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Research article

Multiplicative view point analysis of Hahn calculus and their applications to inequality theory

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Abstract: Hahn multiplicative calculus is the generalization of quantum multiplicative (q-multiplicative) calculus. In this manuscript, we defined novel definitions for derivative and definite integral called left Hahn multiplicative derivative and definite integral in the Hahn multiplicative calculus. In addition, we derived fundamental results for this newly defined integral. Furthermore, we constructed left Hahn multiplicative Hermite-Hadamard inequalities. Additionally, we defined new definitions for the derivative and definite integral in the Hahn calculus, which enabled us to define further definitions in Hahn multiplicative calculus called the right Hahn multiplicative derivative and definite integral. Moreover, we construct the power rule of the newly defined definite integral in the Hahn calculus, which assisted us in deriving the right Hahn multiplicative Hermite-Hadamard inequalities. Finally, we gave an application of the newly established Hermite-Hadamard inequalities through an example wherein it could be seen that these inequalities were crucial for finding the lower and upper bounds of the range of those functions whose Hahn multiplicative definite integrals were very difficult to find.

Keywords: Hahn multiplicative calculus; Hahn calculus; multiplicative calculus; Hermite-Hadamard inequality
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1. Introduction

One of the most significant contributions to convex analysis is the Hermite-Hadamard inequality, which has been broadly generalized and applied to other modern calculi (fractional calculus [1], q-calculus [2–4], interval-valued calculus [5], conformable calculus [6], Hahn calculus [7], q-symmetric calculus [8, 9], and Hahn symmetric calculus [10]). In these generalized structures, the inequality gives better approximations of the convex and generalized convex functions, in connection with classical integral inequalities and the non-standard operators. These inequalities are used in numerical integration, optimization theory, and applied sciences to provide bounds on means, error estimates, and stability [1, 11]. Moreover, they are important in the approximation methods and quantum theory of non-uniform lattices and in q-calculus and Hahn calculus in discrete models [12, 13]. Fractional calculus Hermite-Hadamard type inequalities are applied to estimate solutions of fractional differential equations and integral transforms [14].

In the 20th century, Grossman and Katz defined a novel kind of integral and derivative named as multiplicative integral and multiplicative derivative, respectively. It substituted multiplication and division for addition and subtraction functions. This recently developed calculus has been called multiplicative calculus. In the literature, it is applicable only to positive functions and is also referred to as a non-Newtonian calculus. Recently, researchers have contributed to many areas of mathematics regarding this calculus. For instance, the fundamental theorem of multiplicative calculus [15], multiplicative stochastic integrals [16], complex multiplicative calculus [17], multiplicative differential equations [18], and the non-Newtonian calculus literature contain multiplicative Hermite-Hadamard inequalities for many kinds of convex functions [19–22]. The multiplicative derivative and integral are defined below.

Definition 1. [15] For any positive function $\mathcal{Z}$ with $\mathcal{Z}(\varsigma) \neq 0$, then the multiplicative derivative or * derivative of a function $\mathcal{Z}$ at $\varsigma \in \Re$ is defined as:

$$\mathcal{Z}^*(\varsigma) = \mathfrak{D}^*\mathcal{Z}(\varsigma) = \lim_{\mathsf{h}\to 0}\left(\frac{\mathcal{Z}(\varsigma+\mathsf{h})}{\mathcal{Z}(\varsigma)}\right)^{\frac{1}{\mathsf{h}}}.$$

Additionally,

$$\mathcal{Z}^*(\varsigma) = exp\left(\frac{\mathcal{Z}'(\varsigma)}{\mathcal{Z}(\varsigma)}\right) = exp\left((\ln \mathcal{Z}(\varsigma))'\right),$$

where $\mathcal{Z}'(\varsigma)$ is the classical derivative.

Definition 2. [15] For any positive and Riemann integrable function $\mathcal{Z} : I = [\nu, \ell] \longrightarrow \Re$, then *integral or multiplicative integral of $\mathcal{Z}$ from ν to ℓ is defined as:

$$\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\,d\varsigma} = exp\left(\int_{\nu}^{\ell} (\ln \mathcal{Z}(\varsigma))d\varsigma\right)$$

for $\varsigma \in [\nu, \ell]$.

Properties related to multiplicative integrals of a function are described as.

Proposition 1. [15] For any positive and Riemann integrable functions $\mathcal{Z}$ and $\mathcal{Y}$ that are defined as $\mathcal{Z} : I \longrightarrow \Re$ and $\mathcal{Y} : I \longrightarrow \Re$ with $\nu \leq \mathsf{c} \leq \ell$ and $\mathsf{n} \in \Re$, then we have

- $\int_{\nu}^{\nu} \mathcal{Z}(\varsigma)^{\,d\varsigma} = 1.$
- $\dfrac{1}{\int_{\ell}^{\nu} \mathcal{Z}(\varsigma)^{\,d\varsigma}} = \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\,d\varsigma}.$

- $\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)\, d\varsigma = \ln\left(\int_{\nu}^{\ell} (exp(\mathcal{Z}(\varsigma)))^{d\varsigma}\right)$.
- $\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma)^{\mathsf{n}})^{\ d\varsigma} = \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{\ d\varsigma}\right)^{\mathsf{n}}$.
- $\int_{\nu}^{\mathsf{c}} \mathcal{Z}(\varsigma)^{\ d\varsigma} = \int_{\nu}^{\mathsf{c}} \mathcal{Z}(\varsigma)^{\ d\varsigma} \cdot \int_{\mathsf{c}}^{\ell} \mathcal{Z}(\varsigma)^{\ d\varsigma}$.
- $\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma)\mathcal{Y}(\varsigma))^{\ d\varsigma} = \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ d\varsigma} \cdot \int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ d\varsigma}$.
- $\int_{\nu}^{\ell} \left(\frac{\mathcal{Z}(\varsigma)}{\mathcal{Y}(\varsigma)}\right)^{d\varsigma} = \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ d\varsigma} \cdot \left(\int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ d\varsigma}\right)^{-1}$.

Quantum calculus has established a strong connection between mathematics and physics, and has found wide applications in number theory, quantum physics, combinatorics, orthogonal polynomials, hypergeometric functions, relativity theory, and other scientific fields. Kac et al. book [23] covers a lot of the fundamental ideas of quantum calculus. While some notable developments regarding q-analog of the Laplace transform and summation-integral type operators can be seen in [24, 25] respectively.

The generalized version of quantum calculus called Hahn quantum or simply Hahn calculus was introduced by Hahn in 1949 [26]. Numerous mathematical and physical problems from classical and q-calculus are explored in Hahn calculus. The fundamental concept of Hahn differential equations and their applications have been studied by Hamza and Ahmed [27, 28]. While, the Hahn Sturm-Liouville problem was put forward by Annaby et al. [29], and the sampling theory for the same problem has been determined by Annaby and Hassan [30]. Moreover, its symmetrical version can also be seen in [10]. The definitions of Hahn derivative and integral are defined as below. In this context, we will assume that $w > 0$ and $0 < q < 1$ across the rest of the article. Also, we set $\acute{w} = \frac{w}{1-q} \in I$.

Definition 3. [26] For any function $\mathcal{Z}$, the Hahn derivative of $\mathcal{Z}$ is defined as

$$\mathfrak{D}_{q,w}\mathcal{Z}(\varsigma) = \begin{cases} \frac{\mathcal{Z}(q\varsigma + w) - \mathcal{Z}(\varsigma)}{w + (q-1)\varsigma} & \text{if } \varsigma \neq \acute{w}, \\ \mathcal{Z}'(\acute{w}) & \text{if } \varsigma = \acute{w}, \end{cases}$$

furnished that $\mathcal{Z}$ is differentiable at $\acute{w}$.

Definition 4. [31] For any function $\mathcal{Z}$ and $\nu_1, \ell_1 \in I$, the Hahn definite integral of $\mathcal{Z}$ can be derived as

$$\int_{\nu_1}^{\ell_1} \mathcal{Z}(\varsigma)\, d_{q,w}\varsigma = \int_{\acute{w}}^{\ell_1} \mathcal{Z}(\varsigma) d_{q,w}\varsigma - \int_{\acute{w}}^{\nu_1} \mathcal{Z}(\varsigma) d_{q,w}\varsigma,$$

where

$$\int_{\acute{w}}^{\mathsf{u}} \mathcal{Z}(\varsigma) d_{q,w}\varsigma = ((1-q)\mathsf{u} - w) \sum_{\mathsf{t}=0}^{\infty} q^{\mathsf{t}} \mathcal{Z}(\mathsf{u}q^{\mathsf{t}} + (1-q^{\mathsf{t}})\acute{w})$$

given the fact that $\mathsf{u} = \nu_1$ and $\mathsf{u} = \ell_1$ are the points at which the series converges. If the function $\mathcal{Z}$ is Hahn integrable from ν_1 to ℓ_1, for every $\nu_1, \ell_1 \in I$, then it is Hahn integrable over I.

In 2016 [32], Tariboon et al. defined the new definitions of derivative and integral in Hahn calculus called left Hahn derivative and integral. For this, they defined $\breve{w} = \frac{w}{1-q} + \nu \in I$. The definition of the left Hahn derivative is written as below.

Definition 5. [32] For any function $\mathcal{Z}$, then the left Hahn derivative of $\mathcal{Z}$ is defined as

$$ {}_{\nu}\mathfrak{D}_{q,w}\mathcal{Z}(\varsigma) = \begin{cases} \dfrac{\mathcal{Z}(q\varsigma + (1-q)\nu + w) - \mathcal{Z}(\varsigma)}{w + (\varsigma - \nu)(q-1)} & \text{if } \varsigma \neq \breve{w}, \\ \mathcal{Z}'(\breve{w}) & \text{if } \varsigma = \breve{w}, \end{cases} $$

furnished that $\mathcal{Z}$ is differentiable at $\breve{w}$.

The definition of the left Hahn definite integral is defined as below.

Definition 6. [32] For any function $\mathcal{Z}$ and $\nu_1, \ell_1 \in I$, then the left Hahn definite integral of $\mathcal{Z}$ is written as

$$ \int_{\nu_1}^{\ell_1} \mathcal{Z}(\varsigma)\ {}_{\nu}d_{q,w}\varsigma = \int_{\breve{w}}^{\ell_1} \mathcal{Z}(\varsigma)\ {}_{\nu}d_{q,w}\varsigma - \int_{\breve{w}}^{\nu_1} \mathcal{Z}(\varsigma)\ {}_{\nu}d_{q,w}\varsigma, $$

where

$$ \int_{\breve{w}}^{\mathsf{u}} \mathcal{Z}(\varsigma)\ {}_{\nu}d_{q,w}\varsigma = ((1-q)(\mathsf{u}-\nu) - w) \sum_{\mathsf{t}=0}^{\infty} q^{\mathsf{t}} \mathcal{Z}(\mathsf{u}q^{\mathsf{t}} + (1-q^{\mathsf{t}})\breve{w}) $$

given the fact that $\mathsf{u} = \nu_1$ and $\mathsf{u} = \ell_1$ are the points at which the series converges. If the function $\mathcal{Z}$ is Hahn integrable from ν_1 to ℓ_1, for every ν_1, $\ell_1 \in I$, then it is Hahn integrable over I.

Example 1. [33] Let a function $\mathcal{Z}(\varsigma) = \varsigma$ over I, then the left Hahn integral of $\mathcal{Z}$ is stated as:

$$ \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)\ {}_{\nu}d_{q,w}\varsigma = (\ell - \nu)\left(\frac{\ell + \nu q - w}{q+1}\right). \tag{1.1} $$

The Eq (1.1) is crucial to derive the new Hermite-Hadamard inequalities in the left Hahn multiplicative calculus. In 2023, using the theory of multiplicative and Hahn calculus, Allahverdiev et al. introduced a novel type of calculus called Hahn multiplicative calculus and derived its fundamental properties [34]. Moreover, the definitions of the Hahn multiplicative derivative and integral are given below.

