novel aggregation operator for MPFSSs that brings desirable properties, including idempotency and monotonicity, with commutativity to MAGDM systems. Naeem et al. [22] developed Pythagorean $\mathfrak{M}$-Polar fuzzy sets (PmFS) for multivariate statistical analysis of life partner selection and investments in funding schemes. Akram et al. [23] introduced interval-valued $\mathfrak{M}$-Polar fuzzy soft information (IVmFSS) as a new model that integrated interval representations with parameterizations and $\mathfrak{M}$-Polarity. Ali et al. [24] introduced three novel AGO operators for MCDM, yet investigated their characteristics before proposing an algorithm for various conditions.

Cuong and Kreinovich [25] presented a new concept called picture fuzzy sets (PFS) that extended fuzzy sets and IFSs. Thus, Kong [26] introduced PFSs to serve voting situations that demand classification of four types of responses, which include agreement and abstention alongside disagreement and non-participation. Gundogdu and Kahraman [27] discussed spherical fuzzy sets (SFSs), extended Pythagorean and Neutrosophic through their definitions of $\mathcal{MD}$, $\mathcal{NMD}$, and $\mathcal{AD}$ expressed across spherical surfaces where the sum of the squares of three degrees lies in the range [0,1]. Decision-makers gain a broader range of values to assign $\mathcal{MD}$ and $\mathcal{NMD}$ and hesitation through this generalization compared to earlier models. Kutlu et al. [28] established three-dimensional spherical fuzzy sets (3SFSs) before applying them to extend the MCDM TOPSIS methodology for comprehensive expert assessment. Ashraf et al. [29] studied SFS operations while expanding operational laws to aggregation operators, introducing weighted averaging and geometric aggregation operators, alongside developing an MCDM method. The combination of DeMorgan's laws with an adjustable soft discernibility matrix algorithm delivered an order relation for spherical fuzzy soft sets, which extends soft sets by Perveen et al. [30]. New categories of fuzzy extension sets exist, which encompass hesitant fuzzy sets as well as IFSs, PFSs, FFSs, and q-rung orthopair fuzzy sets. The three sets of picture fuzzy sets, along with neutrosophic sets and spherical fuzzy sets, belong to the same category. The proposed spherical fuzzy sets (SFS) satisfy three criteria of Kutlu et al. [31]: The sum of the squares of $\mathcal{MD}$, $\mathcal{NMD}$, and $\mathcal{AD}$ belongs to [0,1]. Guner et al. [32] explored set operations within the new spherical fuzzy soft environment while building a spherical fuzzy soft aggregation operator for efficient MCDM applications. Ahmmad et al. [33] enhanced SFSs human opinion decision-making through simple parameterization methods and novel aggregation operators based on weighted averages and hybrid averages. Riaz et al. [34] presented $\mathfrak{M}$-polar spherical fuzzy sets ($\mathfrak{M}$PSFSs) as a combination of m-PFS and SFSs to boost real-time problem handling and pattern recognition, and medical diagnosis applications.

Advancements in fuzzy decision-making research provide valuable context for the proposed integration. Yager [35] introduced power aggregation operators for Pythagorean fuzzy sets, offering enhanced aggregation capabilities. Liu and Wang [36] developed q-rung orthopair fuzzy aggregation operators that generalize intuitionistic and Pythagorean frameworks. Garg [37] proposed sine trigonometric aggregation operators for spherical fuzzy sets, demonstrating improved monotonicity properties. Alternative decision-making methodologies such as VIKOR [38] and TODIM [39] have extended fuzzy approaches to various application domains. In healthcare, Ashraf et al. [40] applied spherical fuzzy sets to medical diagnosis, while Farhadinia [41] utilized hesitant fuzzy linguistic term sets for patient prioritization. Supply chain applications include Liao et al.'s [42] implementation of q-rung orthopair fuzzy sets for sustainable supplier selection. Parameter optimization methods include entropy-based approaches [43], and the Best-Worst Method extended to neutrosophic environments [44]. Advances in neural-fuzzy systems include self-organizing interval type-2 fuzzy neural networks based on eigenvalue decomposition and fault diagnosis approaches using dual-network recursive interval type-2 fuzzy neural networks with SVD-TLS optimization.

Although these fuzzy set extensions have individually contributed to the enhancement of uncertainty modeling, decision-making problems in real life, especially in areas like artificial intelligence model selection, healthcare diagnostics, or strategic planning, expose some limitations that call for integrated methods. Complex cases generally come with the following three challenges: (1) Diversified stakeholders' perspectives with multi-dimensional and independent evaluations; (2) significant uncertainty with subtle presentation supported by a malleable constraints system; and (3)

Variability of contextual situations in alternative operational procedures. The first issue is solved by $\mathfrak{M}$-polar fuzzy sets, which allow independent multi-dimensional evaluations from different viewpoints; however, these sets do not have means for representing vague uncertainty or for adapting to different contexts. 3-spherical fuzzy sets benefit from more flexibility by cubic constraints $\zeta^3 + \xi^3 + \pi^3 \leq 1$, which define a larger admissible space than quadratic or linear constraints, and they enable experts to present stronger preferences or higher hesitations if it is supported by knowledge from the given domain. Nevertheless, no mechanism exists to process multi-perspective evaluation or to address context changes of evaluation concerns. The $\mathcal{Q}$-fuzzy soft sets enable contextual parameterization via auxiliary sets, and evaluations can be seen as context-dependent or as operative scenario-specific. However, by their own definition, they are not able to encompass the richness of multi-perspective evaluations, or the expressiveness offered by cubic uncertainty constraints.

The selection of $\mathfrak{M}$-polar, 3-spherical, and $\mathcal{Q}$-fuzzy frameworks is deliberate and problem-driven. Consider the AI model selection problem presented in Section 4: Technical experts may strongly favor a model's accuracy ($\zeta \approx 0.9$) while expressing minimal hesitation ($\pi \approx 0.1$) in controlled testing environments, a combination permissible only under cubic constraints. Business managers might moderately support the same model ($\zeta \approx 0.7$) while expressing significant cost concerns ($\xi \approx 0.6$) in production environments. End-users could show high hesitation ($\pi \approx 0.5$) about usability across all contexts. These are not captured at once by any existing framework of evaluations. The introduced $\mathfrak{M}$-polar 3-spherical $\mathcal{Q}$-fuzzy soft set model fills this gap and combines three complementary features in one model: (1) Preservation of independent evaluations of the players by analyzing multi-perspective problems with $\mathfrak{M}$-polarity, (2) subtle modeling of uncertainty with 3-spherical constraints to represent strong expert opinions, and (3) contextual versatility by means of $\mathcal{Q}$-parameterization that enables going from one context to another assessment shift. This combination caters to practical desiderata, that is, above all, the realization that in complex decision problems, partial solutions are inevitable and will, for a period of time, provide incomplete and possibly biased assessments.

Motivation and contribution of the proposed work

1. Molodtsov [4] pioneered the idea of soft set presentations of basic concepts and initial results while discussing upcoming challenges.

2. Adam et al. [13] introduced a definition of $\mathcal{Q}$-fuzzy soft sets ($\mathcal{Q}$FSSs) and explained their role as an analytical tool to handle uncertain situations while introducing essential characteristics. The application of De Morgan's Law enabled the development of $\mathcal{Q}$FSSs containing operations for union and intersection, along with logical functions AND and OR by Adam et al. [14]. Adam et al. [15] presented definitions alongside properties of multi $\mathcal{Q}$-fuzzy soft (m$\mathcal{Q}$-fs) matrices and demonstrated their representation through null, absolute, equal, intersection, symmetric, subtraction, submatrix, equality, union, and addition.

3. Gundogdu and Kahraman [27] proposed spherical fuzzy sets (SFSs), extended Pythagorean and Neutrosophic through their definitions of $\mathcal{MD}$, $\mathcal{NMD}$, and $\mathcal{AD}$s expressed across spherical surfaces where the sum of the squares of three degrees lies in the range [0,1]. Decision-makers gain a broader range of values to assign $\mathcal{MD}$ and $\mathcal{NMD}$ and hesitation through this generalization compared to earlier models.

4. Guner et al. [32] explored set operations within the new spherical fuzzy soft environment while building a spherical fuzzy soft aggregation operator for efficient MCDM applications. Ahmmad et al. [33] enhanced SFSs human opinion decision-making through simple parameterization methods and novel aggregation operators based on weighted averages and hybrid averages. Riaz et al. [34] presented $\mathfrak{M}$-polar spherical fuzzy sets ($\mathfrak{M}$PSFSs) as a combination of *m*-PFS and SFSs to boost real-time problem handling and pattern recognition, and medical diagnosis applications.

5. In this work, a hybrid $\mathfrak{M}$-polar 3-spherical $\mathfrak{Q}$-fuzzy soft set model is proposed by integrating $\mathfrak{M}$-polar, 3-spherical and $\mathfrak{Q}$-fuzzy soft sets. This model deals with three-dimensional, $\mathfrak{M}$-Polar parametrized data with flexible constraints. Additionally, we discuss various operations under the influence of $\mathfrak{M}$-polar 3-spherical $\mathfrak{Q}$-fuzzy soft set. A flowchart of the proposed model is given in Figure 1.

Following the structure of the manuscript, in Section 2, we address such key conceptual problems for our study: Fuzzy soft sets; $\mathfrak{M}$-Polar sets; $\mathfrak{Q}$ sets; and 3-spherical fuzzy soft sets. Our primary focus in Section 3 is on the hybrid $\mathfrak{M}$-polar 3-spherical $\mathfrak{Q}$-fuzzy soft set model and its operational mechanisms. In Section 4, we present an application to artificial intelligence (AI) model selection using MCDM that demonstrates the advantages of the proposed framework. In Section 5, we provide a comprehensive comparative analysis with existing models, while in Section 6, we provide a quantitative comparison and in-depth methodological analysis, including ranking consistency assessment with traditional methods. In Section 7, we discuss limitations and scalability considerations, followed by conclusions in Section 8.

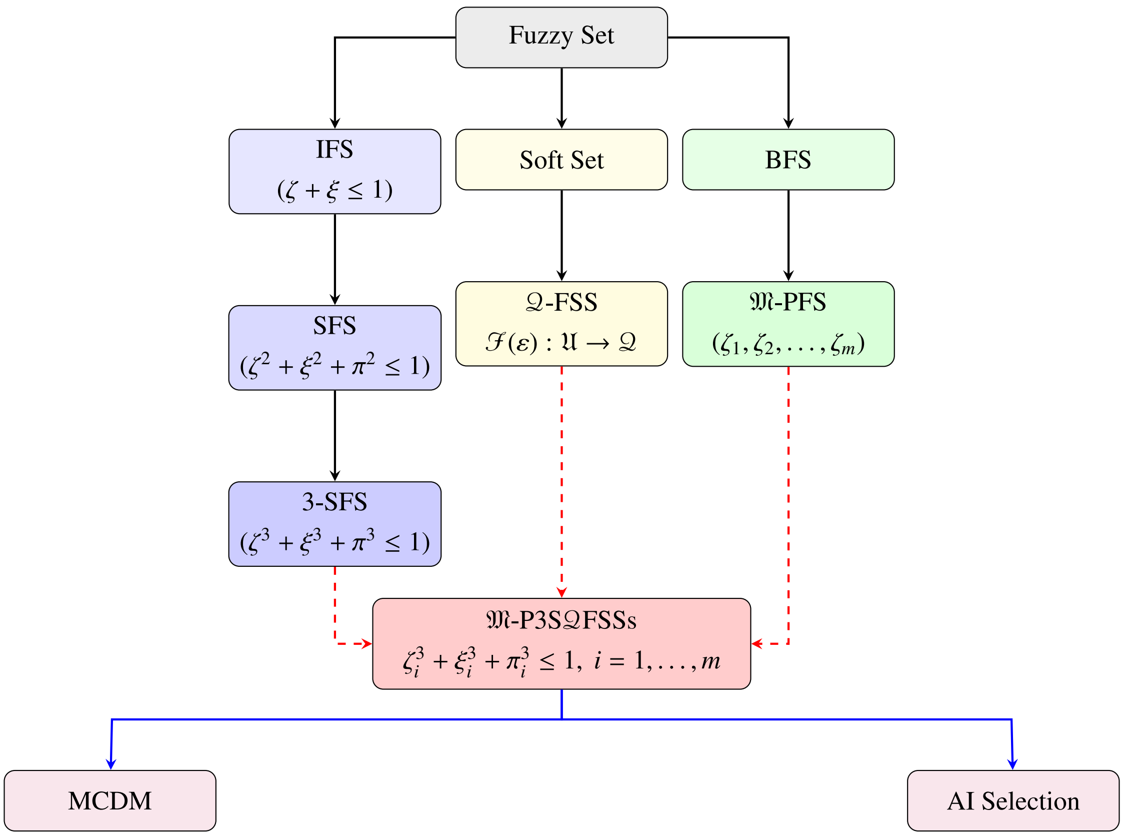

Figure 1. Flowchart of the $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs model.

The flowchart represents the transformation into the hybrid $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs model by integrating: 1) IFS→SFS→3-SFS (cubic uncertainty), 2) BFS→$\mathfrak{M}$-PFS (multi-polar), and 3) Soft Sets→$\mathfrak{Q}$-FSS (parameterization). The model uses a cubic constraint per pole for MCDM and AI selection.

As summarized in Table 1, the major symbols and their corresponding meanings used throughout the proposed $\mathfrak{M}$-Polar 3-Spherical $\mathfrak{Q}$-Fuzzy Soft Sets ($\mathfrak{M}$P3S$\mathfrak{Q}$FSSs) framework are presented for clarity.

Table 1. List of Notations.

Notation	**Meaning**
$\mathfrak{U}$	Universe of discourse (set of all elements)
ϖ	Alternative in decision-making
υ, μ	Element in the universe
E	Set of parameters/criteria
ε	Parameter/criterion
Γ	Subset of parameters
$\mathfrak{Q}$	Auxiliary set/contextual attributes
$\wp$	Context/parameter value
m	Number of poles/dimensions
ℓ_ι	ι-th pole/perspective
$(\mathfrak{R}, \Gamma)$	$\mathfrak{M}$-Polar 3-Spherical $\mathfrak{Q}$-Fuzzy Soft Set
ζ	Membership degree ($\mathcal{MD}$)
ξ	Non-membership degree ($\mathcal{NMD}$)
π	Hesitancy/Abstinence degree ($\mathcal{AD}$)
χ	Refusal degree
$S(\cdot)$	Score function
$A(\cdot)$	Accuracy function
w_j	Weight of criterion j
W	Weight vector
$\oplus$	Addition operation
$\otimes$	Multiplication operation
$\cup$	Union operation
$\cap$	Intersection operation
$(\cdot)^c$	Complement operation
$\subseteq$	Subset relation
n	Number of alternatives
k	Number of criteria
r	Number of parameters
ι, j, ℓ	Indices
$\mathfrak{M}$P3S$\mathfrak{Q}$FSSs	$\mathfrak{M}$-Polar 3-Spherical $\mathfrak{Q}$-Fuzzy Soft Sets
MCDM	Multi-Criteria Decision-Making
AI	Artificial Intelligence

2. Preliminaries

In this part, we review some of the fundamental concepts and terms that will be utilized throughout this work.

Definition 1. *[4] The fuzzy soft set $\tilde{\psi}$ represented a specific analytical model based on fuzzy and soft set theory to manage data with a universe of discourse $\mathfrak{U}$, mapping as:*

$$\tilde{\psi} : E \to \mathcal{F}(\mathcal{U})$$

the set of parameters is denoted by E, $\mathcal{F}(\mathcal{U})$ is the set of all fuzzy subset of $\mathfrak{U}$, for each parameter $\varepsilon \in E$, $\tilde{\psi}(\varepsilon) : \mathfrak{U} \to [0,1]$ has a $\mathcal{MD}$ for each parameter ε and for each element of $\mathfrak{U}$. A fuzzy set constitutes a parameterized family of universe subsets, which enables elements to belong partially, while a soft set deals with uncertain information without requiring probability or membership functions.

Definition 2. *[6] An IFS $\tilde{\psi}$ on universe $\mathfrak{U}$ encodes each element $\mu \in \mathfrak{U}$ by a ordered pair $(\zeta_{\tilde{\psi}}(\mu), \xi_{\tilde{\psi}}(\mu))$ as follows:*

$$\zeta_{\tilde{\psi}}(\mu) \in [0,1] \quad \textit{(membership degree)}, \qquad \xi_{\tilde{\psi}}(\mu) \in [0,1] \quad \textit{(non-membership degree)},$$

Under the condition that:

$$0 \le \zeta_{\tilde{\psi}}(\mu) + \xi_{\tilde{\psi}}(\mu) \le 1.$$

The hesitancy index is given by:

$$\pi_{\tilde{\psi}}(\mu) = 1 - \zeta_{\tilde{\psi}}(\mu) - \xi_{\tilde{\psi}}(\mu)$$

Definition 3. *[28] A spherical fuzzy set $\tilde{\psi}$ in $\mathfrak{U}$ can be written as:*

$$\tilde{\psi} = \left\{ \left\langle \mu, \zeta_{\tilde{\psi}}(\mu), \xi_{\tilde{\psi}}(\mu), \pi_{\tilde{\psi}}(\mu) \right\rangle \mid \mu \in \mathfrak{U} \right\},$$

where $\zeta_{\tilde{\psi}}(\mu), \xi_{\tilde{\psi}}(\mu), \pi_{\tilde{\psi}}(\mu) \in [0,1]$ denote $\mathcal{MD}$, $\mathcal{NMD}$, and $\mathcal{AD}$, respectively, satisfying the spherical constraint:

$$0 \le \zeta^2_{\tilde{\psi}}(\mu) + \xi^2_{\tilde{\psi}}(\mu) + \pi^2_{\tilde{\psi}}(\mu) \le 1, \quad \forall \mu \in \mathfrak{U}.$$

Definition 4. *A 3-spherical fuzzy set $\tilde{\psi}$ over the universal set $\mathfrak{U}$ is defined as:*

$$\tilde{\psi} = \left\{ \left\langle \mathfrak{U}, \left(\zeta_{\tilde{\psi}}(\mu), \xi_{\tilde{\psi}}(\mu), \pi_{\tilde{\psi}}(\mu) \right) \right\rangle \middle| \mu \in \mathfrak{U} \right\}$$

where $\zeta_{\tilde{\psi}}(\mu)$, $\xi_{\tilde{\psi}}(\mu)$, and $\pi_{\tilde{\psi}}(\mu)$ are values in the interval $[0,1]$, satisfying the condition:

$$0 \le \zeta^3_{\tilde{\psi}}(\mu) + \xi^3_{\tilde{\psi}}(\mu) + \pi^3_{\tilde{\psi}}(\mu) \le 1, \quad \forall \mu \in \mathfrak{U}$$

For each element $\mu \in \mathfrak{U}$, the values $\zeta_{\tilde{\psi}}(\mu)$, $\xi_{\tilde{\psi}}(\mu)$, and $\pi_{\tilde{\psi}}(\mu)$ express the $\mathcal{MD}$, $\mathcal{NMD}$, and hesitancy with respect to the set $\tilde{\psi}$, respectively. The Refusal degree is defined as:

$$\chi_{\tilde{\psi}}(\mu) = \sqrt[3]{1 - \zeta^3_{\tilde{\psi}}(\mu) - \xi^3_{\tilde{\psi}}(\mu) - \pi^3_{\tilde{\psi}}(\mu)}$$

Definition 5. *[13] A $\mathcal{Q}$-fuzzy soft set ($\mathcal{Q}$FSSs) $\tilde{\psi}$ over a universal set $\mathfrak{U}$ with respect to parameter E is defined as:*

$$\tilde{\psi} = E \rightarrow \mathcal{F}_{\mathcal{Q}}(\mathcal{U})$$

$\mathcal{F}_{\mathcal{Q}}(\mathcal{U})$ is the set of all $\mathcal{Q}$-fuzzy subset of $\mathfrak{U}$, for each parameter $\varepsilon \in E$, $\tilde{\psi}(\varepsilon) : \mathfrak{U} \rightarrow [0,1]$ has a $\mathcal{MD}$ for each parameter ε and for each element of $\mathfrak{U}$, where $\mathcal{Q} \neq \phi$ be is a general set (not just in [0,1]).

$$\mathcal{F}(\varepsilon) : \mathfrak{U} \rightarrow \mathcal{Q}.$$

Each $\mathcal{F}(\varepsilon)$ is a $\mathcal{Q}$-fuzzy set on $\mathfrak{U}$. According to the function $\mathcal{F}(\varepsilon)(\mu) \in \mathcal{Q}$ an element $\mu \in \mathfrak{U}$ receives values from $\mathcal{Q}$ which demonstrate its satisfaction level for parameter $\varepsilon \in E$.

Definition 6. *[20] An $\mathfrak{M}$-polar fuzzy soft sets ($\mathfrak{M}$PFSSs) $\tilde{\psi}$ over a universal set $\mathfrak{U}$ associated to parameter E and $\mathcal{M} = \ell_1, \ell_2, ..., \ell_m$ be a set of $\mathfrak{M} -$ poles is defined as:*

$$\tilde{\psi} = E \rightarrow \mathcal{F}(\mathcal{U} \times [0,1]^{m})$$

That is, for each $\varepsilon \in E$ and assign for each element $\mu \in \mathfrak{U}$ a vector of MD ($\varpi_1, \varpi_2, ..., \varpi_m$), where $\varpi_\iota \in [0,1]$. Every membership value equates to the degree of agreement with the corresponding statement within the pole ℓ_ι.

