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Research article

Refining and extending the theoretical foundations of r-near topology

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Abstract: The present paper aims to refine and extend the theoretical foundations of r-near topology. For this reason, it first redefines the concept of r-near neighborhoods to address inconsistencies in previous studies and clarifies the relationship between r-near open neighborhoods and r-near closure. This study then elaborates on the fundamental properties of r-near closed sets, r-near interior, r-near closure, and r-near neighborhoods. Subsequently, it introduces four novel concepts within r-near topology: r-near accumulation points, r-near isolated points, r-near exterior points, and r-near boundary points. Furthermore, this study explores some of their basic properties and provides illustrative examples regarding the aforesaid concepts. Additionally, it researches the relationships between r-near interior, r-near closure, and r-near exterior in the r-near topological spaces and their classical topological counterparts. Lastly, the study highlights the theoretical significance of r-near topology and suggests potential directions for further research.

Keywords: near sets; r-near topology; r-near open sets; r-near accumulation points; r-near boundary points
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1. Introduction

Pawlak [1–3] has propounded the concept of rough sets and researched some of their basic properties. This concept is based on an approach to classifying objects as an alternative to fuzzy sets [4]. Approximation spaces have a significant role in this concept, defined by lower and upper approximations. Afterward, Peters [5, 6] has put forward the concepts of nearness approximation spaces and near sets. These concepts utilize probe functions representing objects' features to describe the nearness of the objects, as generalizations of approximation spaces and rough sets, respectively. Moreover, Peters and Wasilewski [7] have introduced a formal foundation for near sets. Later, Peters [8] has defined two basic types of near sets: Spatially near sets and descriptively near sets.

Besides, numerous theoretical studies have been conducted on these concepts in various topics, such as groups [9–11], rings [12, 13], modules [14], vector spaces [15], and nearness measures [16].

Recent applied studies have signified that rough sets and near sets play a significant role in pattern recognition, decision-making, medical diagnosis, and face recognition. For instance, Henry and Peters [17] have proposed a classifier based on near sets in pattern recognition. Later, Zhang et al. [18] have manifested that the concept of fuzzy rough sets based on fuzzy α-neighborhoods can improve performance in multi-criteria decision-making. Moreover, Ali et al. [19] have introduced novel types of soft rough sets constructed by near-open sets and indicated their effectiveness for uncertainty modeling and information systems analysis. Afterward, Khedgaonkar and Singh [20] have suggested a classification approach for face recognition based on near sets. El-Bably et al. [21] have demonstrated that initial-neighborhood structures and their associated approximation operators significantly enhance diagnostic accuracy for COVID-19 variants. These developments exhibit that the successful application of these sets requires, first and foremost, a consistent theoretical foundation. Therefore, the present paper aims to provide a more consistent mathematical basis for the aforementioned applications by redefining the concept of r-near neighborhoods. Such theoretical refinements will enhance the correspondence between the underlying mathematical framework and its applicability to real-world decision-making processes.

Recently, researchers have focused on investigating topological structures of near sets. Atmaca [22] has defined the concept of r-near topology and introduced the basic r-near topological concepts, e.g., r-near open sets, r-near closed sets, r-near neighborhoods, r-near open neighborhoods, r-near interior, and r-near closure. Study [22] is the first study to tackle topological concepts concerning near sets. Subsequently, Atmaca and Zorlutuna [23] have defined r-near continuity and explored some of its basic properties. Later, Sarıkaya and Atmaca [24] have introduced the concepts of r-near base and r-near subspace and investigated some of their key properties. Therefore, the primary motivation of the present study is to extend the existing research on the topological structures of near sets by introducing novel concepts and exploring their fundamental properties.

This study aims to make a theoretical contribution to the r-near topology. The major contributions of the current study can be listed as follows:

- Redefining the concept of r-near neighborhoods.
- Introducing a characterization of r-near closure points via the concept of r-near open neighborhoods, in a manner consistent with the definition of r-near closure.
- Revisiting the inclusion relations between the r-near interior/closure and their classical counterparts, as presented in [22], and providing counterexamples showing that these relations do not hold in general.
- Defining four novel concepts in r-near topology—r-near accumulation points, r-near isolated points, r-near exterior points, and r-near boundary points—by establishing some of their fundamental properties, and exemplifying them.
- Introducing novel theorems, including the characterization of r-near closed sets via r-near accumulation points and of r-near open sets via r-near open neighborhoods.
- Propounding the conditions under which the inclusion relations between the concepts of r-near

interior, r-near closure, and r-near exterior and those in the classical topological spaces hold, thereby clarifying the relationships between r-near and classical topologies.

Table 1 summarizes the basic notations of the classical topological spaces and their descriptions that will be used in the following sections.

Table 1. Notations and their descriptions used in the classical topological spaces.

Notations	**Descriptions**
(X, τ)	A topological space
τ	A topology on the set X
τ^c	The family of all closed sets in the set X
$N(x)$	The family of all neighborhoods of the point $x \in X$ in the topology τ
$\text{int}(A)$	The interior of the set $A \subseteq X$ in the topology τ
$\text{cl}(A)$	The closure of the set $A \subseteq X$ in the topology τ
$\text{acc}(A)$	The set of all accumulation points of the set $A \subseteq X$ in the topology τ
$\text{iso}(A)$	The set of all isolated points of the set $A \subseteq X$ in the topology τ
$\text{ext}(A)$	The set of all exterior points of the set $A \subseteq X$ in the topology τ

The concepts of r-near interior and r-near closure in [22] were defined analogously to their classical counterparts. However, the concept of r-near neighborhoods in [22] did not preserve the classical interpretation of neighborhoods. Particularly, the r-near neighborhood of a point was defined based not only on r-near open sets, but also on classical open sets. This situation has led to structural inconsistencies among the related concepts. For instance, for any element of an r-near open set, there may not always exist a classical open set that contains this element and is contained by the r-near open set. Thereby, according to the definition in [22], an r-near open set may not be an r-near open neighborhood of its own points. Thus, the r-near neighborhood system induced by this definition does not satisfy the fundamental property stating that, "Every open set is an open neighborhood of its points", in the classical topology. This inconsistency arises because, while the r-near interior is defined independently of classical open sets, this definition was not. For this reason, this definition is redefined as in Definition 11 in Section 4 by removing the dependence on classical open sets. Thus, Definition 11 provides a natural generalization of the classical neighborhood concept within the r-near topology. As a result, Definition 11 provides the structural consistency necessary to establish the characterizations and properties introduced in Section 4, including the characterization of r-near open sets via r-near neighborhoods.

Furthermore, the inconsistency inherent in the original definition of r-near neighborhood is not merely conceptual. This definition also affects the validity of subsequent implications concerning r-near topological concepts. Several fundamental characterizations of the concepts open sets, closed sets, closure, accumulation points, boundary points, and continuity in the classical topology are based on the concept of neighborhood. Therefore, when the r-near neighborhood system does not satisfy a fundamental property, these characterizations cannot be established in the r-near topology. Consequently, the concept of r-near neighborhood must be redefined to obtain consistent analogues of these concepts in the r-near topology. Hence, correcting this inconsistency is essential for establishing a mathematically meaningful and consistent theoretical foundation for r-near topology.

The rest of the handled study is organized as follows: Section 2 presents some basic definitions and properties related to near sets, nearness approximation spaces, and r-near topological spaces. Section 3 redefines r-near neighborhoods and proves a related theorem. Section 4 contains a grounding study concerning the concepts of r-near closed sets, r-near interior, r-near closure, and r-near neighborhoods. Besides, it propounds the concepts of r-near accumulation points, r-near isolated points, r-near exterior points, and r-near boundary points and studies some of their basic properties. Section 5 investigates and exemplifies the relationships between some of the concepts in the r-near topological spaces and their counterparts in the classical topological spaces. Section 6 proposes further research on r-near topological spaces.