Definition 7. [34] For any positive function $\mathcal{Z}$ with $\mathcal{Z}(\varsigma) \neq 0$, then the Hahn multiplicative derivative or Hahn *derivative of function $\mathcal{Z}$ at $\varsigma \in \mathfrak{R}$ is written as:

$$ \mathfrak{D}^{*}_{q,w}\mathcal{Z}(\varsigma) = \left(\frac{\mathcal{Z}(q\varsigma + w)}{\mathcal{Z}(\varsigma)}\right)^{\frac{1}{w+(q-1)\varsigma}}. $$

Additionally,

$$ \mathfrak{D}^{*}_{q,w}\mathcal{Z}(\varsigma) = exp\left(\mathfrak{D}_{q,w}(\ln \mathcal{Z}(\varsigma))\right) $$

holds for $w = 0$ and $\mathcal{Z}(\varsigma)$ is monomial, here $\mathfrak{D}_{q,w}$ is a Hahn derivative defined in [26].

Definition 8. [34] For any positive function $\mathcal{Z} : I \longrightarrow \mathfrak{R}$, then the Hahn *integral ($(q,w)^*$-integral) or Hahn multiplicative integral of $\mathcal{Z}$ from ν to ℓ can be defined as:

$$ \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ d_{q,w}\varsigma} = exp\left(\int_{\nu}^{\ell} (\ln \mathcal{Z}(\varsigma)) d_{q,w}\varsigma\right) = exp\left(\int_{\nu}^{\acute{w}} (\ln \mathcal{Z}(\varsigma)) d_{q,w}\varsigma - \int_{\acute{w}}^{\ell} (\ln \mathcal{Z}(\varsigma)) d_{q,w}\varsigma\right) $$

for $\varsigma \in [\nu, \ell]$.

Properties of the Hahn multiplicative integral of a function are mentioned below.

Theorem 1. [34] For any positive and Hahn multiplicative integrable functions $\mathcal{Z}$ and $\mathcal{Y}$ that are defined on I with $\nu \leq \mathsf{c} \leq \ell$ and $\mathsf{n} \in \Re$, then we have

- $\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma)^{\mathsf{n}})^{\ d_{q,w}\varsigma} = \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{\ d_{q,w}\varsigma}\right)^{\mathsf{n}}$.
- $\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ d_{q,w}\varsigma} = \int_{\nu}^{\mathsf{c}} \mathcal{Z}(\varsigma)^{\ d_{q,w}\varsigma} . \int_{\mathsf{c}}^{\ell} \mathcal{Z}(\varsigma)^{\ d_{q,w}\varsigma}$.
- $\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma)\mathcal{Y}(\varsigma))^{\ d_{q,w}\varsigma} = \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ d_{q,w}\varsigma} . \int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ d_{q,w}\varsigma}$.
- $\int_{\nu}^{\ell} \left(\frac{\mathcal{Z}(\varsigma)}{\mathcal{Y}(\varsigma)}\right)^{d_{q,w}\varsigma} = \frac{\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ d_{q,w}\varsigma}}{\int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ d_{q,w}\varsigma}}$.

2. Left Hahn multiplicative calculus with respect to the point ν and related Hermite-Hadamard inequalities

Alp et al. in 2018 introduced the first corrected version of the q-Hermite-Hadamard inequality in [2], considering support lines and the geometrical concept of convex functions by fixing the left end point of the interval. Later in 2020, Bermudo et al. introduced another useful approach by considering the right endpoint of the interval in the context of quantum calculus in [3]. They not only provided new definitions of the right quantum derivative and quantum integrals but also employed these notions to obtain a new interpretation of the Hermite-Hadamard inequality. In the recent past, the notions of symmetric quantum calculus [8] and the symmetric Hahn calculus [10, 35] have been extended to the left point ν and the right point ℓ of the interval I. Motivated by the above studies, we are extending the Hahn multiplicative calculus to the left point ν and the right point ℓ to further generalize the results of the quantum multiplicative calculus [36]. Inspired by Definitions 7 and 8, we introduce the new definitions called the left Hahn multiplicative derivative and integral which are denoted by ${}_{\nu}\mathfrak{D}^{*}_{(q,w)}$ and ${}_{\nu}(q, w)^{*}$-integral respectively.

Definition 9. *For any positive function $\mathcal{Z}$ with $\mathcal{Z}(\varsigma) \neq 0$, then the left Hahn multiplicative derivative or left Hahn *derivative of function $\mathcal{Z}$ at $\varsigma \in \Re$ is written as:*

$$ {}_{\nu}\mathfrak{D}^{*}_{(q,w)}\mathcal{Z}(\varsigma) = \left(\frac{\mathcal{Z}(q\varsigma + (1-q)\nu + w)}{\mathcal{Z}(\varsigma)}\right)^{\frac{1}{w+(q-1)(\varsigma-\nu)}}. $$

Additionally,

$$ {}_{\nu}\mathfrak{D}^{*}_{(q,w)}\mathcal{Z}(\varsigma) = exp\left({}_{\nu}\mathfrak{D}_{(q,w)}(\ln \mathcal{Z}(\varsigma))\right) $$

holds for $w = \nu = 0$ and $\mathcal{Z}(\varsigma)$ is monomial, here ${}_{\nu}\mathfrak{D}_{(q,w)}$ is a left Hahn derivative that is defined in [32].

Definition 10. *For any positive function $\mathcal{Z} : I \longrightarrow \Re$, then the left Hahn *integral (${}_{\nu}(q, w)^{*}$-integral) or the left Hahn multiplicative integral of $\mathcal{Z}$ from ν to ℓ can be defined as:*

$$ \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma} = exp\left(\int_{\nu}^{\ell} (\ln \mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\right) = exp\left(\int_{\nu}^{\breve{w}} (\ln \mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma - \int_{\breve{w}}^{\ell} (\ln \mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\right) $$

for $\varsigma \in [\nu, \ell]$.

Some basic results for the left Hahn multiplicative integral are shown below.

Theorem 2. *For any positive and left Hahn multiplicative integrable functions $\mathcal{Z}$ and $\mathcal{Y}$ that are defined on I with $\nu \leq \mathsf{c} \leq \ell$ and $\mathsf{n} \in \mathfrak{R}$, then we have*

(1) $\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma)^{\mathsf{n}})^{\ {}_{\nu}d_{q,w}\varsigma} = \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{\ {}_{\nu}d_{q,w}\varsigma}\right)^{\mathsf{n}}$.

(2) $\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma} = \int_{\nu}^{\mathsf{c}} \mathcal{Z}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}. \int_{\mathsf{c}}^{\ell} \mathcal{Z}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}$.

(3) $\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma)\mathcal{Y}(\varsigma))^{\ {}_{\nu}d_{q,w}\varsigma} = \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}. \int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}$.

(4) $\int_{\nu}^{\ell} \left(\frac{\mathcal{Z}(\varsigma)}{\mathcal{Y}(\varsigma)}\right)^{{}_{\nu}d_{q,w}\varsigma} = \frac{\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}}{\int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}}$.

Proof. As (1) and (2) are trivial.
Using Definition 10 to prove (3),

$$\begin{aligned}
\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma)\mathcal{Y}(\varsigma))^{\ {}_{\nu}d_{q,w}\varsigma} &= exp\left(\int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma)\cdot\mathcal{Y}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\right) \\
&= exp\left(\int_{\nu}^{\ell} \{\ln(\mathcal{Z}(\varsigma)) + \ln(\mathcal{Y}(\varsigma))\}\ {}_{\nu}d_{q,w}\varsigma\right) \\
&= exp\left(\int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\right)\cdot exp\left(\int_{\nu}^{\ell} \ln(\mathcal{Y}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\right) \\
&= \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}\cdot\int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}.
\end{aligned}$$

Now, to prove (4), again using Definition 10,

$$\begin{aligned}
\int_{\nu}^{\ell} \left(\frac{\mathcal{Z}(\varsigma)}{\mathcal{Y}(\varsigma)}\right)^{{}_{\nu}d_{q,w}\varsigma} &= exp\left(\int_{\nu}^{\ell} \ln(\frac{\mathcal{Z}(\varsigma)}{\mathcal{Y}(\varsigma)})\ {}_{\nu}d_{q,w}\varsigma\right) \\
&= exp\left(\int_{\nu}^{\ell} \{\ln(\mathcal{Z}(\varsigma)) - \ln(\mathcal{Y}(\varsigma))\}\ {}_{\nu}d_{q,w}\varsigma\right) \\
&= \frac{exp\left(\int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\right)}{exp\left(\int_{\nu}^{\ell} \ln(\mathcal{Y}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\right)} \\
&= \frac{\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}}{\int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma}}.
\end{aligned}$$

□

Example 2. *Suppose $\mathcal{Z}(\varsigma) = exp(\varsigma)$, then using the left Hahn integral, Definition* 10, and (1.1)

$$\int_{\nu}^{\ell} exp(\varsigma)^{\ {}_{\nu}d_{q,w}\varsigma} = exp\left(\int_{\nu}^{\ell} \ln(exp(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\right)$$

$$= exp\left(\int_{\nu}^{\ell} \varsigma \ {}_{\nu}d_{q,w}\varsigma\right)$$
$$= exp\left((\ell-\nu)\left(\frac{\ell+\nu q-w}{q+1}\right)\right). \tag{2.1}$$

The multiplicative Hermite-Hadamard inequality is mentioned in [19] and is written below.

Theorem 3. *If $\mathcal{Z}: I \longrightarrow \Re$ is a positive and multiplicative convex function on $[\nu, \ell]$, then*

$$\mathcal{Z}\left(\frac{\nu+\ell}{2}\right) \leq \int_{\nu}^{\ell} \left((\mathcal{Z}(\varsigma))^{d\varsigma}\right)^{\frac{1}{\ell-\nu}} \leq \sqrt{\mathcal{Z}(\nu)\mathcal{Z}(\ell)} \tag{2.2}$$

holds.

Now, we will employ the newly introduced concept of left Hahn multiplicative calculus and derive the corresponding variant of Hermite-Hadamard inequality (2.2) and related results.

Theorem 4. *If $\mathcal{Z}: I \longrightarrow \Re$ is a positive and log-convex differentiable function on $[\nu, \ell]$, then for $w \geq 0$ and $0 < q < 1$*

$$\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right) \leq \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \leq (\mathcal{Z}(\nu))^{\frac{w+(\ell-\nu)q}{(q+1)(\ell-\nu)}} \cdot (\mathcal{Z}(\ell))^{\frac{\ell-\nu-w}{(q+1)(\ell-\nu)}} \tag{2.3}$$

holds.