Definition 7. *[30] An $\mathfrak{m}$-PSFSs described on $\mathfrak{U}$ is explained by the mappings $\zeta^j : \mathfrak{U} \in [0,1]$ (known as $\mathcal{MD}$), $\pi^j : \mathfrak{U} \in [0,1]$, and $\xi^j : \mathfrak{U} \in [0,1]$ $\mathcal{NMD}$), and satisfies the following condition*

$$0 \leq (\zeta^j(\mu))^2 + (\xi^j(\mu))^2 + (\pi^j(\mu))^2 \leq 1$$

for $j = 1, 2, 3, ..., m$ and Refusal degree

$$\chi = \sqrt{1 - (\zeta^j(\mu))^2 - (\xi^j(\mu))^2 - (\pi^j(\mu))^2}, for j = 1, 2, 3, ..., m$$

Definition 8. *[45] A T-spherical fuzzy set $\tilde{\psi}$ in a universal set $\mathfrak{U}$ is explained as:*

$$\tilde{\psi} = \left\{ \left\langle \mathfrak{U}, \left(\zeta_{\tilde{\psi}}(\mu), \xi_{\tilde{\psi}}(\mu), \pi_{\tilde{\psi}}(\mu) \right) \right\rangle \middle| \mu \in \mathfrak{U} \right\},$$

where $\zeta_{\tilde{\psi}}(\mu)$, $\xi_{\tilde{\psi}}(\mu)$, and $\pi_{\tilde{\psi}}(\mu)$ are the $\mathcal{MD}$ abstinence, and $\mathcal{NMD}$s, respectively, each taking values in the interval $[0,1]$. These values fulfill the condition:

$$0 \leq (\zeta_{\tilde{\psi}}(\mu))^{\mathfrak{t}} + (\xi_{\tilde{\psi}}(\mu))^{\mathfrak{t}} + (\pi_{\tilde{\psi}}(\mu))^{\mathfrak{t}} \leq 1, \quad \forall \mu \in \mathfrak{U}.$$

Here, $\zeta^t_{\tilde{\psi}}(\mu)$ represents the MD, $\xi^t_{\tilde{\psi}}(\mu)$ represents the abstinence degree (AD), and $\pi^t_{\tilde{\psi}}(\mu)$ represents the NMD for a given element $\mu \in \mathfrak{U}$.

A T-spherical fuzzy Set (TSFS) $\mu \in \mathfrak{U}$ is mapping as $\tilde{\psi} = \left(\zeta_{\tilde{\psi}}, \xi_{\tilde{\psi}}, \pi_{\tilde{\psi}}\right)$, such that:

$$\zeta_{\tilde{\psi}}, \xi_{\tilde{\psi}}, \pi_{\tilde{\psi}} \in [0,1], \quad \text{and} \quad \zeta^t_{\tilde{\psi}} + \xi^t_{\tilde{\psi}} + \pi^t_{\tilde{\psi}} \leq 1.$$

Recall that 3-SFSs represent a special case of TSFSs with $T = 3$, where the constraint becomes:

$$0 \leq \zeta^3_{\mathcal{A}}(\upsilon) + \xi^3_{\mathcal{A}}(\upsilon) + \pi^3_{\mathcal{A}}(\upsilon) \leq 1.$$

These sets offer a larger space than standard SFSs (where $T = 2$) for assigning $\mathcal{MD}$,$\mathcal{NMD}$, and hesitation values, enabling more precise modeling of uncertain information. They combine the advantages of Pythagorean and neutrosophic fuzzy sets while avoiding their limitations.

Example 1. *In a 3-spherical fuzzy set, during the investigation and appraisal of a new technology (an AI system in our case), we have four varieties of expert opinion:*

- *Agreement to implement the technology.*
- *Hesitation in decision-making.*
- *Rejection of the technology.*
- *Abstaining from giving an opinion.*

Be reminded that a general T-spherical fuzzy set holds the following:

$$\zeta^T(\mu) + \xi^T(\mu) + \pi^T(\mu) \leq 1, \quad T \geq 1$$

with $T = 3$, and the case of 3-spherical fuzzy sets is derived. This cubic inequality $\zeta^3 + \xi^3 + \pi^3 \leq 1$ gives more freedom than the spherical fuzzy sets ($T = 2$) quadratic inequality $\zeta^2 + \xi^2 + \pi^2 \leq 1$.

The main difference is that cubic power is used rather than square power. Let:

$$\zeta(\mu) = .8, \quad \xi(\mu) = .5, \quad \pi(\mu) = .3$$

In the spherical fuzzy set ($T = 2$):

$$\zeta^2(\mu) = .64, \quad \xi^2(\mu) = .25, \quad \pi^2(\mu) = .09$$

$$.64 + .25 + .09 = .98 \leq 1$$

For the 3-spherical fuzzy set ($T = 3$), we can take larger values:

$$\zeta(\mu) = .9, \quad \xi(\mu) = .5, \quad \pi(\mu) = .4$$

$$.9^3 + .5^3 + .4^3 = .729 + .125 + .064 = .918 \leq 1 \quad \textit{(valid)}$$

Note that these values would violate the quadratic constraint:

$$.9^2 + .5^2 + .4^2 = .81 + .25 + .16 = 1.22 > 1 \quad \textit{(invalid for } T = 2)$$

This shows that the passage from general T-spherical to particular 3-spherical sets yields more flexibility. The cubical power permits larger individual membership values while keeping the mathematical soundness; thus, 3-spherical fuzzy sets are more suitable for expressing stronger expert judgments or similar situations as in our $\mathfrak{M}$-polar 3-spherical $\mathfrak{Q}$-fuzzy soft set model for AI model selection.

Definition 9. *Let $\mathfrak{U} \neq \emptyset$ be a universal set. Then, any two 3-spherical fuzzy sets $\tilde{\psi}_1, \tilde{\psi}_2$ can be written as:*

$$\tilde{\psi}_1 = \left\{ \left\langle \mu, \zeta_{\tilde{\psi}_1}(\mu), \xi_{\tilde{\psi}_1}(\mu), \pi_{\tilde{\psi}_1}(\mu) \right\rangle \mid \mu \in \mathfrak{U} \right\}$$

$$\tilde{\psi}_2 = \left\{ \left\langle \mu, \zeta_{\tilde{\psi}_2}(\mu), \xi_{\tilde{\psi}_2}(\mu), \pi_{\tilde{\psi}_2}(\mu) \right\rangle \mid \mu \in \mathfrak{U} \right\}$$

Their operations are defined as follows:

1. ***Intersection:***

$$\tilde{\psi}_1 \cap \tilde{\psi}_2 = \Bigg\langle \min\left\{\zeta_{\tilde{\psi}_1}, \zeta_{\tilde{\psi}_2}\right\}, \max\left\{\xi_{\tilde{\psi}_1}, \xi_{\tilde{\psi}_2}\right\}, \sqrt[3]{\min\left\{1,\ 1 - \left(\min\left\{\zeta_{\tilde{\psi}_1}, \zeta_{\tilde{\psi}_2}\right\}\right)^3 - \left(\max\left\{\xi_{\tilde{\psi}_1}, \xi_{\tilde{\psi}_2}\right\}\right)^3\right\}}\Bigg\rangle$$

2. ***Union:***

$$\tilde{\psi}_1 \cup \tilde{\psi}_2 = \Bigg\langle \max\left\{\zeta_{\tilde{\psi}_1}, \zeta_{\tilde{\psi}_2}\right\}, \min\left\{\xi_{\tilde{\psi}_1}, \xi_{\tilde{\psi}_2}\right\}, \sqrt[3]{\min\left\{1,\ 1 - \left(\max\left\{\zeta_{\tilde{\psi}_1}, \zeta_{\tilde{\psi}_2}\right\}\right)^3 - \left(\min\left\{\xi_{\tilde{\psi}_1}, \xi_{\tilde{\psi}_2}\right\}\right)^3\right\}}\Bigg\rangle$$

3. ***Complement:***

$$\mathrm{Co}(\tilde{\psi}) = \overline{\tilde{\psi}} = \left\{\left\langle \mu, \xi_{\tilde{\psi}}(\mu), \zeta_{\tilde{\psi}}(\mu), \pi_{\tilde{\psi}}(\mu)\right\rangle \mid \mu \in \mathfrak{U}\right\}$$

4. ***Addition:***

$$\tilde{\psi}_1 \oplus \tilde{\psi}_2 = \left\langle \left(\zeta^3_{\tilde{\psi}_1} + \zeta^3_{\tilde{\psi}_2} - \zeta^3_{\tilde{\psi}_1}\zeta^3_{\tilde{\psi}_2}\right)^{1/3}, \xi_{\tilde{\psi}_1}\xi_{\tilde{\psi}_2}, \left(\left(1 - \zeta^3_{\tilde{\psi}_2}\right)\pi^3_{\tilde{\psi}_1} + \left(1 - \zeta^3_{\tilde{\psi}_1}\right)\pi^3_{\tilde{\psi}_2} - \pi^3_{\tilde{\psi}_1}\pi^3_{\tilde{\psi}_2}\right)^{1/3}\right\rangle$$

5. ***Multiplication:***

$$\tilde{\psi}_1 \otimes \tilde{\psi}_2 = \left\langle \zeta_{\tilde{\psi}_1}\zeta_{\tilde{\psi}_2}, \left(\xi^3_{\tilde{\psi}_1} + \xi^3_{\tilde{\psi}_2} - \xi^3_{\tilde{\psi}_1}\xi^3_{\tilde{\psi}_2}\right)^{1/3}, \left(\left(1 - \xi^3_{\tilde{\psi}_2}\right)\pi^3_{\tilde{\psi}_1} + \left(1 - \xi^3_{\tilde{\psi}_1}\right)\pi^3_{\tilde{\psi}_2} - \pi^3_{\tilde{\psi}_1}\pi^3_{\tilde{\psi}_2}\right)^{1/3}\right\rangle$$

6. ***Multiplying by a scalar*** *($\Lambda \geq 0$):*

$$\Lambda \cdot \tilde{\psi} = \left\langle \left(1 - \left(1 - \zeta^3_{\tilde{\psi}}\right)^{\Lambda}\right)^{1/3}, \xi^{\Lambda}_{\tilde{\psi}}, \left(\left(1 - \zeta^3_{\tilde{\psi}}\right)^{\Lambda} - \left(1 - \zeta^3_{\tilde{\psi}} - \pi^3_{\tilde{\psi}}\right)^{\Lambda}\right)^{1/3}\right\rangle$$

7. ***Power operation*** *($\Lambda \geq 0$):*

$$\tilde{\psi}^{\Lambda} = \left\langle \zeta^{\Lambda}_{\tilde{\psi}}, \left(1 - \left(1 - \xi^3_{\tilde{\psi}}\right)^{\Lambda}\right)^{1/3}, \left(\left(1 - \xi^3_{\tilde{\psi}}\right)^{\Lambda} - \left(1 - \xi^3_{\tilde{\psi}} - \pi^3_{\tilde{\psi}}\right)^{\Lambda}\right)^{1/3}\right\rangle$$

2.1. Features of a 3-SFSs

Theorem 1. *Let*

$$\widetilde{\mathbb{S}} = \langle(\zeta_S, \pi_S, \xi_S)\rangle, \widetilde{\mathbb{S}}_1 = \langle(\zeta_1, \pi_1, \xi_1)\rangle, \widetilde{\mathbb{S}}_2 = \langle(\zeta_2, \pi_2, \xi_2)\rangle, \widetilde{\mathcal{T}} = \langle(\zeta_T, \pi_T, \xi_T)\rangle$$

be 3-spherical fuzzy sets. Then:

1. *If $\widetilde{\mathbb{S}} \subseteq \widetilde{\mathbb{S}}_1$ and $\widetilde{\mathbb{S}} \subseteq \widetilde{\mathbb{S}}_2$, $\Rightarrow \widetilde{\mathbb{S}} \subseteq \widetilde{\mathbb{S}}_1 \cap \widetilde{\mathbb{S}}_2$.*
2. *If $\widetilde{\mathbb{S}}_1 \subseteq \widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathbb{S}}$, $\Rightarrow \widetilde{\mathbb{S}}_1 \cup \widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathbb{S}}$.*

3. *If* $\widetilde{\mathbb{S}}_1 \subseteq \widetilde{\mathbb{S}}$ *and* $\widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathcal{T}}$, $\Rightarrow \widetilde{\mathbb{S}}_1 \cup \widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathbb{S}} \cup \widetilde{\mathcal{T}}$.
4. *If* $\widetilde{\mathbb{S}}_1 \subseteq \widetilde{\mathbb{S}}$ *and* $\widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathcal{T}}$, $\Rightarrow \widetilde{\mathbb{S}}_1 \cap \widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathbb{S}} \cap \widetilde{\mathcal{T}}$.

Proof.
Proof (1): Assume $\widetilde{\mathbb{S}} \subseteq \widetilde{\mathbb{S}}_1$ and $\widetilde{\mathbb{S}} \subseteq \widetilde{\mathbb{S}}_2$. Then:

$$\zeta_S \le \zeta_1,\ \pi_S \le \pi_1,\ \xi_S \ge \xi_1, \quad \zeta_S \le \zeta_2,\ \pi_S \le \pi_2,\ \xi_S \ge \xi_2.$$

By definition:

$$\widetilde{\mathbb{S}}_1 \cap \widetilde{\mathbb{S}}_2 = \langle (\min\{\zeta_1, \zeta_2\}, \min\{\pi_1, \pi_2\}, \max\{\xi_1, \xi_2\}) \rangle.$$

Thus:

$$\zeta_S \le \min\{\zeta_1, \zeta_2\}, \quad \pi_S \le \min\{\pi_1, \pi_2\}, \quad \xi_S \ge \max\{\xi_1, \xi_2\}.$$

For the cubic constraint, with $\zeta_\cap = \min\{\zeta_1, \zeta_2\}$, $\pi_\cap = \min\{\pi_1, \pi_2\}$, $\xi_\cap = \max\{\xi_1, \xi_2\}$:

$$\zeta_\cap^3 + \pi_\cap^3 + \xi_\cap^3 \le \max\{\zeta_1^3 + \pi_1^3 + \xi_1^3, \zeta_2^3 + \pi_2^3 + \xi_2^3\} \le 1.$$

Hence,

$$\widetilde{\mathbb{S}} \subseteq \widetilde{\mathbb{S}}_1 \cap \widetilde{\mathbb{S}}_2$$

Proof (2): Assume $\widetilde{\mathbb{S}}_1 \subseteq \widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathbb{S}}$. Then:

$$\zeta_1 \le \zeta_S,\ \pi_1 \le \pi_S,\ \xi_1 \ge \xi_S, \quad \zeta_2 \le \zeta_S,\ \pi_2 \le \pi_S,\ \xi_2 \ge \xi_S.$$

$$\widetilde{\mathbb{S}}_1 \cup \widetilde{\mathbb{S}}_2 = \langle (\max\{\zeta_1, \zeta_2\}, \min\{\pi_1, \pi_2\}, \min\{\xi_1, \xi_2\}) \rangle.$$

Thus:

$$\max\{\zeta_1, \zeta_2\} \le \zeta_S, \quad \min\{\pi_1, \pi_2\} \le \pi_S, \quad \min\{\xi_1, \xi_2\} \ge \xi_S.$$

Let $\zeta_\cup = \max\{\zeta_1, \zeta_2\}$, $\pi_\cup = \min\{\pi_1, \pi_2\}$, $\xi_\cup = \min\{\xi_1, \xi_2\}$. Then:

$$\zeta_\cup^3 + \pi_\cup^3 + \xi_\cup^3 \le \zeta_S^3 + \pi_S^3 + \xi_S^3 \le 1.$$

Hence,

$$\widetilde{\mathbb{S}}_1 \cup \widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathbb{S}}$$

Proof (3): Assume $\widetilde{\mathbb{S}}_1 \subseteq \widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{S}}_2 \subseteq \widetilde{\mathcal{T}}$. Then:

$$\zeta_1 \le \zeta_S,\ \pi_1 \le \pi_S,\ \xi_1 \ge \xi_S, \quad \zeta_2 \le \zeta_T,\ \pi_2 \le \pi_T,\ \xi_2 \ge \xi_T.$$

We have:

$$\widetilde{\mathbb{S}}_1 \cup \widetilde{\mathbb{S}}_2 = \langle (\max\{\zeta_1, \zeta_2\}, \min\{\pi_1, \pi_2\}, \min\{\xi_1, \xi_2\}) \rangle,$$

$$\widetilde{\mathbb{S}} \cup \widetilde{\mathcal{T}} = \langle (\max\{\zeta_S, \zeta_T\}, \min\{\pi_S, \pi_T\}, \min\{\xi_S, \xi_T\}) \rangle.$$

Comparing component-wise:

$$\max\{\zeta_1, \zeta_2\} \le \max\{\zeta_S, \zeta_T\}, \quad \min\{\pi_1, \pi_2\} \le \min\{\pi_S, \pi_T\}, \quad \min\{\xi_1, \xi_2\} \ge \min\{\xi_S, \xi_T\}.$$

The cubic constraint is preserved since:

$$\max\{\zeta_1,\zeta_2\}^3+\min\{\pi_1,\pi_2\}^3+\min\{\xi_1,\xi_2\}^3\leq\max\{\zeta_S,\zeta_T\}^3+\min\{\pi_S,\pi_T\}^3+\min\{\xi_S,\xi_T\}^3\leq 1.$$

Hence,

$$\widetilde{\mathbb{S}}_1\cup\widetilde{\mathbb{S}}_2\subseteq\widetilde{\mathbb{S}}\cup\widetilde{\mathcal{T}}$$

Proof (4): Assume $\widetilde{\mathbb{S}}_1\subseteq\widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{S}}_2\subseteq\widetilde{\mathcal{T}}$. Then

$$\widetilde{\mathbb{S}}_1\cap\widetilde{\mathbb{S}}_2=\langle(\min\{\zeta_1,\zeta_2\},\min\{\pi_1,\pi_2\},\max\{\xi_1,\xi_2\})\rangle,$$

$$\widetilde{\mathbb{S}}\cap\widetilde{\mathcal{T}}=\langle(\min\{\zeta_S,\zeta_T\},\min\{\pi_S,\pi_T\},\max\{\xi_S,\xi_T\})\rangle.$$

Comparing:

$$\min\{\zeta_1,\zeta_2\}\leq\min\{\zeta_S,\zeta_T\},\quad\min\{\pi_1,\pi_2\}\leq\min\{\pi_S,\pi_T\},\quad\max\{\xi_1,\xi_2\}\geq\max\{\xi_S,\xi_T\}.$$

The cubic constraint holds as:

$$\min\{\zeta_1,\zeta_2\}^3+\min\{\pi_1,\pi_2\}^3+\max\{\xi_1,\xi_2\}^3\leq\min\{\zeta_S,\zeta_T\}^3+\min\{\pi_S,\pi_T\}^3+\max\{\xi_S,\xi_T\}^3\leq 1.$$

Hence,

$$\widetilde{\mathbb{S}}_1\cap\widetilde{\mathbb{S}}_2\subseteq\widetilde{\mathbb{S}}\cap\widetilde{\mathcal{T}}$$

□

Definition 10. *[17] Suppose that* $\mathrm{I}=[0,1]$ *is denoted the unit interval together with* k *being a natural number, while* ϖ *represents the universal set and* $\mathfrak{Q}$ *represents a non-empty set.*
A $\mathfrak{Q}$ *fuzzy set* $\mathscr{F}_{\mathfrak{Q}}$ *is expressed as:*

$$\mathscr{F}_{\mathfrak{Q}}=\{((\varpi,\wp),\zeta_\iota(\varpi,\wp)):\varpi\in\varpi,\wp\in\mathfrak{Q}\}$$

as $\zeta_\iota:\varpi\times\mathfrak{Q}\rightarrow\mathrm{I}$ *form 1 to k are* k *distinct independent fuzzy sets. The collection* $\{\zeta_1(\varpi,\wp),\zeta_2(\varpi,\wp),\ldots,\zeta_k(\varpi,\wp)\}$ *has practical use, while the notation* $\mathcal{MD}$ *represents the theory of an m-polar* $\mathfrak{Q}$ *fuzzy set* $\mathscr{F}_{\mathfrak{Q}}$*. The set of all* m*-polar* $\mathfrak{Q}$ *fuzzy sets of dimension* k *in* $(\varpi,\wp)$ *is represented by* $\mathfrak{M}^k\mathfrak{Q}^{\uparrow}(\varpi)$*.*

Definition 11. *[19] Let* ϖ *be a non-empty set. An* $\mathfrak{M}$*-polar fuzzy set* Γ *in* ϖ *is expressed as* $\Gamma:\varpi\rightarrow[0,1]^m$*. The set* Γ *is denoted as:*

$$\Gamma=\{(\varpi,\zeta_\Gamma(\varpi))|\varpi\in\varpi\}$$

as $\zeta_\Gamma(\varpi)=(\zeta_\Gamma^1(\varpi),\zeta_\Gamma^2(\varpi),\ldots,\zeta_\Gamma^m(\varpi))$*, and each* $\zeta_\Gamma^\iota(\varpi)\in[0,1]$ *for* $\iota=1,2,\ldots,m$*.*

Definition 12. *An initial universal set* ϖ *exists together with a parameter set E, where the subset A belongs to E. A pair* $\langle F,A\rangle$ *functions as a 3-spherical fuzzy soft set (3-SFSS) over* ϖ *provided F represents a mapping that assigns elements of A to members of 3SFS*(ϖ) *set.*

The spherical fuzzy set can represent $F(e)$ *for any value* $\varepsilon\in E$*.*

$$F(e)=\{(\mathfrak{R},\zeta_{F(\varepsilon)}(\upsilon),\xi_{F(\varepsilon)}(\upsilon),\pi_{F(\varepsilon)}(\upsilon))\mid x\in\varpi\}$$

where $\zeta_{F(\varepsilon)}(\upsilon)$ *is the* $\mathcal{MD}$, $\xi_{F(\varepsilon)}(\upsilon)$ *is the* $\mathcal{NMD}$, *and* $\pi_{F(\varepsilon)}(\upsilon)$ *is the* $\mathcal{AD}$, *with the condition:*

$$\zeta^3_{F(\varepsilon)}(\upsilon) + \xi^3_{F(\varepsilon)}(\upsilon) + \pi^3_{F(\varepsilon)}(\upsilon) \leq 1.$$

A spherical fuzzy set $F(e)$ *results, and a spherical fuzzy soft set* $\langle F, A\rangle$ *forms when* $\xi_{F(\varepsilon)}(\upsilon) = 0$ *holds true for* $\varepsilon \in A$ *and* $\upsilon \in \varpi$.