Table 2 summarizes the notations concerning r-near topological spaces and their descriptions used in the paper.

Table 2. Notations and their descriptions used in the r-near topological spaces.

Notations	Descriptions
O	A set of perceptual objects
$X, Y \subseteq O$	Sets of sample objects
$B_r \subseteq B \subseteq F$ such that $r \leq \|B\|$	A set of real-valued probe functions representing features of perceptual objects
$\sim_{B_r}$	An indiscernibility relation defined relative to $B_r \subseteq B \subseteq F$
$[x]_{B_r}$	The nearness classes of $x \in O$
$O/\sim_{B_r}$	The quotient sets (The partitions of the set O)
$N_r(B)$	The collections of all partitions
$\left(O, F, \sim_{B_r}, N_r(B)\right)$	A nearness approximation space
$N_r(B)_*(X)$	$N_r(B)$-lower approximation of the set X
$N_r(B)^*(X)$	$N_r(B)$-upper approximation of the set X
τ_r^*	An r-near topology on X
$\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$	An r-near topological space
G_r^*	An r-near open set
τ_r^{*c}	The family of all r-near closed sets
F_r^*	An r-near closed set
$\mathrm{int}_r^*(A)$	The r-near interior of the set $A \subseteq X$
$\mathrm{cl}_r^*(A)$	The r-near closure of the set $A \subseteq X$
$N_r^*(x)$	The family of all r-near neighborhoods of $x \in X$ (for Definition 10)
$\tau_r^*(x)$	The family of all r-near open neighborhoods of $x \in X$ (for Definition 10)
$\mathcal{N}_r^*(x)$	The family of all r-near neighborhoods of $x \in X$ (for Definition 11)
$\mathcal{T}_r^*(x)$	The family of all r-near open neighborhoods of $x \in X$ (for Definition 11)
$\mathrm{acc}_r^*(A)$	The set of all r-near accumulation points of the set $A \subseteq X$
$\mathrm{iso}_r^*(A)$	The set of all r-near isolated points of the set $A \subseteq X$
$\mathrm{ext}_r^*(A)$	The r-near exterior of the set $A \subseteq X$
$\mathrm{bnd}_r^*(A)$	The r-near boundary of the set $A \subseteq X$

2. Preliminaries

This section first presents near sets [5] and nearness approximation spaces [5]. Throughout this study, the sets O and $X, Y \subseteq O$ denote the sets of perceptual and sample objects, respectively. Moreover, the set F indicates a set of real-valued probe functions representing object features.

Definition 1. *[5] Let X and Y be two sets. If there exist objects $x \in X$ and $y \in Y$ such that the objects x and y have matching descriptions, then the sets X and Y are called near sets.*

In [5], a "description" is expressed as a tuple of values of probe functions representing an object's features. In other words, an "object description" is defined by a tuple of function values $\phi(x)$ associated with a perceptual object $x \in O$. Consider $B \subseteq F$, a set of probe functions representing features of perceptual objects. Here, probe function $\phi_i \in B$ such that $\phi_i : O \to \mathbb{R}$. Hence, the object description $\phi : O \to \mathbb{R}^L$ is defined as follows:

$$\phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_i(x), \ldots, \phi_L(x)),$$

where $|\phi| = L$ denotes a description length. For more details, see [5].

Definition 2. *[1] Let $x, y \in O$ and $B \subseteq F$. Then, the indiscernibility relation on the set O is defined as follows:*

$$\sim_B := \{(x, y) \in O \times O : \forall \phi_i \in B,\ \phi_i(x) = \phi_i(y)\}.$$

Here, $i \leq |\phi|$ denotes a description length.

Definition 3. *[5] Let $X, Y \subseteq O$ and $B \subseteq F$. If there exist $x \in X$, $y \in Y$, and $\phi_i \in B$ such that $x \sim_{\{\phi_i\}} y$, then the set X is called near to the set Y.*

Definition 4. *[5] Let $x, y \in O$ and $B \subseteq F$. If there exists a $\phi_i \in B$ such that $x \sim_{\{\phi_i\}} y$, then the objects x and y are called minimally near each other.*

In this study, the set B_r indicates a set of probe functions in $B \subseteq F$ such that $r \leq |B|$. Therefore, the indiscernibility relation can be defined for each subset B_r where $B_r \subseteq B \subseteq F$ and $|B_r| = r$. Here, the indiscernibility relation defined using the set B_r is denoted by $\sim_{B_r}$. Thus, $\sim_{B_r}$ forms different decomposition for each r on the set O and separates the set O to nearness classes $[x]_{B_r}$. In fact, these nearness classes are equivalence classes defined as follows:

$$[x]_{B_r} := \{y \in O : x \sim_{B_r} y\}.$$

Hence, the quotient sets, partitions of the set O, are obtained as follows:

$$O/\sim_{B_r} := \{[x]_{B_r} : x \in O\}.$$

Consequently, the collections of all partitions $O/\sim_{B_r}$ are as follows:

$$N_r(B) := \{O/\sim_{B_r} : B_r \subseteq B\}.$$

The collections of partitions are obtained for each combination of probe functions in the set B using $\binom{|B|}{r}$. For more details, see [5].

Definition 5. *[5] Let O be a set of perceptual objects, F be a set of real-valued probe functions representing object features, $\sim_{B_r}$ be an indiscernibility relation defined relative to $B_r \subseteq B \subseteq F$, and $N_r(B)$ be a collection of partitions. Then, the quadruple $\left(O, F, \sim_{B_r}, N_r(B)\right)$ is called a nearness approximation space.*

Definition 6. *[5] Let $\left(O, F, \sim_{B_r}, N_r(B)\right)$ be a nearness approximation space and $X \subseteq O$. Then,*

i. $N_r(B)$-lower approximation of the set X is defined as follows:

$$N_r(B)_*(X) := \bigcup_{[x]_{B_r} \subseteq X} [x]_{B_r}.$$

ii. $N_r(B)$-upper approximation of the set X is defined as follows:

$$N_r(B)^*(X) := \bigcup_{[x]_{B_r} \cap X \neq \emptyset} [x]_{B_r}.$$

Secondly, this section presents r-near topology and some of its basic properties provided in [22].

Definition 7. *[22] Let $\left(O, F, \sim_{B_r}, N_r(B)\right)$ be a nearness approximation space, $X \subseteq O$, and (X, τ) be a topological space. Then, the family $\{N_r(B)^*(G) : G \in \tau\}$ is called an r-near topology generated by $\left(O, F, \sim_{B_r}, N_r(B)\right)$ on the set X (or briefly r-near topology on X) and denoted by τ_r^*.*

From now on, the ordered septuple $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ is referred to as an r-near topological space.

Definition 8. *[22] Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space. Then, every element of τ_r^* is called an r-near open set in the set X (or briefly r-near open set). Moreover, the complement of an r-near open set is called an r-near closed set in the set X (or briefly r-near closed set).*

Hereinafter, if it causes no confusion, then an r-near open set is denoted by G_r^* instead of $N_r(B)^*(G)$. Besides, the family of all r-near closed sets is denoted by τ_r^{*c}.

Theorem 1. *[22] Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space. Then, $\emptyset$ and X are r-near open sets.*

Theorem 2. *[22] Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space. For all $i \in I$, if $(G_i)_r^* \in \tau_r^*$, then $\bigcup\limits_{i \in I} (G_i)_r^* \in \tau_r^*$. Here, I is an arbitrary index set.*

Theorem 3. *[22] Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $r \leq s \leq |B|$, $G, H \in \tau$, and $x_0 \in X$. Then, the following properties are valid:*

i. If $B_r \subseteq B_s$, then $[x]_{B_s} \subseteq [x]_{B_r}$.

ii. If $G \subseteq H$, then $G_r^* \subseteq H_r^*$.

iii. $G \subseteq G_r^*$, for all $G \in \tau$.

iv. $G_s^* \subseteq G_r^*$, *for all* $G \in \tau$.

v. If $G \in \tau$ *and* $x_0 \in G$, *then* G_r^* *is a neighborhood of* x_0.