Proof. As we can write that the supporting line of the function $\mathcal{Z}$ at the point $\frac{\ell+\nu q-w}{q+1} \in (\nu, \ell)$ is

$$\mathsf{P}(\varsigma) = \mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\left(\mathcal{Z}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)^{\left(\varsigma-\frac{\ell+\nu q-w}{q+1}\right)}.$$

By the log-convexity of $\mathcal{Z}$ and for $\nu \leq \varsigma \leq \ell$, we can write

$$\mathsf{P}(\varsigma) = \mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\left(\mathcal{Z}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)^{\left(\varsigma-\frac{\ell+\nu q-w}{q+1}\right)} \leq \mathcal{Z}(\varsigma)$$
$$\ln \mathsf{P}(\varsigma) = \ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) + \left(\varsigma - \frac{\ell+\nu q-w}{q+1}\right)\ln\left(\mathcal{Z}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) \leq \ln(\mathcal{Z}(\varsigma)).$$

Using the left Hahn integral from ν to ℓ and using (1.1),

$$\int_{\nu}^{\ell} \ln \mathsf{P}(\varsigma) \ {}_{\nu}d_{q,w}\varsigma = (\ell-\nu)\ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) + \ln\left(\mathcal{Z}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)$$
$$\times\left(\int_{\nu}^{\ell} \varsigma \ {}_{\nu}d_{q,w}\varsigma - (\ell-\nu)\frac{\ell+\nu q-w}{q+1}\right) \leq \int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma)) \ {}_{\nu}d_{q,w}\varsigma$$
$$= (\ell-\nu)\ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) + \ln\left(\mathcal{Z}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)$$

$$\times\left(\frac{\ell-\nu}{q+1}(\ell+\nu q-w)-\frac{\ell-\nu}{q+1}(\ell+\nu q-w)\right)\leq\int_{\nu}^{\ell}\ln(\mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma$$
$$=(\ell-\nu)\ln\left(\mathcal{Z}\left(\frac{q\nu+\ell-w}{q+1}\right)\right)\leq\int_{\nu}^{\ell}\ln(\mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma$$
$$=\ln\left(\mathcal{Z}\left(\frac{\nu q+\ell-w}{q+1}\right)\right)\leq\int_{\nu}^{\ell}\ln(\mathcal{Z}(\varsigma))^{\frac{1}{\ell-\nu}}\ {}_{\nu}d_{q,w}\varsigma.$$

Taking the exponential of both sides,

$$\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\leq exp\left(\int_{\nu}^{\ell}\ln(\mathcal{Z}(\varsigma))^{\frac{1}{\ell-\nu}}\ {}_{\nu}d_{q,w}\varsigma\right)$$
$$\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\leq\left(\int_{\nu}^{\ell}(\mathcal{Z}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}}. \tag{2.4}$$

Similarly, the equation of the line segment joining the points $(\nu,\mathcal{Z}(\nu))$ and $(\ell,\mathcal{Z}(\ell))$ can be written as a function of the following form

$$\mathsf{p}(\varsigma)=\left\{(\mathcal{Z}(\nu))^{-1}\mathcal{Z}(\ell)\right\}^{\frac{\varsigma-\nu}{\ell-\nu}}\cdot\mathcal{Z}(\nu).$$

Since $\mathcal{Z}$ is log-convex, therefore, $\forall\ \varsigma\in[\nu,\ell]$, we have

$$\mathcal{Z}(\varsigma)\leq\mathsf{p}(\varsigma)=\left\{\frac{\mathcal{Z}(\ell)}{\mathcal{Z}(\nu)}\right\}^{\frac{\varsigma-\nu}{\ell-\nu}}\cdot\mathcal{Z}(\nu).$$

Also

$$\ln(\mathcal{Z}(\varsigma))\leq\ln(\mathsf{p}(\varsigma))=\left(\frac{\varsigma-\nu}{\ell-\nu}\right)[\ln(\mathcal{Z}(\ell))-\ln(\mathcal{Z}(\nu))]+\ln(\mathcal{Z}(\nu)).$$

Taking the left Hahn integral from ν to ℓ and using (1.1),

$$\int_{\nu}^{\ell}\ln(\mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma\leq\int_{\nu}^{\ell}\ln(\mathsf{p}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma=(\ell-\nu)\ln(\mathcal{Z}(\nu))+\frac{\ln(\mathcal{Z}(\ell))-\ln(\mathcal{Z}(\nu))}{\ell-\nu}$$
$$\times\left(\int_{\nu}^{\ell}\varsigma\ {}_{\nu}d_{q,w}\varsigma+(\nu-\ell)\nu\right)$$
$$\leq(\ell-\nu)\ln(\mathcal{Z}(\nu))+(\ln(\mathcal{Z}(\ell))-\ln(\mathcal{Z}(\nu)))\left(\frac{\ell+\nu q-w}{q+1}-\nu\right)$$
$$=(\ell-\nu)\ln(\mathcal{Z}(\nu))+(\ln(\mathcal{Z}(\ell))-\ln(\mathcal{Z}(\nu)))\left(\frac{\ell-(w+\nu)}{q+1}\right)$$
$$=(\ell-\nu)\left\{\ln(\mathcal{Z}(\nu))+(\ln(\mathcal{Z}(\ell))-\ln(\mathcal{Z}(\nu)))\left(\frac{1}{q+1}-\frac{w}{(q+1)(\ell-\nu)}\right)\right\}$$
$$=(\ell-\nu)\left[\ln(\mathcal{Z}(\nu))+\left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))-\frac{\ln(\mathcal{Z}(\nu))}{q+1}+\frac{w\ln(\mathcal{Z}(\nu))}{(q+1)(\ell-\nu)}\right]$$
$$=(\ell-\nu)\left[\frac{q\ln(\mathcal{Z}(\nu))}{q+1}+\frac{w\ln(\mathcal{Z}(\nu))}{(q+1)(\ell-\nu)}+\left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))\right]$$

$$= (\ell - \nu)\left[\left(\frac{-w + (\nu - \ell)q}{(q+1)(\nu - \ell)}\right)\ln(\mathcal{Z}(\nu)) + \left(\frac{\ell - (w+\nu)}{(q+1)(\ell - \nu)}\right)\ln(\mathcal{Z}(\ell))\right]$$

$$= (\ell - \nu)\ln\left[\{\mathcal{Z}(\nu)\}^{w+(\ell-\nu)q}\cdot\{\mathcal{Z}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}$$

$$\frac{1}{\ell - \nu}\int_{\nu}^{\ell}\ln(\mathcal{Z}(\varsigma))\ {}_{\nu}d_{q,w}\varsigma \leq \ln\left[\{\mathcal{Z}(\nu)\}^{w+(\ell-\nu)q}\cdot\{\mathcal{Z}(\ell)\}^{\ell-\nu-w}\right]^{\frac{1}{(q+1)(\ell-\nu)}}.$$

By using the property of the log function and the first case of Theorem 2, we will get

$$\left(\int_{\nu}^{\ell}(\mathcal{Z}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \leq (\mathcal{Z}(\nu))^{\frac{w+(\ell-\nu)q}{(q+1)(\ell-\nu)}}\cdot(\mathcal{Z}(\ell))^{\frac{\ell-\nu-w}{(q+1)(\ell-\nu)}}. \tag{2.5}$$

From (2.4) and (2.5), the desired result has become. □

Theorem 5. *If $\mathcal{Z} : I \longrightarrow \Re$ and $\mathcal{Y} : I \longrightarrow \Re$ are two positive and log convex differentiable functions on $[\nu, \ell]$, then*

$$\mathcal{Z}\left(\frac{\ell + \nu q - w}{q+1}\right)\cdot\mathcal{Y}\left(\frac{\ell + \nu q - w}{q+1}\right) \leq \left(\int_{\nu}^{\ell}(\mathcal{Z}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma}\cdot\int_{\nu}^{\ell}(\mathcal{Y}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}}$$
$$\leq \left[(\mathcal{Z}(\nu)\cdot\mathcal{Y}(\nu))^{w+q(\ell-\nu)q}(\mathcal{Z}(\ell)\cdot\mathcal{Y}(\ell))^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}} \tag{2.6}$$

holds for $q \in (0, 1)$ and $w \geq 0$.

Proof. As we know that the supporting line of the function $\mathcal{Z}$ and $\mathcal{Y}$ at the point $\frac{\ell + \nu q - w}{q+1} \in (\nu, \ell)$ can be written as:

$$\mathsf{p}_1(\varsigma) = \mathcal{Z}\left(\frac{\ell + \nu q - w}{q+1}\right)\left(\mathcal{Z}^*\left(\frac{\ell + \nu q - w}{q+1}\right)\right)^{\left(\varsigma - \frac{\ell + \nu q - w}{q+1}\right)}$$

and

$$\mathsf{p}_2(\varsigma) = \mathcal{Y}\left(\frac{\ell + \nu q - w}{q+1}\right)\left(\mathcal{Y}^*\left(\frac{\ell + \nu q - w}{q+1}\right)\right)^{\left(\varsigma - \frac{\ell + \nu q - w}{q+1}\right)}.$$

By the log-convexity of $\mathcal{Z}$ and $\mathcal{Y}$ for $\nu \leq \varsigma \leq \ell$, we can write

$$\mathsf{p}_1(\varsigma) = \mathcal{Z}\left(\frac{\ell + \nu q - w}{q+1}\right)\left(\mathcal{Z}^*\left(\frac{\ell + \nu q - w}{q+1}\right)\right)^{\left(\varsigma - \frac{\ell + \nu q - w}{q+1}\right)} \leq \mathcal{Z}(\varsigma) \tag{2.7}$$

and

$$\mathsf{p}_2(\varsigma) = \mathcal{Y}\left(\frac{\ell + \nu q - w}{q+1}\right)\left(\mathcal{Y}^*\left(\frac{\ell + \nu q - w}{q+1}\right)\right)^{\left(\varsigma - \frac{\ell + \nu q - w}{q+1}\right)} \leq \mathcal{Y}(\varsigma). \tag{2.8}$$

Multiplying (2.7) and (2.8) then applying ln on both sides, we get

$$\begin{aligned}\ln \mathsf{p}_1(\varsigma)+\ln \mathsf{p}_2(\varsigma) &= \ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)+\left(\varsigma-\frac{\ell+\nu q-w}{q+1}\right)\ln\left(\mathcal{Z}^*\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\\ &+\ln\left(\mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)+\left(\varsigma-\frac{\ell+\nu q-w}{q+1}\right)\ln\left(\mathcal{Y}^*\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\\ &\leq \ln(\mathcal{Z}(\varsigma))+\ln(\mathcal{Y}(\varsigma)).\end{aligned}$$

Applying the left Hahn integral over I and using (1.1)

$$\begin{aligned}&\int_\nu^\ell \ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\,{}_\nu d_{q,w}\varsigma+\ln\left(\mathcal{Z}^*\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\int_\nu^\ell\left(\varsigma-\frac{\ell+\nu q-w}{q+1}\right)\,{}_\nu d_{q,w}\varsigma\\ &+\int_\nu^\ell \ln\left(\mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\,{}_\nu d_{q,w}\varsigma+\ln\left(\mathcal{Y}^*\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\int_\nu^\ell\left(\varsigma-\frac{\ell+\nu q-w}{q+1}\right)\,{}_\nu d_{q,w}\varsigma\\ &\leq \int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\,{}_\nu d_{q,w}\varsigma+\int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\,{}_\nu d_{q,w}\varsigma\\ &=(\ell-\nu)\ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)+\ln\left(\mathcal{Z}^*\left(\frac{\ell+\nu q-w}{q+1}\right)\right)(0)\\ &+(\ell-\nu)\ln\left(\mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)+\ln\left(\mathcal{Y}^*\left(\frac{\ell+\nu q-w}{q+1}\right)\right)(0)\\ &\leq \int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\,{}_\nu d_{q,w}\varsigma+\int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\,{}_\nu d_{q,w}\varsigma\\ &=\ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\cdot\mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\\ &\leq \frac{1}{\ell-\nu}\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\,{}_\nu d_{q,w}\varsigma+\int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\,{}_\nu d_{q,w}\varsigma\right).\end{aligned}$$

Applying the exponential function

$$\begin{aligned}\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\cdot\mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right) &\leq exp\left(\frac{1}{\ell-\nu}\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\,{}_\nu d_{q,w}\varsigma+\int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\,{}_\nu d_{q,w}\varsigma\right)\right)\\ &=\left(exp\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\,{}_\nu d_{q,w}\varsigma+\int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\,{}_\nu d_{q,w}\varsigma\right)\right)^{\frac{1}{\ell-\nu}}\\ &=\left(exp\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\,{}_\nu d_{q,w}\varsigma\right)\cdot exp\left(\int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\,{}_\nu d_{q,w}\varsigma\right)\right)^{\frac{1}{\ell-\nu}}\\ &=\left(\int_\nu^\ell (\mathcal{Z}(\varsigma))^{{}_\nu d_{q,w}\varsigma}\cdot\int_\nu^\ell (\mathcal{Y}(\varsigma))^{{}_\nu d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}}. \qquad (2.9)\end{aligned}$$