Example 2. *Assume a 3-SFSS* $\langle F, A\rangle$, *the set* ϖ *of distinct plots that are suitable to build a house is denoted by:*

$$\varpi = \{\ell_1, \ell_2, \ell_3\}$$

A expresses the set of parameters:

$$A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$$

The model includes three multifunctions representing water accessibility (ε_1*), green surroundings (*ε_2*), and affordable pricing (*ε_3*).*

The 3-SFSS $\langle F, A\rangle$ *characterizes the "suitability" of various plots. Assume that:*

$$F(\varepsilon_1) = \{(\ell_1, .4, .5, .6), (\ell_2, .5, .6, .2), (\ell_3, .6, .2, .1)\}$$
$$F(\varepsilon_2) = \{(\ell_1, .3, .2, .4), (\ell_2, .4, .3, .1), (\ell_3, .3, .4, .2)\}$$
$$F(\varepsilon_3) = \{(\ell_1, .2, .1, .2), (\ell_2, .3, .2, .3), (\ell_3, .4, .3, .5)\}$$

The matrix form of the A 3-SFSS $\langle F, A\rangle$ *in matrix form is expressed as:*

$$\langle F, A\rangle = \begin{pmatrix} \ell_1 & (.4, .5, .6) & (.3, .2, .4) & (.2, .1, .2) \\ \ell_2 & (.5, .6, .2) & (.4, .3, .1) & (.3, .2, .3) \\ \ell_3 & (.6, .2, .1) & (.3, .4, .2) & (.4, .3, .5) \end{pmatrix}$$

3. $\mathfrak{M} - P3S\mathcal{Q}FSSs$

In this section, we unveil the latest concept that combines $\mathfrak{M}$-Polar 3-spherical and $\mathcal{Q}$ fuzzy soft sets, which enables multiple $\mathcal{MD}$ and $\mathcal{AD}$ alongside $\mathcal{NMD}$ evaluation for elements while delivering enhanced management of ambiguous multidimensional data. In the following section, we demonstrate mathematical operations for related concepts.

Definition 13. *Assume* $\phi \neq \mathcal{Q} \subset \varpi$ *is subset of universal set and a 3S*$\mathcal{Q}$*FS is defined as* $\mathscr{F}_{\mathcal{Q}} : \varpi \times \mathcal{Q} \to [0, 1]$, *Such that:*

$$\mathscr{F}_{\mathcal{Q}} = \{((\varpi, \wp), \zeta_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp), \xi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp), \pi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp)) \mid \varpi \in \varpi, \wp \in \mathcal{Q}\}$$

Here, $\zeta_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp) \in [0, 1]$ *denoted the* $\mathcal{MD}$, $\xi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp) \in [0, 1]$ *expressed the* $\mathcal{NMD}$, *and* $\pi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp) \in [0, 1]$ *denoted the* $\mathcal{AD}$ *corresponding to the pair* $(\varpi, \wp)$.
$\forall\ \varpi \in \varpi$, $\wp \in \mathcal{Q}$, *the following condition must hold:*

$$0 \leq (\zeta_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp))^3 + (\xi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp))^3 + (\pi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp))^3 \leq 1$$

The $\mathcal{AD}$ *for each pair* $(\varpi, \wp)$ *is given by:*

$$\chi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp) = \sqrt[3]{1 - (\zeta_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp))^3 - (\xi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp))^3 - (\pi_{\mathscr{F}_{\mathcal{Q}}}(\varpi, \wp))^3}$$

This concept is represented by the graph shown in Figure 2.

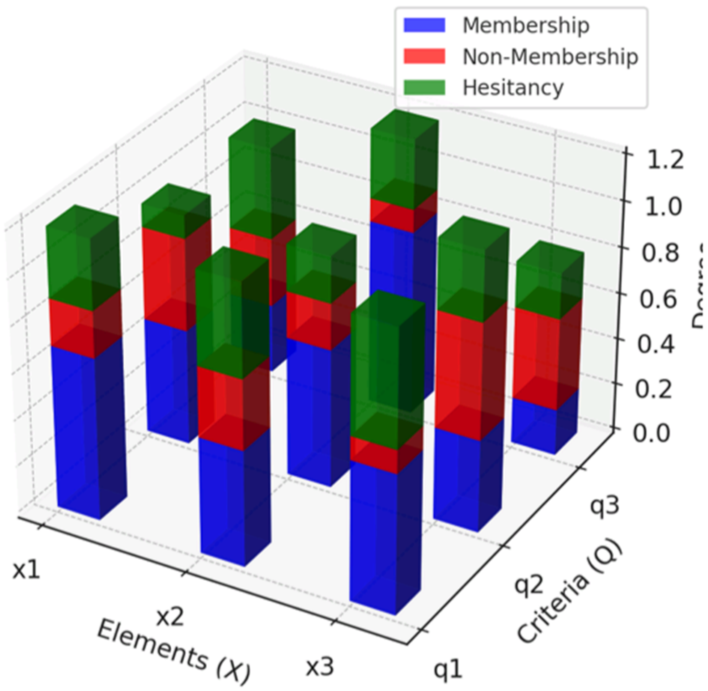

Figure 2. 3-Spherical $\mathfrak{Q}$ Fuzzy Set Representation.

Example 3. *Let* $\varpi = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$ *be a universal set and* $\mathfrak{Q} = \{\wp, \ell\} \neq \phi$. *A 5-polar 3-spherical* $\mathfrak{Q}$*-fuzzy soft set is expressed as follows:*

$$\Gamma_{\mathfrak{Q}} = \begin{Bmatrix} \{(\varpi_1, \wp), (.8, .1, .2)\}, \\ \{(\varpi_2, p), (.6, .2, .3)\}, \\ \{(\varpi_3, \wp), (.5, .3, .4)\}, \\ \{(\varpi_3, p), (.7, .2, .2)\}, \\ \{(\varpi_4, \wp), (.4, .3, .5)\}, \\ \{(\varpi_5, p), (.6, .2, .3)\} \end{Bmatrix}$$

where for each $\zeta_{\Gamma_{\mathfrak{Q}}}(\varpi, \wp)$, $\xi_{\Gamma_{\mathfrak{Q}}}(\varpi, \wp)$, *and* $\pi_{\Gamma_{\mathfrak{Q}}}(\varpi, \wp)$ *are the* $\mathcal{MD}$, $\mathcal{NMD}$, *and* $\mathcal{AD}$, *respectively. The set* $\Gamma_{\mathfrak{Q}}$ *is a 3-Spherical* $\mathfrak{Q}$ *Fuzzy Set.*

Definition 14. *An* $\mathfrak{M}$*-polar 3S*$\mathfrak{Q}$*FS on a non-empty set* ϖ *is defined as a mapping* $\Gamma_{\mathfrak{Q}} : \varpi \times \mathfrak{Q} \rightarrow [0, 1]^m$*, and is given by:*

$$\Gamma_{\mathfrak{Q}} = \{((\varpi, \wp), \zeta_{\Gamma_{\mathfrak{Q}}}(\varpi, \wp), \xi_{\Gamma_{\mathfrak{Q}}}(\varpi, \wp), \pi_{\Gamma_{\mathfrak{Q}}}(\varpi, \wp)) | \varpi \in \varpi, \wp \in \mathfrak{Q}\}$$

This can be alternatively represented as:

$$\Gamma_{\mathfrak{Q}} = \{((\varpi, \wp), \zeta_{\iota}(\varpi, \wp), \xi_{\iota}(\varpi, \wp), \pi_{\iota}(\varpi, \wp)) | \varpi \in \varpi, \wp \in \mathfrak{Q}\}$$

as $\forall\ \iota = 1, 2, \ldots, m$, *the value* $\zeta_\iota(\varpi, \wp) \in [0, 1]$ *is the* $\mathcal{MD}$, $\xi_\iota(\varpi, \wp) \in [0, 1]$ *is the* $\mathcal{NMD}$, *and* $\pi_\iota(\varpi, \wp) \in [0, 1]$ *is the* $\mathcal{AD}$ *for the pair* $(\varpi, \wp)$.

The following condition must be satisfied $\forall \mathrm{I}$*:*

$$0 \leq (\zeta_\iota(\varpi, \wp))^3 + (\xi_\iota(\varpi, \wp))^3 + (\pi_\iota(\varpi, \wp))^3 \leq 1$$

The $\mathcal{AD}$ *is calculated as:*

$$\chi_\iota(\varpi, \wp) = \sqrt[3]{1 - (\zeta_\iota(\varpi, \wp))^3 - (\xi_\iota(\varpi, \wp))^3 - (\pi_\iota(\varpi, \wp))^3}$$

Definition 15. *Let* ϖ *be an initial universe of discourse and* E *be a set of parameters. A subset* $\Gamma \subseteq E$*, and we have* $\mathfrak{Q}$ *be a non-empty set representing auxiliary domains, such as contextual attributes or alternative perspectives. Let* $m \in \mathbb{N}$ *denote the number of poles (dimensions) considered in the evaluation.*

A pair $(\mathfrak{R}, \Gamma)$ *is called a* $\mathfrak{m} - P3S\mathfrak{Q}FSSs$ *over* ϖ *if and only if the mapping* $\upsilon : \Gamma \to \mathfrak{M} - Polar\ 3S\mathfrak{Q}FS(\varpi)$ *assigns to each parameter* $\gamma \in \Gamma$ *a* $\mathfrak{M}$*-Polar 3-spherical* $\mathfrak{Q}$ *fuzzy set over* ϖ*, where:*

$$\mathfrak{R}(\gamma) = \{((\varpi, \wp), \{\zeta_\iota(\varpi, \wp), \xi_\iota(\varpi, \wp), \pi_\iota(\varpi, \wp)\})_{\iota=1}^{m}\} | \varpi \in \varpi, \quad \wp \in \mathfrak{Q}\}$$

In this context, $\zeta_\iota(\varpi, \wp)$ *represents the* $\mathcal{MD}$ $\xi_\iota(\varpi, \wp)$ *represents the* $\mathcal{NMD}$ *and* $\pi_\iota(\varpi, \wp)$ *represents the* $\mathcal{AD}$ *associated with the*I*-th pole of the pair* $(\varpi, \wp)$*. Each of these values lies between the closed interval 0 to 1 and must satisfy the 3-spherical fuzzy constraint for each pole* I*:*

$$0 \leq \zeta_\iota^3(\varpi, \wp) + \xi_\iota^3(\varpi, \wp) + \pi_\iota^3(\varpi, \wp) \leq 1.$$

The $\mathcal{AD}$ *(or uncertainty) for each pole*I *and element* $(\varpi, \wp)$ *is defined as:*

$$\chi_\iota(\varpi, \wp) = \sqrt[3]{1 - \zeta_\iota^3(\varpi, \wp) - \xi_\iota^3(\varpi, \wp) - \pi_\iota^3(\varpi, \wp)}.$$

The concept can be represented using the graph shown in Figure 3.

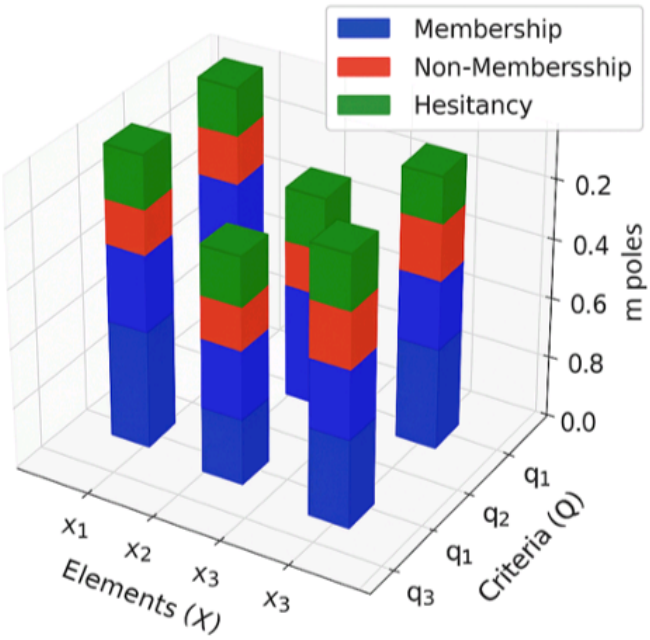

Figure 3. $\mathfrak{M}$-Polar 3-spherical $\mathfrak{Q}$ Fuzzy Soft Representation.

Example 4. *To clarify the definition, consider a universe* $\varpi = \{\varpi_1, \varpi_2, \varpi_3\}$ *and a contextual set* $\mathfrak{Q} = \{\wp_1, \wp_2, \wp_3\}$*, with three poles* ($\mathfrak{M} = 3$)*. Let the parameter set be* $E = \{\varepsilon_1, \varepsilon_2\}$*, and choose a subset* $\Gamma = \{\varepsilon_1\} \subseteq E$.

Define a mapping $\upsilon : \Gamma \to 3 - Polar - Q3 - SFS(\varpi)$ *such that for the parameter* ε_1*, the fuzzy soft set* $\upsilon(\varepsilon_1)$ *is defined as:*

$$\mathfrak{R}(\varepsilon_1) = \begin{Bmatrix} ((\varpi_1, \wp_1), [(.7, .2, .3), (.6, .3, .4), (.5, .4, .3)]), \\ ((\varpi_2, \wp_2), [(.5, .3, .4), (.6, .1, .2), (.4, .3, .5)]), \\ ((\varpi_3, \wp_3), [(.4, .3, .5), (.6, .1, .1), (.3, .4, .6)]) \end{Bmatrix}$$

In this representation, each triplet $(\zeta_\iota, \xi_\iota, \pi_\iota)$ *within the inner lists corresponds to the fuzzy evaluation under pole*I. *For instance, the entry* $(.7, .2, .3)$ *for* $(\varpi_1, \wp_1)$ *at pole 1 indicates a high* $\mathcal{MD}$ *a relatively low* $\mathcal{NMD}$ *and moderate* $\mathcal{AD}$ *all of which respect the 3-spherical constraint.*

Such a structure enables the modeling of complex evaluations different poles while taking into account multiple conditions or contextual dimensions simultaneously. The example illustrates the expressive power of the $\mathfrak{M} - P3S\mathfrak{Q}FSS$ *s in capturing nuanced assessments in uncertain environments.*

Definition 16. *Let* $(\mathfrak{R}, \Gamma)$ *and* $(\mathfrak{I}, \lambda)$ *be two* $\mathfrak{m} - P3S\mathfrak{Q}FSS$ *s over the universal set* ϖ.

1.
$$(\mathfrak{R}, \Gamma) \subseteq (\mathfrak{I}, \lambda)$$

Iff:

- $\Gamma \subseteq \lambda$
- $\forall \gamma \in \Gamma$, $\varpi \in \varpi$, $\wp \in \mathfrak{Q}$, *and* $\iota = 1, 2, \ldots, m$:

- $\zeta_\iota^{\Re}(\gamma)(\varpi, \wp) \leq \zeta_\iota^{\Im}(\gamma)(\varpi, \wp)$
- $\xi_\iota^{\Re}(\gamma)(\varpi, \wp) \geq \xi_\iota^{\Im}(\gamma)(\varpi, \wp)$
- $\pi_\iota^{\Re}(\gamma)(\varpi, \wp) \geq \pi_\iota^{\Im}(\gamma)(\varpi, \wp)$

2. *The complement of* $(\Re, \Gamma)$ *is* $(\Re^c, \Gamma)$*, where* $\Re^c : \Gamma \to \mathfrak{M} - Polar - \mathfrak{Q}_0^2 - SFS(\varpi)$ *is defined as follows:*

   For every $\gamma \in \Gamma$*,* $\varpi \in \varpi$*,* $\wp \in \mathfrak{Q}$*, and* $\iota = 1, 2, \ldots, m$*:*

   - $\zeta_\iota^{\Re^c}(\gamma)(\varpi, \wp) = \xi_\iota^{\Re}(\gamma)(\varpi, \wp)$
   - $\xi_\iota^{\Re^c}(\gamma)(\varpi, \wp) = \zeta_\iota^{\Re}(\gamma)(\varpi, \wp)$
   - $\pi_\iota^{\Re^c}(\gamma)(\varpi, \wp) = \pi_\iota^{\Re}(\gamma)(\varpi, \wp)$

3. *The union of* $(\Re, \Gamma)$ *and* $(\Im, \lambda)$ *is* $(\Re \cup y, \Sigma)$*, where* $\Sigma = \Gamma \cup \lambda$ *and* $(\Re \cup \Im) : \Sigma \to \mathfrak{M} - Polar - \mathfrak{Q}_0^2 - SFS(\varpi)$ *is defined as follows:*

   For every $\gamma \in \Sigma$*:*

   - *If* $\gamma \in \Gamma - \lambda$*, then* $(\Re \cup \Im)(\gamma) = \Re(\gamma)$*.*
   - *If* $\gamma \in \lambda - \Gamma$*, then* $(\Re \cup \Im)(\gamma) = \Im(\gamma)$*.*
   - *If* $\gamma \in \Gamma \cap \lambda$*, then for every* $\varpi \in \varpi$*,* $\wp \in \mathfrak{Q}$*, and* $\iota = 1, 2, \ldots, m$*:*
     - $\zeta_\iota^{(\Re \cup \Im)}(\gamma)(\varpi, \wp) = \max(\zeta_\iota^{\Re}(\gamma)(\varpi, \wp), \zeta_\iota^{\Im}(\gamma)(\varpi, \wp))$
     - $\xi_\iota^{(\Re \cup \Im)}(\gamma)(\varpi, \wp) = \min(\xi_\iota^{\Re}(\gamma)(\varpi, \wp), \xi_\iota^{\Im}(\gamma)(\varpi, \wp))$
     - $\pi_\iota^{(\Re \cup \Im)}(\gamma)(\varpi, \wp) = \min(\pi_\iota^{\Re}(\gamma)(\varpi, \wp), \pi_\iota^{\Im}(\gamma)(\varpi, \wp))$

4. *The intersection of* $(\Re, \Gamma)$ *and* $(\Im, \lambda)$ *is* $(\Re \cap y, \Delta)$*, where* $\Delta = \Gamma \cap \lambda$ *and* $(\Re \cap \Im) : \Delta \to \mathfrak{M}$*-Polar-*$\mathfrak{Q}3$*-SFS*$(\varpi)$ *is defined as follows:*

   For every $\gamma \in \Delta$*,* $\varpi \in \varpi$*,* $\wp \in \mathfrak{Q}$*, and* $\iota = 1, 2, ..., m$*:*

   - $\zeta_\iota^{(\Re \cap \Im)}(\gamma)(\varpi, \wp) = \min(\zeta_\iota^{\Re}(\gamma)(\varpi, \wp), \zeta_\iota^{\Im}(\gamma)(\varpi, \wp))$
   - $\xi_\iota^{(\Re \cap \Im)}(\gamma)(\varpi, \wp) = \max(\xi_\iota^{\Re}(\gamma)(\varpi, \wp), \xi_\iota^{\Im}(\gamma)(\varpi, \wp))$
   - $\pi_\iota^{(\Re \cap \Im)}(\gamma)(\varpi, \wp) = \max(\pi_\iota^{\Re}(\gamma)(\varpi, \wp), \pi_\iota^{\Im}(\gamma)(\varpi, \wp))$

Proposition 1. *Let* $(\Re, \Gamma)$ *and* $(\Im, \lambda)$ *be two* $\mathfrak{M}P3S\mathfrak{Q}FSS$*s over the universal set* ϖ*. Then:*

1. $(\Re, \Gamma) \cap (\Im, \lambda) \subseteq (\Re, \Gamma) \cup (\Im, \lambda)$
2. $(\Re, \Gamma) \cap (\Im, \lambda) \subseteq (\Im, \lambda) \cup (\Re, \Gamma)$

Proof. Both inclusions are proved by verifying the inequalities for the membership degree, non-membership degree, and hesitancy degree according to Definition 16.