Definition 9. *[22] Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A \subseteq X$. *Then, the r-near interior of the set A is defined by*

$$\bigcup \{G_r^* \subseteq X : G_r^* \in \tau_r^* \text{ and } G_r^* \subseteq A\},$$

and denoted by $\mathrm{int}_r^*(A)$. *Moreover, the r-near closure of the set A is defined by*

$$\bigcap \{F_r^* \subseteq X : F_r^* \in \tau_r^{*c} \text{ and } A \subseteq F_r^*\},$$

and denoted by $\mathrm{cl}_r^*(A)$.

Theorem 4. *[22] Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A_1, A_2 \subseteq X$. *Then, the following properties are valid:*

i. If $A_1 \subseteq A_2$, *then* $\mathrm{int}_r^*(A_1) \subseteq \mathrm{int}_r^*(A_2)$.

ii. If $A_1 \subseteq A_2$, *then* $\mathrm{cl}_r^*(A_1) \subseteq \mathrm{cl}_r^*(A_2)$.

Proposition 1. *[22] Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A \subseteq X$. *Then, the following properties are valid:*

i. $X \setminus \mathrm{int}_r^*(A) = \mathrm{cl}_r^*(X \setminus A)$.

ii. $X \setminus \mathrm{cl}_r^*(A) = \mathrm{int}_r^*(X \setminus A)$.

Example 1. *Let* $X = \{a, b, c, d, e, f\}$ *be a set of sample objects and* $B = \{\phi_1, \phi_2, \phi_3\}$ *be a set of probe functions representing object features such that for* $i \in \{1, 2, 3\}$, $\phi_i : X \to \mathbb{R}$ *is defined by*

	a	b	c	d	e	f
ϕ_1	1	0	2	0	0	0
ϕ_2	1	2	0	1	2	1
ϕ_3	1	1	2	2	0	2

Consider the topology $\tau = \{\emptyset, X, \{d\}, \{a, b\}, \{a, b, d\}, \{c, d, e, f\}\}$ *on the set X. Then, for each r combination, the nearness classes of the set X are as follows:*
For $r = 1$,

$$[a]_{\{\phi_1\}} = \{a\}, \quad [b]_{\{\phi_1\}} = \{b, d, e, f\}, \quad [c]_{\{\phi_1\}} = \{c\},$$
$$[a]_{\{\phi_2\}} = \{a, d, f\}, \quad [b]_{\{\phi_2\}} = \{b, e\}, \quad [c]_{\{\phi_2\}} = \{c\},$$
$$[a]_{\{\phi_3\}} = \{a, b\}, \quad [c]_{\{\phi_3\}} = \{c, d, f\}, \quad [e]_{\{\phi_3\}} = \{e\}.$$

For $r = 2$,

$$[a]_{\{\phi_1,\phi_2\}} = \{a\}, \quad [b]_{\{\phi_1,\phi_2\}} = \{b, e\}, \quad [c]_{\{\phi_1,\phi_2\}} = \{c\}, \quad [d]_{\{\phi_1,\phi_2\}} = \{d, f\},$$
$$[a]_{\{\phi_1,\phi_3\}} = \{a\}, \quad [b]_{\{\phi_1,\phi_3\}} = \{b\}, \quad [c]_{\{\phi_1,\phi_3\}} = \{c\}, \quad [d]_{\{\phi_1,\phi_3\}} = \{d, f\}, \quad [e]_{\{\phi_1,\phi_3\}} = \{e\},$$
$$[a]_{\{\phi_2,\phi_3\}} = \{a\}, \quad [b]_{\{\phi_2,\phi_3\}} = \{b\}, \quad [c]_{\{\phi_2,\phi_3\}} = \{c\}, \quad [d]_{\{\phi_2,\phi_3\}} = \{d, f\}, \quad [e]_{\{\phi_2,\phi_3\}} = \{e\}.$$

For r = 3*,*

$$[a]_{\{\phi_1,\phi_2,\phi_3\}} = \{a\}, \quad [b]_{\{\phi_1,\phi_2,\phi_3\}} = \{b\}, \quad [c]_{\{\phi_1,\phi_2,\phi_3\}} = \{c\}, \quad [d]_{\{\phi_1,\phi_2,\phi_3\}} = \{d, f\}, \quad [e]_{\{\phi_1,\phi_2,\phi_3\}} = \{e\}.$$

Thus, for r = 1*, the partitions of the set X are as follows:*

$$\begin{aligned} X/\sim_{\{\phi_1\}} &= \left\{[a]_{\{\phi_1\}}, [b]_{\{\phi_1\}}, [c]_{\{\phi_1\}}\right\} \\ &= \{\{a\}, \{b, d, e, f\}, \{c\}\}, \end{aligned}$$

$$\begin{aligned} X/\sim_{\{\phi_2\}} &= \left\{[a]_{\{\phi_2\}}, [b]_{\{\phi_2\}}, [c]_{\{\phi_2\}}\right\} \\ &= \{\{a, d, f\}, \{b, e\}, \{c\}\}, \end{aligned}$$

and

$$\begin{aligned} X/\sim_{\{\phi_3\}} &= \left\{[a]_{\{\phi_3\}}, [c]_{\{\phi_3\}}, [e]_{\{\phi_3\}}\right\} \\ &= \{\{a, b\}, \{c, d, f\}, \{e\}\}. \end{aligned}$$

For r = 2*, the partitions of the set X are as follows:*

$$\begin{aligned} X/\sim_{\{\phi_1,\phi_2\}} &= \left\{[a]_{\{\phi_1,\phi_2\}}, [b]_{\{\phi_1,\phi_2\}}, [c]_{\{\phi_1,\phi_2\}}, [d]_{\{\phi_1,\phi_2\}}\right\} \\ &= \{\{a\}, \{b, e\}, \{c\}, \{d, f\}\}, \end{aligned}$$

$$\begin{aligned} X/\sim_{\{\phi_1,\phi_3\}} &= \left\{[a]_{\{\phi_1,\phi_3\}}, [b]_{\{\phi_1,\phi_3\}}, [c]_{\{\phi_1,\phi_3\}}, [d]_{\{\phi_1,\phi_3\}}, [e]_{\{\phi_1,\phi_3\}}\right\} \\ &= \{\{a\}, \{b\}, \{c\}, \{d, f\}, \{e\}\}, \end{aligned}$$

and

$$\begin{aligned} X/\sim_{\{\phi_2,\phi_3\}} &= \left\{[a]_{\{\phi_2,\phi_3\}}, [b]_{\{\phi_2,\phi_3\}}, [c]_{\{\phi_2,\phi_3\}}, [d]_{\{\phi_2,\phi_3\}}, [e]_{\{\phi_2,\phi_3\}}\right\} \\ &= \{\{a\}, \{b\}, \{c\}, \{d, f\}, \{e\}\}. \end{aligned}$$

For r = 3*, the partition of the set X is as follows:*

$$\begin{aligned} X/\sim_{\{\phi_1,\phi_2,\phi_3\}} &= \left\{[a]_{\{\phi_1,\phi_2,\phi_3\}}, [b]_{\{\phi_1,\phi_2,\phi_3\}}, [c]_{\{\phi_1,\phi_2,\phi_3\}}, [d]_{\{\phi_1,\phi_2,\phi_3\}}, [e]_{\{\phi_1,\phi_2,\phi_3\}}\right\} \\ &= \{\{a\}, \{b\}, \{c\}, \{d, f\}, \{e\}\}. \end{aligned}$$