Additionally, it is possible to describe the lines that connect points $(\nu,\mathcal{Z}(\nu))$, $(\ell,\mathcal{Z}(\ell))$, and $(\nu,\mathcal{Y}(\nu))$, $(\ell,\mathcal{Y}(\ell))$ as functions are

$$\mathsf{p}(\varsigma)=\mathcal{Z}(\nu)\left[\frac{\mathcal{Z}(\ell)}{\mathcal{Z}(\nu)}\right]^{\left(\frac{\varsigma-\nu}{\ell-\nu}\right)}$$

and

$$\mathsf{p}_3(\varsigma) = \mathcal{Y}(\nu)\left[\frac{\mathcal{Y}(\ell)}{\mathcal{Y}(\nu)}\right]^{(\frac{\varsigma-\nu}{\ell-\nu})}$$

respectively. Since $\mathcal{Z}$ and $\mathcal{Y}$ are log-convex, then

$$\mathcal{Z}(\varsigma) \leq \mathsf{p}(\varsigma) = \mathcal{Z}(\nu)\left[\frac{\mathcal{Z}(\ell)}{\mathcal{Z}(\nu)}\right]^{(\frac{\varsigma-\nu}{\ell-\nu})} \tag{2.10}$$

and

$$\mathcal{Y}(\varsigma) \leq \mathsf{p}_3(\varsigma) = \mathcal{Y}(\nu)\left[\frac{\mathcal{Y}(\ell)}{\mathcal{Y}(\nu)}\right]^{(\frac{\varsigma-\nu}{\ell-\nu})}. \tag{2.11}$$

Multiplying (2.10) and (2.11) then applying ln, we get

$$\ln\mathcal{Z}(\varsigma) + \ln\mathcal{Y}(\varsigma) \leq \ln\mathcal{Z}(\nu) + \frac{\varsigma-\nu}{\ell-\nu}[\ln\mathcal{Z}(\ell) - \ln\mathcal{Z}(\nu)] + \ln\mathcal{Y}(\nu) + \frac{\varsigma-\nu}{\ell-\nu}[\ln\mathcal{Y}(\ell) - \ln\mathcal{Y}(\nu)].$$

Applying the left Hahn integral over I and using (1.1)

$$\begin{aligned}
\int_\nu^\ell \ln\mathcal{Z}(\varsigma)\, {}_\nu d_{q,w}\varsigma + \int_\nu^\ell \ln\mathcal{Y}(\varsigma)\, {}_\nu d_{q,w}\varsigma &\leq (\ell-\nu)\ln\mathcal{Z}(\nu) + (\ln\mathcal{Z}(\ell) - \ln\mathcal{Z}(\nu))\left(\frac{1}{\ell-\nu}\int_\nu^\ell \varsigma\, {}_\nu d_{q,w}\varsigma - \nu\right)\\
&+ (\ell-\nu)\ln\mathcal{Y}(\nu) + \frac{\ln\mathcal{Y}(\ell) - \ln\mathcal{Y}(\nu)}{\ell-\nu}\left(\int_\nu^\ell \varsigma\, {}_\nu d_{q,w}\varsigma - \nu(\ell-\nu)\right)\\
&= (\ell-\nu)\ln(\mathcal{Z}(\nu)) + (\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu)))\left(\frac{\ell+\nu q-w}{q+1} - \nu\right)\\
&+ (\ell-\nu)\ln(\mathcal{Y}(\nu)) + (\ln(\mathcal{Y}(\ell)) - \ln(\mathcal{Y}(\nu)))\left(\frac{\ell+\nu q-w}{q+1} - \nu\right)\\
&= (\ell-\nu)\ln(\mathcal{Z}(\nu)) + (\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu)))\left(\frac{\ell-(w+\nu)}{q+1}\right)\\
&+ (\ell-\nu)\ln(\mathcal{Y}(\nu)) + (\ln(\mathcal{Y}(\ell)) - \ln(\mathcal{Y}(\nu)))\left(\frac{\ell-(w+\nu)}{q+1}\right)\\
&= (\ell-\nu)\ln(\mathcal{Z}(\nu)) + (\ell-\nu)(\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu)))\\
&\times \frac{1}{q+1}\left(1 - \frac{w}{\ell-\nu}\right) + (\ell-\nu)\ln(\mathcal{Y}(\nu))\\
&+ \frac{(\ell-\nu)(\ln(\mathcal{Y}(\ell)) - \ln(\mathcal{Y}(\nu)))}{q+1}\left(1 - \frac{w}{\ell-\nu}\right)\\
&= (\ell-\nu)\left[\ln(\mathcal{Z}(\nu)) + \left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell)) - \frac{\ln(\mathcal{Z}(\nu))}{q+1}\right.\\
&\left.+\frac{w\ln(\mathcal{Z}(\nu))}{(q+1)(\ell-\nu)}\right] + (\ell-\nu)\left[\ln(\mathcal{Y}(\nu)) + \left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Y}(\ell))\right.\\
&\left.-\frac{\ln(\mathcal{Y}(\nu))}{q+1} + \frac{w\ln(\mathcal{Y}(\nu))}{(q+1)(\ell-\nu)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= (\ell - \nu)\left[\frac{q\ln(\mathcal{Z}(\nu))}{q+1} + \frac{w\ln(\mathcal{Z}(\nu))}{(q+1)(\ell-\nu)} + \left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))\right] \\
&+ (\ell - \nu)\left[\frac{q\ln(\mathcal{Y}(\nu))}{q+1} + \frac{w\ln(\mathcal{Y}(\nu))}{(q+1)(\ell-\nu)} + \left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Y}(\ell))\right] \\
&= (\ell - \nu)\left[\left(\frac{(\ell-\nu)q+w}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\nu)) + \left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))\right] \\
&+ (\ell - \nu)\left[\left(\frac{w+(\ell-\nu)q}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Y}(\nu)) + \left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Y}(\ell))\right] \\
&= (\ell-\nu)\ln\left[\{\mathcal{Z}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Z}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}} \\
&+ (\ell-\nu)\ln\left[\{\mathcal{Y}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Y}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\ell-\nu}\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\, {}_\nu d_{q,w}\varsigma + \int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\, {}_\nu d_{q,w}\varsigma\right) &\le \ln\left[\{\mathcal{Z}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Z}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}} \\
&+ \ln\left[\{\mathcal{Y}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Y}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}} \\
\frac{1}{\ell-\nu}\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\, {}_\nu d_{q,w}\varsigma + \int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\, {}_\nu d_{q,w}\varsigma\right) &\le \ln\left\{\left[\{\mathcal{Z}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Z}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\right. \\
&\left.\times\left[\{\mathcal{Y}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Y}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\right\}.
\end{aligned}
$$

Taking the exponential, we get

$$
\left(\int_\nu^\ell (\mathcal{Z}(\varsigma))^{{}_\nu d_{q,w}\varsigma} \cdot \int_\nu^\ell (\mathcal{Y}(\varsigma))^{{}_\nu d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \le \left[(\mathcal{Z}(\nu)\cdot\mathcal{Y}(\nu))^{w+(\ell-\nu)q}\,(\mathcal{Z}(\ell)\cdot\mathcal{Y}(\ell))^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}. \tag{2.12}
$$

Combining (2.9) and (2.12), then the (2.6) has been established. □

Theorem 6. *If $\mathcal{Z}$ and $\mathcal{Y}$ are two positive and log-convex differentiable functions on $[\nu, \ell]$, then*

$$
\begin{aligned}
\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right) : \mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right) &\le \left(\int_\nu^\ell (\mathcal{Z}(\varsigma))^{{}_\nu d_{q,w}\varsigma} : \int_\nu^\ell (\mathcal{Y}(\varsigma))^{{}_\nu d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \\
&\le \left[(\mathcal{Z}(\nu) : \mathcal{Y}(\nu))^{w+(\ell-\nu)q}\cdot(\mathcal{Z}(\ell) : \mathcal{Y}(\ell))^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}
\end{aligned} \tag{2.13}
$$

holds.

Proof. Dividing (2.7) from (2.8) and then applying ln, we have

$$
\begin{aligned}
\ln \mathsf{p}_1(\varsigma) - \ln \mathsf{p}_2(\varsigma) &= \ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) + \left(\varsigma - \frac{\ell+\nu q-w}{q+1}\right)\ln\left(\mathcal{Z}^*\left(\frac{\ell+\nu q-w}{q+1}\right)\right) \\
&- \ln\left(\mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) + \left(\varsigma - \frac{\ell+\nu q-w}{q+1}\right)\ln\left(\mathcal{Y}^*\left(\frac{\ell+\nu q-w}{q+1}\right)\right) \\
&\le \ln(\mathcal{Z}(\varsigma)) - \ln(\mathcal{Y}(\varsigma)).
\end{aligned}
$$

Applying left Hahn integral over I and using (1.1)

$$\begin{aligned}
&\int_{\nu}^{\ell} \ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) {}_{\nu}d_{q,w}\varsigma + \ln\left(\mathcal{Z}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\int_{\nu}^{\ell}\left(\varsigma-\frac{\ell+\nu q-w}{q+1}\right) {}_{\nu}d_{q,w}\varsigma \\
&-\int_{\nu}^{\ell} \ln\left(\mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) {}_{\nu}d_{q,w}\varsigma + \ln\left(\mathcal{Y}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)\int_{\nu}^{\ell}\left(\varsigma-\frac{\ell+\nu q-w}{q+1}\right) {}_{\nu}d_{q,w}\varsigma \\
&\leq \int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma - \int_{\nu}^{\ell} \ln(\mathcal{Y}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma \\
&= (\ell-\nu)\ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) + \ln\left(\mathcal{Z}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)(0) \\
&-(\ell-\nu)\ln\left(\mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) + \ln\left(\mathcal{Y}^{*}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)(0) \\
&\leq \int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma - \int_{\nu}^{\ell} \ln(\mathcal{Y}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma \\
&= \ln\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right) : \mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right)\right) \\
&\leq \frac{1}{\ell-\nu}\left(\int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma - \int_{\nu}^{\ell} \ln(\mathcal{Y}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma\right).
\end{aligned}$$

Applying an exponential function

$$\begin{aligned}
\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right) : \mathcal{Y}\left(\frac{\ell+\nu q-w}{q+1}\right) &\leq exp\left(\frac{1}{\ell-\nu}\left(\int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma - \int_{\nu}^{\ell} \ln(\mathcal{Y}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma\right)\right) \\
&= \left(exp\left(\int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma - \int_{\nu}^{\ell} \ln(\mathcal{Y}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma\right)\right)^{\frac{1}{\ell-\nu}} \\
&= \left(exp\left(\int_{\nu}^{\ell} \ln(\mathcal{Z}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma\right) : exp\left(\int_{\nu}^{\ell} \ln(\mathcal{Y}(\varsigma)) \, {}_{\nu}d_{q,w}\varsigma\right)\right)^{\frac{1}{\ell-\nu}} \\
&= \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma} : \int_{\nu}^{\ell} (\mathcal{Y}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}}. \qquad (2.14)
\end{aligned}$$

Likewise, dividing (2.10) and (2.11) then applying ln, then

$$\ln\mathcal{Z}(\varsigma) - \ln\mathcal{Y}(\varsigma) \leq \ln\mathcal{Z}(\nu) + \frac{\varsigma-\nu}{\ell-\nu}[\ln\mathcal{Z}(\ell) - \ln\mathcal{Z}(\nu)] - \ln\mathcal{Y}(\nu) - \frac{\varsigma-\nu}{\ell-\nu}[\ln\mathcal{Y}(\ell) - \ln\mathcal{Y}(\nu)].$$