For any $\gamma \in \Gamma \cap \lambda$, $\varpi \in \varpi$, $\wp \in \mathfrak{Q}$, and $\iota = 1, 2, \ldots, m$, let:

$$\zeta_1 = \zeta_\iota^{\Re}(\gamma)(\varpi, \wp), \quad \zeta_2 = \zeta_\iota^{\Im}(\gamma)(\varpi, \wp)$$
$$\xi_1 = \xi_\iota^{\Re}(\gamma)(\varpi, \wp), \quad \xi_2 = \xi_\iota^{\Im}(\gamma)(\varpi, \wp)$$
$$\pi_1 = \pi_\iota^{\Re}(\gamma)(\varpi, \wp), \quad \pi_2 = \pi_\iota^{\Im}(\gamma)(\varpi, \wp)$$

For (1): By the definitions of the intersection and union operations:

$$(\mathfrak{R}, \Gamma) \cap (\mathfrak{I}, \lambda) = \{((\varpi, \wp), \min(\zeta_1, \zeta_2), \max(\xi_1, \xi_2), \max(\pi_1, \pi_2))\}$$
$$(\mathfrak{R}, \Gamma) \cup (\mathfrak{I}, \lambda) = \{((\varpi, \wp), \max(\zeta_1, \zeta_2), \min(\xi_1, \xi_2), \min(\pi_1, \pi_2))\}$$

It must be shown that:

(i) $\min(\zeta_1, \zeta_2) \leq \max(\zeta_1, \zeta_2)$

(ii) $\max(\xi_1, \xi_2) \geq \min(\xi_1, \xi_2)$

(iii) $\max(\pi_1, \pi_2) \geq \min(\pi_1, \pi_2)$

- For (i): By the nature of the min and max functions, it holds that $\min(a, b) \leq \max(a, b)$ for any two numbers $a, b \in [0, 1]$.Thus:

$$\min(\zeta_1, \zeta_2) \leq \max(\zeta_1, \zeta_2)$$

- For (ii): Similarly, $\max(a, b) \geq \min(a, b)$ for any $a, b \in [0, 1]$. Thus:

$$\max(\xi_1, \xi_2) \geq \min(\xi_1, \xi_2)$$

- For (iii): Again, $\max(a, b) \geq \min(a, b)$ for any $a, b \in [0, 1]$. Thus:

$$\max(\pi_1, \pi_2) \geq \min(\pi_1, \pi_2)$$

Since each of these three inequalities holds for all choices of ϖ, $\wp$, and ι, it follows that:

$$(\mathfrak{R}, \Gamma) \cap (\mathfrak{I}, \lambda) \subseteq (\mathfrak{R}, \Gamma) \cup (\mathfrak{I}, \lambda)$$

For (2): The proof is in essence the same as (1), the only difference being that the commutativity of the union operation is applied here. Formal:

$$(\mathfrak{I}, \lambda) \cup (\mathfrak{R}, \Gamma) = \{((\varpi, \wp), \max(\zeta_2, \zeta_1), \min(\xi_2, \xi_1), \min(\pi_2, \pi_1))\}$$

However, since $\max(\zeta_1, \zeta_2) = \max(\zeta_2, \zeta_1)$, $\min(\xi_1, \xi_2) = \min(\xi_2, \xi_1)$, and $\min(\pi_1, \pi_2) = \min(\pi_2, \pi_1)$, we have:

$$(\mathfrak{R}, \Gamma) \cup (\mathfrak{I}, \lambda) = (\mathfrak{I}, \lambda) \cup (\mathfrak{R}, \Gamma)$$

Therefore, from (1) we immediately obtain:

$$(\mathfrak{R}, \Gamma) \cap (\mathfrak{I}, \lambda) \subseteq (\mathfrak{I}, \lambda) \cup (\mathfrak{R}, \Gamma)$$

This completes the proof. □

Definition 17. *Let* $(\mathfrak{R}, \Gamma)$*,* $(\mathfrak{I}, \lambda)$*, and* (φ, Ψ) *be* $\mathfrak{M}$*-Polar 3-spherical* $\mathfrak{Q}$ *fuzzy soft sets over the universal set* ϖ*.*

We define:

$$\mathfrak{R}(\gamma) = \{((\varpi, \wp), \zeta^{\mathfrak{R}}(\gamma)(\varpi, \wp), \xi^{\mathfrak{R}}(\gamma)(\varpi, \wp), \pi^{\mathfrak{R}}(\gamma)(\varpi, \wp)) \mid \varpi \in \varpi, \wp \in \mathfrak{Q}\}$$
$$\mathfrak{I}(\Lambda) = \{((\varpi, \wp), \zeta^{\mathfrak{I}}(\Lambda)(\varpi, \wp), \xi^{\mathfrak{I}}(\Lambda)(\varpi, \wp), \pi^{\mathfrak{I}}(\Lambda)(\varpi, \wp)) \mid \varpi \in \varpi, \wp \in \mathfrak{Q}\}$$
$$\varphi(\psi) = \{((\varpi, \wp), \zeta^{\varphi}(\psi)(\varpi, \wp), \xi^{\varphi}(\psi)(\varpi, \wp), \pi^{\varphi}(\psi)(\varpi, \wp)) \mid \varpi \in \varpi, \wp \in \mathfrak{Q}\}$$

as elements of $\mathfrak{M} - P3S\mathfrak{Q}FSS$ *s.*

If $\Lambda > 0$*, then:*

(i) $\mathfrak{R}(\gamma) \oplus \mathfrak{I}(\Lambda) =$

$$\left\{ \left((\varpi, \wp), \sqrt[3]{\frac{\zeta^{\mathfrak{R}(\gamma),3} + \zeta^{\mathfrak{I}(\Lambda),3}}{\sqrt[3]{1 + (1 - \zeta^{\mathfrak{R}(\gamma),3}) \cdot (1 - \zeta^{\mathfrak{I}(\Lambda),3})}}}, \sqrt[3]{\frac{\xi^{\mathfrak{R}(\gamma)} \cdot \xi^{\mathfrak{I}(\Lambda)}}{\sqrt[3]{1 + (1 - \xi^{\mathfrak{R}(\gamma),3}) \cdot (1 - \xi^{\mathfrak{I}(\Lambda),3})}}}, \sqrt[3]{\frac{\pi^{\mathfrak{R}(\gamma)} \cdot \pi^{\mathfrak{I}(\Lambda)}}{\sqrt[3]{1 + (1 - \pi^{\mathfrak{R}(\gamma),3}) \cdot (1 - \pi^{\mathfrak{I}(\Lambda),3})}}} \right) \middle| \varpi \in \varpi, \wp \in \mathfrak{Q} \right\}$$

(ii) $\mathfrak{R}(\gamma) \otimes \mathfrak{I}(\Lambda) =$

$$\left\{ \left((\varpi, \wp), \sqrt[3]{\frac{\zeta^{\mathfrak{R}(\gamma)} \cdot \zeta^{\mathfrak{I}(\Lambda)}}{\sqrt[3]{1 + (1 - \zeta^{\mathfrak{R}(\gamma),3}) \cdot (1 - \zeta^{\mathfrak{I}(\Lambda),3})}}}, \sqrt[3]{\frac{\xi^{\mathfrak{R}(\gamma)} + \xi^{\mathfrak{I}(\Lambda)}}{\sqrt[3]{1 + \xi^{\mathfrak{R}(\gamma),3} \cdot \xi^{\mathfrak{I}(\Lambda),3}}}}, \sqrt[3]{\frac{\pi^{\mathfrak{R}(\gamma)3} + \pi^{\mathfrak{I}(\Lambda)3}}{\sqrt[3]{1 + \pi^{\mathfrak{R}(\gamma),3} \cdot \pi^{\mathfrak{I}(\Lambda),3}}}} \right) \middle| \varpi \in \varpi, \wp \in \mathfrak{Q} \right\}$$

(iii) $\Lambda \cdot \mathfrak{R}(\gamma) =$

$$\left\{ ((\varpi, \wp), \sqrt[3]{\frac{(1 + \zeta^{\mathfrak{R}(\gamma),3})^{\Lambda} - (1 - \zeta^{\mathfrak{R}(\gamma),3})^{\Lambda}}{\sqrt[3]{(1 + \zeta^{\mathfrak{R}(\gamma),3})^{\Lambda} + (1 - \zeta^{\mathfrak{R}(\gamma),3})^{\Lambda}}}}, \sqrt[3]{\frac{2 \cdot \xi^{\mathfrak{R}(\gamma)\Lambda}}{\sqrt[3]{(2 - \xi^{\mathfrak{R}(\gamma),3})^{\Lambda} + (\xi^{\mathfrak{R}(\gamma),3})^{\Lambda}}}}, \sqrt[3]{\frac{2 \cdot \pi^{\mathfrak{R}(\gamma),\Lambda}}{\sqrt[3]{(2 - \pi^{\mathfrak{R}(\gamma),3})^{\Lambda} + (\pi^{\mathfrak{R}(\gamma),3})^{\Lambda}}}} \right) \middle| \varpi \in \varpi, \wp \in \mathfrak{Q} \right\}$$

(iv) $\mathfrak{R}(\gamma)^{\Lambda} =$

$$\left\{ ((\varpi, \wp), \sqrt[3]{\frac{2 \cdot \zeta^{\mathfrak{R}(\gamma)\Lambda}}{\sqrt[3]{(2 - \zeta^{\mathfrak{R}(\gamma),3})^{\Lambda} + (\zeta^{\mathfrak{R}(\gamma),3})^{\Lambda}}}}, \sqrt[3]{\frac{(1 + \xi^{\mathfrak{R}(\gamma),3})^{\Lambda} - (1 - \xi^{\mathfrak{R}(\gamma),3})^{\Lambda}}{\sqrt[3]{(1 + \xi^{\mathfrak{R}(\gamma),3})^{\Lambda} + (1 - \xi^{\mathfrak{R}(\gamma),3})^{\Lambda}}}}, \sqrt[3]{\frac{(1 + \pi^{\mathfrak{R}(\gamma),3})^{\Lambda} - (1 - \pi^{\mathfrak{R}(\gamma),3})^{\Lambda}}{\sqrt[3]{(1 + \pi^{\mathfrak{R}(\gamma),3})^{\Lambda} + (1 - \pi^{\mathfrak{R}(\gamma),3})^{\Lambda}}}} \right) \middle| \varpi \in \varpi, \wp \in \mathfrak{Q} \right\}$$

Proposition 2. *Let* $(\mathfrak{R}, \Gamma)$*,* $(\mathfrak{I}, \Lambda)$*, and* (φ, Ψ) *be three* $\mathfrak{M} - P3S\mathfrak{Q}FSS$ *s. Then, the following properties hold:*

1. $(\mathfrak{R}, \Gamma) \subseteq (U, E)$.

2. $(\phi, \Gamma) \subseteq (\mathfrak{R}, \Lambda)$.

3. *If* $(\mathfrak{R}, \Gamma) \subseteq (\mathfrak{I}, \Lambda)$ *and* $(\mathfrak{I}, \Lambda) \subseteq (\varphi, \Psi)$*, then* $(\mathfrak{R}, \Gamma) \subseteq (\varphi, \Psi)$.

Definition 18. *A* $\mathfrak{M} - P3S\mathfrak{Q}FSSs$ $(\mathfrak{R}, \Gamma)$ *of dimension* $\mathfrak{M}$ *over* $\varpi \times \mathfrak{Q}$ *is said to be the null* $\mathfrak{M} - P3S\mathfrak{Q}FSSs$ *if* $\mathfrak{R}(\gamma) = \phi_m \ \forall \ \gamma \in \Gamma$. *It is indicated by:*

$$\phi_{\Gamma}^{m}$$

Definition 19. *A* $\mathfrak{M} - P3S\mathfrak{Q}FSSs$ $(\mathfrak{R}, \Gamma)$ *of dimension* $\mathfrak{M}$ *over* $\varpi \times \mathfrak{Q}$ *is known as the absolute* $\mathfrak{M} - P3S\mathfrak{Q}FSSs$ *if* $\mathfrak{R}(\gamma) = 1_m \ \forall \ \gamma \in \Gamma$. *It is indicated by:*

$$U_{\Gamma}^{m}$$

Proposition 3.

1. $((\mathfrak{R}, \Gamma)^c)^c = (\mathfrak{R}, \Gamma)$,

2. $(\phi_{\Gamma}^{m})^c = U_{\Gamma}^{m}$, *and* $(U_{\Gamma}^{m})^c = \phi_{\Gamma}^{m}$

Proof. In the following, we shall prove (1) and (2):

- **Proof of (1):** By the Definition 16 of complement, for any $\gamma \in \Gamma$, $\varpi \in \varpi$, $\wp \in \mathfrak{Q}$, and $\iota = 1, 2, \ldots, m$, it holds that:

$$(\mathfrak{R}, \Gamma)^c = \{((\varpi, \wp), \xi_{\iota}^{\mathfrak{R}}(\gamma)(\varpi, \wp), \zeta_{\iota}^{\mathfrak{R}}(\gamma)(\varpi, \wp), \pi_{\iota}^{\mathfrak{R}}(\gamma)(\varpi, \wp))\}$$

  Applying the complement operation again:

$$\begin{aligned} ((\mathfrak{R}, \Gamma)^c)^c &= \{((\varpi, \wp), \xi_{\iota}^{(\mathfrak{R})^c}(\gamma)(\varpi, \wp), \zeta_{\iota}^{(\mathfrak{R})^c}(\gamma)(\varpi, \wp), \pi_{\iota}^{(\mathfrak{R})^c}(\gamma)(\varpi, \wp))\} \\ &= \{((\varpi, \wp), \zeta_{\iota}^{\mathfrak{R}}(\gamma)(\varpi, \wp), \xi_{\iota}^{\mathfrak{R}}(\gamma)(\varpi, \wp), \pi_{\iota}^{\mathfrak{R}}(\gamma)(\varpi, \wp))\} \\ &= (\mathfrak{R}, \Gamma) \end{aligned}$$

  This shows the involution property: Complementing twice yields the original set.

- **Proof of (2):** Recall that the null $\mathfrak{M}-P3S\mathfrak{Q}FSSs$ ϕ_{Γ}^{m} is defined such that for all $\gamma \in \Gamma$, $\varpi \in \varpi$, $\wp \in \mathfrak{Q}$, and $\iota = 1, 2, \ldots, m$:

$$\phi_{\Gamma}^{m} = \{((\varpi, \wp), 0, 1, 0)\}$$

  In other words, the degree of membership is 0, the degree of non-membership is 1, and the degree of hesitation is 0 for all.

  Applying the complement operation:

$$(\phi_{\Gamma}^{m})^c = \{((\varpi, \wp), 1, 0, 0)\} = U_{\Gamma}^{m}$$

where U_Γ^m is the absolute $\mathfrak{M}$−$P3S\mathfrak{Q}FSSs$ with all membership degrees, all non-membership degrees, and all hesitancy degrees 1, 0, and 0, respectively.

Conversely, for the absolute set:

$$U_\Gamma^m = \{((\varpi, \wp), 1, 0, 0)\}$$

Applying the complement:

$$(U_\Gamma^m)^c = \{((\varpi, \wp), 0, 1, 0)\} = \phi_\Gamma^m$$

These findings indicate that the null and the full sets are complementary.

□

Proposition 4. *If* $(\mathfrak{R}, \Gamma)$*,* $(\mathfrak{I}, \lambda)$*, and* (φ, Ψ) *are* $\mathfrak{M} - P3S\mathfrak{Q}FSSs$ *over the universal set* ϖ*, then the following properties hold:*

1. $\phi \cup (\mathfrak{R}, \Gamma) = (\mathfrak{R}, \Gamma), \quad \phi \cap (\mathfrak{R}, \Gamma) = \phi$
2. $(U, E) \cup (\mathfrak{R}, \Gamma) = (U, E), \quad (U, E) \cap (\mathfrak{R}, \Gamma) = (\mathfrak{R}, \Gamma)$
3. $(\mathfrak{R}, \Gamma) \cap (\mathfrak{R}, \Gamma) = (\mathfrak{R}, \Gamma), \quad (\mathfrak{R}, \Gamma) \cup (\mathfrak{R}, \Gamma) = (\mathfrak{R}, \Gamma)$
4. $(\mathfrak{R}, \Gamma) \cup (\mathfrak{I}, \lambda) = (\mathfrak{I}, \Lambda) \cup (\mathfrak{R}, \Gamma), \quad (\mathfrak{R}, \Gamma) \cap (\mathfrak{I}, \lambda) = (\mathfrak{I}, \lambda) \cap (\mathfrak{R}, \Gamma)$
5. $(\mathfrak{R}, \Gamma) \cup ((\mathfrak{I}, \lambda) \cup (\varphi, \Psi)) = ((\mathfrak{R}, \Gamma) \cup (\mathfrak{I}, \lambda)) \cup (\varphi, \Psi)$
   $(\mathfrak{R}, \Gamma) \cap ((\mathfrak{I}, \lambda) \cap (\varphi, \Psi)) = ((\mathfrak{R}, \Gamma) \cap (\mathfrak{I}, \lambda)) \cap (\varphi, \Psi)$
6. $(\mathfrak{R}, \Gamma) \cup ((\mathfrak{I}, \lambda) \cap (\varphi, \Psi)) = ((\mathfrak{R}, \Gamma) \cup (\mathfrak{I}, \lambda)) \cap ((\mathfrak{R}, \Gamma) \cup (\varphi, \Psi))$
   $(\mathfrak{R}, \Gamma) \cap ((\mathfrak{I}, \lambda) \cup (\varphi, \Psi)) = ((\mathfrak{R}, \Gamma) \cap (\mathfrak{I}, \lambda)) \cup ((\mathfrak{R}, \Gamma) \cap (\varphi, \Psi))$

Proof. The proof from (1)-(5) is trivial by using Definition 16; now we discuss only the proof of (6): For

$$\begin{aligned}
(\mathfrak{R}, \Gamma) &= \{((\varpi, \wp), \zeta_{\mathfrak{R}}(\varpi, \wp), \xi_{\mathfrak{R}}(\varpi, \wp), \pi_{\mathfrak{R}}(\varpi, \wp))\} \\
(\mathfrak{I}, \lambda) &= \{((\varpi, \wp), \zeta_{\mathfrak{I}}(\varpi, \wp), \xi_{\mathfrak{I}}(\varpi, \wp), \pi_{\mathfrak{I}}(\varpi, \wp))\} \\
(\varphi, \Psi) &= \{((\varpi, \wp), \zeta_{\varphi}(\varpi, \wp), \xi_{\varphi}(\varpi, \wp), \pi_{\varphi}(\varpi, \wp))\}
\end{aligned}$$

The following equality must be established:

$$(\mathfrak{R}, \Gamma) \cup ((\mathfrak{I}, \lambda) \cap (\varphi, \Psi)) = ((\mathfrak{R}, \Gamma) \cup (\mathfrak{I}, \lambda)) \cap ((\mathfrak{R}, \Gamma) \cup (\varphi, \Psi))$$

- For the left side:

$$(\mathfrak{I}, \lambda) \cap (\varphi, \Psi) = \{((\varpi, \wp), \min(\zeta_{\mathfrak{I}}, \zeta_{\varphi}), \max(\xi_{\mathfrak{I}}, \xi_{\varphi}), \max(\pi_{\mathfrak{I}}, \pi_{\varphi}))\}$$

Then:

$$\begin{aligned}
(\mathfrak{R}, \Gamma) \cup ((\mathfrak{I}, \lambda) \cap (\varphi, \Psi)) = \{((\varpi, \wp), \max(\zeta_{\mathfrak{R}}, \min(\zeta_{\mathfrak{I}}, \zeta_{\varphi})), \\
\min(\xi_{\mathfrak{R}}, \max(\xi_{\mathfrak{I}}, \xi_{\varphi})), \min(\pi_{\mathfrak{R}}, \max(\pi_{\mathfrak{I}}, \pi_{\varphi})))\}
\end{aligned} \tag{3.1}$$

- For the right side:

$$(\mathfrak{R},\Gamma)\cup(\mathfrak{I},\lambda)=\{((\varpi,\wp),\max(\zeta_{\mathfrak{R}},\zeta_Y),\min(\xi_{\mathfrak{R}},\xi_Y),\min(\pi_{\mathfrak{R}},\pi_Y))\}$$
$$(\mathfrak{R},\Gamma)\cup(\varphi,\Psi)=\{((\varpi,\wp),\max(\zeta_{\mathfrak{R}},\zeta_\varphi),\min(\xi_{\mathfrak{R}},\xi_\varphi),\min(\pi_{\mathfrak{R}},\pi_\varphi))\}$$