Therefore, for r $\in \{1, 2, 3\}$*, the collections of all partitions are as follows:*

$$\begin{aligned} N_1(B) &= \left\{X/\sim_{\{\phi_1\}}, X/\sim_{\{\phi_2\}}, X/\sim_{\{\phi_3\}}\right\} \\ &= \{\{\{a\}, \{b, d, e, f\}, \{c\}\}, \{\{a, d, f\}, \{b, e\}, \{c\}\}, \{\{a, b\}, \{c, d, f\}, \{e\}\}\}, \end{aligned}$$

$$\begin{aligned} N_2(B) &= \left\{X/\sim_{\{\phi_1,\phi_2\}}, X/\sim_{\{\phi_1,\phi_3\}}, X/\sim_{\{\phi_2,\phi_3\}}\right\} \\ &= \{\{\{a\}, \{b, e\}, \{c\}, \{d, f\}\}, \{\{a\}, \{b\}, \{c\}, \{d, f\}, \{e\}\}\}, \end{aligned}$$

and

$$\begin{aligned} N_3(B) &= \left\{X/\sim_{\{\phi_1,\phi_2,\phi_3\}}\right\} \\ &= \{\{\{a\}, \{b\}, \{c\}, \{d, f\}, \{e\}\}\}. \end{aligned}$$

Hence, for $r \in \{1, 2, 3\}$*, the r-near topologies on X are as follows:*

$$\tau_1^* = \{\emptyset, X, \{a, b, d, e, f\}\},$$

$$\tau_2^* = \{\emptyset, X, \{d, f\}, \{a, b, e\}, \{a, b, d, e, f\}, \{b, c, d, e, f\}\},$$

and

$$\tau_3^* = \{\emptyset, X, \{a, b\}, \{d, f\}, \{a, b, d, f\}, \{c, d, e, f\}\}.$$

To exemplify, the 1*-near topology* τ_1^* *is obtained as follows:*

$$\tau_1^* = \{N_1(B)^*(\emptyset), N_1(B)^*(X), N_1(B)^*(\{d\}), N_1(B)^*(\{a, b\}), N_1(B)^*(\{a, b, d\}), N_1(B)^*(\{c, d, e, f\})\}.$$

Here,

$$N_1(B)^*(\emptyset) = \emptyset,$$

$$N_1(B)^*(X) = X,$$

$$\begin{aligned} N_1(B)^*(\{d\}) &= \bigcup_{[x]_{B_1} \cap \{d\} \neq \emptyset} [x]_{B_1} \\ &= \bigcup \{\{b, d, e, f\}, \{a, d, f\}, \{c, d, f\}\} \\ &= X, \end{aligned}$$

$$\begin{aligned} N_1(B)^*(\{a, b\}) &= \bigcup_{[x]_{B_1} \cap \{a,b\} \neq \emptyset} [x]_{B_1} \\ &= \bigcup \{\{a\}, \{b, d, e, f\}, \{a, d, f\}, \{b, e\}, \{a, b\}\} \\ &= \{a, b, d, e, f\}, \end{aligned}$$

$$\begin{aligned} N_1(B)^*(\{a, b, d\}) &= \bigcup_{[x]_{B_1} \cap \{a,b,d\} \neq \emptyset} [x]_{B_1} \\ &= \bigcup \{\{a\}, \{b, d, e, f\}, \{a, d, f\}, \{b, e\}, \{a, b\}, \{c, d, f\}\} \\ &= X, \end{aligned}$$

and

$$\begin{aligned} N_1(B)^*(\{c, d, e, f\}) &= \bigcup_{[x]_{B_1} \cap \{c,d,e,f\} \neq \emptyset} [x]_{B_1} \\ &= \bigcup \{\{b, d, e, f\}, \{c\}, \{a, d, f\}, \{b, e\}, \{c, d, f\}, \{e\}\} \\ &= X. \end{aligned}$$

Remark 1. *It is observed that every r-near topology is not a classical topology from Example 1. If an r-near topology is a classical topology, then every r-near open set is an open set. Similarly, every r-near closed set is a closed set. Thus, the r-near topological concepts coincide with their classical counterparts.*

3. Clarifying previous definitions and properties

This section clarifies the definitions of specific concepts and their associated properties as presented in [22]. To this end, it presents counterexamples demonstrating that some inclusion relations do not hold in general. Moreover, the section redefines the r-near neighborhoods and revisits an associated theorem to ensure conceptual and structural consistency.

Proposition 2. *[22, Proposition 4] Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A \subseteq X$*. Then, the following properties are valid:*

i. $\mathrm{int}_r^*(A) \subseteq \mathrm{int}(A)$.

ii. $\mathrm{cl}(A) \subseteq \mathrm{cl}_r^*(A)$.

With the help of Definition 9, Examples 2 and 3 manifest that the inclusions in Proposition 2 may not be valid.

Example 2. ***(Counterexample to Proposition 2** (i)) Consider Example 1 and the set* $A = \{a, b, e\}$*. Then,* $\mathrm{int}(A) = \{a, b\}$*,* $\mathrm{int}_1^*(A) = \emptyset$*,* $\mathrm{int}_2^*(A) = A$*, and* $\mathrm{int}_3^*(A) = \{a, b\}$*. Therefore,* $\mathrm{int}_1^*(A) \subseteq \mathrm{int}(A)$ *and* $\mathrm{int}_3^*(A) \subseteq \mathrm{int}(A)$ *but* $\mathrm{int}_2^*(A) \nsubseteq \mathrm{int}(A)$*. Hence, for all r, the inclusion* $\mathrm{int}_r^*(A) \subseteq \mathrm{int}(A)$ *is not always valid.*

Example 3. ***(Counterexample to Proposition 2** (ii)) From Example 1,*

$$\tau^c = \{\emptyset, X, \{a, b\}, \{c, e, f\}, \{c, d, e, f\}, \{a, b, c, e, f\}\},$$

$$\tau_1^{*c} = \{\emptyset, X, \{c\}\},$$

$$\tau_2^{*c} = \{\emptyset, X, \{a\}, \{c\}, \{c, d, f\}, \{a, b, c, e\}\},$$

and

$$\tau_3^{*c} = \{\emptyset, X, \{a, b\}, \{c, e\}, \{a, b, c, e\}, \{c, d, e, f\}\}.$$

Thus, for $A = \{a, b, e\}$*,* $\mathrm{cl}(A) = \{a, b, c, e, f\}$*,* $\mathrm{cl}_1^*(A) = X$*,* $\mathrm{cl}_2^*(A) = \{a, b, c, e\}$*, and* $\mathrm{cl}_3^*(A) = \{a, b, c, e\}$*. Therefore,* $\mathrm{cl}(A) \subseteq \mathrm{cl}_1^*(A)$ *but* $\mathrm{cl}(A) \nsubseteq \mathrm{cl}_2^*(A)$ *and* $\mathrm{cl}(A) \nsubseteq \mathrm{cl}_3^*(A)$*. Hence, for all r, the inclusion* $\mathrm{cl}(A) \subseteq \mathrm{cl}_r^*(A)$ *is not always valid.*

The following corollaries are derived from the above counterexamples.

Corollary 1. *Every point in the r-near interior of a set may not be an interior point of the set.*

Corollary 2. *Every closure point of a set may not be a point in the r-near closure of the set.*

Additionally, the following examples demonstrate that for all r and s such that $r \leq s \leq |B|$, the inclusions $\mathrm{int}_r^*(A) \subseteq \mathrm{int}_s^*(A)$, $\mathrm{int}_s^*(A) \subseteq \mathrm{int}_r^*(A)$, $\mathrm{cl}_r^*(A) \subseteq \mathrm{cl}_s^*(A)$, and $\mathrm{cl}_s^*(A) \subseteq \mathrm{cl}_r^*(A)$ may not be valid.

