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Research article

An a priori error analysis of the porous thermoelastic Coleman–Gurtin model

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Abstract: In this work, we study, from the numerical point of view, a poro-thermoelastic problem where the heat conduction is modeled by using the Coleman–Gurtin law. This is written as a linear system of partial differential equations written in terms of the displacements, the porosity (or volume fraction), and the temperature. Then, we introduce a fully discrete approximation of a weak form of the thermomechanical problem, based on the classical finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. We prove a discrete stability property and a main a priori error estimates result, which allows us to conclude the linear convergence of the approximations under suitable additional regularity. Finally, we present some numerical simulations to demonstrate the convergence and the decay of the discrete energy.

Keywords: Coleman–Gurtin model; porosity; finite history; finite elements; stability; error estimates; numerical simulations
Mathematics Subject Classification: 65M60, 74F05, 74K10, 74F10, 65M15, 65M12

1. Introduction

It is known that, when considering a thermoelastic problem, the usual way to determine the heat conduction is through the linear relationship between heat flux and temperature gradient (attributed to Fourier, also of Type I by Green and Naghdi). Unfortunately, this hypothesis has some theoretical and empirical consequences that are not well accepted by the scientific community. Some of these are:

(1) It does not fit well with the description of heat conduction in low-temperature processes [1, 2].

(2) The mathematical study of this hypothesis allows us to demonstrate the instantaneous propagation of thermal waves [3]. This fact is not compatible with the hypothesis of maximum bounded velocity and, consequently, it is not compatible with the principle of causality.

(3) Considering only the first order of temperature development, it does not appear to be adequate to describe the effects of heat over long distances.

(4) It also fails to take into account the possible effects on the past history of the solid or fluid.

These difficulties have naturally motivated many researchers to explore alternatives to Fourier's classical description of heat conduction. One of the best-known examples is the Cattaneo–Maxwell proposal [4], which successfully addresses the second issue by introducing a dissipative hyperbolic equation for heat flux. This idea gave rise to two foundational theories of thermoelasticity: the Lord–Shulman model [5] and the Green–Lindsay model [6].

Green and Naghdi also developed an alternative axiomatic framework for continuum thermodynamics [7–9]. Besides the previously mentioned Type I theory, they introduced Type II and Type III formulations. Type II again leads to a hyperbolic (non-dissipative) heat equation, whereas Type III encompasses Types I and II as limiting cases. However, the heat equation associated with Type III is parabolic and therefore subject to the same difficulties outlined in point 2 [10].

For this reason, following a line of thought parallel to the Cattaneo–Maxwell refinement of Fourier's law, a relaxation parameter can be incorporated, leading to the Moore–Gibson–Thompson equation [11], which is once again hyperbolic. It is also worth highlighting the substantial contributions of Dorin Ieşan (see, for example, [12–14]), who, building on the Green–Naghdi axioms, proposed several thermoelastic theories in which the heat equation includes higher-order spatial derivatives, thereby addressing the third difficulty.

Regarding the last point, it is important to recall that, between 1965 and 1975, several thermoelastic theories were developed in which the concept of material history played a central role. Among them, we may cite the proposal of Gurtin and Pipkin [15], that of Gurtin [16], and the theory introduced by Coleman and Gurtin [17]. More recently, the work [18] attempted to incorporate a history-dependent structure into the Moore–Gibson–Thompson equation, although it ultimately reconnects with Gurtin's earlier proposal. In this paper, we focus specifically on the formulation by Coleman and Gurtin.

On the other hand, it is widely recognized that classical elasticity theory is insufficient for describing the full behavior of real materials, as it does not account for phenomena such as microstructure or material mixtures. For this reason, during the second half of the last century, several generalized theories of materials were proposed to incorporate these effects, thereby extending the classical framework. Among the most notable contributions are those of Eringen [19] and Ieşan [20]. Terms such as micropolarity and microstretch are commonly used to refer to some of these generalizations. One particularly relevant class is that of porous materials or materials with voids.

The key idea behind these models is that the elastic medium contains pores distributed throughout its structure. As a consequence, describing the material requires not only the classical deformation variables but also the volume fraction, which quantifies the amount of solid present at each material point. This theory was introduced by Cowin and Nunziato [21–23] and has been widely applied to the study of biological tissues, construction materials, volcanic rocks, and related media. It is also worth mentioning that, from a purely mathematical standpoint, the governing equations for porous materials coincide with those of elastic materials with stretch.

These models have attracted significant attention in recent years [24–26]. In particular, the analysis of solution decay in the one-dimensional setting, under various dissipation mechanisms, has been the subject of sustained research over the past two decades [27–30]. We specifically highlight [31], where the system of equations considered in this work was studied analytically, establishing results on existence and decay of solutions.

In this article, we study numerical solutions of the system of equations from both mechanical and thermal perspectives. From the mechanical standpoint, the analysis is carried out within the framework of porous elastic materials, while from the thermal viewpoint, we adopt the Coleman—Gurtin theory. To the best of our knowledge, this work is the first to address the numerical approximation of Coleman—Gurtin-type poroelasticity. The main objective is to quantify the error introduced by the spatial discretization using the finite element method and by the temporal discretization of the time derivatives via the backward Euler scheme. To this end, we establish a discrete stability property and derive a priori error estimates. Finally, we remark that the Coleman—Gurtin model has been employed in mathematical physics to analyze complex systems such as porous materials, microbeams, and laminated structures.

The paper is organized as follows: In Section 2 we present the weak formulation of the one-dimensional poro-thermoelastic problem and recall an existence and uniqueness result, together with a discussion of the exponential decay of solutions. Section 3 introduces the fully discrete approximation, where the spatial variable is discretized by the finite element method and the time derivatives by the backward Euler scheme. We also establish a discrete stability property and provide an a priori error analysis, which yields linear convergence under suitable additional regularity assumptions. Section 4 contains numerical simulations illustrating the accuracy of the proposed approximations and the decay of the discrete energy. Finally, we present some conclusions in Section 5.

2. The poro-thermoelastic model

In what follows, we present the porous thermoelastic problem obtained by using the Coleman–Gurtin model. Therefore, let $[0,\ell]$, $\ell > 0$ be the poro-thermoelastic bar, and denote by $x \in [0,\ell]$ and $t \in [0,\infty)$, the spatial and time variables, respectively.

Our aim in this paper is to solve the following thermomechanical problem, which is written in terms of the transverse displacement of the solid elastic material u, the volume fraction ϕ, and the temperature difference θ.