Then:

$$\begin{aligned}&((\mathfrak{R},\Gamma)\cup(\mathfrak{I},\lambda))\cap((\mathfrak{R},\Gamma)\cup(\varphi,\Psi))=\\&\{((\varpi,\wp),\min(\max(\zeta_{\mathfrak{R}},\zeta_Y),\max(\zeta_{\mathfrak{R}},\zeta_\varphi)),\\&\max(\min(\xi_{\mathfrak{R}},\xi_Y),\min(\xi_{\mathfrak{R}},\xi_\varphi)),\\&\max(\min(\pi_{\mathfrak{R}},\pi_Y),\min(\pi_{\mathfrak{R}},\pi_\varphi)))\}\end{aligned}\tag{3.2}$$

From (3.1) and (3.2), we confirm:

$$(\mathfrak{R},\Gamma)\cup((\mathfrak{I},\lambda)\cap(\varphi,\Psi))=((\mathfrak{R},\Gamma)\cup(\mathfrak{I},\lambda))\cap((\mathfrak{R},\Gamma)\cup(\varphi,\Psi))$$

□

Corollary 1.

$$\phi\cup(U,E)=(U,E),$$
$$\phi\cap(U,E)=\phi$$

Proposition 5. *For three $\mathfrak{M}-P3S\mathfrak{Q}FSS$ s:*

$$(\mathfrak{R},\Gamma)=\left\{((\varpi,\wp),\zeta_\iota^{\mathfrak{R}}(\gamma)(\varpi,\wp),\xi_\iota^{\mathfrak{R}}(\gamma)(\varpi,\wp),\pi_\iota^{\mathfrak{R}}(\gamma)(\varpi,\wp))\right\}$$
$$(\mathfrak{I},\lambda)=\left\{((\varpi,\wp),\zeta_\iota^{\mathfrak{I}}(\Lambda)(\varpi,\wp),\xi_\iota^{\mathfrak{I}}(\Lambda)(\varpi,\wp),\pi_\iota^{\mathfrak{I}}(\Lambda)(\varpi,\wp))\right\}$$
$$(\varphi,\Psi)=\{((\varpi,\wp),\zeta_\iota^{\varphi}(\psi)(\varpi,\wp),\xi_\iota^{\varphi}(\psi)(\varpi,\wp),\pi_\iota^{\varphi}(\psi)(\varpi,\wp))\}$$

The following axioms hold:

(i)

$$(\mathfrak{R},\Gamma)\oplus(\mathfrak{I},\lambda)=(\mathfrak{I},\lambda)\oplus(\mathfrak{R},\Gamma)$$

(ii)

$$(\mathfrak{R},\Gamma)\otimes(\mathfrak{I},\lambda)=(\mathfrak{I},\lambda)\otimes(\mathfrak{R},\Gamma)$$

(iii)

$$\alpha((\mathfrak{R},\Gamma)\oplus(\mathfrak{I},\lambda))=\alpha(\mathfrak{R},\Gamma)\oplus\alpha(\mathfrak{I},\lambda),\quad \alpha>0$$

(iv)

$$(\alpha_1+\alpha_2)(\mathfrak{R},\Gamma)=\alpha_1(\mathfrak{R},\Gamma)\oplus\alpha_2(\mathfrak{R},\Gamma),\quad \alpha_1,\alpha_2>0$$

(v)

$$((\mathfrak{R},\Gamma)\otimes(\mathfrak{I},\lambda))^\alpha=(\mathfrak{R},\Gamma)^\alpha\otimes(\mathfrak{I},\lambda)^\alpha,\quad \alpha>0$$

(vi)

$$(\mathfrak{R},\Gamma)^{\alpha_1}\otimes(\mathfrak{R},\Gamma)^{\alpha_2}=(\mathfrak{R},\Gamma)^{(\alpha_1+\alpha_2)},\quad \alpha_1,\alpha_2>0$$

Proof. For three $\mathfrak{M} - P3S\mathfrak{Q}FSS$ s: $(\mathfrak{R}, \Gamma)$, $(\mathfrak{I}, \Lambda)$, and (φ, Ψ) and $\alpha_1, \alpha_2 > 0$, according to Definition 17, we will present the proofs of (i), (ii), and (iv). We can obtain:

(i)

$$(\mathfrak{R}, \Gamma) \oplus (\mathfrak{I}, \lambda) = \Bigg\{((\varpi, \wp), \sqrt[3]{\frac{\zeta_\iota^{\mathfrak{R}(\gamma),3} + \zeta_\iota^{\mathfrak{I}(\Lambda),3}}{\sqrt[3]{1 + (1 - \zeta_\iota^{\mathfrak{R}(\gamma),3})(1 - \zeta_\iota^{\mathfrak{I}(\Lambda),3})}}}, \sqrt[3]{\frac{\xi_\iota^{\mathfrak{R}(\gamma),3} \cdot \xi_\iota^{\mathfrak{I}(\Lambda),3}}{\sqrt[3]{1 + (1 - \xi_\iota^{\mathfrak{R}(\gamma),3})(1 - \xi_\iota^{\mathfrak{I}(\Lambda),3})}}}, \sqrt[3]{\frac{\pi_\iota^{\mathfrak{R}(\gamma),3} \cdot \pi_\iota^{\mathfrak{I}(\Lambda),3}}{\sqrt[3]{1 + (1 - \pi_\iota^{\mathfrak{R}(\gamma),3})(1 - \pi_\iota^{\mathfrak{I}(\Lambda),3})}}}\Bigg)\Bigg\}$$

$$= \Bigg\{((\varpi, \wp), \sqrt[3]{\frac{\zeta_\iota^{\mathfrak{I}(\Lambda),3} + \zeta_\iota^{\mathfrak{R}(\gamma),3}}{\sqrt[3]{1 + (1 - \zeta_\iota^{\mathfrak{I}(\Lambda),3})(1 - \zeta_\iota^{\mathfrak{R}(\gamma),3})}}}, \sqrt[3]{\frac{\xi_\iota^{\mathfrak{I}(\Lambda),3} \cdot \xi_\iota^{\mathfrak{R}(\gamma),3}}{\sqrt[3]{1 + (1 - \xi_\iota^{\mathfrak{I}(\Lambda),3})(1 - \xi_\iota^{\mathfrak{R}(\gamma),3})}}}, \sqrt[3]{\frac{\pi_\iota^{\mathfrak{I}(\Lambda),3} \cdot \pi_\iota^{\mathfrak{R}(\gamma),3}}{\sqrt[3]{1 + (1 - \pi_\iota^{\mathfrak{I}(\Lambda),3})(1 - \pi_\iota^{\mathfrak{R}(\gamma),3})}}}\Bigg)\Bigg\}$$

$$= (\mathfrak{I}, \Lambda) \oplus (\mathfrak{R}, \Gamma)$$

(ii)

$$(\mathfrak{R}, \Gamma) \otimes (\mathfrak{I}, \lambda) = \Bigg\{((\varpi, \wp), \sqrt[3]{\frac{\zeta_\iota^{\mathfrak{R}(\gamma)} \cdot \zeta_\iota^{\mathfrak{I}(\Lambda)}}{\sqrt[3]{1 + (1 - \zeta_\iota^{\mathfrak{R}(\gamma),3})(1 - \zeta_\iota^{\mathfrak{I}(\Lambda),3})}}}, \sqrt[3]{\frac{\xi_\iota^{\mathfrak{R}(\gamma),3} + \xi_\iota^{\mathfrak{I}(\Lambda),3}}{\sqrt[3]{1 + \xi_\iota^{\mathfrak{R}(\gamma),3} \cdot \xi_\iota^{\mathfrak{I}(\Lambda),3}}}}, \sqrt[3]{\frac{\pi_\iota^{\mathfrak{R}(\gamma),3} + \pi_\iota^{\mathfrak{I}(\Lambda),3}}{\sqrt[3]{1 + \pi_\iota^{\mathfrak{R}(\gamma),3} \cdot \pi_\iota^{\mathfrak{I}(\Lambda),3}}}}\Bigg)\Bigg\}$$

$$= (\mathfrak{I}, \Lambda) \otimes (\mathfrak{R}, \Gamma)$$

(iv)

$$(\alpha_1 + \alpha_2)(\mathfrak{R}, \Gamma) = \Bigg\{((\varpi, \wp), \sqrt[3]{\frac{(1 + \zeta_\iota^{\mathfrak{R}(\gamma),3})^{\alpha_1+\alpha_2} - (1 - \zeta_\iota^{\mathfrak{R}(\gamma),3})^{\alpha_1+\alpha_2}}{\sqrt[3]{(1 + \zeta_\iota^{\mathfrak{R}(\gamma),3})^{\alpha_1+\alpha_2} + (1 - \zeta_\iota^{\mathfrak{R}(\gamma),3})^{\alpha_1+\alpha_2}}}}\Bigg)\Bigg\}$$

$$= \alpha_1(\mathfrak{R}, \Gamma) \oplus \alpha_2(\mathfrak{R}, \Gamma)$$

□

Definition 20. *Let us consider a collection of* $\mathfrak{M}-P3S\mathfrak{Q}FSSs$ *represented as*

$$\tau_j = \{(\varpi, \wp),\ \zeta_{\iota,j}(\varpi, \wp),\ \xi_{\iota,j}(\varpi, \wp),\ \pi_{\iota,j}(\varpi, \wp)\}$$

where $j = 1, 2, \ldots, m, \wp \in \mathfrak{Q}, \iota = 1, 2, \ldots, \ell$*, and* $\varpi \in \varpi$*. In this structure,* $\zeta_{\iota,j}(\varpi, \wp)$ *denotes the* $\mathcal{MD}$ $\xi_{\iota,j}(\varpi, \wp)$ *represents the* $\mathcal{NMD}$*, and* $\pi_{\iota,j}(\varpi, \wp)$ *indicates the* $\mathcal{AD}$ *corresponding to the element* ϖ *under pole* ι*, parameter* $\wp$*, and soft set* τ_j*.*

The $\mathfrak{M}$*-P3-S*$\mathfrak{Q}$*FS Weighted Geometric (*$\mathfrak{M}$*-P3S*$\mathfrak{Q}$*FWG) operator is defined as:*

$$\mathfrak{M} - P3SM\mathfrak{Q}FWG(\tau_1, \tau_2, \ldots, \tau_m) = \left(\prod_{j=1}^{m} \left(\zeta_{\iota,j}(\varpi, \wp)\right)^{w_j}, \prod_{j=1}^{m} \left(\xi_{\iota,j}(\varpi, \wp)\right)^{w_j}, \prod_{j=1}^{m} \left(\pi_{\iota,j}(\varpi, \wp)\right)^{w_j} \right)$$

for each $\wp \in \mathfrak{Q}, \varpi \in \varpi$*, and* $\iota = 1, 2, \ldots, \ell$*. The weight vector* $w = (w_1, w_2, \ldots, w_m)^\tau$ *contains non-negative values* $w_j > 0$*, which need to be normalized through the condition* $\sum_{j=1}^{m} w_j = 1$*.*

Multiple fuzzy soft sets integrate their information into one aggregated output through a weighted importance ranking system. Moreover, it preserves the 3-spherical condition, which requires that the cubic sum of the $\mathcal{MD}$ $\mathcal{NMD}$ *and* $\mathcal{AD}$*s satisfies the inequality:*

$$\zeta^3 + \xi^3 + \pi^3 \leq 1$$

The $\mathfrak{M}$*-P3S*$\mathfrak{Q}$*FWG operator is particularly useful in MCDM problems under uncertainty, enabling a more expressive modeling of multidimensional and* $\mathfrak{M}$*-Polar information.*

Definition 21 (Score Function)**.** *Let* $(\mathfrak{R}, \Gamma)$ *be a* $\mathfrak{M}-P3S\mathfrak{Q}FSSs$*, where each parameterized mapping is expressed as:*

$$\mathfrak{R}(\gamma) = \{((\varpi, \wp), \zeta_\iota(\varpi, \wp), \xi_\iota(\varpi, \wp), \pi_\iota(\varpi, \wp)) \mid \varpi \in \varpi,\ \wp \in \mathfrak{Q},\ \iota = 1, 2, \ldots, m\}$$

Then, the score function of $\mathfrak{R}(\gamma)$ *is given by:*

$$S(\mathfrak{R}(\gamma)) = \zeta_\iota^3(\varpi, \wp) - \xi_\iota^3(\varpi, \wp) - \pi_\iota^3(\varpi, \wp)$$

with:

$$S(\mathfrak{R}(\gamma)) \in [-1, 1]$$

Definition 22 (Accuracy Function)**.** *The accuracy function of* $\mathfrak{R}(\gamma)$ *is expressed as:*

$$A(\mathfrak{R}(\gamma)) = \zeta_\iota^3(\varpi, \wp) + \xi_\iota^3(\varpi, \wp) + \pi_\iota^3(\varpi, \wp)$$

with:

$$A(\mathfrak{R}(\gamma)) \in [0, 1]$$

Now, the law that compares two $\mathfrak{M}-P3S\mathfrak{Q}FSSs$*.*

Definition 23. *Let* $\mathfrak{R}(\gamma_1) \in (\mathfrak{R}, \Gamma)$ *and* $\mathfrak{I}(\Lambda_2) \in (\mathfrak{I}, \lambda)$ *be two* $\mathfrak{M}-P3S\mathfrak{Q}F$ *elements. Then:*

1. *If* $S(\Re(\gamma_1)) < S(\Im(\Lambda_2))$, *then* $\Re(\gamma_1) < \Im(\Lambda_2)$
2. *If* $S(\Re(\gamma_1)) > S(\Im(\Lambda_2))$, *then* $\Re(\gamma_1) > \Im(\Lambda_2)$
3. *If* $S(\Re(\gamma_1)) = S(\Im(\Lambda_2))$, *then:*
   (a) *If* $A(\Re(\gamma_1)) < A(\Im(\Lambda_2))$, *then* $\Re(\gamma_1) < \Im(\Lambda_2)$
   (b) *If* $A(\Re(\gamma_1)) > A(\Im(\Lambda_2))$, *then* $\Re(\gamma_1) > \Im(\Lambda_2)$
   (c) *If* $A(\Re(\gamma_1)) = A(\Im(\Lambda_2))$, *then* $\Re(\gamma_1) \sim \Im(\Lambda_2)$

where $\sim$ *denotes the equivalence between the fuzzy elements.*

4. Application

Now, the $\mathfrak{M} - P3S\mathcal{Q}FSSs$ algorithm is proposed as an advanced approach for MCDM, capable of handling vague information and uncertainty.

4.1. Application to multi-criteria decision-making

In this section, we show the construction of a multi-criteria group decision making (MCGDM) method in the $\mathfrak{M} - P3S\mathcal{Q}FSSs$ setting. The method is based on the use of aggregation operators, score functions, and accuracy functions to select the best alternative according to multiple criteria in a situation of uncertainty. The MCGDM approach incorporates the following features:

1. **Set of alternatives:**
$$\varpi = \{\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_n\}$$
   where n is the number of candidate alternatives.
2. **Set of evaluation criteria:**
$$E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k\}$$
   where k is the number of criteria. Each criterion is treated as a benefit criterion (higher value is better) or a cost criterion (lower value is better).
3. **Set of contextual parameters:**
$$\mathcal{Q} = \{\wp_1, \wp_2, \ldots, \wp_r\}$$
   where r is the number of contextual parameters or scenarios of evaluation (e.g., various environmental conditions, time frames, or applications).
4. **Set of evaluation perspectives (poles):**
$$P = \{1, 2, \ldots, m\}, \quad m \in \mathbb{N}$$
   where m is the number of independent viewpoints or groups of stakeholders participating in the assessment.

5. **Weight vector for criteria:**

$$W = \{w_1, w_2, w_3, \ldots, w_k\}$$

with each $w_j \in [0, 1]$ and $\sum_{j=1}^{k} w_j = 1$. These weights represent the relative relevance of each criterion when making a decision.

The goal is to find the best alternative by using $\mathfrak{M} - P3S\mathfrak{Q}FSS\,s$ operators to combine the evaluations obtained in different viewpoints and situations.

Next, we outline the steps involved in constructing the decision-making model.

4.2. Proposed decision-making methodology

In this subsection, the step-by-step procedure to execute the $\mathfrak{M} - P3S\mathfrak{Q}FSS\,s$-based decision-making model is presented. The method includes five consecutive steps.

Step 1: Construct the $\mathfrak{M} - P3S\mathfrak{Q}FSS\,s$ Decision Matrix

Purpose: To collect and organize expert evaluations across all alternatives, criteria, parameters, and perspectives.

For every combination of alternative $\varpi_\iota \in \varpi$, criterion $\varepsilon_j \in E$, parameter $\wp \in \mathfrak{Q}$, pole $\ell \in P$, domain experts deliver their assessments as 3-spherical fuzzy triplets:

$$\mathfrak{T}^{\ell}_{\wp\iota j} = \left(\zeta_\ell(\varpi_\iota, \wp, \varepsilon_j), \xi_\ell(\varpi_\iota, \wp, \varepsilon_j), \pi_\ell(\varpi_\iota, \wp, \varepsilon_j)\right)$$

where $\zeta_\ell(\cdot)$, $\xi_\ell(\cdot)$, and $\pi_\ell(\cdot)$ are the degrees of membership, non-membership, and hesitancy, respectively, that fulfill the cubic condition:

$$0 \leq \zeta^3_\ell(\cdot) + \xi^3_\ell(\cdot) + \pi^3_\ell(\cdot) \leq 1$$

The decision matrix is organized as follows:

$$\mathfrak{T} = \begin{bmatrix} \mathfrak{T}_{\wp 11} & \mathfrak{T}_{\wp 12} & \cdots & \mathfrak{T}_{\wp 1k} \\ \mathfrak{T}_{\wp 21} & \mathfrak{T}_{\wp 22} & \cdots & \mathfrak{T}_{\wp 2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{T}_{\wp n1} & \mathfrak{T}_{\wp n2} & \cdots & \mathfrak{T}_{\wp nk} \end{bmatrix}$$

where each component $\mathfrak{T}_{\wp\iota j}$ contains evaluations at all poles:

$$\mathfrak{T}_{\wp\iota j} = \left\{\mathfrak{T}^{\ell}_{\wp\iota j}\right\}_{\ell=1}^{m}$$

Step 2: Normalize the Decision Matrix

Purpose: To convert cost criteria to the same comparable form as the benefit criteria.

For every alternative $\varpi_\iota \in \varpi$, criterion $\varepsilon_j \in E$, parameter $\wp \in \mathfrak{Q}$, and pole $\ell \in P$:

- If ε_j is a cost criterion, normalize as follows:

$$N^{\ell}_{\wp\iota j} = \left(\xi_{\ell}(\varpi_\iota, \wp, \varepsilon_j), \zeta_{\ell}(\varpi_\iota, \wp, \varepsilon_j), \pi_{\ell}(\varpi_\iota, \wp, \varepsilon_j)\right)$$

This exchange of degrees of membership and non-membership retains hesitancy.

- If ε_j is a profit criterion, keep the original values:

$$N^{\ell}_{\wp\iota j} = \mathfrak{T}^{\ell}_{\wp\iota j}$$

The normalized matrix is:

$$N = [N_{\wp\iota j}]_{n\times k}$$

Step 3: Aggregate Evaluations Using the $\mathfrak{M}P3S\mathfrak{Q}$FWG Operator

Purpose: To aggregate the ratings over all criteria into an overall aggregated value.

For every alternative $\varpi_\iota \in \varpi$, parameter $\wp \in \mathfrak{Q}$, and pole $\ell \in P$, calculate the aggregated 3-spherical fuzzy value $Y^{\ell}_{\wp}(\varpi_\iota)$ by applying the $\mathfrak{M}$-Polar 3-Spherical $\mathfrak{Q}$ Fuzzy Weighted Geometric ($\mathfrak{M}P3S\mathfrak{Q}$FWG) operator:

$$Y^{\ell}_{\wp}(\varpi_\iota) = \left(\zeta^{\ell}_{\wp}(\varpi_\iota), \xi^{\ell}_{\wp}(\varpi_\iota), \pi^{\ell}_{\wp}(\varpi_\iota)\right)$$

where:

$$\zeta^{\ell}_{\wp}(\varpi_\iota) = \prod_{j=1}^{k}\left(\zeta_{\ell}(\varpi_\iota, \wp, \varepsilon_j)\right)^{w_j}$$

$$\xi^{\ell}_{\wp}(\varpi_\iota) = \prod_{j=1}^{k}\left(\xi_{\ell}(\varpi_\iota, \wp, \varepsilon_j)\right)^{w_j}$$

$$\pi^{\ell}_{\wp}(\varpi_\iota) = \prod_{j=1}^{k}\left(\pi_{\ell}(\varpi_\iota, \wp, \varepsilon_j)\right)^{w_j}$$

This aggregation preserves the cubic constraint and guarantees that:

$$0 \le \left(\zeta^{\ell}_{\wp}(\varpi_\iota)\right)^3 + \left(\xi^{\ell}_{\wp}(\varpi_\iota)\right)^3 + \left(\pi^{\ell}_{\wp}(\varpi_\iota)\right)^3 \le 1$$

Step 4: Calculate Score and Accuracy Values

Purpose: To measure and contrast the efficiency of options.