Example 4. *In Example 2,* $\mathrm{int}_2^*(A) \nsubseteq \mathrm{int}_3^*(A)$ *and* $\mathrm{int}_2^*(A) \nsubseteq \mathrm{int}_1^*(A)$*. Hence, every* 2*-near interior point of the set A is neither a* 3*-near interior point nor a* 1*-near interior point of it.*

Example 5. *Consider Example 1 and the set* $A = \{c, d, f\}$*. Then,* $\mathrm{cl}_1^*(A) = X$*,* $\mathrm{cl}_2^*(A) = A$*, and* $\mathrm{cl}_3^*(A) = \{c, d, e, f\}$*. Therefore,* $\mathrm{cl}_1^*(A) \nsubseteq \mathrm{cl}_3^*(A)$ *and* $\mathrm{cl}_3^*(A) \nsubseteq \mathrm{cl}_2^*(A)$*. Hence, every* 1*-near closure point of the set A is not a* 3*-near closure point of it, and every* 3*-near closure point of the set A is not a* 2*-near closure point of it.*

Definition 10. *[22, Definition 9] Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $N \subseteq X$, $G \in \tau$, and $x \in X$. If there exists a $G_r^* \in \tau_r^*$ such that $x \in G$ and $G \subseteq G_r^* \subseteq N$, then the set N is called an r-near neighborhood of the point x. If the set N is an r-near open set, then the set N is called an r-near open neighborhood of the point x.*

Henceforth, the family of all r-near neighborhoods of the point x and the family of all r-near open neighborhoods of the point x are denoted by $N_r^*(x)$ and $\tau_r^*(x)$, respectively.

Example 6. *Consider Example 1 and Definition 10. Then, the families of all* 1*-near neighborhoods of the elements in the set X are as follows:*

$$N_1^*(a) = \{\{a,b,d,e,f\}, X\}, \quad N_1^*(b) = N_1^*(a), \quad N_1^*(c) = \{X\},$$
$$N_1^*(d) = N_1^*(a), \quad N_1^*(e) = N_1^*(c), \quad N_1^*(f) = N_1^*(c).$$

The families of all 1*-near open neighborhoods of the elements in the set X are as follows:*

$$\tau_1^*(a) = \{\{a,b,d,e,f\}, X\}, \quad \tau_1^*(b) = \tau_1^*(a), \quad \tau_1^*(c) = \{X\},$$
$$\tau_1^*(d) = \tau_1^*(a), \quad \tau_1^*(e) = \tau_1^*(c), \quad \tau_1^*(f) = \tau_1^*(c).$$

The families of all 2*-near neighborhoods of the elements in the set X are as follows:*

$$N_2^*(a) = \{N \subseteq X : \{a,b,e\} \subseteq N\}, \quad N_2^*(b) = N_2^*(a),$$
$$N_2^*(c) = \{\{b,c,d,e,f\}, X\}, \quad N_2^*(d) = \{N \subseteq X : \{d,f\} \subseteq N\},$$
$$N_2^*(e) = N_2^*(c), \quad N_2^*(f) = N_2^*(c).$$

The families of all 2*-near open neighborhoods of the elements in the set X are as follows:*

$$\tau_2^*(a) = \{\{a,b,e\}, \{a,b,d,e,f\}, X\}, \quad \tau_2^*(b) = \tau_2^*(a),$$
$$\tau_2^*(c) = \{\{b,c,d,e,f\}, X\}, \quad \tau_2^*(d) = \{\{d,f\}, \{a,b,d,e,f\}, \{b,c,d,e,f\}, X\},$$
$$\tau_2^*(e) = \tau_2^*(c), \quad \tau_2^*(f) = \tau_2^*(c).$$

The families of all 3*-near neighborhoods of the elements in the set X are as follows:*

$$N_3^*(a) = \{N \subseteq X : \{a,b\} \subseteq N\}, \quad N_3^*(b) = N_3^*(a),$$
$$N_3^*(c) = \{\{c,d,e,f\}, \{a,c,d,e,f\}, \{b,c,d,e,f\}, X\}, \quad N_3^*(d) = \{N \subseteq X : \{d,f\} \subseteq N\},$$
$$N_3^*(e) = N_3^*(c), \quad N_3^*(f) = N_3^*(c).$$

The families of all 3*-near open neighborhoods of the elements in the set X are as follows:*

$$\tau_3^*(a) = \{\{a,b\}, \{a,b,d,f\}, X\}, \quad \tau_3^*(b) = \tau_3^*(a),$$
$$\tau_3^*(c) = \{\{c,d,e,f\}, X\}, \quad \tau_3^*(d) = \{\{d,f\}, \{a,b,d,f\}, \{c,d,e,f\}, X\},$$
$$\tau_3^*(e) = \tau_3^*(c), \quad \tau_3^*(f) = \tau_3^*(c).$$

For instance, $e \in \{a,b,d,e,f\}$ and $\{a,b,d,e,f\} \in \tau_1^*$, but $\{a,b,d,e,f\} \notin \tau_1^*(e)$. Similarly, $f \in \{d,f\}$ and $\{d,f\} \in \tau_2^*$, but $\{d,f\} \notin \tau_2^*(f)$. Finally, $f \in \{a,b,d,f\}$ and $\{a,b,d,f\} \in \tau_3^*$, but $\{a,b,d,f\} \notin \tau_3^*(f)$. This case results from the definition of r-near open neighborhoods, i.e., Definition 10.

Corollary 3. *According to Definition 10, in an r-near topological space, every r-near open set may not be an r-near open neighborhood of all elements in this set.*

Theorem 5. *[22, Theorem 6] Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $A \subseteq X$, and $x \in X$. Then, $x \in \mathrm{cl}_r^*(A)$ if and only if $N \cap A \neq \emptyset$, for all $N \in N_r^*(x)$.*

Example 7. ***(Counterexample to Theorem 5)*** *Consider Examples 1 and 6 and the set $A = \{a, b, e\}$. Thus, $\mathrm{cl}_2^*(A) = \{a, b, c, e\}$ from Definition 9. Besides, $\mathrm{cl}_2^*(A) = \{a, b, c, e, f\}$ from Theorem 5. Here,*

- *For all $N \in N_2^*(a)$, $A \subseteq N$. Thus, $a \in \mathrm{cl}_2^*(A)$ since $N \cap A \neq \emptyset$ for all $N \in N_2^*(a)$.*
- *For all $N \in N_2^*(b)$, $A \subseteq N$. Thus, $b \in \mathrm{cl}_2^*(A)$ since $N \cap A \neq \emptyset$ for all $N \in N_2^*(b)$.*
- *For $\{b, c, d, e, f\}, X \in N_2^*(c)$, since $\{b, c, d, e, f\} \cap A \neq \emptyset$ and $X \cap A \neq \emptyset$, then $c \in \mathrm{cl}_2^*(A)$.*
- *For $\{d, f\} \in N_2^*(d)$, since $\{d, f\} \cap A = \emptyset$, then $d \notin \mathrm{cl}_2^*(A)$.*
- *For $\{b, c, d, e, f\}, X \in N_2^*(e)$, since $\{b, c, d, e, f\} \cap A \neq \emptyset$ and $X \cap A \neq \emptyset$, then $e \in \mathrm{cl}_2^*(A)$.*
- *For $\{b, c, d, e, f\}, X \in N_2^*(f)$, since $\{b, c, d, e, f\} \cap A \neq \emptyset$ and $X \cap A \neq \emptyset$, then $f \in \mathrm{cl}_2^*(A)$.*

Consequently, the result obtained from Definition 9 contradicts that of Theorem 5. This case results from the definition of r-near neighborhoods i.e., Definition 10.