Find the displacements $u : [0,\ell]\times[0,\infty) \to \mathbb{R}$, the volume fraction $\phi : [0,\ell]\times[0,\infty) \to \mathbb{R}$, and the temperature $\theta : [0,\ell]\times[0,\infty) \to \mathbb{R}$ such that

$$\rho u_{tt} - \mu u_{xx} - b\phi_x + \beta\theta_x = 0 \quad \text{in} \quad (0,\ell)\times(0,\infty), \tag{2.1}$$

$$J\phi_{tt} - d\phi_{xx} + bu_x + \xi\phi - m\theta = 0 \quad \text{in} \quad (0,\ell)\times(0,\infty), \tag{2.2}$$

$$\rho_1\theta_t - \frac{1}{\beta_1}\Big[(1-\alpha)\theta_{xx} + \alpha\int_0^\infty g(s)\theta_{xx}(t-s)\,ds\Big] + \beta u_{tx} + m\phi_t = 0 \quad \text{in} \quad (0,\ell)\times(0,\infty), \tag{2.3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x) \quad \text{for a.e. } x \in (0,\ell), \tag{2.4}$$

$$\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \psi_0(x) \quad \text{for a.e. } x \in (0,\ell), \tag{2.5}$$

$$\theta(x,0) = \theta_0(x) \quad \text{for a.e. } x \in (0,\ell), \tag{2.6}$$

$$\theta(x,t) = 0 \quad \text{for a.e. } x \in (0,\ell), \quad t < 0, \tag{2.7}$$

$$u(x,t) = \phi(x,t) = \theta(x,t) = 0 \quad \text{for } x = 0, \ell \text{ and } t \geq 0, \tag{2.8}$$

where ρ, J, μ, b, d, m, ρ_1, ξ, b, β, and β_1 are constitutive coefficients. In this case, in order to have couplings among the equations, we need to assume that β and m are different from zero, although we do not have any restriction on their sign.

From condition (2.7), we can conclude that the memory vanishes at the initial condition. So, in order to simplify the problem, we will include it into the integral term, leading to the following problem:

Find the displacements $u : [0,\ell] \times [0,\infty) \to \mathbb{R}$*, the volume fraction* $\phi : [0,\ell] \times [0,\infty) \to \mathbb{R}$*, and the temperature* $\theta : [0,\ell] \times [0,\infty) \to \mathbb{R}$ *such that*

$$\begin{aligned}
&\rho u_{tt} - \mu u_{xx} - b\phi_x + \beta\theta_x = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty),\\
&J\phi_{tt} - d\phi_{xx} + bu_x + \xi\phi - m\theta = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty),\\
&\rho_1\theta_t - \frac{1}{\beta_1}\Big[(1-\alpha)\theta_{xx} + \alpha\int_0^t g(s)\theta_{xx}(t-s)\,ds\Big] + \beta u_{tx} + m\phi_t = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty),\\
&u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x) \quad \text{for a.e. } x \in (0,\ell),\\
&\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \psi_0(x) \quad \text{for a.e. } x \in (0,\ell),\\
&\theta(x,0) = \theta_0(x) \quad \text{for a.e. } x \in (0,\ell),\\
&u(x,t) = \phi(x,t) = \theta(x,t) = 0 \quad \text{for } x = 0,\ell \text{ and } t \geq 0.
\end{aligned}$$

Finally, making a change of variable within the integral terms, the previous problem can be written in the following form:

Find the displacements $u : [0,\ell] \times [0,\infty) \to \mathbb{R}$*, the volume fraction* $\phi : [0,\ell] \times [0,\infty) \to \mathbb{R}$*, and the temperature* $\theta : [0,\ell] \times [0,\infty) \to \mathbb{R}$ *such that*

$$\rho u_{tt} - \mu u_{xx} - b\phi_x + \beta\theta_x = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty), \tag{2.9}$$

$$J\phi_{tt} - d\phi_{xx} + bu_x + \xi\phi - m\theta = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty), \tag{2.10}$$

$$\rho_1\theta_t - \frac{1}{\beta_1}\Big[(1-\alpha)\theta_{xx} + \alpha\int_0^t g(t-s)\theta_{xx}(s)\,ds\Big] + \beta u_{tx} + m\phi_t = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty), \tag{2.11}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x) \quad \text{for a.e. } x \in (0,\ell), \tag{2.12}$$

$$\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \psi_0(x) \quad \text{for a.e. } x \in (0,\ell), \tag{2.13}$$

$$\theta(x,0) = \theta_0(x) \quad \text{for a.e. } x \in (0,\ell), \tag{2.14}$$

$$u(x,t) = \phi(x,t) = \theta(x,t) = 0 \quad \text{for } x = 0,\ell \text{ and } t \geq 0. \tag{2.15}$$

In the rest of this paper, if we denote $\kappa(s) = -\alpha g_s(s)$, we will assume that

$$\rho > 0, \quad \mu > 0, \quad \alpha \in (0,1), \quad J > 0, \quad d > 0, \quad \xi > 0, \quad \rho_1 > 0, \quad \beta_1 > 0, \quad \mu\xi > b^2, \tag{2.16}$$

$$\text{the relaxation function satisfies } g(\infty) = 0, \tag{2.17}$$

$$\kappa \in C^1(0,\infty), \quad \kappa(s) \geq 0, \quad \kappa_s(s) \leq 0 \text{ for every } s \in (0,\infty), \tag{2.18}$$

$$\text{and } \int_0^\infty \kappa(s)\,ds, \int_0^\infty \kappa(s)s\,ds > 0, \tag{2.19}$$

$$\text{there exists a constant } \nu > 0 \text{ such that } \kappa_s(s) \leq -\nu\kappa(s)\,\forall s > 0 \text{ and } \kappa(0) < \infty. \tag{2.20}$$

The following existence and uniqueness result can be proved after the arguments used in [32].

Theorem 2.1. *Assume that the constitutive coefficients and the relaxation kernel g satisfy conditions* (2.16)–(2.20). *Then, problem* (2.9)–(2.15) *has a unique solution with the regularity*

$$u, \phi \in C^2([0,\infty); L^2(0,\ell)) \cap C^1([0,\infty); H_0^1(0,\ell)),$$
$$\theta \in C^1([0,\infty); L^2(0,\ell)) \cap C([0,\infty); H_0^1(0,\ell)).$$

Moreover, if we also suppose that the stability number $\Xi = \dfrac{\rho}{\mu} - \dfrac{J}{d}$ equals to zero, then the semigroup generated by the operator associated to problem (2.9)–(2.15) *is exponentially stable.*

In the rest of this section, we obtain the variational formulation of problem (2.9)–(2.15). Let us define the variational spaces $Y = L^2(0,\ell)$ and $V = H_0^1(0,\ell)$ and consider a finite interval of time $[0,T]$, $T > 0$ being the final time, where we will study the deformation of the thermoelastic bar. Moreover, let us denote by $(\cdot,\cdot)$ and $\|\cdot\|$, the usual inner product and norm, respectively, given in the space Y.

Therefore, applying integration by parts and using the boundary conditions (2.15), it leads to the following weak problem written in terms of the velocity field $v = u_t$, the volume fraction speed $\psi = \phi_t$, and the temperature θ.

Find the velocity $v : [0,T] \to V$, the volume fraction speed $\psi : [0,T] \to V$, and the temperature $\theta : [0,T] \to V$ such that $v(0) = v_0$, $\psi(0) = \psi_0$, $\theta(0) = \theta_0$ and, for a.e. $t \in [0,T]$ and for all $w, r, z \in V$,

$$\rho(v_t(t), w) + \mu(u_x(t), w_x) - b(\phi_x(t), w) + \beta(\theta_x(t), w) = 0, \tag{2.21}$$

$$J(\psi_t(t), r) + d(\phi_x(t), r_x) + b(u_x(t), r) + \xi(\phi(t), r) - m(\theta(t), r) = 0, \tag{2.22}$$

$$\rho_1(\theta_t(t), z) + \alpha_1(\theta_x(t), z_x) + \alpha_2\left(\int_0^t g(t-s)\theta_x(s)\,ds, z_x\right) + \beta(v_x(t), z) + m(\psi(t), z) = 0, \tag{2.23}$$

where we note that the displacements u and the volume fraction ϕ are recovered by the expressions

$$u(t) = \int_0^t v(s)\,ds + u_0, \quad \phi(t) = \int_0^t \psi(s)\,ds + \phi_0. \tag{2.24}$$

3. Numerical analysis of the variational problem

In this section, we will numerically study the variational problem (2.21)–(2.24). Therefore, we will first provide a numerical scheme based on the classical finite element method and the implicit Euler scheme, and then we will obtain a discrete stability property and a main a priori error estimates result.