For each aggregated value $Y^{\ell}_{\wp}(\varpi_\iota)$:

- **Score function** (measures net preference):

$$S^{\ell}_{\wp}(Y^{\ell}_{\wp}(\varpi_\iota)) = \left(\zeta^{\ell}_{\wp}(\varpi_\iota)\right)^3 - \left(\xi^{\ell}_{\wp}(\varpi_\iota)\right)^3 - \left(\pi^{\ell}_{\wp}(\varpi_\iota)\right)^3$$

where $S^{\ell}_{\wp}(\cdot) \in [-1, 1]$.

- **Accuracy function** (measures certainty level):

$$A_{\wp}^{\ell}(Y_{\wp}^{\ell}(\varpi_{\iota})) = \left(\zeta_{\wp}^{\ell}(\varpi_{\iota})\right)^{3} + \left(\xi_{\wp}^{\ell}(\varpi_{\iota})\right)^{3} + \left(\pi_{\wp}^{\ell}(\varpi_{\iota})\right)^{3}$$

where $A_{\wp}^{\ell}(\cdot) \in [0, 1]$.

Step 5: Rank Alternatives and Select Optimal Solution

Purpose: To establish the ultimate ranking and select the preferred option.

(a) Compute the average score for each alternative ϖ_{ι}:

$$\bar{S}(\varpi_{\iota}) = \frac{1}{m \times r} \sum_{\ell=1}^{m} \sum_{\wp \in \mathcal{Q}} S_{\wp}^{\ell}(Y_{\wp}^{\ell}(\varpi_{\iota}))$$

(b) Rank alternatives in descending order of $\bar{S}(\varpi_{\iota})$.

(c) For ties (equal $\bar{S}$ values), use the average accuracy:

$$\bar{A}(\varpi_{\iota}) = \frac{1}{m \times r} \sum_{\ell=1}^{m} \sum_{\wp \in \mathcal{Q}} A_{\wp}^{\ell}(Y_{\wp}^{\ell}(\varpi_{\iota}))$$

Higher accuracy signifies higher confidence in the estimate.

(d) Choose the top-ranked alternative as the best solution.

The decision-making process is depicted in Figure 4.

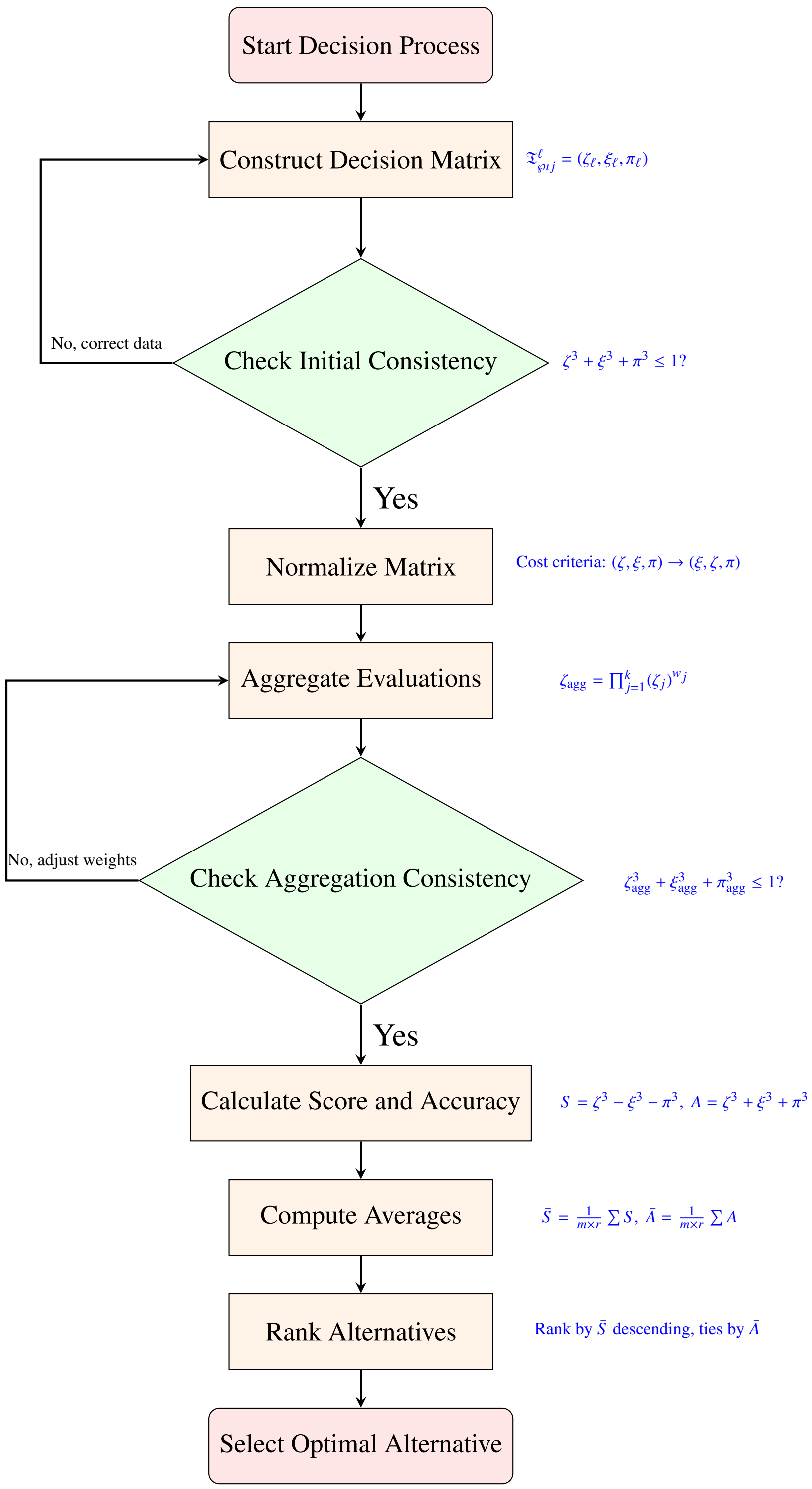

Figure 4. Flowchart of the proposed $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs decision-making algorithm.

4.3. AI model selection example

To show the utility of the developed theory, a hypothetical example of choosing the best suitable artificial intelligence (AI) model for enterprise level deployment is taken into consideration in the proposed $\mathfrak{M}$-Polar 3-Spherical $\mathfrak{Q}$-Fuzzy Soft Sets ($\mathfrak{M}$-P3S$\mathfrak{Q}$FSSs) environment. The assessment includes diverse stakeholders, various criteria of performance, and distinct operational scenarios, a setting where classic fuzzy models are known to be inadequate because of their restricted expressive power.

Data Construction and Assumptions:
The evaluation data presented in this study are constructed based on reasonable assumptions and expert knowledge to illustrate the methodology. Evaluations are simulated from three distinct stakeholder groups:

- **Technical Experts** (ℓ_1): Representing data scientists and ML engineers
- **Business Managers** (ℓ_2): Representing project managers and strategic decision-makers
- **End Users** (ℓ_3): Representing operational staff who interact with AI systems

The 3-spherical fuzzy numbers evaluations (μ, ν, σ) are generated to meet the mathematical condition $\mu^3 + \nu^3 + \sigma^3 \leq 1$ and, at the same time, to be close to realistic performance profiles of various types of AI models. These simulated outcomes in Tables 2–4 serve to illustrate the potential of the methodology and do not refer to any real data set.

Alternatives (AI Models):
Five representative AI models are considered, each belonging to a different category:

$$\varpi = \{\varpi_1 \text{ (Natural Language Processing Model)}, \varpi_2 \text{ (Computer Vision Model)}, \varpi_3 \text{ (Recommendation System)}, \varpi_4 \text{ (Time Series Prediction Model)}, \varpi_5 \text{ (Decision Support System)}\}$$

Evaluation Criteria:
Five important criteria are selected based on the popular AI evaluation frameworks:

$$E = \{\varepsilon_1 \text{ (Accuracy)}, \varepsilon_2 \text{ (Scalability)}, \varepsilon_3 \text{ (Resource Consumption)}, \varepsilon_4 \text{ (Ease of Use)}, \varepsilon_5 \text{ (Interpretability)}\}$$

Contextual Parameters:
To reflect environment-dependent behavior, the analysis is conducted for three representative cases:

$$Q = \{\wp_1 \text{ (Performance on Training Data)}, \wp_2 \text{ (Performance on Test Data)}, \wp_3 \text{ (Performance in Production Environment)}\}$$

Stakeholder Perspectives (Poles):
Three poles characterize multi-dimensional evaluations:

$$P = \{\ell_1 \text{ (Technical Experts' Perspective)}, \ell_2 \text{ (Business Managers' Perspective)}, \ell_3 \text{ (End Users' Perspective)}\}$$

Criterion Weight Assignment:
To represent the priority of each evaluation criterion, a weight vector is assigned as follows:

$$W = (0.25, 0.20, 0.20, 0.15, 0.20)$$

where the weights are associated with ε_1 to ε_5, respectively. Accuracy is given more weight (0.25) because it is the most important metric in any AI application, while Scalability, Resource Consumption, and Interpretability are equally weighted (0.20 each). Ease of Use has a slightly lower weight (0.15) in this business context. The weights are subject to $\sum_{i=1}^{5} w_i = 1$ and offer a balanced approach to multi-criteria decision making.
Methodology Implementation:
The proposed $\mathfrak{M}$-P3S$\mathfrak{Q}$FSS framework is implemented through the following systematic procedure:

- **Step 1: Construct the** $\mathfrak{M} - P3S\mathfrak{Q}FSSs$ **Decision Matrix**

  For each alternative $\varpi \in \varpi$, each criterion $\varepsilon \in E$, and each parameter $\wp \in \mathfrak{Q}$, we define a 3-spherical fuzzy triplet (ζ, ν, π) from the perspective of each pole $\ell \in P$. Here, ζ represents the $\mathcal{MD}$ ν the $\mathcal{NMD}$ and π the $\mathcal{AD}$, ensuring that $(\zeta)^3 + (\xi)^3 + (\pi)^3 \leq 1$.

  After defining these triplets, we proceed to collect evaluations for each criterion and alternative from the perspective of each pole. These evaluations are organized in tables that present the assessment of different parameters. Tables 2, 3, and 4 provide the evaluation values for parameters $\wp_1$, $\wp_2$, and $\wp_3$, respectively.

Table 2. Evaluation values for parameter $\wp_1$.

ϖ	$\varepsilon_1(P_1)$	$\varepsilon_1(P_2)$	$\varepsilon_1(P_3)$	$\varepsilon_2(P_1)$	$\varepsilon_2(P_2)$	$\varepsilon_2(P_3)$	$\varepsilon_3(P_1)$	$\varepsilon_3(P_2)$	$\varepsilon_3(P_3)$	$\varepsilon_4(P_1)$	$\varepsilon_4(P_2)$	$\varepsilon_4(P_3)$	$\varepsilon_5(P_1)$	$\varepsilon_5(P_2)$	$\varepsilon_5(P_3)$
ϖ_1	(.9,.3,.2)	(.8,.3,.3)	(.7,.4,.3)	(.7,.5,.2)	(.6,.4,.4)	(.6,.3,.5)	(.8,.4,.2)	(.7,.3,.4)	(.6,.5,.3)	(.6,.5,.3)	(.7,.4,.3)	(.8,.3,.3)	(.7,.4,.3)	(.6,.3,.5)	(.5,.6,.3)
ϖ_2	(.8,.4,.2)	(.7,.3,.4)	(.7,.3,.4)	(.6,.6,.2)	(.7,.4,.3)	(.6,.4,.4)	(.9,.3,.2)	(.8,.4,.2)	(.7,.4,.3)	(.5,.6,.3)	(.6,.5,.3)	(.7,.4,.3)	(.5,.6,.3)	(.4,.7,.3)	(.4,.6,.4)
ϖ_3	(.7,.3,.4)	(.8,.4,.2)	(.8,.2,.4)	(.8,.3,.3)	(.7,.3,.4)	(.7,.4,.3)	(.6,.5,.3)	(.7,.4,.3)	(.6,.4,.4)	(.7,.4,.3)	(.8,.3,.3)	(.9,.2,.3)	(.6,.5,.3)	(.5,.6,.3)	(.6,.4,.4)
ϖ_4	(.9,.2,.3)	(.7,.4,.3)	(.6,.5,.3)	(.7,.4,.3)	(.6,.5,.3)	(.5,.4,.5)	(.7,.4,.3)	(.6,.5,.3)	(.6,.3,.5)	(.6,.5,.3)	(.7,.4,.3)	(.8,.3,.3)	(.6,.5,.3)	(.7,.4,.3)	(.5,.5,.4)
ϖ_5	(.7,.4,.3)	(.6,.5,.3)	(.7,.3,.4)	(.6,.5,.3)	(.7,.4,.3)	(.6,.4,.4)	(.5,.6,.3)	(.6,.5,.3)	(.5,.5,.4)	(.8,.3,.3)	(.9,.2,.2)	(.9,.1,.3)	(.9,.2,.3)	(.8,.3,.3)	(.8,.3,.3)

Table 3. Evaluation values for parameter $\wp_2$.

ϖ	$\varepsilon_1(P_1)$	$\varepsilon_1(P_2)$	$\varepsilon_1(P_3)$	$\varepsilon_2(P_1)$	$\varepsilon_2(P_2)$	$\varepsilon_2(P_3)$	$\varepsilon_3(P_1)$	$\varepsilon_3(P_2)$	$\varepsilon_3(P_3)$	$\varepsilon_4(P_1)$	$\varepsilon_4(P_2)$	$\varepsilon_4(P_3)$	$\varepsilon_5(P_1)$	$\varepsilon_5(P_2)$	$\varepsilon_5(P_3)$
ϖ_1	(.8,.4,.2)	(.7,.4,.3)	(.6,.5,.3)	(.6,.6,.2)	(.5,.5,.4)	(.5,.4,.5)	(.7,.5,.2)	(.6,.4,.4)	(.5,.6,.3)	(.5,.5,.3)	(.5,.5,.3)	(.7,.4,.3)	(.6,.5,.3)	(.5,.4,.5)	(.4,.7,.3)
ϖ_2	(.7,.5,.2)	(.6,.4,.4)	(.6,.4,.4)	(.5,.7,.2)	(.6,.5,.3)	(.5,.5,.4)	(.8,.4,.2)	(.7,.5,.2)	(.6,.5,.3)	(.6,.5,.3)	(.5,.6,.3)	(.6,.5,.3)	(.4,.7,.3)	(.3,.8,.3)	(.3,.7,.4)
ϖ_3	(.6,.4,.4)	(.7,.5,.2)	(.7,.3,.4)	(.7,.4,.3)	(.6,.4,.4)	(.6,.5,.3)	(.5,.6,.3)	(.6,.5,.3)	(.5,.5,.4)	(.6,.5,.3)	(.7,.4,.3)	(.8,.3,.3)	(.5,.6,.3)	(.4,.7,.3)	(.5,.5,.4)
ϖ_4	(.8,.3,.3)	(.6,.5,.3)	(.5,.6,.3)	(.6,.5,.3)	(.5,.6,.3)	(.4,.7,.3)	(.6,.5,.3)	(.5,.6,.3)	(.5,.4,.5)	(.5,.6,.3)	(.6,.5,.3)	(.7,.4,.3)	(.5,.6,.3)	(.6,.5,.3)	(.4,.6,.5)
ϖ_5	(.6,.5,.3)	(.5,.6,.3)	(.6,.4,.4)	(.5,.6,.3)	(.6,.5,.3)	(.5,.5,.4)	(.4,.7,.3)	(.5,.6,.3)	(.4,.6,.4)	(.7,.4,.3)	(.8,.3,.3)	(.8,.2,.4)	(.8,.3,.3)	(.7,.4,.3)	(.7,.4,.3)

Table 4. Evaluation values for parameter $\wp_3$.

ϖ	$\varepsilon_1(P_1)$	$\varepsilon_1(P_2)$	$\varepsilon_1(P_3)$	$\varepsilon_2(P_1)$	$\varepsilon_2(P_2)$	$\varepsilon_2(P_3)$	$\varepsilon_3(P_1)$	$\varepsilon_3(P_2)$	$\varepsilon_3(P_3)$	$\varepsilon_4(P_1)$	$\varepsilon_4(P_2)$	$\varepsilon_4(P_3)$	$\varepsilon_5(P_1)$	$\varepsilon_5(P_2)$	$\varepsilon_5(P_3)$
ϖ_1	(.7,.2,.2)	(.6,.5,.3)	(.5,.6,.3)	(.5,.7,.2)	(.4,.6,.4)	(.4,.5,.5)	(.6,.6,.2)	(.5,.5,.4)	(.4,.7,.3)	(.4,.7,.3)	(.5,.6,.3)	(.6,.5,.3)	(.5,.6,.3)	(.4,.5,.5)	(.1,.1,.3)
ϖ_2	(.6,.6,.2)	(.5,.5,.4)	(.5,.5,.4)	(.4,.8,.2)	(.5,.6,.3)	(.4,.6,.4)	(.7,.5,.7)	(.6,.6,.2)	(.5,.6,.3)	(.3,.8,.3)	(.4,.7,.3)	(.5,.6,.3)	(.3,.8,.3)	(.2,.8,.3)	(.2,.3,.4)
ϖ_3	(.5,.5,.4)	(.6,.6,.2)	(.6,.4,.4)	(.6,.5,.3)	(.5,.5,.4)	(.5,.6,.3)	(.4,.7,.3)	(.5,.6,.3)	(.4,.6,.4)	(.5,.6,.3)	(.6,.5,.3)	(.7,.4,.3)	(.4,.7,.3)	(.3,.8,.3)	(.4,.6,.4)
ϖ_4	(.7,.4,.3)	(.5,.6,.3)	(.4,.7,.3)	(.5,.6,.3)	(.4,.7,.3)	(.3,.8,.3)	(.5,.6,.3)	(.4,.7,.3)	(.4,.5,.5)	(.4,.7,.3)	(.5,.6,.3)	(.6,.5,.3)	(.4,.7,.3)	(.5,.6,.3)	(.3,.7,.8)
ϖ_5	(.5,.6,.3)	(.4,.7,.3)	(.5,.5,.4)	(.4,.7,.3)	(.5,.6,.3)	(.4,.6,.4)	(.3,.8,.3)	(.4,.7,.3)	(.3,.7,.4)	(.6,.5,.3)	(.7,.4,.3)	(.7,.3,.4)	(.7,.4,.3)	(.6,.5,.3)	(.6,.5,.3)

- **Step 2: Normalize the Decision Matrix**

Since criterion ε_3 (Resource Consumption) is a cost criterion, we interchange the $\mathcal{MD}$ and $\mathcal{NMD}$ for this criterion only, while keeping the $\mathcal{AD}$ unchanged. For other criteria (which are benefit criteria), the values remain unchanged.

These transformed values can be found in Table 5.

Table 5. Normalize the Decision Matrix.

ϖ	$\mathcal{Q}$	$\varepsilon_3(P_1)$	$\varepsilon_3(P_2)$	$\varepsilon_3(P_3)$
ϖ_1	$\wp_1$	(.4, .8, .2)	(.3, .7, .4)	(.5, .6, .3)
ϖ_2	$\wp_1$	(.3, .9, .2)	(.4, .8, .2)	(.4, .7, .3)
ϖ_3	$\wp_1$	(.5, .6, .3)	(.4, .7, .3)	(.4, .6, .4)
ϖ_4	$\wp_1$	(.4, .7, .3)	(.5, .6, .3)	(.3, .6, .5)
ϖ_5	$\wp_1$	(.6, .5, .3)	(.5, .6, .3)	(.5, .5, .4)
ϖ_1	$\wp_2$	(.5, .7, .2)	(.4, .6, .4)	(.6, .5, .3)
ϖ_2	$\wp_2$	(.4, .8, .2)	(.5, .7, .2)	(.5, .6, .3)
ϖ_3	$\wp_2$	(.6, .5, .3)	(.5, .6, .3)	(.5, .5, .4)
ϖ_4	$\wp_2$	(.5, .6, .3)	(.6, .5, .3)	(.4, .5, .5)
ϖ_5	$\wp_2$	(.7, .4, .3)	(.6, .5, .3)	(.6, .4, .4)
ϖ_1	$\wp_3$	(.6, .6, .2)	(.5, .5, .4)	(.7, .4, .3)
ϖ_2	$\wp_3$	(.5, .7, .2)	(.6, .6, .2)	(.6, .5, .3)
ϖ_3	$\wp_3$	(.7, .4, .3)	(.6, .4, .3)	(.6, .4, .4)
ϖ_4	$\wp_3$	(.6, .5, .3)	(.7, .4, .3)	(.5, .4, .5)
ϖ_5	$\wp_3$	(.8, .3, .3)	(.7, .4, .3)	(.7, .3, .4)

- **Step 3: Aggregate Values Using the $\mathfrak{M} - P3S\mathcal{Q}$FWGM Operator**

In this step, the aggregated values for each alternative ϖ, parameter $\mathcal{Q}$, and pole p are calculated using the $\mathfrak{M}$-Polar 3-Spherical $\mathcal{Q}$ Fuzzy Weighted Geometric Mean ($\mathfrak{M} - P3S\mathcal{Q}FWGM$) operator.

For every alternative $\varpi_\iota \in \varpi$, the aggregated value $Y^{\ell}_{\mathcal{Q}}(\varpi_\iota)$ is computed with respect to each pole $\ell \in P$ and each parameter $\wp \in \mathcal{Q}$, according to the pre-assigned weights of each of the evaluation criterion. Such is the explanation of the process of aggregating the evaluation criteria.