Example 7 demonstrates that the conclusion obtained from Definition 9 does not coincide with that obtained from the characterization of the r-near closure provided in Theorem 5. This inconsistency indicates that the r-near closure is not uniquely determined by the r-near neighborhood defined in Definition 10. For this reason, this section redefines Definition 10 as in Definition 11 and revises Theorem 5 as in Theorem 6 to ensure compatibility with classical intuition. From now on, this study uses Definition 11 and Theorem 6.

Definition 11. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $N \subseteq X$, and $x \in X$. If there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq N$, then the set N is called an r-near neighborhood of the point x. Besides, if the set N is an r-near open set, then the set N is called an r-near open neighborhood of the point x.*

Henceforth, the family of all r-near neighborhoods of the point x and the family of all r-near open neighborhoods of the point x are denoted by $\mathcal{N}_r^*(x)$ and $\mathcal{T}_r^*(x)$, respectively.

Example 8. *Consider Example 1 and Definition 11. Then, the families of all* 1*-near neighborhoods of the elements in the set X are as follows:*

$$\begin{aligned} &\mathcal{N}_1^*(a) = \{\{a, b, d, e, f\}, X\}, && \mathcal{N}_1^*(b) = \mathcal{N}_1^*(a), && \mathcal{N}_1^*(c) = \{X\}, \\ &\mathcal{N}_1^*(d) = \mathcal{N}_1^*(a), && \mathcal{N}_1^*(e) = \mathcal{N}_1^*(a), && \mathcal{N}_1^*(f) = \mathcal{N}_1^*(a). \end{aligned}$$

The families of all 1*-near open neighborhoods of the elements in the set X are as follows:*

$$\begin{aligned} &\mathcal{T}_1^*(a) = \{\{a, b, d, e, f\}, X\}, && \mathcal{T}_1^*(b) = \mathcal{T}_1^*(a), && \mathcal{T}_1^*(c) = \{X\}, \\ &\mathcal{T}_1^*(d) = \mathcal{T}_1^*(a), && \mathcal{T}_1^*(e) = \mathcal{T}_1^*(a), && \mathcal{T}_1^*(f) = \mathcal{T}_1^*(a). \end{aligned}$$

The families of all 2*-near neighborhoods of the elements in the set X are as follows:*

$$\mathcal{N}_2^*(a) = \{N \subseteq X : \{a,b,e\} \subseteq N\}, \quad \mathcal{N}_2^*(b) = \{N \subseteq X : \{a,b,e\} \subseteq N\} \cup \{\{b,c,d,e,f\}\},$$
$$\mathcal{N}_2^*(c) = \{\{b,c,d,e,f\}, X\}, \quad \mathcal{N}_2^*(d) = \{N \subseteq X : \{d,f\} \subseteq N\},$$
$$\mathcal{N}_2^*(e) = \mathcal{N}_2^*(b), \quad \mathcal{N}_2^*(f) = \mathcal{N}_2^*(d).$$

The families of all 2*-near open neighborhoods of the elements in the set X are as follows:*

$$\mathcal{T}_2^*(a) = \{\{a,b,e\}, \{a,b,d,e,f\}, X\}, \quad \mathcal{T}_2^*(b) = \{\{a,b,e\}, \{a,b,d,e,f\}, \{b,c,d,e,f\}, X\},$$
$$\mathcal{T}_2^*(c) = \{\{b,c,d,e,f\}, X\}, \quad \mathcal{T}_2^*(d) = \{\{d,f\}, \{a,b,d,e,f\}, \{b,c,d,e,f\}, X\},$$
$$\mathcal{T}_2^*(e) = \mathcal{T}_2^*(b), \quad \mathcal{T}_2^*(f) = \mathcal{T}_2^*(d).$$

The families of all 3*-near neighborhoods of the elements in the set X are as follows:*

$$\mathcal{N}_3^*(a) = \{N \subseteq X : \{a,b\} \subseteq N\}, \quad \mathcal{N}_3^*(b) = \mathcal{N}_3^*(a),$$
$$\mathcal{N}_3^*(c) = \{\{c,d,e,f\}, \{a,c,d,e,f\}, \{b,c,d,e,f\}, X\}, \quad \mathcal{N}_3^*(d) = \{N \subseteq X : \{d,f\} \subseteq N\},$$
$$\mathcal{N}_3^*(e) = \mathcal{N}_3^*(c), \quad \mathcal{N}_3^*(f) = \mathcal{N}_3^*(d).$$

The families of all 3*-near open neighborhoods of the elements in the set X are as follows:*

$$\mathcal{T}_3^*(a) = \{\{a,b\}, \{a,b,d,f\}, X\}, \quad \mathcal{T}_3^*(b) = \mathcal{T}_3^*(a),$$
$$\mathcal{T}_3^*(c) = \{\{c,d,e,f\}, X\}, \quad \mathcal{T}_3^*(d) = \{\{d,f\}, \{a,b,d,f\}, \{c,d,e,f\}, X\},$$
$$\mathcal{T}_3^*(e) = \mathcal{T}_3^*(c), \quad \mathcal{T}_3^*(f) = \mathcal{T}_3^*(d).$$

Theorem 6 introduces the relationship between the r-near closure and r-near open neighborhoods.

Theorem 6. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $A \subseteq X$, and $x \in X$. Then, $x \in \mathrm{cl}_r^*(A)$ if and only if $N \cap A \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $A \subseteq X$, and $x \in X$.
($\Rightarrow$): Assume that $x \in \mathrm{cl}_r^*(A)$ and $N \cap A = \emptyset$, for an $N \in \mathcal{T}_r^*(x)$. Then, $A \subseteq X \setminus N$, for an $N \in \tau_r^*$. Therefore, there exists an $X \setminus N \in \tau_r^{*c}$ such that $A \subseteq X \setminus N$ and $x \notin X \setminus N$. Thus, $x \notin \mathrm{cl}_r^*(A)$ from Definition 9. This is a contradiction. Hence, $N \cap A \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$.
($\Leftarrow$): Assume that $N \cap A \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$, and $x \notin \mathrm{cl}_r^*(A)$. From Definition 9, there exists a $F_r^* \in \tau_r^{*c}$ such that $A \subseteq F_r^*$ and $x \notin F_r^*$. Therefore, $(X \setminus F_r^*) \cap A = \emptyset$, for an $X \setminus F_r^* \in \mathcal{T}_r^*(x)$. This is a contradiction. Hence, $x \in \mathrm{cl}_r^*(A)$. □

Example 9. *Consider Examples 1 and 8 and the set $A = \{a,b,e\}$. Thus, $\mathrm{cl}_2^*(A) = \{a,b,c,e\}$ from Definition 9. Besides, $\mathrm{cl}_2^*(A) = \{a,b,c,e\}$ from Theorem 6. Here,*

- *For $\{a,b,e\}, \{a,b,d,e,f\}, X \in \mathcal{T}_2^*(a)$, since $\{a,b,e\} \cap A \neq \emptyset$, $\{a,b,d,e,f\} \cap A \neq \emptyset$, and $X \cap A \neq \emptyset$, then $a \in \mathrm{cl}_2^*(A)$.*
- *For $\{a,b,e\}, \{a,b,d,e,f\}, \{b,c,d,e,f\}, X \in \mathcal{T}_2^*(b)$, since $\{a,b,e\} \cap A \neq \emptyset$, $\{a,b,d,e,f\} \cap A \neq \emptyset$, $\{b,c,d,e,f\} \cap A \neq \emptyset$, and $X \cap A \neq \emptyset$, then $b \in \mathrm{cl}_2^*(A)$.*