Applying the left Hahn integral over I and using (1.1)

$$\begin{aligned}
\int_{\nu}^{\ell} \ln\mathcal{Z}(\varsigma) \, {}_{\nu}d_{q,w}\varsigma - \int_{\nu}^{\ell} \ln\mathcal{Y}(\varsigma) \, {}_{\nu}d_{q,w}\varsigma &\leq (\ell-\nu)\ln\mathcal{Z}(\nu) + \frac{\ln\mathcal{Z}(\ell)-\ln\mathcal{Z}(\nu)}{\ell-\nu}\left(\int_{\nu}^{\ell} \varsigma \, {}_{\nu}d_{q,w}\varsigma - \nu(\ell-\nu)\right) \\
&- (\ell-\nu)\ln\mathcal{Y}(\nu) - \frac{\ln\mathcal{Y}(\ell)-\ln\mathcal{Y}(\nu)}{\ell-\nu}\left(\int_{\nu}^{\ell} \varsigma \, {}_{\nu}d_{q,w}\varsigma - \nu(\ell-\nu)\right) \\
&= (\ell-\nu)\ln(\mathcal{Z}(\nu)) + (\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu)))\left(\frac{\ell+\nu q-w}{q+1} - \nu\right)
\end{aligned}$$

$$
\begin{aligned}
&-(\ell-\nu)\ln(\mathcal{Y}(\nu))-(\ln(\mathcal{Y}(\ell))-\ln(\mathcal{Y}(\nu)))\left(\frac{\ell+\nu q-w}{q+1}-\nu\right)\\
&=(\ell-\nu)\ln(\mathcal{Z}(\nu))+(\ln(\mathcal{Z}(\ell))-\ln(\mathcal{Z}(\nu)))\left(\frac{\ell-(w+\nu)}{q+1}\right)\\
&-(\ell-\nu)\ln(\mathcal{Y}(\nu))-(\ln(\mathcal{Y}(\ell))-\ln(\mathcal{Y}(\nu)))\left(\frac{\ell-\nu-w}{q+1}\right)\\
&=(\ell-\nu)\ln(\mathcal{Z}(\nu))+(\ell-\nu)\left(\ln(\mathcal{Z}(\ell))-\ln(\mathcal{Z}(\nu))\right)\\
&\times\left(\frac{1}{q+1}-\frac{w}{(q+1)(\ell-\nu)}\right)-(\ell-\nu)\ln(\mathcal{Y}(\nu))\\
&-(\ell-\nu)\left(\ln(\mathcal{Y}(\ell))-\ln(\mathcal{Y}(\nu))\right)\left(\frac{1}{q+1}-\frac{w}{(q+1)(\ell-\nu)}\right)\\
&=(\ell-\nu)\left[\ln(\mathcal{Z}(\nu))+\left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))-\frac{\ln(\mathcal{Z}(\nu))}{q+1}\right.\\
&\left.+\frac{w\ln(\mathcal{Z}(\nu))}{(q+1)(\ell-\nu)}\right]-(\ell-\nu)\left[\ln(\mathcal{Y}(\nu))+\left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Y}(\ell))\right.\\
&\left.-\frac{\ln(\mathcal{Y}(\nu))}{q+1}+\frac{w\ln(\mathcal{Y}(\nu))}{(q+1)(\ell-\nu)}\right]\\
&=(\ell-\nu)\left[\frac{q\ln(\mathcal{Z}(\nu))}{q+1}+\frac{w\ln(\mathcal{Z}(\nu))}{(q+1)(\ell-\nu)}+\left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))\right]\\
&-(\ell-\nu)\left[\frac{q\ln(\mathcal{Y}(\nu))}{q+1}+\frac{w\ln(\mathcal{Y}(\nu))}{(q+1)(\ell-\nu)}+\left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Y}(\ell))\right]\\
&=(\ell-\nu)\left[\left(\frac{w+(\ell-\nu)q}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\nu))+\left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))\right]\\
&-(\ell-\nu)\left[\left(\frac{w+(\ell-\nu)q}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Y}(\nu))+\left(\frac{\ell-(w+\nu)}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Y}(\ell))\right]\\
&=(\ell-\nu)\ln\left[\{\mathcal{Z}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Z}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\\
&-(\ell-\nu)\ln\left[\{\mathcal{Y}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Y}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\ell-\nu}\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\,{}_\nu d_{q,w}\varsigma-\int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\,{}_\nu d_{q,w}\varsigma\right)&\le \ln\left[\{\mathcal{Z}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Z}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\\
&-\ln\left[\{\mathcal{Y}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Y}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\\
\frac{1}{\ell-\nu}\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\,{}_\nu d_{q,w}\varsigma-\int_\nu^\ell \ln(\mathcal{Y}(\varsigma))\,{}_\nu d_{q,w}\varsigma\right)&\le \ln\left\{\left[\{\mathcal{Z}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Z}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\right.\\
&\left.:\ \left[\{\mathcal{Y}(\nu)\}^{w+(\ell-\nu)q}\{\mathcal{Y}(\ell)\}^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\right\}.
\end{aligned}
$$

Taking the exponential, we get

$$
\left(\int_\nu^\ell (\mathcal{Z}(\varsigma))^{{}_\nu d_{q,w}\varsigma}:\ \int_\nu^\ell (\mathcal{Y}(\varsigma))^{{}_\nu d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}}\le\left[(\mathcal{Z}(\nu):\mathcal{Y}(\nu))^{w+(\ell-\nu)q}\cdot(\mathcal{Z}(\ell):\mathcal{Y}(\ell))^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}. \qquad (2.15)
$$

Combining (2.14) and (2.15), then the (2.13) has been accomplished. □

3. New notions in Hahn calculus

In this section, we will define the new definitions for derivative and definite integral at a point ℓ called the right Hahn derivative and definite integral. For this, we define a new number $\mathring{w} = \ell - \dfrac{w}{1-q} \in I$. The definition of the right Hahn derivative can be defined as.

Definition 11. *For any function $\mathcal{Z}$, then the right Hahn derivative of $\mathcal{Z}$ is defined as*

$$ {}^{\ell}\mathfrak{D}_{(q,w)}\mathcal{Z}(\varsigma) = \begin{cases} \dfrac{\mathcal{Z}(q\varsigma + (1-q)\ell - w) - \mathcal{Z}(\varsigma)}{(1-q)(\ell-\varsigma) - w} & \text{if } \varsigma \neq \mathring{w}, \\ \mathcal{Z}'(\mathring{w}) & \text{if } \varsigma = \mathring{w}, \end{cases} $$

furnished that $\mathcal{Z}$ is differentiable at $\mathring{w}$.

The basic properties of the right Hahn derivative can be proved easily.

Theorem 7. *For any $\mathcal{Z}$ and $\mathcal{Y}$ be two right Hahn differentiable functions on $[\nu, \ell]$, then the following results exist:*

(1) ${}^{\ell}\mathfrak{D}_{(q,w)}(\mathcal{Z}(\varsigma) \pm \mathcal{Y}(\varsigma)) = {}^{\ell}\mathfrak{D}_{(q,w)}\mathcal{Z}(\varsigma) \pm {}^{\ell}\mathfrak{D}_{(q,w)}\mathcal{Y}(\varsigma)$;

(2) *For any constant ν_1,*
${}^{\ell}\mathfrak{D}_{(q,w)}(\nu_1\mathcal{Z}(\varsigma)) = \nu_1 {}^{\ell}\mathfrak{D}_{(q,w)}\mathcal{Z}(\varsigma)$;

(3) ${}^{\ell}\mathfrak{D}_{(q,w)}(\mathcal{Z}(\varsigma) \cdot \mathcal{Y}(\varsigma)) = \mathcal{Z}(q\varsigma + (1-q)\ell - w)\, {}^{\ell}\mathfrak{D}_{q,w)}\mathcal{Y}(\varsigma) + \mathcal{Y}(\varsigma)\, {}^{\ell}\mathfrak{D}_{(q,w)}\mathcal{Z}(\varsigma)$;

(4) *For any $\mathcal{Y}(\varsigma)\mathcal{Y}(q\varsigma + (1-q)\ell - w) \neq 0$,*
$${}^{\ell}\mathfrak{D}_{(q,w)}\left(\frac{\mathcal{Z}(\varsigma)}{\mathcal{Y}(\varsigma)}\right) = \frac{\mathcal{Y}(\varsigma)\, {}^{\ell}\mathfrak{D}_{q,w)}\mathcal{Z}(\varsigma) - \mathcal{Z}(\varsigma)\, {}^{\ell}\mathfrak{D}_{(q,w)}\mathcal{Y}(\varsigma)}{\mathcal{Y}(\varsigma)\mathcal{Y}(q\varsigma + (1-q)\ell - w)}.$$

Proof. Using Definition 11 of the right Hahn derivative, all of these can be proved. □

The right Hahn definite integral of a function can be defined as.

Definition 12. *For any function $\mathcal{Z}$ and $\nu_1, \ell_1 \in I$, then the right Hahn definite integral of $\mathcal{Z}$ is defined as*

$$\int_{\nu_1}^{\ell_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma = \int_{\mathring{w}}^{\ell_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma - \int_{\mathring{w}}^{\nu_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma,$$

where

$$\int_{\mathring{w}}^{\mathsf{u}} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma = ((1-q)(\mathsf{u} - \ell) + w) \sum_{\mathsf{t}=0}^{\infty} q^{\mathsf{t}} \mathcal{Z}(\mathsf{u}q^{\mathsf{t}} + (1-q^{\mathsf{t}})\mathring{w})$$

given the fact that $\mathsf{u} = \nu_1$ and $\mathsf{u} = \ell_1$ are the points at which the series converges. If the function $\mathcal{Z}$ is the right Hahn integrable from ν_1 to ℓ_1, for every ν_1, $\ell_1 \in I$, then it is the right Hahn integrable over I.

The fundamental characteristics of the right Hahn integral can be proved easily. So, their proofs are omitted.

Theorem 8. *For any $\mathcal{Z}$ and $\mathcal{Y}$ to be two right Hahn differentiable functions on $[\nu, \ell]$, then the following results exist for $\nu_1, \ell_1 \in I$:*

(1) $\int_{\nu_1}^{\ell_1} (\mathcal{Z}(\varsigma) \pm \mathcal{Y}(\varsigma))\, {}^{\ell}d_{q,w}\varsigma = \int_{\nu_1}^{\ell_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma \pm \int_{\nu_1}^{\ell_1} \mathcal{Y}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma$;

(2) For any constant ν_2,
$\int_{\nu_1}^{\ell_1} \nu_2\, \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma = \nu_2 \int_{\nu_1}^{\ell_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma$;

(3) $\int_{\nu_1}^{\nu_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma = 0$;

(4) $\int_{\nu_1}^{\ell_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma = -\int_{\ell_1}^{\nu_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma$;

(5) For any constant ℓ_2 such that $\nu_1 \le \ell_2 \le \ell_1$,
$\int_{\nu_1}^{\ell_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma = \int_{\nu_1}^{\ell_2} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma + \int_{\ell_2}^{\ell_1} \mathcal{Z}(\varsigma)\, {}^{\ell}d_{q,w}\varsigma$.

Proof. Using Definition 12 of the right Hahn integral, all of these can be proved. □

Lemma 1. *For any two real numbers ν_3, ℓ_3 such that $\ell_3 \in \mathfrak{R} \setminus \{-1\}$, then*

$$\int_{\mathring{w}}^{\mathsf{u}} (\varsigma - \mathring{w})^{\ell_3}\, {}^{\ell}d_{q,w}\varsigma = \frac{(1-q)(\mathsf{u} - \mathring{w})^{\ell_3+1}}{1 - q^{\ell_3+1}} \tag{3.1}$$

equality holds.