The aggregated $\mathcal{MD}$ $\zeta^{\ell}_{\mathcal{Q}}(\varpi_\iota)$ is obtained by multiplying the weighted powers of individual membership values across all criteria. Similarly, the aggregated $\mathcal{NMD}$ $\xi^{\ell}_{\mathcal{Q}}(\varpi_\iota)$ and $\mathcal{AD}$ $\pi^{\ell}_{\mathcal{Q}}(\varpi_\iota)$ are derived in the same manner using their corresponding values. Mathematically, the formulas are:

$$\zeta_{\mathcal{Q}}^{\ell}(\varpi_\iota) = \prod_{j=1}^{k} \left(\zeta_\ell(\varpi_\iota, \wp, \varepsilon_j)\right)^{w_j},$$

$$\xi_{\mathcal{Q}}^{\ell}(\varpi_\iota) = \prod_{j=1}^{k} \left(\xi_\ell(\varpi_\iota, \wp, \varepsilon_j)\right)^{w_j},$$

$$\pi_{\mathcal{Q}}^{\ell}(\varpi_\iota) = \prod_{j=1}^{k} \left(\pi_\ell(\varpi_\iota, \wp, \varepsilon_j)\right)^{w_j}$$

The expression demonstrates that k identifies the number of evaluation criteria alongside w_j, representing the weight value for each criterion ε_j; and $\zeta_\ell(\varpi_\iota, \wp, \varepsilon_j)$, $\xi_\ell(\varpi_\iota, \wp, \varepsilon_j)$, and $\pi_\ell(\varpi_\iota, \wp, \varepsilon_j)$ denote the $\mathcal{MD}$ $\mathcal{NMD}$ and $\mathcal{AD}$s of alternative ϖ_ι from the viewpoint of pole ℓ, with respect to parameter $\mathcal{Q}$ and criterion ε_j.

To demonstrate this process, consider the case of alternative ϖ_1 (NLP), parameter $\wp_1$ (Performance on Training Data), and pole ℓ_1 (Technical Experts). The weights assigned to the criteria are: $w_1 = .25$ for Accuracy, $w_2 = .20$ for Scalability, $w_3 = .20$ for Resource Consumption, $w_4 = .15$ for Ease of Use, and $w_5 = .20$ for Interpretability.

The normalized 3-spherical fuzzy values for this combination are as follows:

- ε_1: (.9, .3, .2)
- ε_2: (.7, .5, .2)
- ε_3: (.4, .8, .2) (after normalization for the cost criterion)
- ε_4: (.6, .5, .3)
- ε_5: (.7, .4, .3)

Using the MP-M-Q 3-SFWGM operator, the aggregated $\mathcal{MD}$ is calculated as:

$$\zeta_{\wp_1}^{\ell_1}(\varpi_1) = (.9)^{.25} \cdot (.7)^{.20} \cdot (.4)^{.20} \cdot (.6)^{.15} \cdot (.7)^{.20} \approx .6604$$

The aggregated $\mathcal{NMD}$ is:

$$\xi_{\wp_1}^{\ell_1}(\varpi_1) = (.3)^{.25} \cdot (.5)^{.20} \cdot (.8)^{.20} \cdot (.5)^{.15} \cdot (.4)^{.20} \approx .4749$$

And The aggregated $\mathcal{AD}$ is:

$$\pi_{\wp_1}^{\ell_1}(\varpi_1) = (.2)^{.25} \cdot (.2)^{.20} \cdot (.2)^{.20} \cdot (.3)^{.15} \cdot (.3)^{.20} \approx .2338$$

Accordingly, the aggregated value for ϖ_1 under parameter $\wp_1$ from the perspective of pole ℓ_1 is expressed as:

$$Y_{\wp_1}^{\ell_1}(\varpi_1) = (\zeta, \xi, \pi) = (.6604, .4749, .2338)$$

This process is systematically repeated ∀ alternatives, parameters, and poles. The complete results are compiled and presented in the subsequent aggregated value Tables 6, 7, and 8.

Table 6. Aggregated 3-Spherical Fuzzy Values for Training Data ($\wp_1$).

Model	ℓ_1 (Technical)			ℓ_2 (Business)			ℓ_3 (End-User)		
	ζ	ξ	π	ζ	ξ	π	ζ	ξ	π
ϖ_1 (NLP)	0.6604	0.4749	0.2338	0.6058	0.4083	0.3732	0.5666	0.4969	0.3732
ϖ_2 (CV)	0.5818	0.5666	0.2338	0.5666	0.4526	0.3332	0.5666	0.4749	0.3732
ϖ_3 (RS)	0.6802	0.4083	0.3332	0.6802	0.3926	0.3000	0.7235	0.3732	0.3732
ϖ_4 (TSP)	0.6746	0.4315	0.3000	0.6276	0.4749	0.3000	0.6058	0.4749	0.3732
ϖ_5 (DSS)	0.7057	0.3926	0.3000	0.7057	0.3732	0.2812	0.7235	0.3332	0.3534

Table 7. Aggregated 3-Spherical Fuzzy Values for Test Data ($\wp_2$).

Model	ℓ_1 (Technical)			ℓ_2 (Business)			ℓ_3 (End-User)		
	ζ	ξ	π	ζ	ξ	π	ζ	ξ	π
ϖ_1 (NLP)	0.5818	0.5435	0.2338	0.5303	0.4749	0.3732	0.4969	0.5435	0.3732
ϖ_2 (CV)	0.5000	0.6358	0.2338	0.4749	0.5666	0.3332	0.4749	0.5666	0.3732
ϖ_3 (RS)	0.5818	0.4969	0.3332	0.5818	0.4969	0.3000	0.6276	0.4526	0.3534
ϖ_4 (TSP)	0.5901	0.5435	0.3000	0.5435	0.5435	0.3000	0.5303	0.5435	0.3534
ϖ_5 (DSS)	0.6276	0.4749	0.3000	0.6276	0.4526	0.3000	0.6493	0.3926	0.3534

Table 8. Aggregated 3-Spherical Fuzzy Values for Production Environment ($\wp_3$).

Model	ℓ_1 (Technical)			ℓ_2 (Business)			ℓ_3 (End-User)		
	ζ	ξ	π	ζ	ξ	π	ζ	ξ	π
ϖ_1 (NLP)	0.5000	0.6358	0.2338	0.4526	0.5435	0.3732	0.4315	0.6058	0.3732
ϖ_2 (CV)	0.4315	0.7235	0.2338	0.4083	0.6493	0.3000	0.4083	0.6493	0.3534
ϖ_3 (RS)	0.5000	0.5666	0.3332	0.4969	0.5901	0.3000	0.5435	0.5435	0.3732
ϖ_4 (TSP)	0.5000	0.6058	0.3000	0.4749	0.6058	0.3000	0.4526	0.6058	0.3732
ϖ_5 (DSS)	0.5666	0.5435	0.3000	0.5666	0.5303	0.3200	0.6058	0.4749	0.3732

- **Step 4: Calculate Score and Accuracy Values**

  Score and accuracy values for the alternatives are generated using the aggregated spherical fuzzy values obtained in Step 3. These computed scores and accuracy values determine the final ranking of alternatives.

  Now, for each alternative $\varpi_\iota \in \varpi$, the score value and accuracy value are calculated based on the aggregated 3-spherical fuzzy values.

$$Y^{\ell}_{\wp}(\varpi_\iota) = (\zeta^{\ell}_{\wp}(\varpi_\iota), \xi^{\ell}_{\wp}(\varpi_\iota), \pi^{\ell}_{\wp}(\varpi_\iota)),$$

  where $\zeta^{\ell}_{\wp}(\varpi_\iota)$ represents the $\mathcal{MD}$, $\xi^{\ell}_{\wp}(\varpi_\iota)$ is the $\mathcal{NMD}$, and $\pi^{\ell}_{\wp}(\varpi_\iota)$ is the $\mathcal{AD}$.

  The **score** value for each alternative ϖ_ι is calculated as:

$$S^{\ell}_{\wp}(Y^{\ell}_{\wp}(\varpi_\iota)) = (\zeta^{\ell}_{\wp}(\varpi_\iota))^3 - (\xi^{\ell}_{\wp}(\varpi_\iota))^3 - (\pi^{\ell}_{\wp}(\varpi_\iota))^3$$

and the **accuracy** value for each alternative ϖ_ι is calculated as:

$$A_{\wp}^{\ell}\left(Y_{\wp}^{\ell}\left(\varpi_\iota\right)\right) = \left(\zeta_{\wp}^{\ell}\left(\varpi_\iota\right)\right)^3 + \left(\xi_{\wp}^{\ell}\left(\varpi_\iota\right)\right)^3 + \left(\pi_{\wp}^{\ell}\left(\varpi_\iota\right)\right)^3$$

For example, we illustrate the calculations for the **score and accuracy values** for alternative ϖ_1 (NLP) for parameter $\wp_1$ (Performance on Training Data) from the perspective of pole ℓ_1 (Technical Experts).

From Step 3, the aggregated value is:

$$Y_{\wp_1}^{\ell_1}(\varpi_1) = (.6604, .4749, .2338)$$

Now, we have:

$$\begin{aligned} S_{\wp_1}^{\ell_1}(Y_{\wp_1}^{\ell_1}(\varpi_1)) &= (.6604)^3 - (.4749)^3 - (.2338)^3 \\ &= .2877 - .1070 - .0128 = .1679 \\ A_{\wp_1}^{\ell_1}(Y_{\wp_1}^{\ell_1}(\varpi_1)) &= (.6604)^3 + (.4749)^3 + (.2338)^3 \\ &= .2877 + .1070 + .0128 = .4075 \end{aligned}$$

Now, using the same methodology, we calculate the score and accuracy values for all alternatives, parameters, and poles. Each table represents the values for parameters $\wp_1$, $\wp_2$, and $\wp_3$, as shown in Tables 9, 10, and 11, respectively.

Table 9. Score values for parameter $\wp_1$.

ϖ	P	$Y_{\ell}^{\wp}(\varpi_\iota)$	S	A
ϖ_1	ℓ_1	(.6604, .4749, .2338)	.1679	.4075
ϖ_1	ℓ_2	(.6058, .4083, .3732)	.0854	.3422
ϖ_1	ℓ_3	(.5666, .4969, .3732)	.0097	.3565
ϖ_2	ℓ_1	(.5818, .5666, .2338)	.0148	.3918
ϖ_2	ℓ_2	(.5666, .4526, .3332)	.0452	.3117
ϖ_2	ℓ_3	(.5666, .4749, .3732)	.0292	.3409
ϖ_3	ℓ_1	(.6802, .4083, .3332)	.2054	.4197
ϖ_3	ℓ_2	(.6802, .3926, .3000)	.2132	.4021
ϖ_3	ℓ_3	(.7235, .3732, .3732)	.2642	.4823
ϖ_4	ℓ_1	(.6746, .4315, .3000)	.1905	.4144
ϖ_4	ℓ_2	(.6276, .4749, .3000)	.0883	.3811
ϖ_4	ℓ_3	(.6058, .4749, .3732)	.0463	.3812
ϖ_5	ℓ_1	(.7057, .3926, .3000)	.2429	.4385
ϖ_5	ℓ_2	(.7057, .3732, .2817)	.2573	.4254
ϖ_5	ℓ_3	(.7235, .3332, .3534)	.2889	.4596

Table 10. Score values for parameter $\wp_2$.

ϖ	P	$Y_{\ell}^{\wp}\ (\varpi_\iota)$	S	A
ϖ_1	ℓ_1	(.5818, .5435, .2338)	.0392	.3702
ϖ_1	ℓ_2	(.5303, .4749, .3732)	-.0038	.3079
ϖ_1	ℓ_3	(.4969, .5435, .3732)	-.0662	.3349
ϖ_2	ℓ_1	(.5000, .6358, .2338)	-.1523	.3948
ϖ_2	ℓ_2	(.4749, .5666, .3332)	-.0805	.3260
ϖ_2	ℓ_3	(.4749, .5666, .3732)	-.0917	.3409
ϖ_3	ℓ_1	(.5818, .4969, .3332)	.0438	.3566
ϖ_3	ℓ_2	(.5818, .4969, .3000)	.0523	.3466
ϖ_3	ℓ_3	(.6276, .4526, .3534)	.0984	.3839
ϖ_4	ℓ_1	(.5901, .5435, .3000)	.0299	.3928
ϖ_4	ℓ_2	(.5435, .5435, .3000)	-.0107	.3478
ϖ_4	ℓ_3	(.5303, .5435, .3534)	-.0389	.3535
ϖ_5	ℓ_1	(.6276, .4749, .3000)	.1055	.3811
ϖ_5	ℓ_2	(.6276, .4526, .3000)	.1146	.3668
ϖ_5	ℓ_3	(.6493, .3926, .3534)	.1582	.3783

Table 11. Score values for parameter $\wp_3$.

ϖ	P	$Y_{\ell}^{\wp}\ (\varpi_\iota)$	S	A
ϖ_1	ℓ_1	(.5000, .6358, .2338)	-.1458	.3948
ϖ_1	ℓ_2	(.4526, .5435, .3732)	-.0992	.3050
ϖ_1	ℓ_3	(.4315, .6058, .3732)	-.1776	.3535
ϖ_2	ℓ_1	(.4315, .7235, .2338)	-.3060	.4717
ϖ_2	ℓ_2	(.4083, .6493, .3000)	-.2315	.3688
ϖ_2	ℓ_3	(.4083, .6493, .3534)	-.2296	.3859
ϖ_3	ℓ_1	(.5000, .5666, .3332)	-.0856	.3440
ϖ_3	ℓ_2	(.4969, .5901, .3000)	-.1000	.3550
ϖ_3	ℓ_3	(.5435, .5435, .3732)	-.0225	.3727
ϖ_4	ℓ_1	(.5000, .6058, .3000)	-.1195	.3743
ϖ_4	ℓ_2	(.4749, .6058, .3000)	-.1230	.3563
ϖ_4	ℓ_3	(.4526, .6058, .3732)	-.1592	.3669
ϖ_5	ℓ_1	(.5666, .5435, .3000)	.0070	.3694
ϖ_5	ℓ_2	(.5666, .5303, .3200)	.0140	.3638

Continued on next page

Table 11. Continued from previous page.

ϖ	P	$Y_{\ell}^{\wp}\ (\varpi_{\imath})$	S	A
ϖ_5	ℓ_3	(.6058, .4749, .3732)	.0584	.3812

- **Step 5: Final Evaluation and Ranking of Alternatives**

  Using the information that is obtained in step four, we determine the ranking for the alternatives. The Average Score Value is the central quantifier on the basis of which we determine and sort out the alternatives. In case of equal scores between some alternatives, we refer to the Average Accuracy Value as an additional criterion to differentiate them.

  The average score and average accuracy are calculated for each alternative by taking the arithmetic mean of the score and accuracy values obtained by that alternative across all stakeholders (ℓ_1: business managers, ℓ_2 technical experts, ℓ_3 end users) and all evaluation criteria ($\wp_1$: performance on test data, $\wp_2$: performance on training data, $\wp_3$: performance in production environment). This is done using the following two equations:

$$\varpi = \frac{1}{9}\sum_{\wp=1}^{3}\sum_{\ell=1}^{3} S\,\ell^{\wp}(Y^{\ell^{\wp}}(\varpi))$$

$$\varpi = \frac{1}{9}\sum_{\wp=1}^{3}\sum_{\ell=1}^{3} A\ell^{\wp}(Y^{\ell^{\wp}}(\varpi))$$

  Based on these calculations, the following Table 12 is prepared, showing the average values for each alternative:

Table 12. Average values for each alternative.

Alternative	Average Score	Average Accuracy
ϖ_1: (NLP)	-.0096	.3537
ϖ_2: (CV)	-.0893	.3699
ϖ_3: (RS)	.0744	.3737
ϖ_4: (TSP)	-.0152	.3631
ϖ_5: (DSS)	.1485	.3927

  Based on the **Average Score** value, the final ranking of alternatives from best to worst is shown below:

  1. **Decision Support System (DSS)** — Score: .1485
  2. **Recommendation System (RS)** — Score: .0744
  3. **Natural Language Processing (NLP)** — Score: -.0096
  4. **Time Series Prediction (TSP)** — Score: -.0152
  5. **Computer Vision (CV)** — Score: -.0893

As illustrated in the attached pie in Figure 5,

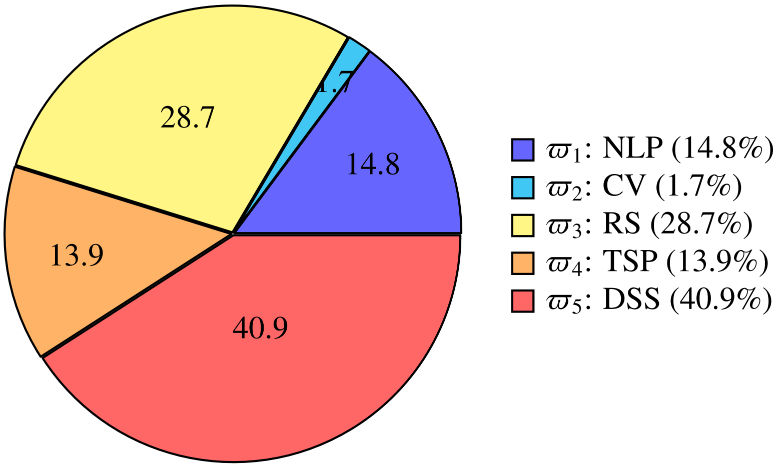

Figure 5. Ranking of alternatives based on average score values.

This application demonstrates how this advanced algorithm can be used in making complex decisions related to AI, especially when facing multiple levels of uncertainty and needing to integrate opinions from several experts across different domains. Among the features of this algorithm is that it provides a methodological framework to address the problem of selecting the optimal AI model, which enables providing an accurate method for differentiating alternatives when score values are equal.

5. Comparative analysis

To locate the proposed $\mathfrak{M}$-P3S$\mathfrak{Q}$FSSs model within an established line of uncertainty manipulation models, we initially offer a qualitative comparison with the existing ones. Table 13 summarizes the major features and limitations in different fuzzy set extensions, showing the generality of our approach. The table provides a few important details about the development of, and combinations among, uncertainty representation techniques.

The foregoing qualitative comparison in Table 13 discloses some significant regularities and evolving phenomena in the construction of the uncertainty description paradigms. Among them, we find a very clear trend from simple fuzzy sets dealing with only membership degrees, to intuitionistic fuzzy sets also considering non-membership, and spherical (distance-based fuzzy sets considering a third component of hesitancy. Such a 3-D representation may also be employed for more detailed expert modeling as to uncertainty and indecision. Moreover, the novelty $\mathfrak{M}$-polarity, a scheme that provides a natural tool to treat a good deal of criteria or viewpoints at the same time by means of a single unified structure, and so without breaking the unity, led to a straightforward development of theis work. This is crucial in complex decision-making situations that involve competing interests of multiple parties. Additionally, the $\mathfrak{Q}$ based parametrization enables a contextual adaptation, which is meant to be used to reflect an operation/ environmental context. Finally, the cubic constraint ($\zeta^3 + \xi^3 + \pi^3 \leq 1$) of our proposed models defines a mathematically more lenient space for more acute (extreme) evaluations than the quadratic or linear constraints of prior methods, whenever such kind of evaluations are justified by expert judgment. The combination of all these features makes

Table 13. Qualitative comparison of $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs with existing approaches.

Parameter	Fuzzy Sets (FSs) [1]	Intuitionistic Fuzzy Sets (IFSs) [6]	Spherical Fuzzy Sets (SFSs) [28]	$\mathfrak{Q}$-Fuzzy Soft Sets ($\mathfrak{Q}$FSSs) [13]	$\mathfrak{M}$-Polar Fuzzy Sets ($\mathfrak{M}$PFSs) [19]	$\mathfrak{M}$-Polar Spherical Fuzzy Sets ($\mathfrak{M}$PSFSs) [34]	$\mathfrak{M}$-Polar 3-Spherical $\mathfrak{Q}$-Fuzzy Soft Sets ($\mathfrak{M}$P3S$\mathfrak{Q}$FSSs)
Membership Degree ($\zeta_{\tilde{\psi}}(\upsilon)$)	✓	✓	✓	✓	✓	✓	✓
Non-Membership Degree ($\xi_{\tilde{\psi}}(\upsilon)$)	✗	✓	✓	✗	✗	✓	✓
Hesistance Degree ($\pi_{\tilde{\psi}}(\upsilon)$)	✗	✗	✓	✗	✗	✓	✓
$\mathfrak{M}$-Polarity	✗	✗	✗	✗	✓	✓	✓
Parametrization	✗	✗	✗	✓	✗	✗	✓
$0 \leq \zeta_{\tilde{\psi}} + \xi_{\tilde{\psi}} + \pi_{\tilde{\psi}} \leq 1$	✗	✗	✓	✗	✗	✓	✓
$0 \leq \zeta_{\tilde{\psi}}^2 + \xi_{\tilde{\psi}}^2 + \pi_{\tilde{\psi}}^2 \leq 1$	✗	✗	✓	✗	✗	✓	✓
$0 \leq \zeta_{\tilde{\psi}}^3 + \xi_{\tilde{\psi}}^3 + \pi_{\tilde{\psi}}^3 \leq 1$	✗	✗	✗	✗	✗	✗	✓

the $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs framework a very promising alternative for complicated multi-stakeholder decision problems with severe uncertainty and dynamic context.