- *For* $\{b,c,d,e,f\}, X \in \mathcal{T}_2^*(c)$*, since* $\{b,c,d,e,f\} \cap A \neq \emptyset$ *and* $X \cap A \neq \emptyset$*, then* $c \in \mathrm{cl}_2^*(A)$.
- *For* $\{d,f\} \in \mathcal{T}_2^*(d)$*, since* $\{d,f\} \cap A = \emptyset$*, then* $d \notin \mathrm{cl}_2^*(A)$.
- *For* $\{a,b,e\}, \{a,b,d,e,f\}, \{b,c,d,e,f\}, X \in \mathcal{T}_2^*(e)$*, since* $\{a,b,e\} \cap A \neq \emptyset$*,* $\{a,b,d,e,f\} \cap A \neq \emptyset$*,* $\{b,c,d,e,f\} \cap A \neq \emptyset$*, and* $X \cap A \neq \emptyset$*, then* $e \in \mathrm{cl}_2^*(A)$.
- *For* $\{d,f\} \in \mathcal{T}_2^*(f)$*, since* $\{d,f\} \cap A = \emptyset$*, then* $f \notin \mathrm{cl}_2^*(A)$.

Consequently, the result obtained from Definition 9 is the same as that of Theorem 6.

4. Basic concepts and their related properties in r-near topology

To make a theoretical contribution to the conceptualization of r-near topology, this section first explores some properties related to the concepts of r-near closed sets, r-near interior, r-near closure, and r-near neighborhoods.

Theorem 7. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space. Then,* $\emptyset$ *and* X *are r-near closed sets.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space. From Theorem 1, $\emptyset$ and X are r-near open sets. Since $X \setminus \emptyset = X$ and $X \setminus X = \emptyset$, then $\emptyset$ and X are r-near closed sets by Definition 8. □

Theorem 8. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space. For all* $i \in I$*, if* $(F_i)_r^* \in \tau_r^{*c}$*, then* $\bigcap\limits_{i \in I} (F_i)_r^* \in \tau_r^{*c}$*. Here, I is an arbitrary index set.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $(F_i)_r^* \in \tau_r^{*c}$, for all $i \in I$. From Definition 8, $X \setminus (F_i)_r^* \in \tau_r^*$, for all $i \in I$. Thus, $\bigcup\limits_{i \in I} X \setminus (F_i)_r^* \in \tau_r^*$ from Theorem 2. Besides, since $\bigcup\limits_{i \in I} X \setminus (F_i)_r^* = X \setminus \bigcap\limits_{i \in I} (F_i)_r^*$, then $X \setminus \bigcap\limits_{i \in I} (F_i)_r^* \in \tau_r^*$. Hence, $\bigcap\limits_{i \in I} (F_i)_r^* \in \tau_r^{*c}$. □

Corollary 4. *In an r-near topological space, the arbitrary intersection of r-near closed sets is also an r-near closed set.*

Example 10. *Consider Example 1. Since*

$$\tau_2^* = \{\emptyset, X, \{d,f\}, \{a,b,e\}, \{a,b,d,e,f\}, \{b,c,d,e,f\}\},$$

then

$$\tau_2^{*c} = \{\emptyset, X, \{a\}, \{c\}, \{c,d,f\}, \{a,b,c,e\}\}.$$

Therefore, $\{a\} \cup \{c\} = \{a,c\} \notin \tau_2^{*c}$*, for* $\{a\}, \{c\} \in \tau_2^{*c}$*. Hence, in the* 2*-near topology* τ_2^**, the union of two* 2*-near closed sets is not a* 2*-near closed set.*

Corollary 5. *In an r-near topological space, the finite union of r-near closed sets may not be an r-near closed set.*

Definition 12. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space,* $A \subseteq X$*, and* $x \in A$*. If* $x \in \mathrm{int}_r^*(A)$*, then the point x is called an r-near interior point of the set A. In other words, if there exists a* $G_r^* \in \tau_r^*$ *such that* $x \in G_r^*$ *and* $G_r^* \subseteq A$*, then the point x is called an r-near interior point of the set A.*

Proposition 3. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A \subseteq X$. *Then, the following properties are valid:*

i. The set $\text{int}_r^*(A)$ *is an r-near open set.*

ii. The set $\text{int}_r^*(A)$ *is the biggest r-near open set contained by the set A.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.

i. Since $\text{int}_r^*(A) = \bigcup \{G_r^* \subseteq X : G_r^* \in \tau_r^* \text{ and } G_r^* \subseteq A\}$, then the set $\text{int}_r^*(A)$ is an r-near open set from Theorem 2.

ii. Since $\text{int}_r^*(A) = \bigcup \{G_r^* \subseteq X : G_r^* \in \tau_r^* \text{ and } G_r^* \subseteq A\}$, then $G_r^* \subseteq \text{int}_r^*(A)$ such that $G_r^* \subseteq A$, for all $G_r^* \in \tau_r^*$. Hence, the set $\text{int}_r^*(A)$ is the biggest r-near open set contained by the set A.

□

Proposition 4. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A, A_1, A_2 \subseteq X$. *Then, the following properties are valid:*

i. $\text{int}_r^*(A) \subseteq A$.

ii. $A \in \tau_r^* \Leftrightarrow \text{int}_r^*(A) = A$.

iii. $\text{int}_r^*(\text{int}_r^*(A)) = \text{int}_r^*(A)$.

iv. $\text{int}_r^*(A_1 \cap A_2) \subseteq \text{int}_r^*(A_1) \cap \text{int}_r^*(A_2)$.

v. $\text{int}_r^*(A_1) \cup \text{int}_r^*(A_2) \subseteq \text{int}_r^*(A_1 \cup A_2)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A, A_1, A_2 \subseteq X$.

i. Assume that $x \in \text{int}_r^*(A)$. From Definition 12, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq A$. Therefore, $x \in A$. Hence, $\text{int}_r^*(A) \subseteq A$.

ii. ($\Rightarrow$): Assume that $A \in \tau_r^*$. Since the set $\text{int}_r^*(A)$ is the biggest r-near open set contained by the set A, then $\text{int}_r^*(A) = A$.
($\Leftarrow$): Assume that $\text{int}_r^*(A) = A$. Therefore, $A \in \tau_r^*$ from Proposition 3 (*i*).

iii. Assume that $\text{int}_r^*(A) = A_1$. Therefore, $A_1 \in \tau_r^*$ from Proposition 3 (*i*). Thus, $\text{int}_r^*(A_1) = A_1$ from Proposition 4 (*ii*). Hence, $\text{int}_r^*(\text{int}_r^*(A)) = \text{int}_r^*(A)$.

iv. Since $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$, then $\text{int}_r^*(A_1 \cap A_2) \subseteq \text{int}_r^*(A_1)$ and $\text{int}_r^*(A_1 \cap A_2) \subseteq \text{int}_r^*(A_2)$ from Theorem 4 (*i*). Hence, $\text{int}_r^*(A_1 \cap A_2) \subseteq \text{int}_r^*(A_1) \cap \text{int}_r^*(A_2)$.

v. Since $A_1 \subseteq A_1 \cup A_2$ and $A_2 \subseteq A_1 \cup A_2$, then $\text{int}_r^*(A_1) \subseteq \text{int}_r^*(A_1 \cup A_2)$ and $\text{int}_r^*(A_2) \subseteq \text{int}_r^*(A_1 \cup A_2)$ from Theorem 4 (*i*). Hence, $\text{int}_r^*(A_1) \cup \text{int}_r^*(A_2) \subseteq \text{int}_r^*(A_1 \cup A_2)$.