Proof. Using the definition of right Hahn definite integral

$$\begin{aligned}
\int_{\mathring{w}}^{\mathsf{u}} (\varsigma - \mathring{w})^{\ell_3}\, {}^{\ell}d_{q,w}\varsigma &= ((1-q)(\mathsf{u} - \ell) + w) \sum_{\mathsf{t}=0}^{\infty} q^{\mathsf{t}} (\mathsf{u}q^{\mathsf{t}} + (1-q^{\mathsf{t}})\mathring{w} - \mathring{w})^{\ell_3} \\
&= ((1-q)(\mathsf{u} - \ell) + w) \sum_{\mathsf{t}=0}^{\infty} q^{\mathsf{t}} (q^{\mathsf{t}}(\mathsf{u} - \mathring{w}))^{\ell_3} \\
&= (1-q)(\mathsf{u} - \mathring{w}) \sum_{\mathsf{t}=0}^{\infty} q^{(\ell_3+1)\mathsf{t}} (\mathsf{u} - \mathring{w})^{\ell_3} \\
&= (1-q)(\mathsf{u} - \mathring{w})^{\ell_3+1} \sum_{\mathsf{t}=0}^{\infty} q^{(\ell_3+1)\mathsf{t}} \\
&= \frac{(1-q)(\mathsf{u} - \mathring{w})^{\ell_3+1}}{1 - q^{\ell_3+1}}.
\end{aligned}$$

□

Corollary 1. *For any $\nu_1, \ell_1 \in I$, then the* (3.1) *becomes*

$$\int_{\nu_1}^{\ell_1} (\varsigma - \mathring{w})^{\ell_3}\, {}^{\ell}d_{q,w}\varsigma = \frac{(1-q)\{(\ell_1 - \mathring{w})^{\ell_3+1} - (\nu_1 - \mathring{w})^{\ell_3+1}\}}{1 - q^{\ell_3+1}}. \tag{3.2}$$

(1) Put $\ell_3 = 0$ *in* (3.2)*, then*

$$\int_{\nu_1}^{\ell_1} (\varsigma - \mathring{w})^0 \, {}^{\ell}d_{q,w}\varsigma = \ell_1 - \nu_1.$$

(2) Put $\ell_3 = 1$ *in* (3.2)*, then*

$$\int_{\nu_1}^{\ell_1} (\varsigma - \mathring{w}) \, {}^{\ell}d_{q,w}\varsigma = \frac{(\ell_1 - \nu_1)(\nu_1 + \ell_1 - 2\mathring{w})}{q+1}. \tag{3.3}$$

(3) Using (3.3)*,*

$$\begin{aligned}
\int_{\nu_1}^{\ell} (\varsigma - \ell) \, {}^{\ell}d_{q,w}\varsigma &= \int_{\nu_1}^{\ell} (\varsigma - \mathring{w}) \, {}^{\ell}d_{q,w}\varsigma + \int_{\nu_1}^{\ell} (\mathring{w} - \ell) \, {}^{\ell}d_{q,w}\varsigma \\
&= \frac{(\ell - \nu_1)(\nu_1 + \ell - 2\mathring{w})}{q+1} + (\mathring{w} - \ell)(\ell - \nu_1) \\
&= \frac{\ell - \nu_1}{q+1}[\nu_1 + \ell - 2\mathring{w} + \mathring{w} - \ell + q\mathring{w} - q\ell] \\
&= \frac{\ell - \nu_1}{q+1}[\ell(1-q) + \nu_1 - \ell - (1-q)\mathring{w}] \\
&= -\frac{\ell - \nu_1}{q+1}[\ell - \nu_1 + (1-q)(\mathring{w} - \ell)] \\
&= \frac{-1}{q+1}[(\ell - \nu_1)^2 - (\ell - \nu_1)w].
\end{aligned} \tag{3.4}$$

(4) Finally, using (3.4)*, we have*

$$\begin{aligned}
\int_{\nu}^{\ell} \varsigma \, {}^{\ell}d_{q,w}\varsigma &= \int_{\nu}^{\ell} (\varsigma - \ell) \, {}^{\ell}d_{q,w}\varsigma + \int_{\nu}^{\ell} \ell \, {}^{\ell}d_{q,w}\varsigma \\
&= \frac{-1}{q+1}[(\ell - \nu)^2 - (\ell - \nu)w] + \ell(\ell - \nu) \\
&= \frac{-(\ell - \nu)}{q+1}[\ell - \nu - w - \ell q - \ell] \\
&= \frac{(\ell - \nu)(\nu + \ell q + w)}{q+1}.
\end{aligned} \tag{3.5}$$

The Eq (3.5) is crucial to derive the new Hermite-Hadamard inequalities in the right Hahn multiplicative calculus.

4. Right Hahn multiplicative calculus with respect to the point ℓ and related Hermite-Hadamard type inequalities

Motivated by Definitions 9–12, we introduce the new definitions called the right Hahn multiplicative derivative and integral, which are denoted by ${}^{\ell}\mathfrak{D}^{*}_{(q,w)}$ and ${}^{\ell}(q, w)^{*}$-integral respectively.

Definition 13. *For any positive function $\mathcal{Z}$ with $\mathcal{Z}(\varsigma) \neq 0$, then the right Hahn multiplicative derivative or right Hahn *derivative of function $\mathcal{Z}$ at $\varsigma \in \Re$ is written as:*

$$^{\ell}\mathfrak{D}^{*}_{(q,w)}\mathcal{Z}(\varsigma) = \left(\frac{\mathcal{Z}(q\varsigma + (1-q)\ell - w)}{\mathcal{Z}(\varsigma)}\right)^{\frac{1}{(1-q)(\ell-\varsigma)-w}}.$$

Additionally,

$$^{\ell}\mathfrak{D}^{*}_{(q,w)}\mathcal{Z}(\varsigma) = exp\left(^{\ell}\mathfrak{D}_{(q,w)}(\ln \mathcal{Z}(\varsigma))\right)$$

holds for $w = \ell = 0$ and $\mathcal{Z}(\varsigma)$ is monomial, here $^{\ell}\mathfrak{D}_{(q,w)}$ is a right Hahn derivative which is defined in Definition11.

Definition 14. *For any positive function $\mathcal{Z}$, then the right Hahn *integral ($^{\ell}(q,w)^*$-integral) or right Hahn multiplicative integral of $\mathcal{Z}$ from ν to ℓ can be defined as:*

$$\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}^{\ell}d_{q,w}\varsigma} = exp\left(\int_{\nu}^{\ell} (\ln \mathcal{Z}(\varsigma))\ {}^{\ell}d_{q,w}\varsigma\right) = exp\left(\int_{\nu}^{\mathring{w}} (\ln \mathcal{Z}(\varsigma))\ {}^{\ell}d_{q,w}\varsigma - \int_{\mathring{w}}^{\ell} (\ln \mathcal{Z}(\varsigma))\ {}^{\ell}d_{q,w}\varsigma\right)$$

for $\varsigma \in [\nu, \ell]$.

Some basic results for the right Hahn multiplicative integral are following as below.

Theorem 9. *For any positive and right Hahn multiplicative integrable functions $\mathcal{Z}$ and $\mathcal{Y}$ that are defined on I with $\nu \leq \mathsf{c} \leq \ell$ and $\mathsf{n} \in \Re$, then we have*

(1) $\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma)^{\mathsf{n}})^{\ {}^{\ell}d_{q,w}\varsigma} = \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{\ {}^{\ell}d_{q,w}\varsigma}\right)^{\mathsf{n}}$.

(2) $\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}^{\ell}d_{q,w}\varsigma} = \int_{\nu}^{\mathsf{c}} \mathcal{Z}(\varsigma)^{\ {}^{\ell}d_{q,w}\varsigma} \cdot \int_{\mathsf{c}}^{\ell} \mathcal{Z}(\varsigma)^{\ {}^{\ell}d_{q,w}\varsigma}$.

(3) $\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma) \cdot \mathcal{Y}(\varsigma))^{\ {}^{\ell}d_{q,w}\varsigma} = \int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}^{\ell}d_{q,w}\varsigma} \cdot \int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ {}^{\ell}d_{q,w}\varsigma}$.

(4) $\int_{\nu}^{\ell} \left(\frac{\mathcal{Z}(\varsigma)}{\mathcal{Y}(\varsigma)}\right)^{{}^{\ell}d_{q,w}\varsigma} = \frac{\int_{\nu}^{\ell} \mathcal{Z}(\varsigma)^{\ {}^{\ell}d_{q,w}\varsigma}}{\int_{\nu}^{\ell} \mathcal{Y}(\varsigma)^{\ {}^{\ell}d_{q,w}\varsigma}}$.

Proof. These can be proved to be the same as Theorem 2.

□

Inspired by (2.2) and Definition 14, we construct the Hermite-Hadamard inequality and related results in the right Hahn multiplicative calculus.

Theorem 10. *If $\mathcal{Z}$ is a positive and log-convex differentiable function on $[\nu, \ell]$, then*

$$\mathcal{Z}\left(\frac{w + \nu + q\ell}{q+1}\right) \leq \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{{}^{\ell}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \leq \left[(\mathcal{Z}(\nu))^{\ell-(w+\nu)} \cdot (\mathcal{Z}(\ell))^{q(\ell-\nu)+w}\right]^{\frac{1}{(q+1)(\ell-\nu)}} \tag{4.1}$$

holds.

Proof. As we can write that the supporting line of the function $\mathcal{Z}$ at the point $\frac{w+\nu+q\ell}{q+1} \in (\nu, \ell)$ is

$$\mathsf{P}_1(\varsigma) = \mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\left(\mathcal{Z}^*\left(\frac{w+\nu+q\ell}{q+1}\right)\right)^{\left(\varsigma-\frac{w+\nu+q\ell}{q+1}\right)}.$$

By the log-convexity of $\mathcal{Z}$ and for $\nu \leq \varsigma \leq \ell$, it can also be written as

$$\mathsf{P}_1(\varsigma) = \mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\left(\mathcal{Z}^*\left(\frac{w+\nu+q\ell}{q+1}\right)\right)^{\left(\varsigma-\frac{w+\nu+q\ell}{q+1}\right)} \leq \mathcal{Z}(\varsigma)$$

$$\ln \mathsf{P}_1(\varsigma) = \ln\left(\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\right) + \left(\varsigma - \frac{w+\nu+q\ell}{q+1}\right)\ln\left(\mathcal{Z}^*\left(\frac{w+\nu+q\ell}{q+1}\right)\right) \leq \ln(\mathcal{Z}(\varsigma)).$$

Using the right Hahn integral from ν to ℓ and using (3.5),

$$\begin{aligned}
\int_\nu^\ell \ln \mathsf{P}_1(\varsigma)\ {}^\ell d_{q,w}\varsigma &= (\ell-\nu)\ln\left(\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\right) + \ln\left(\mathcal{Z}^*\left(\frac{w+\nu+q\ell}{q+1}\right)\right) \\
&\times\left(\int_\nu^\ell \varsigma\ {}^\ell d_{q,w}\varsigma - (\ell-\nu)\frac{w+\nu+q\ell}{q+1}\right) \leq \int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\ {}^\ell d_{q,w}\varsigma \\
&= (\ell-\nu)\ln\left(\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\right) + \ln\left(\mathcal{Z}^*\left(\frac{w+\nu+q\ell}{q+1}\right)\right) \\
&\times\left((\ell-\nu)\frac{w+\nu+q\ell}{q+1} - (\ell-\nu)\frac{w+\nu+q\ell}{q+1}\right) \leq \int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\ {}^\ell d_{q,w}\varsigma \\
&= (\ell-\nu)\ln\left(\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\right) \leq \int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\ {}^\ell d_{q,w}\varsigma \\
&= \ln\left(\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\right) \leq \int_\nu^\ell \ln(\mathcal{Z}(\varsigma))^{\frac{1}{\ell-\nu}}\ {}^\ell d_{q,w}\varsigma.
\end{aligned}$$

Taking exponential,

$$\begin{aligned}
\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right) &\leq exp\left(\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))^{\frac{1}{\ell-\nu}}\ {}^\ell d_{q,w}\varsigma\right) \\
\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right) &\leq \left(\int_\nu^\ell (\mathcal{Z}(\varsigma))^{{}^\ell d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}}.
\end{aligned} \tag{4.2}$$

Similarly, the equation of the line segment joining the points $(\nu, \mathcal{Z}(\nu))$ and $(\ell, \mathcal{Z}(\ell))$ can be written as a function is

$$\mathsf{p}(\varsigma) = \mathcal{Z}(\nu)\left[\frac{\mathcal{Z}(\ell)}{\mathcal{Z}(\nu)}\right]^{\left(\frac{\varsigma-\nu}{\ell-\nu}\right)}.$$

Since $\mathcal{Z}$ is log-convex, therefore $\forall\ \varsigma \in [\nu, \ell]$ we have

$$\mathcal{Z}(\varsigma) \leq \mathsf{p}(\varsigma) = \mathcal{Z}(\nu)\left[\frac{\mathcal{Z}(\ell)}{\mathcal{Z}(\nu)}\right]^{\left(\frac{\varsigma-\nu}{\ell-\nu}\right)}.$$

Also

$$\ln(\mathcal{Z}(\varsigma)) \leq \ln(\mathsf{p}(\varsigma)) = \ln(\mathcal{Z}(\nu)) + \left(\frac{\varsigma-\nu}{\ell-\nu}\right)[\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu))].$$