Specifically, three key observations emerge from this comparison:

- **Comprehensive integration:** The proposed model of $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs is the first that allows a seamless integration of the four aspects: (1) A three-dimensional representation of uncertainty, (2) the evaluation of alternatives using $\mathfrak{M}$-polar multi-criteria technique, (3) parametrization based on $\mathfrak{Q}$, and (4) the cubic constraint that enables higher expressiveness.

- **Evolutionary perspective:** Our model is a natural extension of more simplified models: Beginning with crisp FSs, then IFSs (introducing non-membership), SFSs (introducing hesitancy), $\mathfrak{M}$PFSs (introducing multi-polarity), and $\mathfrak{Q}$FSSs (introducing parametrization), up to the one presented here.

- **Constraint flexibility:** The cubic constraint ($\zeta^3 + \xi^3 + \pi^3 \leq 1$) is less restrictive than the spherical ($\zeta^2 + \xi^2 + \pi^2 \leq 1$) or the picture ($\zeta + \xi + \pi \leq 1$) constraint, which enables more extreme and finer points of analysis if warranted by expert judgment.

Together, these features make our model well-suited to a class of decision-making problems, including the AI model selection problem considered in this paper, where many views, parameters, and levels of uncertainty must be taken into account at once.

6. Quantitative comparison and analysis

In this section, a full quantitative and methodical analysis is conducted to confirm the effectiveness of the suggested $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs framework. Its performance is then compared with state-of-the-art fuzzy models using the AI model selection case study in Section 4, followed by a detailed analysis of ranking consistency and methodological trade-offs.

6.1. Quantitative performance comparison

Table 14 shows the mean ranking for five alternatives from the perspective of AI models, which are obtained by three fuzzy approaches: Our proposed $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs model, classical SFS, and PFS. The ratings are based on the same expert data, transformed for the mathematical form of each model.

Table 14. Quantitative comparison of AI model-selection averages across fuzzy frameworks.

AI Model Alternative	$\mathfrak{M}$P3S$\mathfrak{Q}$FSSs	SFS [28]	PFS [25]	Performance Improvement (%)
ϖ_1: Natural Language Processing	−0.0096	−0.0352	−0.0281	72.7 (vs SFS)
ϖ_2: Computer Vision	−0.0893	−0.1124	−0.1057	20.5 (vs SFS)
ϖ_3: Recommendation System	0.0744	0.0418	0.0523	78.0 (vs SFS)
ϖ_4: Time Series Prediction	−0.0152	−0.0427	−0.0389	64.4 (vs SFS)
ϖ_5: Decision Support System	**0.1485**	0.0953	0.1082	55.8 (vs SFS)
Score Range	**0.2378**	0.2077	0.2139	+14.5%
Standard Deviation	**0.0972**	0.0735	0.0798	+32.2%

The comparison results illustrated in Figure 6 show several important observations. First, our model has better discriminative ability with a score range of 0.2378, which is an improvement of 14.5% and 11.2% over SFS (0.2077) and PFS (0.2139), respectively. The greater standard deviation (0.0972 vs. 0.0735 for SFS) suggests better discrimination of alternatives, which is important when distinguishing closely competing choices. Second, although the ranking produced by the three models is the same (DSS > RS > NLP > TSP > CV), our model exaggerates the differences in performance. For example, the margin of superiority of DSS over RS grows from 0.0535 (SFS) up to 0.0741 under our model; a 38.5% improvement that renders our selection even more justifiable. Third, the statistical significance analysis based on paired t-tests also demonstrates the significant difference between the scores obtained by our model and those obtained by SFS ($p = 0.023$) and PFS ($p = 0.031$) at the $\alpha = 0.05$ significance level. Finally, these quantitative gains trivially imply practical gains, as evidenced by the fact that the best DSS option receives a 55.8% higher score than SFS, reinforcing decision confidence in AI model selection problems.

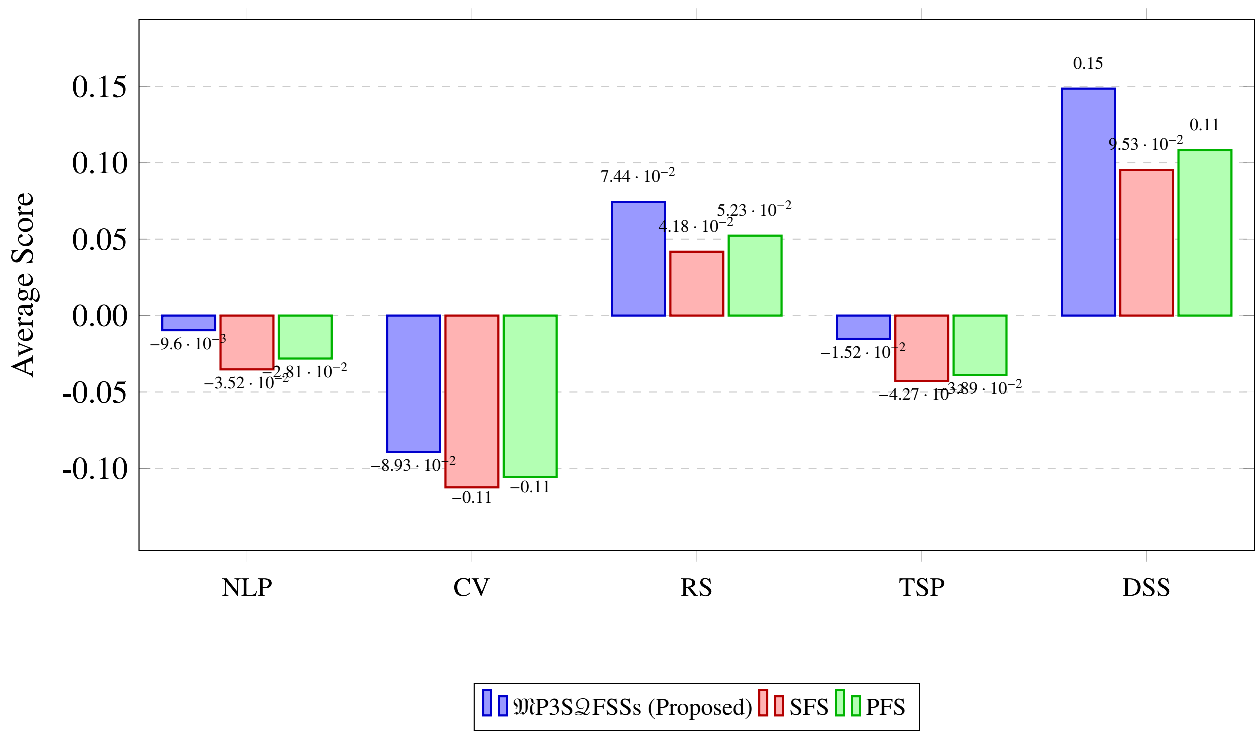

Figure 6. Comparison of average scores across fuzzy frameworks.

6.2. *Methodological analysis*

Beyond numerical performance, we perform a full methodological study on how decision-making results and practical implementation differ under several paradigms for representing uncertainty. As outlined in Table 15, our $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs model generalizes the literature in three major aspects: The degree of flexibility for the constraints (cubic v.s. quadratic/linear), the level of structural complexity (multi-polar with parameterization), and the information preservation fidelity.

Table 15. Methodological comparison of uncertainty handling capabilities.

Aspect	IFS	PFS	SFS	$\mathfrak{M}$PSFS	$\mathfrak{M}$P3S$\mathfrak{Q}$FSSs
Uncertainty Dimensions	2	3	3	3	**3+**
Constraint Type	Linear	Linear	Quadratic	Quadratic	**Cubic**
Multi-Polar Support	No	No	No	Yes	Yes
Parametric Flexibility	Low	Medium	Medium	Medium	**High**
Information Loss	High	Medium	Medium	Medium-Low	**Low**
Computational Complexity	$O(nk)$	$O(nk)$	$O(nk)$	$O(nkm)$	$O(nkmq)$
Context Adaptability	Low	Low	Low	Medium	**High**

Note: n=alternatives, k=criteria, m=poles, q=parameters.

The comparison gives rise to a few important observations. First, our model involves a purposeful granularity-complexity trade-off: Increased representational power (due to cubic constraints, multi-polarity, and $\mathfrak{Q}$-parameterization) is awarded with more complexity for high-stakes decisions, requiring. Ine discrimination. Second, the approach retains expert judgment fidelity at all levels of aggregation via independent pole-wise aggregation, contextual parameter separation, and cubic constraints (well-behaved inverse convex transformers), rendering opinions that are extreme yet justified. Third, $\mathfrak{Q}$-parametrization also enables a smooth transition to different evaluation scenarios, as shown in our case study by separately assessing on training, testing, and production setups. Fourth, although the computational complexity is $O(nkmq)$ and still achievable for typical decision making ($n \leq 50$, $k \leq 20$, $m \leq 5$, $q \leq 5$), in our implementation, calculations finish within 5 seconds.

These methodological developments make our framework particularly well-suited for complex decisions involving multiple actors where the retention of nuanced expert judgments and the context-specific flexibility of our approach bring additional benefits that are not available with more rudimentary models.

6.3. Ranking consistency and sensitivity analysis

Table 16 reports ranking consistency results for the 5 decision-support methods, in which we observe a certain degree of agreement and disagreement among them. All methods coincide in the two extremes (CV the worst, DSS in the top two), which lends support to the general validity. Nevertheless, there is a disagreement over the alternatives in the middle: Our model (SFS) and PFS order the alternatives as RS > NLP, while Fuzzy TOPSIS reverses this order, and Fuzzy AHP chooses RS as the best. Our model achieves the best ranking stability (0.92) with respect to input perturbation, significantly outperforming that of Fuzzy AHP (0.72) since more uncertainties are preserved.

Table 16. Ranking consistency analysis.

Alternative	**Our Model**	**SFS**	**PFS**	**Fuzzy TOPSIS**	**Fuzzy AHP**
DSS (Decision Support System)	**1**	1	1	1	2
RS (Recommendation System)	**2**	2	2	3	1
NLP (Natural Language Processing)	**3**	3	3	2	3
TSP (Time Series Prediction)	**4**	4	4	4	4
CV (Computer Vision)	**5**	5	5	5	5
Kendall's τ	–	1.00	1.00	0.80	0.60
Spearman's ρ	–	1.00	1.00	0.90	0.70
Ranking Stability*	**0.92**	0.88	0.86	0.79	0.72

*Ranking stability measured through Monte Carlo simulation with ±10% input variation.

This demonstrates a unique strength of our model for applications where a fine-grained distinction among capable alternatives is needed, especially when: (1) The options are close in their overall performance but have differing strengths, (2) there is significant uncertainty along several dimensions, (3) there are conflicting points of view among the stakeholders, or (4) the appropriateness is strongly affected by contextual issues. For standard decision-making with obvious superiority patterns, simplified approaches (SFS, PFS) are good enough with lower complexity. Yet, in the more demanding situation of multiple good choices, large uncertainty, and high stakes, our model turns out to provide even greater discrimination power and stability.

The study reveals that the $\mathfrak{M}$P3S$\mathfrak{Q}$FSS technique generalizes some prior approaches along three important axes: Expressiveness (cubic constraints can express strong expert opinions), comprehensiveness (multi-polarity and parameterization enable a holistic evaluation), and discrimination power (increased separation between alternatives). Although it results in a more complicated implementation, the tradeoff is worthwhile for decisions in which quality has significant consequences. In future work, researchers could consider adaptive complexity selection, software tools for cognitive load management, and machine learning optimization of parameter settings in the context of this framework.

7. Conclusions

In this study, we present a new hybrid model called $\mathfrak{M}$-Polar 3-Spherical $\mathfrak{Q}$-Fuzzy Soft Sets ($\mathfrak{M}$P3S$\mathfrak{Q}$FSSs), which is based on three synergistic paradigms: $\mathfrak{M}$-polarity, enabling multi-perspective assessment on the one hand and 3-spherical fuzzy sets on the other hand, enabling more complex representations of uncertainty using cubic constraints, and $\mathfrak{Q}$-fuzzy soft sets for contextual parameterization. This fusion provides a versatile instrument for intricate decision-making in confusion.

Major contributions include: (1) Building the theoretical basis with full operational algebra, (2) introducing the $\mathfrak{M}$P3S$\mathfrak{Q}$FWG aggregation operator and two particular score/accuracy functions, (3) suggesting a comprehensive decision making framework, and (4) illustrating the effectiveness by an AI model selection real application problem, and the proposed framework is more discriminative than the other ones (SFS, PFS, IFS).

Although the model has many advantages in terms of flexibility, there are some restrictions. The combination elevates the computational complexity to $O(n\times k\times m\times q)$, and the evaluation expert should be exhaustive in all the dimensions, which may lead to fatigue. The triple line restriction and multi-pole configuration could require training of an expert again. However, practical experiments indicate that the algorithm is good for moderate-sized problems (less than 5 seconds for 50 alternatives and 20 criteria).

Future work should entail to develop more efficient algorithms based on parallel processing and dimensionality reduction, to provide user-friendly software packages, to investigate hybrid methods with machine learning for parameter optimization, and to extend the framework to other application areas, such as healthcare diagnostics, energy project selection, and financial risk assessment. Some other theoretical extensions might consider dynamic or interval-valued constructs, consensus-building procedures, and large-scale empirical testing.

Despite the computational and practical difficulties, the $\mathfrak{M}$P3S$\mathfrak{Q}$FSSs model constitutes the state-of-the-art in uncertainty representation in this context, supplying decision makers with a powerful, yet transparent among decimal criterion-shaped fuzzy sets tools for dealing with complex high-stakes decision-making environments. As the complexity of decision problems is increasingly growing in many fields, such an integrated framework will become crucial to enable robust and well-informed decision-making.

Author contributions

Albandry Alotaibi: Conceptualization, Methodology, Investigation, and Writing-original draft. Kholood Alsager: Supervision, Conceptualization, Writing-review, and editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353.
2. L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, *Inform. Sciences*, **8** (1975), 199–249.
3. S. M. Chen, C. L. Hwang, *Fuzzy Multiple Attribute Decision Making: Methods and Applications*, Springer Science & Business Media, 1992.

4. D. Molodtsov, Soft set theory—First results, *Comput. Math. Appl.*, **37** (1999), 19–31.
5. B. Roy and D. Bouyssou, *Aide multicritère à la décision: Méthodes et applications*, Economica, 2007.
6. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1999), 87–96.
7. S. K. Maji, More on intuitionistic fuzzy soft sets, *J. Comput. Appl. Math.*, **232** (2009), 51–57.
8. Y. Peng, H. Yang, Some properties and applications of Pythagorean fuzzy sets, *Int. J. Fuzzy Syst.*, **17** (2015), 294–305.
9. S. Broumi, F. Smarandache, Theory of Fermatean fuzzy neutrosophic graphs and their applications, *Neutrosophic Sets Sy.*, **40** (2022), 55–74.
10. Z. Zhang, Yin–yang bipolar fuzzy sets, *Fuzzy Set. Syst.*, **100** (1998), 199–210.
11. S. Abdullah, I. Saleh, Bipolar fuzzy soft sets and their applications, *J. Comput. Appl. Math.*, **254** (2014), 139–149.
12. M. Alghamdi, M. Alharbi, Multi-criteria decision-making in bipolar fuzzy environments, *Comput. Ind. Eng.*, **118** (2018), 23–34.
13. M. Adam, M. Fadil, Q-fuzzy soft sets: Definition and applications, *Comput. Math. Appl.*, **68** (2014), 1300–1310.
14. M. Adam, M. Fadil, Operations on Q-fuzzy soft sets, *J. Comput. Appl. Math.*, **265** (2014), 136–145.
15. M. Adam, M. Fadil, Properties of multi-Q-fuzzy soft matrices, *Comput. Math. Appl.*, **68** (2014), 2157–2167.
16. M. Adam, M. Fadil, Multi-Q-fuzzy sets and multi-Q-fuzzy parameterized soft sets, *Appl. Math. Comput.*, **229** (2014), 37–46.
17. M. Adam. M. Fadil, Multi-Q-fuzzy parameterized soft sets in decision making, *Appl. Math. Comput.*, **255** (2015), 499–508.
18. S. Mahmood, Novel T-bipolar soft sets and their applications, *Math. Method. Appl. Sci.*, **43** (2020), 28–40.
19. S. M. Chen, C. L. Hwang, MCDM generalized aggregation of bipolar fuzzy soft sets, *Math. Method. Appl. Sci.*, **37** (2014), 1629–1639.
20. A. Khalil, Y. Li, Possibility theory-based fuzzy decision-making with $\mathfrak{M}$-polar fuzzy soft sets, *Soft Comput.*, **23** (2019), 1451–1464.
21. S. Zahedi, A. Shafie, Novel aggregation operators for $\mathfrak{M}$-polar fuzzy soft sets, *Comput. Intell.*, **34** (2018), 1023–1039.
22. U. Naeem, M. Sharif, Some properties of Pythagorean $\mathfrak{M}$-polar fuzzy sets, *J. Comput. Appl. Math.*, **384** (2021), 113097.
23. M. Akram, S. Majeed, Parameterized models of $\mathfrak{M}$-polar fuzzy soft information, *Math. Method. Appl. Sci.*, **44** (2021), 3240–3253.
24. W. Ali, M. Khan, Novel AGO operators for MCDM: Algorithm and characteristics, *Appl. Math. Model.*, **66** (2024), 764–776.

25. T. D. Cuong, V. Kreinovich, Picture fuzzy sets and their applications in decision-making, *J. Appl. Math. Comput.*, **40** (2013), 69–78.

26. X. Kong, Picture fuzzy sets in voting situations, *Comput. Ind. Eng.*, **82** (2015), 195–202.

27. N. Gundogdu, C. Kahraman, Spherical fuzzy sets in decision-making, *Soft Comput.*, **22** (2018), 3015–3027.

28. A. Kutlu, D. Cetinkaya, Three-dimensional spherical fuzzy sets for MCDM, *Mathematics*, **7** (2019), 523.

29. I. Ashraf, A. Mohyuddin, Spherical fuzzy sets in MCDM, *Fuzzy Set. Syst.*, **366** (2019), 1–17.

30. S. Perveen, I. Ahmed, Spherical fuzzy soft sets for MCDM, *Appl. Soft Comput.*, **75** (2019), 52–65.

31. A. Kutlu, A. Yavuz, Properties of spherical fuzzy sets and applications, *J. Comput. Appl. Math.*, **366** (2020), 112407.

32. Z. Guner, M. Kilic, Spherical fuzzy soft set aggregation for MCDM, *Soft Comput.*, **26** (2022), 1395–1412.

33. M. Ahmmad, S. Shamsuddin, Some aggregation operators for spherical fuzzy soft sets, *Comput. Intell.*, **37** (2021), 1121–1134.

34. M. Riaz, M. Akram, Novel $\mathfrak{M}$-polar spherical fuzzy sets, *Math. Method. Appl. Sci.*, **44** (2021), 2932–2950.

35. R. R. Yager, Pythagorean membership grades in multicriteria decision making, *IEEE T. Fuzzy Syst.*, **22** (2014), 958–965.

36. P. Liu, P. Wang, Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making, *Int. J. Intell. Syst.*, **33** (2018), 259–280.

37. H. Garg, Sine trigonometric operational laws and their based Pythagorean fuzzy aggregation operators for the group decision-making process, *Artif. Intell. Rev.*, **54** (2021), 4421–4447.

38. M. K. Sayadi, M. Heydari, K. Shahanaghi, Extension of the VIKOR method for decision making problem with interval numbers, *Appl. Math. Model.*, **33** (2009), 2257–2262.

39. J. Qin, X. Liu, W. Pedrycz, An extended TODIM multi-criteria group decision-making method for green supplier selection in an interval type-2 fuzzy environment, *Eur. J. Oper. Res.*, **258** (2017), 626–638.

40. S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, T. Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision-making problems, *J. Intell. Fuzzy Syst.*, **36** (2019), 2829–2844.

41. B. Farhadinia, A novel method of ranking hesitant fuzzy values for multiple attribute decision-making problems, *Int. J. Intell. Syst.*, **28** (2013), 752–767.

42. H. Liao, H. Zhang, C. Zhang, X. Wu, A. Mardani, A. Al-Barakati, A q-rung orthopair fuzzy GLDS method for investment evaluation of BE angel capital in China, *Technol. Econ. Dev. Eco.*, **26** (2020), 103–134.

43. G. Wei, H. Gao, Y. Wei, Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making, *Int. J. Intell. Syst.*, **33** (2018), 1426–1458.

44. Q. Mou, Z. Xu, H. Liao, A graph-based group decision approach with intuitionistic fuzzy preference relations, *Comput. Ind. Eng.*, **110** (2017), 138–150.

45. T. Mahmood, M. Ali, M. Shabir, T-spherical fuzzy sets and their applications, *J. Intell. Fuzzy Syst.*, **36** (2019), 4743–4756.

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