3.1. Fully discrete approximation

In this first subsection, we introduce some fully discrete approximations of the variational problem (2.21)–(2.24). We will proceed doing it by approximating both in space and time. Regarding the spatial approximation, we construct a uniform partition of the bar by using the nodes $a_0 = 0 < \ldots < a_M = \ell$. Now, we can define the finite element space

$$V^h = \{w^h \in C([0,\ell]) \cap V\,;\, w^h_{[a_i,a_{i+1}]} \in P_1([a_i,a_{i+1}]) \text{ for } i = 0,\ldots,M-1\}.$$

Here, $P_1([a_i,a_{i+1}])$ denotes the space of polynomials with a degree less than or equal to one in the subinterval $[a_i,a_{i+1}]$. Moreover, let $h = a_{i+1} - a_i = \ell/M$ be the spatial discretization parameter.

Now, we can define the discrete initial conditions as

$$u_0^h = P^h u_0, \quad v_0^h = P^h v_0, \quad \phi_0^h = P^h \phi_0, \quad \psi_0^h = P^h \psi_0, \quad \theta_0^h = P^h \theta_0,$$

where P^h is the finite element interpolation operator over the finite element space V^h (see, for details, [33]).

Regarding the discretization of the time derivatives, let $t_0 = 0 < \ldots < t_N = T$ be a uniform partition of the time interval [0,T] with nodes $t_n = nk$ for $n = 0, \ldots, N$ and a time step $k = t_1 - t_0 = T/N$. Furthermore, let $w_n = w(t_n)$ be the value of a continuous function $w(t)$ at time $t = t_n$, and, for a sequence $\{w_n\}_{n=0}^N$, let $\delta w_n = (w_n - w_{n-1})/k$ be its divided differences.

Applying the well-known implicit Euler scheme, we have the following fully discrete problem:

Find the discrete velocity $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$, the discrete volume fraction speed $\psi^{hk} = \{\psi_n^{hk}\}_{n=0}^N \subset V^h$, and the discrete temperature $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset V^h$ such that $v_0^{hk} = v_0^h$, $\psi_0^{hk} = \psi_0^h$, $\theta_0^{hk} = \theta_0^h$ and, for $n = 1, \ldots, N$ and for all $w^h, r^h, z^h \in V^h$,

$$\rho(\delta v_n^{hk}, w^h) + \mu((u_n^{hk})_x, w_x^h) - b((\phi_n^{hk})_x, w^h) + \beta((\theta_n^{hk})_x, w^h) = 0, \tag{3.1}$$

$$J(\delta \psi_n^{hk}, r^h) + d((\phi_n^{hk})_x, r_x^h) + b((u_n^{hk})_x, r^h) + \xi(\phi_n^{hk}, r^h) - m(\theta_n^{hk}, r^h) = 0, \tag{3.2}$$

$$\rho_1(\delta \theta_n^{hk}, z^h) + \alpha_1((\theta_n^{hk})_x, z_x^h) + \alpha_2\left(I_n^{hk}, z_x^h\right) + \beta((v_n^{hk})_x, z^h) + m(\psi_n^{hk}, z^h) = 0, \tag{3.3}$$

where we note that the discrete displacements u_n^{hk} and the discrete volume fraction ϕ_n^{hk} are recovered by the expressions

$$u_n^{hk} = k\sum_{j=1}^{n} v_j^{hk} + u_0^h, \quad \phi_n^{hk} = k\sum_{j=1}^{n} \psi_j^{hk} + \phi_0^h, \tag{3.4}$$

and I_n^{hk} is an approximation of the integral of the relaxation function defined as

$$I_n^{hk} = k\sum_{j=1}^{n} g_{n-j}(\theta_j^{hk})_x.$$

It is easy to show that the above fully discrete problem has a unique solution by using the well-known Lax–Milgram lemma.

3.2. An a priori error analysis

In this subsection, we will show a discrete stability property and a main a priori error estimates result, from which we will conclude the linear convergence of the approximations.

First, we have the following.

Lemma 3.1. *Under the assumptions* (2.16)–(2.20)*, we have the following stability estimates:*

$$\|v_n^{hk}\|^2 + \|(u_n^{hk})_x\|^2 + \|\psi_n^{hk}\|^2 + \|(\phi_n^{hk})_x\|^2 + \|\theta_n^{hk}\|^2 \le C \quad \textit{for} \quad n = 1, \ldots, N,$$

where $C > 0$ is a constant that is independent of the discretization parameters.

Proof. First, we obtain some estimates for the discrete velocity. Taking as a test function $w^h = v_n^{hk}$ in (3.1), it leads

$$\rho(\delta v_n^{hk}, v_n^{hk}) + \mu((u_n^{hk})_x, (v_n^{hk})_x) - b((\phi_n^{hk})_x, v_n^{hk}) + \beta((\theta_n^{hk})_x, v_n^{hk}) = 0.$$

Taking into account the estimates

$$\begin{aligned}
&\rho(\delta v_n^{hk}, v_n^{hk}) \geq \frac{\rho}{2k}\Big[\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\Big],\\
&\mu((u_n^{hk})_x, (v_n^{hk})_x) \geq \frac{\mu}{2k}\Big[\|(u_n^{hk})_x\|^2 - \|(u_{n-1}^{hk})_x\|^2\Big],\\
&b|((\phi_n^{hk})_x, v_n^{hk})| \leq b(\|(\phi_n^{hk})_x\|^2 + \|v_n^{hk}\|^2),
\end{aligned}$$

it follows that

$$\frac{\rho}{2k}\Big[\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\Big] + \frac{\mu}{2k}\Big[\|(u_n^{hk})_x\|^2 - \|(u_{n-1}^{hk})_x\|^2\Big] + \beta((\theta_n^{hk})_x, v_n^{hk}) \leq b(\|(\phi_n^{hk})_x\|^2 + \|v_n^{hk}\|^2). \tag{3.5}$$

Now, we proceed with the estimates for the volume fraction speed. By using as a test function $r^h = \psi_n^{hk}$ in (3.2), we have

$$J(\delta\psi_n^{hk}, \psi_n^{hk}) + d((\phi_n^{hk})_x, (\psi_n^{hk})_x) + b((u_n^{hk})_x, \psi_n^{hk}) + \xi(\phi_n^{hk}, \psi_n^{hk}) - m(\theta_n^{hk}, \psi_n^{hk}) = 0.$$