Taking the right Hahn integral from ν to ℓ and using (3.5),

$$\begin{aligned}\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\, {}^\ell d_{q,w}\varsigma \leq \int_\nu^\ell \ln(\mathsf{p}(\varsigma))\, {}^\ell d_{q,w}\varsigma &= (\ell-\nu)\ln(\mathcal{Z}(\nu)) + \frac{\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu))}{\ell-\nu} \\ &\quad\times\left(\int_\nu^\ell \varsigma\, {}^\ell d_{q,w}\varsigma - (\ell-\nu)\nu\right) \\ &\leq (\ell-\nu)\ln(\mathcal{Z}(\nu)) + (\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu)))\left(\frac{w+\nu+q\ell}{q+1} - \nu\right) \\ &= (\ell-\nu)\ln(\mathcal{Z}(\nu)) + (\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu)))\left(\frac{(\ell-\nu)q+w}{q+1}\right) \\ &= (\ell-\nu)\ln(\mathcal{Z}(\nu)) + (\ell-\nu)(\ln(\mathcal{Z}(\ell)) - \ln(\mathcal{Z}(\nu)))\left(\frac{q}{q+1} + \frac{w}{(q+1)(\ell-\nu)}\right) \\ &= (\ell-\nu)\left[\ln(\mathcal{Z}(\nu)) + \left(\frac{(\ell-\nu)q+w}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell)) - \frac{q\ln(\mathcal{Z}(\nu))}{q+1} - \frac{w\ln(\mathcal{Z}(\nu))}{(q+1)(\ell-\nu)}\right] \\ &= (\ell-\nu)\left[\frac{\ln(\mathcal{Z}(\nu))}{q+1} - \frac{w\ln(\mathcal{Z}(\nu))}{(q+1)(\ell-\nu)} + \left(\frac{(\ell-\nu)q+w}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))\right] \\ &= (\ell-\nu)\left[\left(\frac{\ell-\nu-w}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\nu)) + \left(\frac{q(\ell-\nu)+w}{(q+1)(\ell-\nu)}\right)\ln(\mathcal{Z}(\ell))\right] \\ &= (\ell-\nu)\ln\left[\{\mathcal{Z}(\nu)\}^{\ell-\nu-w}\cdot\{\mathcal{Z}(\ell)\}^{q(\ell-\nu)+w}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\end{aligned}$$

$$\frac{1}{\ell-\nu}\int_\nu^\ell \ln(\mathcal{Z}(\varsigma))\, {}_\nu d_{q,w}\varsigma \leq \ln\left[\{\mathcal{Z}(\nu)\}^{\ell-\nu-w}\cdot\{\mathcal{Z}(\ell)\}^{q(\ell-\nu)+w}\right]^{\frac{1}{(q+1)(\ell-\nu)}}.$$

By using the property of the log function and the first case of Theorem 2, we will get

$$\left(\int_\nu^\ell (\mathcal{Z}(\varsigma))^{{}_\nu d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \leq \left[\{\mathcal{Z}(\nu)\}^{\ell-\nu-w}\cdot\{\mathcal{Z}(\ell)\}^{q(\ell-\nu)+w}\right]^{\frac{1}{(q+1)(\ell-\nu)}}. \tag{4.3}$$

From (4.2) and (4.3), the desired result has become. □

Theorem 11. *If $\mathcal{Z}$ and $\mathcal{Y}$ are two positive and log-convex differentiable functions on $[\nu,\ell]$, then*

$$\begin{aligned}\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\cdot\mathcal{Y}\left(\frac{w+\nu+q\ell}{q+1}\right) &\leq \left(\int_\nu^\ell (\mathcal{Z}(\varsigma))^{{}^\ell d_{q,w}\varsigma}\cdot\int_\nu^\ell (\mathcal{Y}(\varsigma))^{{}^\ell d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \\ &\leq \left[(\mathcal{Z}(\nu)\cdot\mathcal{Y}(\nu))^{\ell-\nu-w}\cdot(\mathcal{Z}(\ell)\cdot\mathcal{Y}(\ell))^{q(\ell-\nu)+w}\right]^{\frac{1}{(q+1)(\ell-\nu)}}\end{aligned} \tag{4.4}$$

holds.

Proof. The proof is similar to Theorem 5. □

Theorem 12. *If $\mathcal{Z}$ and $\mathcal{Y}$ are two positive and log-convex differentiable functions on* $[\nu, \ell]$*, then*

$$\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right) : \mathcal{Y}\left(\frac{w+\nu+q\ell}{q+1}\right) \leq \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{^{\ell}d_{q,w}\varsigma} : \int_{\nu}^{\ell} (\mathcal{Y}(\varsigma))^{^{\ell}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}}$$
$$\leq \left[(\mathcal{Z}(\nu) : \mathcal{Y}(\nu))^{\ell-\nu-w} \cdot (\mathcal{Z}(\ell) : \mathcal{Y}(\ell))^{q(\ell-\nu)+w}\right]^{\frac{1}{(q+1)(\ell-\nu)}} \tag{4.5}$$

holds.

Proof. It can be proved as the same as Theorem 6. □

Remark 1. *If we put $w = 0$ in all the newly obtained results, it will reduce to the results established in* [36].

5. Applications

First, we investigate Theorem 4 through an example and give a graphical representation that supports our result.

Example 3. *If we set a positive function $\mathcal{Z}(\varsigma) = exp(\varsigma)$ in Theorem* 4 and use (2.1), then the following cases can be obtained:

Case 1. Let $\nu = 1$, $\ell = 3$, $q = 0.5$, and $1 < w < 3$, then

$$exp\left(\frac{\ell+\nu q-w}{q+1}\right) \leq \left(\int_{\nu}^{\ell} (exp(\varsigma))^{^{\nu}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \leq \left[(exp(\nu))^{w-(\nu-\ell)q} \cdot (exp(\ell))^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}$$
$$exp\left(\frac{3.5-w}{1.5}\right) \leq \left(exp\left(\frac{7-2w}{1.5}\right)\right)^{\frac{1}{2}} \leq \left[(exp(1))^{1+w} \cdot \left(exp(3)^{2-w}\right)\right]^{\frac{1}{3}}$$
$$exp\left(\frac{7-2w}{3}\right) \leq \left(exp\left(\frac{7-2w}{1.5}\right)\right)^{\frac{1}{2}} \leq [(exp(1+w)) \cdot (exp(3(2-w)))]^{\frac{1}{3}}$$
$$exp\left(\frac{7-2w}{3}\right) \leq \left(exp\left(\frac{7-2w}{1.5}\right)\right)^{\frac{1}{2}} \leq [exp(1+w+6-3w)]^{\frac{1}{3}}$$
$$exp\left(\frac{7-2w}{3}\right) \leq exp\left(\frac{7-2w}{3}\right) \leq exp\left(\frac{7-2w}{3}\right). \tag{5.1}$$

Case 2. Let $\nu = 1$, $\ell = 3$, $w = 1.5$, and $0 < q < 1$, then we have

$$exp\left(\frac{\ell+\nu q-w}{q+1}\right) \leq \left(\int_{\nu}^{\ell} (exp(\varsigma))^{^{\nu}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \leq \left[(exp(\nu))^{w-(\nu-\ell)q} \cdot (exp(\ell))^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}$$
$$exp\left(\frac{2q-1}{2+2q}\right) \leq \left(exp\left(2\frac{2q-1}{2+2q}\right)\right)^{\frac{1}{2}} \leq \left[(exp(1))^{\frac{4q+3}{2}} \cdot (exp(3))^{\frac{1}{2}}\right]^{\frac{1}{2+2q}}$$
$$exp\left(\frac{2q-1}{2+2q}\right) \leq \left(exp\left(2\frac{2q-1}{2+2q}\right)\right)^{\frac{1}{2}} \leq \left[exp\left(\frac{4q+3}{2}\right) \cdot exp\left(\frac{3}{2}\right)\right]^{\frac{1}{2+2q}}$$

$$exp\left(\frac{2q-1}{2+2q}\right) \leq exp\left(\frac{2q-1}{2+2q}\right) \leq \left[exp\left(\frac{4q+6}{2}\right)\right]^{\frac{1}{2+2q}}$$
$$exp\left(\frac{2q-1}{2+2q}\right) \leq exp\left(\frac{2q-1}{2+2q}\right) \leq exp\left(\frac{2q+3}{2+2q}\right). \tag{5.2}$$

The pictorial form of (5.2) can be seen in Figure 1.

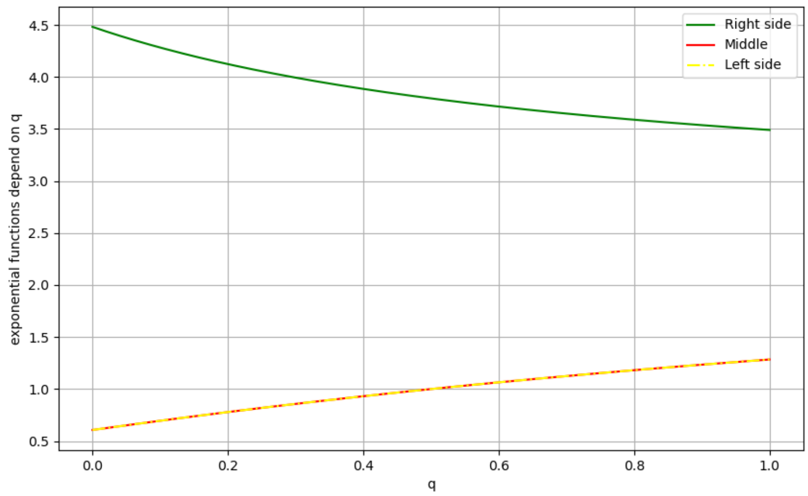

Figure 1. Graphical representation of (5.2) that shows the accuracy of Theorem 4.

Now, we will give an application of the newly obtained inequalities that are used to find the range of Hahn multiplicative integrals of functions that are very difficult to solve. For instance, the left or right Hahn multiplicative integrals of $\mathcal{Z}(\varsigma) = exp(\varsigma^2)$ are difficult to calculate. However, using the Hahn multiplicative Hermite-Hadamard inequalities, we can find the lower and upper bounds of their ranges:

5.1. *The left Hahn multiplicative integral*

From Theorem 4, we have

$$\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right) \leq \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma}\right)^{\frac{1}{\ell-\nu}} \leq \left[(\mathcal{Z}(\nu))^{w+(\ell-\nu)q} \cdot (\mathcal{Z}(\ell))^{\ell-(w+\nu)}\right]^{\frac{1}{(q+1)(\ell-\nu)}}.$$

Taking power $(\ell-\nu)$,

$$\left(\mathcal{Z}\left(\frac{\ell+\nu q-w}{q+1}\right)\right)^{(\ell-\nu)} \leq \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{{}_{\nu}d_{q,w}\varsigma}\right) \leq \left[(\mathcal{Z}(\nu))^{w+(\ell-\nu)q} \cdot (\mathcal{Z}(\ell))^{\ell-(w+\nu)}\right]^{\frac{1}{q+1}}$$
$$\left(exp\left(\left(\frac{\ell+\nu q-w}{q+1}\right)^2\right)\right)^{(\ell-\nu)} \leq \left(\int_{\nu}^{\ell} \left(exp(\varsigma^2)\right)^{{}_{\nu}d_{q,w}\varsigma}\right) \leq \left[\left(exp(\nu^2)\right)^{w+(\ell-\nu)q} \cdot \left(exp(\ell^2)\right)^{\ell-(w+\nu)}\right]^{\frac{1}{q+1}}$$

$$exp\left((\ell-\nu)\left(\frac{\ell+\nu q-w}{q+1}\right)^2\right)\leq\left(\int_\nu^\ell\left(exp(\varsigma^2)\right)^{\nu}d_{q,w}\varsigma\right)\leq\left[exp(\nu^2(w+(\ell-\nu)q))\cdot exp(\ell^2(\ell-(w+\nu)))\right]^{\frac{1}{q+1}}. \tag{5.3}$$

Case 1. Put $\nu=1,\ell=3$ and $q=0.5$ in (5.3), we have

$$exp\left(2\left(\frac{7-2w}{3}\right)^2\right)\leq\left(\int_1^3\left(exp(\varsigma^2)\right)^{1}d_{0.5,w}\varsigma\right)\leq[exp(1+w)\cdot exp(18-9w)]^{\frac{2}{3}}$$
$$exp\left(\frac{98-56w+8w^2}{9}\right)\leq\left(\int_1^3\left(exp(\varsigma^2)\right)^{1}d_{0.5,w}\varsigma\right)\leq exp\left(\frac{38-16w}{3}\right). \tag{5.4}$$

Case 2. Put $\nu=1,\ell=3$ and $w=1.5$ in (5.3), we have

$$exp\left(2\left(\frac{2q+3}{2+2q}\right)^2\right)\leq\left(\int_1^3\left(exp(\varsigma^2)\right)^{1}d_{q,1.5}\varsigma\right)\leq\left[exp\left(\frac{4q+3}{2}\right)\cdot exp\left(\frac{9}{2}\right)\right]^{\frac{1}{q+1}}$$
$$exp\left(\frac{9+12q+4q^2}{2+4q+2q^2}\right)\leq\left(\int_1^3\left(exp(\varsigma^2)\right)^{1}d_{q,1.5}\varsigma\right)\leq exp\left(\frac{2q+6}{q+1}\right). \tag{5.5}$$

Graphic visualizations of the lower and upper bounds of (5.4) and (5.5) can be seen in Figures 2 and 3 respectively.