Keeping in mind the estimates

$$\begin{aligned}
&J(\delta\psi_n^{hk}, \psi_n^{hk}) \geq \frac{J}{2k}\Big[\|\psi_n^{hk}\|^2 - \|\psi_{n-1}^{hk}\|^2\Big],\\
&d((\phi_n^{hk})_x, (\psi_n^{hk})_x) \geq \frac{d}{2k}\Big[\|(\phi_n^{hk})_x\|^2 - \|(\phi_{n-1}^{hk})_x\|^2\Big],\\
&\xi(\phi_n^{hk}, \psi_n^{hk}) \geq \frac{\xi}{2k}\Big[\|\phi_n^{hk}\|^2 - \|\phi_{n-1}^{hk}\|^2\Big],\\
&b((u_n^{hk})_x, \psi_n^{hk}) \leq b(\|(u_n^{hk})_x\|^2 + \|\psi_n^{hk}\|^2),\\
&m(\theta_n^{hk}, \psi_n^{hk}) \leq m(\|\theta_n^{hk}\|^2 + \|\psi_n^{hk}\|^2),
\end{aligned}$$

we obtain the following estimates:

$$\begin{aligned}
&\frac{J}{2k}\Big[\|\psi_n^{hk}\|^2 - \|\psi_{n-1}^{hk}\|^2\Big] + \frac{d}{2k}\Big[\|(\phi_n^{hk})_x\|^2 - \|(\phi_{n-1}^{hk})_x\|^2\Big] + \frac{\xi}{2k}\Big[\|\phi_n^{hk}\|^2 - \|\phi_{n-1}^{hk}\|^2\Big]\\
&\qquad \leq C\Big(\|(u_n^{hk})_x\|^2 + \|\psi_n^{hk}\|^2 + \|\theta_n^{hk}\|^2\Big).
\end{aligned} \tag{3.6}$$

Finally, we derive the estimates for the temperature. Taking as a test function $z^h = \theta_n^{hk}$ in (3.3), we have

$$\rho_1(\delta\theta_n^{hk}, \theta_n^{hk}) + \alpha_1((\theta_n^{hk})_x, (\theta_n^{hk})_x) + \alpha_2\left(I_n^{hk}, (\theta_n^{hk})_x\right) + \beta((v_n^{hk})_x, \theta_n^{hk}) + m(\psi_n^{hk}, \theta_n^{hk}) = 0.$$

Thus, keeping in mind the estimates

$$\begin{aligned}
&\rho_1(\delta\theta_n^{hk}, \theta_n^{hk}) \geq \frac{\rho_1}{2k}\Big[\|\theta_n^{hk}\|^2 - \|\theta_{n-1}^{hk}\|^2\Big],\\
&\alpha_1((\theta_n^{hk})_x, (\theta_n^{hk})_x) = \alpha_1\|(\theta_n^{hk})_x\|^2,\\
&\beta((v_n^{hk})_x, \theta_n^{hk}) = -\beta(v_n^{hk}, (\theta_n^{hk})_x),\\
&m(\psi_n^{hk}, \theta_n^{hk}) \leq m(\|\psi_n^{hk}\|^2 + \|\theta_n^{hk}\|^2),
\end{aligned}$$

it follows that

$$\frac{\rho_1}{2k}\Big[\|\theta_n^{hk}\|^2 - \|\theta_{n-1}^{hk}\|^2\Big] - \beta(v_n^{hk}, (\theta_n^{hk})_x) + \alpha_1\|(\theta_n^{hk})_x\|^2 + \alpha_2\left(I_n^{hk}, (\theta_n^{hk})_x\right) \leq C\Big(\|\psi_n^{hk}\|^2 + \|\theta_n^{hk}\|^2\Big). \tag{3.7}$$

Combining estimates (3.5)–(3.7), we find that

$$\frac{\rho}{2k}\Big[\|v_n^{hk}\|^2-\|v_{n-1}^{hk}\|^2\Big]+\frac{\mu}{2k}\Big[\|(u_n^{hk})_x\|^2-\|(u_{n-1}^{hk})_x\|^2\Big]+\frac{J}{2k}\Big[\|\psi_n^{hk}\|^2-\|\psi_{n-1}^{hk}\|^2\Big]$$
$$+\frac{d}{2k}\Big[\|(\phi_n^{hk})_x\|^2-\|(\phi_{n-1}^{hk})_x\|^2\Big]+\frac{\xi}{2k}\Big[\|\phi_n^{hk}\|^2-\|\phi_{n-1}^{hk}\|^2\Big]+\frac{\rho_1}{2k}\Big[\|\theta_n^{hk}\|^2-\|\theta_{n-1}^{hk}\|^2\Big]$$
$$+\alpha_1\|(\theta_n^{hk})_x\|^2+\alpha_2\left(I_n^{hk},(\theta_n^{hk})_x\right)$$
$$\leq C(\|(\phi_n^{hk})_x\|^2+\|v_n^{hk}\|^2+\|(u_n^{hk})_x\|^2+\|\psi_n^{hk}\|^2+\|\theta_n^{hk}\|^2).$$

Multiplying these estimates by k and summing it until n, we have

$$\|v_n^{hk}\|^2+\|(u_n^{hk})_x\|^2+\|\psi_n^{hk}\|^2+\|(\phi_n^{hk})_x\|^2+\|\phi_n^{hk}\|^2+\|\theta_n^{hk}\|^2$$
$$+k\sum_{j=1}^{n}\|(\theta_j^{hk})_x\|^2+k\sum_{j=1}^{n}\left(I_j^{hk},(\theta_j^{hk})_x\right)$$
$$\leq C\sum_{j=1}^{n}(\|(\phi_j^{hk})_x\|^2+\|v_j^{hk}\|^2+\|(u_j^{hk})_x\|^2+\|\psi_j^{hk}\|^2+\|\theta_j^{hk}\|^2)$$
$$+C(\|(\phi_0^{h})_x\|^2+\|v_0^{h}\|^2+\|(u_0^{h})_x\|^2+\|\psi_0^{h}\|^2+\|\theta_0^{h}\|^2).$$

We observe that

$$k\sum_{j=1}^{n}\left(I_j^{hk},(\theta_j^{hk})_x\right)\leq \varepsilon k\sum_{j=1}^{n}\|(\theta_j^{hk})_x\|^2+Ck\sum_{j=1}^{n}\|I_j^{hk}\|^2,$$
$$k\sum_{j=1}^{n}\|I_j^{hk}\|^2\leq Ck\sum_{j=1}^{n}k\sum_{m=1}^{j}\|(\theta_m^{hk})_x\|^2,$$

where $\varepsilon>0$ is assumed small enough. Thus, using a discrete version of Grönwall's lemma (see, for instance, [34]), we conclude the desired stability estimates. □

Now, we provide the numerical analysis of the approximations to problem (2.21)–(2.24) obtained from the fully discrete problem (3.1)–(3.4). This is summarized in the following.