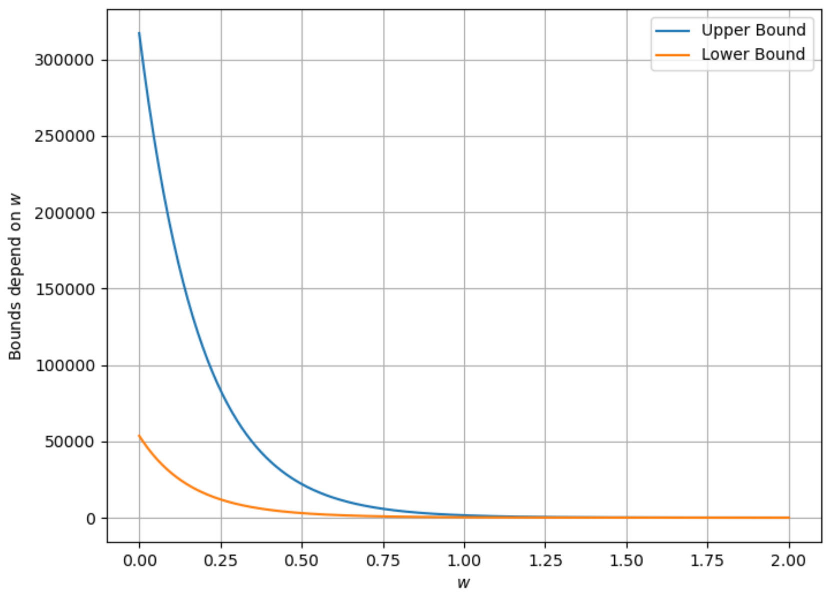

Figure 2. Graphical representation of the lower and upper bounds of (5.4).

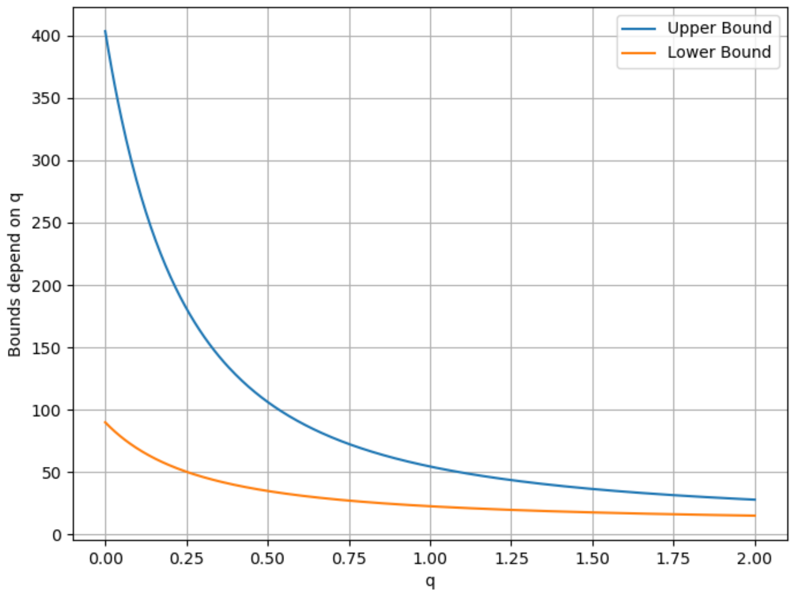

Figure 3. Pictorial form of the lower and upper bounds of (5.5).

5.2. The right Hahn multiplicative integral

From Theorem 10, we have

$$\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right) \leq \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{\ell_{d_{q,w}\varsigma}}\right)^{\frac{1}{\ell-\nu}} \leq \left[(\mathcal{Z}(\nu))^{\ell-\nu-w} \cdot (\mathcal{Z}(\ell))^{q(\ell-\nu)+w}\right]^{\frac{1}{(q+1)(\ell-\nu)}}.$$

Taking power $(\ell-\nu)$,

$$\left(\mathcal{Z}\left(\frac{w+\nu+q\ell}{q+1}\right)\right)^{(\ell-\nu)} \leq \left(\int_{\nu}^{\ell} (\mathcal{Z}(\varsigma))^{\ell_{d_{q,w}\varsigma}}\right) \leq \left[(\mathcal{Z}(\nu))^{\ell-\nu-w} \cdot (\mathcal{Z}(\ell))^{q(\ell-\nu)+w}\right]^{\frac{1}{q+1}}$$

$$\left(exp\left(\left(\frac{w+\nu+q\ell}{q+1}\right)^2\right)\right)^{(\ell-\nu)} \leq \left(\int_{\nu}^{\ell} \left(exp(\varsigma^2)\right)^{\ell_{d_{q,w}\varsigma}}\right) \leq \left[\left(exp(\nu^2)\right)^{\ell-\nu-w} \cdot \left(exp(\ell^2)\right)^{q(\ell-\nu)+w}\right]^{\frac{1}{q+1}}$$

$$exp\left((\ell-\nu)\left(\frac{w+\nu+q\ell}{q+1}\right)^2\right) \leq \left(\int_{\nu}^{\ell} \left(exp(\varsigma^2)\right)^{\ell_{d_{q,w}\varsigma}}\right) \leq \left[exp(\nu^2(\ell-\nu-w)) \cdot exp(\ell^2(q(\ell-\nu)+w))\right]^{\frac{1}{q+1}}. \tag{5.6}$$

Case 1. Put $\nu = 1, \ell = 3$ and $q = 0.5$ in (5.6), we have

$$exp\left(2\left(\frac{5+2w}{3}\right)^2\right) \leq \left(\int_{1}^{3} \left(exp(\varsigma^2)\right)^{3_{d_{0.5,w}\varsigma}}\right) \leq [exp(2-w) \cdot exp(9+9w)]^{\frac{2}{3}}$$

$$exp\left(\frac{50+40w+8w^2}{9}\right) \leq \left(\int_{1}^{3} \left(exp(\varsigma^2)\right)^{3_{d_{0.5,w}\varsigma}}\right) \leq exp\left(\frac{22+16w}{3}\right). \tag{5.7}$$

Case 2. Put $v = 1, \ell = 3$ and $w = 1.5$ in (5.6), we have

$$exp\left(2\left(\frac{6q+5}{2+2q}\right)^2\right) \leq \left(\int_1^3 \left(exp(\varsigma^2)\right)^{3_{d_{q,1.5}\varsigma}}\right) \leq \left[exp\left(\frac{1}{2}\right)\cdot exp\left(\frac{36q+27}{2}\right)\right]^{\frac{1}{q+1}}$$
$$exp\left(\frac{25+60q+36q^2}{2+4q+2q^2}\right) \leq \left(\int_1^3 \left(exp(\varsigma^2)\right)^{3_{d_{q,1.5}\varsigma}}\right) \leq exp\left(\frac{18q+14}{q+1}\right). \tag{5.8}$$

Graphic visualizations of the lower and upper bounds of (5.7) and (5.8) can be seen in Figures 4 and 5 respectively.

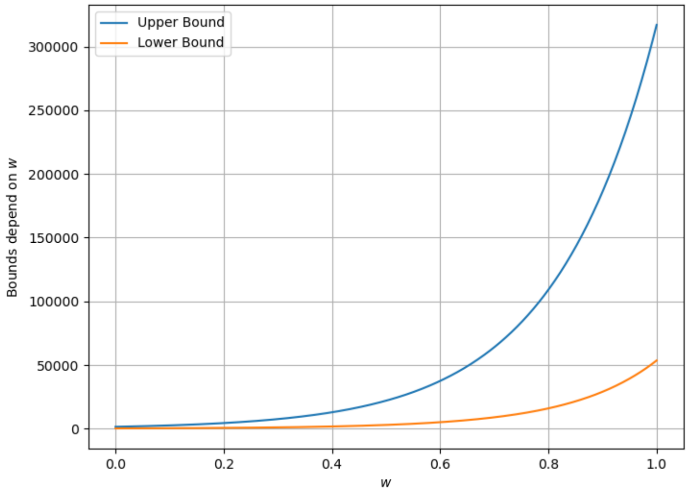

Figure 4. Graphical representation of the lower and upper bounds of (5.7).

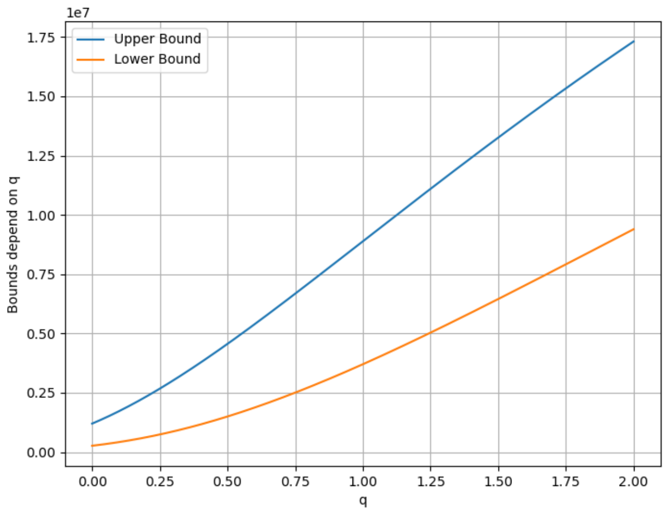

Figure 5. Pictorial form of the lower and upper bounds of (5.8).

6. Concluding remarks

In this article, we established new definitions for derivative and definite integral, known as left Hahn multiplicative derivative and definite integral in the Hahn multiplicative calculus. We also obtained basic results for this recently defined integral. The left Hahn multiplicative Hermite-Hadamard inequality was developed and we gave an example that endorses the correctness of this inequality. After that, we derived the product and quotient of the left Hahn multiplicative Hermite-Hadamard inequalities. Furthermore, we provided new definitions for the derivative and definite integral in the Hahn calculus, allowing us to construct further definitions for the right Hahn multiplicative derivative and the definite integral in the Hahn multiplicative calculus. In addition, in order to help us derive the right Hahn multiplicative Hermite-Hadamard inequalities, we constructed the power rule of the recently defined definite integral in the Hahn calculus. We concluded by providing an example of how the recently developed Hermite-Hadamard inequalities can be used to determine the lower and upper bounds of the range of functions whose Hahn multiplicative definite integrals are extremely challenging to find. These newly obtained definitions, as well as the results in Hahn multiplicative calculus, will play a key role in finding further inequalities. For example, this article can assist in deriving the Hahn multiplicative midpoint-type and trapezoidal-type inequalities in future research. Moreover, it would be helpful to merge the research of generalized quantum multiplicative calculus and advanced research in number theory [37].

Author contributions

S.I.B., M.N.A. and Y.S.: Conceptualization, investigation, methodology, software, writing-original draft, writing-review & editing; M.N.A. and S.I.B.: Formal analysis; M.A: Methodology, writing-review & editing, formal analysis. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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