Theorem 3.1. *Let the assumptions* (2.16)–(2.20) *still hold. Denoting by* (u,v,ϕ,ψ,θ)*, the solution to problem* (2.21)–(2.24)*, and by* $\{u_n^{hk},v_n^{hk},\phi_n^{hk},\psi_n^{hk},\theta_n^{hk}\}_{n=0}^N$*, the solution to problem* (3.1)–(3.4)*, we have the following a priori error estimates that, for all* $\{w_n^h\}_{n=0}^N$, $\{r_n^h\}_{n=0}^N$, $\{z_n^h\}_{n=0}^N\subset V^h$,

$$\max_{0\leq n\leq N}\Big\{\|v_n-v_n^{hk}\|^2+\|(u_n-u_n^{hk})_x\|^2+\|\psi_n-\psi_n^{hk}\|^2+\|\phi_n-\phi_n^{hk}\|^2+\|(\phi_n-\phi_n^{hk})_x\|^2+\|\theta_n-\theta_n^{hk}\|_Y^2\Big\}$$
$$\leq Ck\sum_{j=1}^{N}\Big[\|v_{tj}-\delta v_j\|^2+\|u_{tj}-\delta u_j\|_V^2+\|\psi_{tj}-\delta\psi_j\|^2+\|\phi_{tj}-\delta\phi_j\|_V^2+\|\theta_{tj}-\delta\theta_j\|^2$$
$$+\|v_j-w_j^h\|_V^2+\|\psi_j-r_j^h\|_V^2+\|\theta_j-z_j^h\|_V^2+\|Err_j\|^2\Big]+\frac{C}{k}\sum_{j=1}^{N-1}\Big[\|v_j-w_j^h-(v_{j+1}-w_{j+1}^h)\|^2$$
$$+\|\psi_j-r_j^h-(\psi_{j+1}-r_{j+1}^h)\|^2+\|\theta_j-z_j^h-(\theta_{j+1}-z_{j+1}^h)\|^2\Big]+C\max_{0\leq n\leq N}\Big[\|v_n-w_n^h\|^2+\|\psi_n-r_n^h\|^2$$
$$+\|\theta_n-z_n^h\|^2\Big]+C\Big(\|v_0-v_0^h\|^2+\|u_0-u_0^h\|_V^2+\|\psi_0-\psi_0^h\|^2+\|\phi_0-\phi_0^h\|_V^2+\|\theta_0-\theta_0^h\|^2\Big),$$

where we denote by C, a positive constant independent of h and k, but depending on the continuous solution. Moreover, let $\|\cdot\|_V$ be the usual norm in the space V, and we represent by Err_n, the approximation error of the integral term given by

$$Err_n = \int_0^{t_n} g(t_n - s)\theta_x(s)\, ds - k\sum_{j=1}^{n} g_{n-j}(\theta_j)_x. \tag{3.8}$$

Proof. Subtracting the variational equation (2.21) at time $t = t_n$ and for a test function $w = w^h \in V^h \subset V$ and the discrete variational equation (3.1), we have, for all $w^h \in V^h$,

$$\rho(v_{tn} - \delta v_n^{hk}, w^h) + \mu((u_n - u_n^{hk})_x, w_x^h) - b((\phi_n - \phi_n^{hk})_x, w^h) + \beta((\theta_n - \theta_n^{hk})_x, w^h) = 0,$$

and so, we conclude that

$$\begin{aligned}
&\rho(v_{tn} - \delta v_n^{hk}, v_n - v_n^{hk}) + \mu((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x) - b((\phi_n - \phi_n^{hk})_x, v_n - v_n^{hk})\\
&\quad +\beta((\theta_n - \theta_n^{hk})_x, v_n - v_n^{hk})\\
&= \rho(v_{tn} - \delta v_n^{hk}, v_n - w^h) + \mu((u_n - u_n^{hk})_x, (v_n - w^h)_x) - b((\phi_n - \phi_n^{hk})_x, v_n - w^h)\\
&\quad +\beta((\theta_n - \theta_n^{hk})_x, v_n - w^h).
\end{aligned}$$

By using the estimates

$$\begin{aligned}
&\rho(\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}) \geq \frac{\rho}{2k}\Big[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big],\\
&\mu((u_n - u_n^{hk})_x, (\delta u_n - \delta u_n^{hk})_x) \geq \frac{\mu}{2k}\Big[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big],\\
&b|((\phi_n - \phi_n^{hk})_x, v_n - v_n^{hk})| \leq b(\|(\phi_n - \phi_n^{hk})_x\|^2 + \|v_n - v_n^{hk}\|^2),\\
&\beta((\theta_n - \theta_n^{hk})_x, v_n - w^h) = -\beta(\theta_n - \theta_n^{hk}, (v_n - w^h)_x),
\end{aligned}$$

it follows that, for all $w^h \in V^h$,

$$\begin{aligned}
&\frac{\rho}{2k}\Big[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big] + \frac{\mu}{2k}\Big[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big]\\
&\quad +\beta((\theta_n - \theta_n^{hk})_x, v_n - v_n^{hk})\\
&\leq C\Big(\|v_{tn} - \delta v_n\|^2 + \|u_{tn} - \delta u_n\|_V^2 + \|v_n - w^h\|_V^2 + \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2\\
&\quad +\|\theta_n - \theta_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h)\Big).
\end{aligned} \tag{3.9}$$

Now, we obtain the error estimates for the volume fraction speed. So, subtracting the variational equation (2.22) at time $t = t_n$ and for a test function $r = r^h \in V^h \subset V$ and the discrete variational equation (3.2), it leads

$$J(\psi_{tn} - \delta\psi_n^{hk}, r^h) + d((\phi_n - \phi_n^{hk})_x, r_x^h) + b((u_n - u_n^{hk})_x, r^h) + \xi(\phi_n - \phi_n^{hk}, r^h) - m(\theta_n - \theta_n^{hk}, r^h) = 0,$$

and therefore we obtain, for all $r^h \in V^h$,

$$\begin{aligned}
&J(\psi_{tn} - \delta\psi_n^{hk}, \psi_n - \psi_n^{hk}) + d((\phi_n - \phi_n^{hk})_x, (\psi_n - \psi_n^{hk})_x) + b((u_n - u_n^{hk})_x, \psi_n - \psi_n^{hk})\\
&\quad +\xi(\phi_n - \phi_n^{hk}, \psi_n - \psi_n^{hk}) - m(\theta_n - \theta_n^{hk}, \psi_n - \psi_n^{hk})\\
&= J(\psi_{tn} - \delta\psi_n^{hk}, \psi_n - r^h) + d((\phi_n - \phi_n^{hk})_x, (\psi_n - r^h)_x) + b((u_n - u_n^{hk})_x, \psi_n - r^h)\\
&\quad +\xi(\phi_n - \phi_n^{hk}, \psi_n - r^h) - m(\theta_n - \theta_n^{hk}, \psi_n - r^h).
\end{aligned}$$

Taking into account the estimates

$$\begin{aligned}
&J(\delta\psi_n - \delta\psi_n^{hk}, \psi_n - \psi_n^{hk}) \geq \frac{J}{2k}\Big[\|\psi_n - \psi_n^{hk}\|^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|^2\Big],\\
&d((\phi_n - \phi_n^{hk})_x, (\delta\phi_n - \delta\phi_n^{hk})_x) \geq \frac{d}{2k}\Big[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\Big],\\
&\xi(\phi_n - \phi_n^{hk}, \delta\phi_n - \delta\phi_n^{hk}) \geq \frac{\xi}{2k}\Big[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2\Big],\\
&b((u_n - u_n^{hk})_x, \psi_n - \psi_n^{hk}) \leq b(\|\psi_n - \psi_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2),
\end{aligned}$$

we have the following estimates that for all $r^h \in V^h$,

$$\begin{aligned}
&\frac{J}{2k}\Big[\|\psi_n - \psi_n^{hk}\|^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|^2\Big] + \frac{d}{2k}\Big[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\Big]\\
&\quad + \frac{\xi}{2k}\Big[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2\Big]\\
&\leq C\Big(\|\psi_{tn} - \delta\psi_n\|^2 + \|\phi_{tn} - \delta\phi_n\|_V^2 + \|\psi_n - r^h\|_V^2 + \|\phi_n - \phi_n^{hk}\|^2 + \|\psi_n - \psi_n^{hk}\|^2\\
&\quad + \|(u_n - u_n^{hk})_x\|^2 + \|\theta_n - \theta_n^{hk}\|^2\Big).
\end{aligned} \tag{3.10}$$

Finally, we will obtain the error estimates for the temperature field. We proceed in a similar way as in the previous variables. So, we subtract variational equation (2.21) at time $t = t_n$ and for a test function $z = z^h \in V^h \subset V$ and the discrete variational equation (3.3), and we find that

$$\rho_1(\theta_{tn} - \delta\theta_n^{hk}, z^h) + \alpha_1((\theta_n - \theta_n^{hk})_x, z_x^h) + \alpha_2\left(I_n - I_n^{hk}, z_x^h\right) + \beta((v_n - v_n^{hk})_x, z^h) + m(\psi_n - \psi_n^{hk}, z^h) = 0.$$

Therefore, it follows that for all $z^h \in V^h$,

$$\begin{aligned}
&\rho_1(\theta_{tn} - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) + \alpha_1((\theta_n - \theta_n^{hk})_x, (\theta_n - \theta_n^{hk})_x) + \alpha_2\left(I_n - I_n^{hk}, (\theta_n - \theta_n^{hk})_x\right)\\
&\quad + \beta((v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk}) + m(\psi_n - \psi_n^{hk}, \theta_n - \theta_n^{hk})\\
&= \rho_1(\theta_{tn} - \delta\theta_n^{hk}, \theta_n - z^h) + \alpha_1((\theta_n - \theta_n^{hk})_x, (\theta_n - z^h)_x) + \alpha_2\left(I_n - I_n^{hk}, (\theta_n - z^h)_x\right)\\
&\quad + \beta((v_n - v_n^{hk})_x, \theta_n - z^h) + m(\psi_n - \psi_n^{hk}, \theta_n - z^h).
\end{aligned}$$

We use the estimates

$$\begin{aligned}
&\rho_1(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) \geq \frac{\rho_1}{2k}\Big[\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big],\\
&\alpha_1((\theta_n - \theta_n^{hk})_x, (\theta_n - \theta_n^{hk})_x) = \alpha_1\|(\theta_n - \theta_n^{hk})_x\|^2,\\
&\beta((v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk}) = -\beta(v_n - v_n^{hk}, (\theta_n - \theta_n^{hk})_x),\\
&\beta((v_n - v_n^{hk})_x, \theta_n - z^h) = -\beta(v_n - v_n^{hk}, (\theta_n - z^h)_x),
\end{aligned}$$

which allow us to obtain

$$\begin{aligned}
&\frac{\rho_1}{2k}\Big[\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big] + \alpha_1\|(\theta_n - \theta_n^{hk})_x\|^2 - \beta(v_n - v_n^{hk}, (\theta_n - \theta_n^{hk})_x)\\
&\quad \leq C\Big(\|\theta_{tn} - \delta\theta_n\|^2 + \|\theta_n - z^h\|_V^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|\psi_n - \psi_n^{hk}\|^2 + \|v_n - v_n^{hk}\|^2\\
&\qquad + \|Err_n\|^2 + \|I_n - I_n^{hk}\|^2 + \varepsilon\|(\theta_n - \theta_n^{hk})_x\|^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - z^h)\Big),
\end{aligned} \tag{3.11}$$

where $\varepsilon > 0$ is assumed small enough, Err_n was defined in (3.8), and I_n is an approximation of the integral term given by

$$I_n = k\sum_{j=1}^{n} g_{n-j}(\theta_j)_x.$$

If we combine estimates (3.9)–(3.11), we find that, for all $w^h, r^h, z^h \in V^h$,

$$
\begin{aligned}
&\frac{\rho}{2k}\Big[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big] + \frac{\mu}{2k}\Big[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big]\\
&\quad + \frac{J}{2k}\Big[\|\psi_n - \psi_n^{hk}\|^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|^2\Big] + \frac{\xi}{2k}\Big[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2\Big]\\
&\quad + \frac{d}{2k}\Big[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\Big] + C\|(\theta_n - \theta_n^{hk})_x\|^2\\
&\quad + \frac{\rho_1}{2k}\Big[\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big]\\
&\leq C\Big(\|v_{tn} - \delta v_n\|^2 + \|u_{tn} - \delta u_n\|_V^2 + \|v_n - w^h\|_V^2 + \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2\\
&\quad + \|\theta_n - \theta_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h)\\
&\quad + \|\psi_{tn} - \delta\psi_n\|^2 + \|\phi_{tn} - \delta\phi_n\|_V^2 + \|\psi_n - r^h\|_V^2 + \|\phi_n - \phi_n^{hk}\|^2 + \|\psi_n - \psi_n^{hk}\|^2\\
&\quad + \|\theta_{tn} - \delta\theta_n\|^2 + \|\theta_n - z^h\|_V^2 + \|Err_n\|^2 + \|I_n - I_n^{hk}\|^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - z^h)\Big).
\end{aligned}
$$

Multiplying these estimates by k, by induction we have, for all $\{w_j^h, r_j^h, z_j^h\}_{j=1}^n \subset V^h \times V^h \times V^h$,

$$
\begin{aligned}
&\|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|\psi_n - \psi_n^{hk}\|^2 + \|\phi_n - \phi_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2\\
&\quad + k\sum_{j=1}^n \|(\theta_j - \theta_j^{hk})_x\|^2 + \|\theta_n - \theta_n^{hk}\|^2\\
&\leq Ck\sum_{j=1}^n \Big(\|v_{tj} - \delta v_j\|^2 + \|u_{tj} - \delta u_j\|_V^2 + \|v_j - w_j^h\|_V^2 + \|v_j - v_j^{hk}\|^2\\
&\quad + \|(u_j - u_j^{hk})_x\|^2 + \|\theta_j - \theta_j^{hk}\|^2 + \|(\phi_j - \phi_j^{hk})_x\|^2 + (\delta v_j - \delta v_j^{hk}, v_j - w_j^h)\\
&\quad + \|\psi_{tj} - \delta\psi_j\|^2 + \|\phi_{tj} - \delta\phi_j\|_V^2 + \|\psi_j - r_j^h\|_V^2 + \|\phi_j - \phi_j^{hk}\|^2 + \|\psi_j - \psi_j^{hk}\|^2\\
&\quad + \|\theta_{tj} - \delta\theta_j\|^2 + \|\theta_j - z_j^h\|_V^2 + \|Err_j\|^2 + \|I_j - I_j^{hk}\|^2 + \|(\theta_j - \theta_j^{hk})_x\|^2\\
&\quad + (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - z_j^h)\Big) + C\Big(\|v_0 - v_0^h\|^2 + \|u_0 - u_0^h\|_V^2 + \|\psi_0 - \psi_0^h\|^2\\
&\quad + \|\phi_0 - \phi_0^h\|_V^2 + \|\theta_0 - \theta_0^h\|^2\Big).
\end{aligned}
$$

Finally, using the estimates

$$
\begin{aligned}
k\sum_{j=1}^n(\delta v_j - \delta v_j^{hk}, v_j - w_j^h) &= (v_n - v_n^{hk}, v_n - w_n^h) + (v_0^h - v_0, v_1 - w_1^h)\\
&\quad + \sum_{j=1}^{n-1}(v_j - v_j^{hk}, w_j - w_j^h - (v_{j+1} - w_{j+1}^h)),\\
k\sum_{j=1}^n(\delta\psi_j - \delta\psi_j^{hk}, \psi_j - r_j^h) &= (\psi_n - \psi_n^{hk}, \psi_n - r_n^h) + (\psi_0^h - \psi_0, \psi_1 - r_1^h)\\
&\quad + \sum_{j=1}^{n-1}(\psi_j - \psi_j^{hk}, \psi_j - r_j^h - (\psi_{j+1} - r_{j+1}^h)),
\end{aligned}
$$

$$
\begin{aligned}
k\sum_{j=1}^n(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - z_j^h) &= (\theta_n - \theta_n^{hk}, \theta_n - z_n^h) + (\theta_0^h - \theta_0, \theta_1 - z_1^h)\\
&\quad + \sum_{j=1}^{n-1}(\theta_j - \theta_j^{hk}, \theta_j - z_j^h - (\theta_{j+1} - z_{j+1}^h)),
\end{aligned}
$$

$$k\sum_{j=1}^{n}\|I_j - I_j^{hk}\|^2 \le Ck\sum_{j=1}^{n}k\sum_{m=1}^{j}\|(\theta_m-\theta_m^{hk})_x\|^2,$$

and a discrete version of Grönwall's lemma (see, again, [34]), we obtain the desired a priori error estimates. □

It is worth noting that by applying the error estimates provided in Theorem 3.1, we can analyze the convergence order of the approximations given in the fully discrete problem (3.1)–(3.4). For instance, if we assume that

$$u,\phi \in W^{3,\infty}([0,T];L^2(0,\ell))\cap C^1([0,T];H^2(0,\ell)),$$
$$\theta \in W^{2,\infty}([0,T];L^2(0,\ell))\cap C([0,T];H^2(0,\ell))\cap C^1([0,T];V),$$

then the linear convergence of the approximations can be deduced, that is, there exists a positive constant such that

$$\max_{0\le n\le N}\left\{\|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|\psi_n - \psi_n^{hk}\| + \|\phi_n - \phi_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\|\right\} \le C(h+k).$$

4. Numerical simulations

In this section, we demonstrate, from the numerical point of view, the accuracy of the approximations and the behavior of the discrete energy by solving some numerical examples. In this way, the following linear system, derived from problem (3.1)–(3.4) and implemented in MATLAB is that for all $w^h, r^h, z^h \in V^h$,

$$\begin{aligned}
&\frac{\rho}{k}(v_n^{hk},w^h) + k(\mu(v_n^{hk})_x, w_x^h) = \frac{\rho}{k}(v_{n-1}^{hk},w^h) - (\mu(u_{n-1}^{hk})_x, w_x^h) + b((\phi_n^{hk})_x, w^h)\\
&\qquad -\beta((\theta_n^{hk})_x, w^h) + (F_{1n}, w^h),\\
&\frac{J}{k}(\psi_n^{hk},r^h) + dk((\psi_n^{hk})_x, r_x^h) + \xi k(\psi_n^{hk}, r^h) = \frac{J}{k}(\psi_{n-1}^{hk},r^h) + d((\phi_{n-1}^{hk})_x, r_x^h) - b((u_n^{hk})_x, r^h)\\
&\qquad -\xi(\phi_{n-1}^{hk}, r^h) + m(\theta_n^{hk}, r^h) + (F_{2n}, r^h),\\
&\frac{\rho_1}{k}(\theta_n^{hk},z^h) + \alpha_1((\theta_n^{hk})_x, z_x^h) + k\alpha_2(g(0)(\theta_n^{hk})_x, z_x^h) = \frac{\rho_1}{k}(\theta_{n-1}^{hk},z^h) - \alpha_2(\sum_{j=1}^{n-1} g(n-j)(\theta_j^{hk})_x, z_x^h)\\
&\qquad -\beta((v_n^{hk})_x, z^h) - m(\psi_n^{hk}, z^h) + (F_{3n}, z^h).
\end{aligned}$$

The numerical scheme was implemented on a 3.2 GHz PC, and we note that a typical run, with parameters $h = k = 0.001$, took about 1.62 sec of CPU time.

4.1. First example: numerical convergence

As a simple example, in order to show the accuracy of the approximations, we use the following data:

$$\ell = 1,\quad T = 1,\quad \rho = 2,\quad \beta = 1,\quad J = 1,\quad \alpha_1 = 0.5,\quad \alpha_2 = 0.5,$$
$$\mu = 2,\quad d = 1,\quad \rho_1 = 1,\quad \xi = 2,\quad b = 1,\quad m = 1,\quad g(t) = e^{-2t},$$

and the initial conditions that for all $x \in [0,1]$,

$$u_0(x) = v_0(x) = x(x-1),\quad \phi_0(x) = \psi_0(x) = x(x-1),\quad \theta_0(x) = x(x-1).$$

Considering the supply terms that for all $(x,t) \in [0,1] \times [0,1]$,

$$F_1(x,t) = e^t(2x^2 - 2x - 4),$$
$$F_2(x,t) = e^t(2x^2 - 3),$$
$$F_3(x,t) = e^t(2x^2 + \frac{e^{-3t}}{3} - \frac{7}{3}),$$

the exact solution to the problem can be calculated as

$$u(x,t) = \phi(x,t) = \theta(x,t) = e^t x(x-1) \quad \forall (x,t) \in [0,1] \times [0,1].$$

Thus, the approximation errors estimated by

$$\max_{0 \le n \le N} \left\{ \|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|\psi_n - \psi_n^{hk}\| + \|\phi_n - \phi_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\| \right\}$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h + k$ is plotted in Figure 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence seems to be found when the parameters are small, confirming the theoretical behavior.

Table 1. Numerical errors for some values of h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.284497	0.283968	0.283726	0.283658	0.283626	0.283608	0.283602
$1/2^4$	0.142581	0.141947	0.141718	0.141669	0.141649	0.141639	0.141635
$1/2^5$	0.072243	0.071204	0.070879	0.070824	0.070806	0.070799	0.070797
$1/2^6$	0.037956	0.036093	0.035518	0.035430	0.035406	0.035398	0.035396
$1/2^7$	0.022089	0.018980	0.017920	0.017755	0.017713	0.017700	0.017698
$1/2^8$	0.015393	0.011067	0.009269	0.008958	0.008877	0.008853	0.008850
$1/2^9$	0.012794	0.007733	0.005182	0.004634	0.004479	0.004433	0.004427
$1/2^{10}$	0.011855	0.006439	0.003406	0.002592	0.002317	0.002230	0.002217
$1/2^{11}$	0.011554	0.005973	0.002701	0.001704	0.001296	0.001140	0.001115
$1/2^{12}$	0.011470	0.005824	0.002440	0.001352	0.000852	0.000617	0.000570
$1/2^{13}$	0.011449	0.005783	0.002353	0.001222	0.000676	0.000382	0.000308
$1/2^{14}$	0.011443	0.005772	0.002328	0.001178	0.000611	0.000286	0.000191

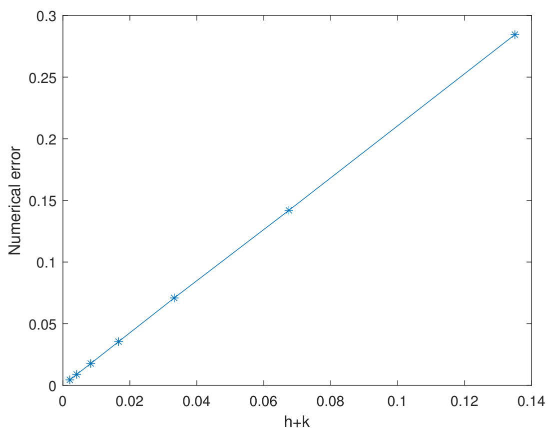

Figure 1. Asymptotic constant error.

4.2. Second example: behavior of the discrete energy decay

In this final section, our aim is to show the behavior of the discrete energy depending on the value of the stability coefficient $\Xi = \frac{\rho}{\mu} - \frac{J}{d}$.

Therefore, using the following data:

$$\ell = 1, \quad T = 80, \quad \rho = 8, \quad \beta = 6, \quad J = 1, \quad \alpha_1 = 0.5, \quad \alpha_2 = 0.5,$$
$$\mu = 8, \quad d = 1, \quad \rho_1 = 6, \quad \xi = 0.1, \quad b = 0.5, \quad m = 1, \quad g(t) = 2e^{-2t},$$

and the initial conditions that for all $x \in [0, 1]$,

$$u_0(x) = v_0(x) = 0, \quad \phi_0(x) = \psi_0(x) = 0, \quad \theta_0(x) = x(x-1),$$

if we define the discrete energy as

$$E_n^{hk} = \frac{1}{2}\Big(\rho\|v_n^{hk}\|^2 + \mu\|(u_n^{hk})_x\|^2 + J\|\psi_n^{hk}\|^2 + d\|(\phi_n^{hk})_x\|^2 + \rho_1\|\theta_n^{hk}\|^2 + 2((u_n^{hk})_x, \phi_n^{hk})\Big),$$

taking the discretization parameters $h = k = 0.001$, its evolution in time is plotted in Figure 2 (in both natural and semi-log scales). As can be seen, it converges to zero, and an exponential decay seems to be achieved. It is worth noting that the values of the constitutive coefficients satisfy conditions (2.16) and also that $\Xi = 0$.

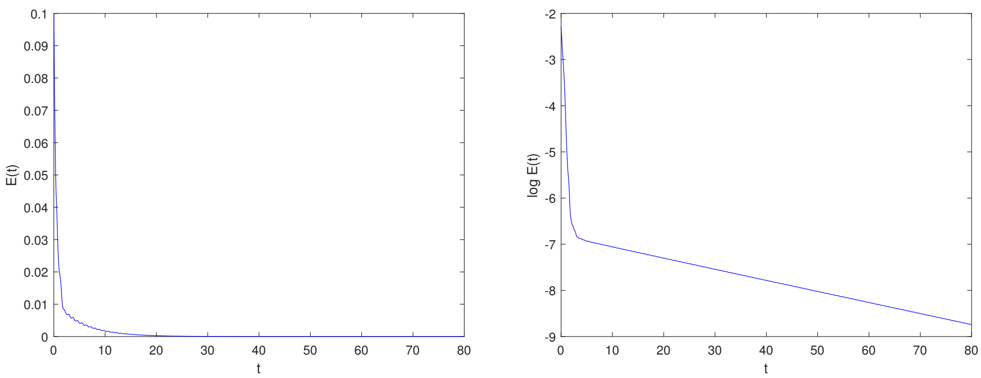

Figure 2. Evolution in time of the discrete energy (natural and semi-log scales) with $\Xi = 0$.

Now, we solve a similar problem, slightly changing the constitutive data

$$\ell = 1, \quad T = 80, \quad \rho = 0.1, \quad \beta = 5, \quad J = 18, \quad \alpha_1 = 0.5, \quad \alpha_2 = 0.5,$$
$$\mu = 2, \quad d = 1, \quad \rho_1 = 6, \quad \xi = 2, \quad b = 1, \quad m = 1, \quad g(t) = 2e^{-2t}.$$

Taking the discretization parameters $h = k = 0.001$, its evolution in time is plotted in Figure 3 (in both natural and semi-log scales). We have also added (in red) the function $4 \times 10^{-4} t^{-1/2}$ to compare the decay. As can be seen, it converges to zero, and a polynomial decay seems to be achieved. In this case, we note that the values of the constitutive coefficients satisfy conditions (2.16), but $\Xi \neq 0$.

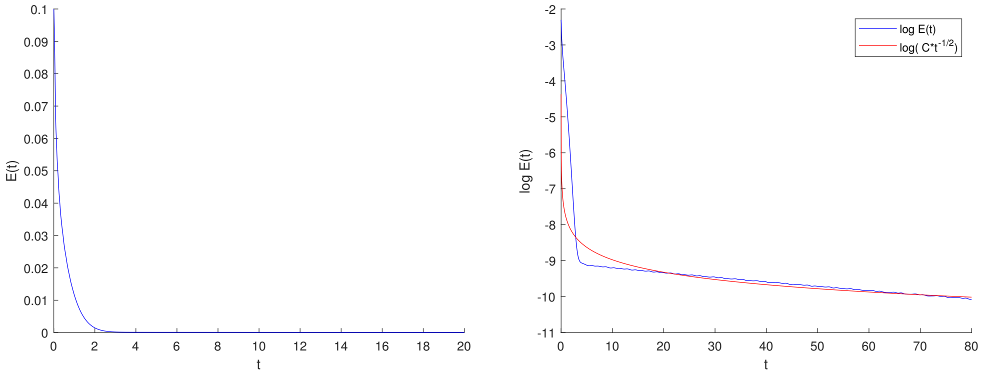

Figure 3. Evolution in time of the discrete energy (natural and semi-log scales) with $\Xi \neq 0$.

5. Conclusions

In this work, we have analyzed, from the numerical point of view, a porous thermoelastic problem where the heat conduction is modeled with the Coleman–Gurtin model. By using an adequate change of variable in the integral of the kernel function, we have obtained its variational formulation. Then, we have introduced a fully discrete problem by using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. By applying a discrete version of Grönwall's lemma, we have proved a discrete stability property (see Lemma 3.1), and a main a priori error estimates result, Theorem 3.1. The linear convergence of the approximations has been concluded under a suitable additional regularity. Finally, we have provided some numerical simulations: a first example that demonstrated the numerical convergence in a simple academical case, and a second one to show the evolution of the discrete energy depending on the constitutive parameters.

Author contributions

All the authors have contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Ramón Quintanilla is an editorial board member for AIMS Mathematics and was not involved in the editorial review and/or the decision to publish this article.

All authors declare no conflicts of interest in this paper.

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