□

Example 11. *Consider τ_1^* and τ_2^* in Example 1 and the sets $A_1 = \{a, b, c, e, f\}$ and $A_2 = \{b, c, d, e, f\}$. Since*

$$\mathrm{int}_1^*(A_1) = \emptyset, \ \ \mathrm{int}_1^*(A_2) = \emptyset, \ \ \mathrm{int}_1^*(A_1) \cap \mathrm{int}_1^*(A_2) = \emptyset, \ \ \textit{and} \ \ \mathrm{int}_1^*(A_1 \cap A_2) = \mathrm{int}_1^*(\{b, c, e, f\}) = \emptyset$$

then $\mathrm{int}_1^*(A_1 \cap A_2) = \mathrm{int}_1^*(A_1) \cap \mathrm{int}_1^*(A_2)$. *However, since*

$$\mathrm{int}_2^*(A_1) = \{a, b, e\}, \ \ \mathrm{int}_2^*(A_2) = A_2, \ \ \mathrm{int}_2^*(A_1) \cap \mathrm{int}_2^*(A_2) = \{b, e\}, \ \ \textit{and} \ \ \mathrm{int}_2^*(A_1 \cap A_2) = \mathrm{int}_2^*(\{b, c, e, f\}) = \emptyset$$

then $\mathrm{int}_2^*(A_1 \cap A_2) \subsetneq \mathrm{int}_2^*(A_1) \cap \mathrm{int}_2^*(A_2)$. *Consequently, for all r,* $\mathrm{int}_r^*(A_1 \cap A_2) = \mathrm{int}_r^*(A_1) \cap \mathrm{int}_r^*(A_2)$ *is not always valid.*

Proposition 5. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then, the following properties are valid:*

i. The set $\mathrm{cl}_r^*(A)$ *is an r-near closed set.*

ii. The set $\mathrm{cl}_r^*(A)$ *is the smallest r-near closed set containing the set A.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.

i. Since $\mathrm{cl}_r^*(A) = \bigcap\{F_r^* \subseteq X : F_r^* \in \tau_r^{*c} \text{ and } A \subseteq F_r^*\}$, then the set $\mathrm{cl}_r^*(A)$ is an r-near closed set from Theorem 8.

ii. Since $\mathrm{cl}_r^*(A) = \bigcap\{F_r^* \subseteq X : F_r^* \in \tau_r^{*c} \text{ and } A \subseteq F_r^*\}$, then $\mathrm{cl}_r^*(A) \subseteq F_r^*$ such that $A \subseteq F_r^*$, for all $F_r^* \in \tau_r^{*c}$. Hence, the set $\mathrm{cl}_r^*(A)$ is the smallest r-near closed set containing the set A.

□

Proposition 6. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A, A_1, A_2 \subseteq X$. Then, the following properties are valid:*

i. $A \subseteq \mathrm{cl}_r^*(A)$.

ii. $A \in \tau_r^{*c} \Leftrightarrow \mathrm{cl}_r^*(A) = A$.

iii. $\mathrm{cl}_r^*(\mathrm{cl}_r^*(A)) = \mathrm{cl}_r^*(A)$.

iv. $\mathrm{cl}_r^*(A_1 \cap A_2) \subseteq \mathrm{cl}_r^*(A_1) \cap \mathrm{cl}_r^*(A_2)$.

v. $\mathrm{cl}_r^*(A_1) \cup \mathrm{cl}_r^*(A_2) \subseteq \mathrm{cl}_r^*(A_1 \cup A_2)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A, A_1, A_2 \subseteq X$.

i. Assume that $x \in A$. Then, $x \in F_r^*$, for all $F_r^* \in \tau_r^{*c}$ such that $A \subseteq F_r^*$. From Definition 9, $x \in \mathrm{cl}_r^*(A)$. Hence, $A \subseteq \mathrm{cl}_r^*(A)$.

ii. (⇒): Assume that $A \in \tau_r^{*c}$. Since the set $\mathrm{cl}_r^*(A)$ is the smallest r-near closed set containing the set A, then $\mathrm{cl}_r^*(A) = A$.
(⇐): Assume that $\mathrm{cl}_r^*(A) = A$. Therefore, $A \in \tau_r^{*c}$ from Proposition 5 (*i*).

iii. Assume that $\mathrm{cl}_r^*(A) = A_1$. Therefore, $A_1 \in \tau_r^{*c}$ from Proposition 5 (*i*). Thus, $\mathrm{cl}_r^*(A_1) = A_1$ from Proposition 6 (*ii*). Hence, $\mathrm{cl}_r^*(\mathrm{cl}_r^*(A)) = \mathrm{cl}_r^*(A)$.

iv. Since $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$, then $\mathrm{cl}_r^*(A_1 \cap A_2) \subseteq \mathrm{cl}_r^*(A_1)$ and $\mathrm{cl}_r^*(A_1 \cap A_2) \subseteq \mathrm{cl}_r^*(A_2)$ from Theorem 4 (*ii*). Hence, $\mathrm{cl}_r^*(A_1 \cap A_2) \subseteq \mathrm{cl}_r^*(A_1) \cap \mathrm{cl}_r^*(A_2)$.

v. Since $A_1 \subseteq A_1 \cup A_2$ and $A_2 \subseteq A_1 \cup A_2$, then $\mathrm{cl}_r^*(A_1) \subseteq \mathrm{cl}_r^*(A_1 \cup A_2)$ and $\mathrm{cl}_r^*(A_2) \subseteq \mathrm{cl}_r^*(A_1 \cup A_2)$ from Theorem 4 (*ii*). Hence, $\mathrm{cl}_r^*(A_1) \cup \mathrm{cl}_r^*(A_2) \subseteq \mathrm{cl}_r^*(A_1 \cup A_2)$.

□

Example 12. *Consider τ_1^* and τ_2^* in Example 1 and the sets $A_1 = \{a\}$ and $A_2 = \{d, f\}$. Then,*

$$\tau_1^{*c} = \{\emptyset, X, \{c\}\} \quad \textit{and} \quad \tau_2^{*c} = \{\emptyset, X, \{a\}, \{c\}, \{c, d, f\}, \{a, b, c, e\}\}.$$

Since

$$\mathrm{cl}_1^*(A_1) = X, \quad \mathrm{cl}_1^*(A_2) = X, \quad \mathrm{cl}_1^*(A_1) \cup \mathrm{cl}_1^*(A_2) = X, \quad \textit{and} \quad \mathrm{cl}_1^*(A_1 \cup A_2) = \mathrm{cl}_1^*(\{a, d, f\}) = X$$

then $\mathrm{cl}_1^*(A_1 \cup A_2) = \mathrm{cl}_1^*(A_1) \cup \mathrm{cl}_1^*(A_2)$*. However, since*

$$\mathrm{cl}_2^*(A_1) = A_1, \;\; \mathrm{cl}_2^*(A_2) = \{c, d, f\}, \;\; \mathrm{cl}_2^*(A_1) \cup \mathrm{cl}_2^*(A_2) = \{a, c, d, f\}, \;\; \textit{and} \;\; \mathrm{cl}_2^*(A_1 \cup A_2) = \mathrm{cl}_2^*(\{a, d, f\}) = X$$

then $\mathrm{cl}_2^*(A_1) \cup \mathrm{cl}_2^*(A_2) \subsetneq \mathrm{cl}_2^*(A_1 \cup A_2)$*. Consequently, for all r,* $\mathrm{cl}_r^*(A_1 \cup A_2) = \mathrm{cl}_r^*(A_1) \cup \mathrm{cl}_r^*(A_2)$ *is not always valid.*

Theorem 9. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $N, N_1, N_2, U \subseteq X$, and $x \in X$. Then, the following properties are valid:*

i. If $N \in \mathcal{N}_r^*(x)$, then $x \in N$.

ii. If $N_1 \in \mathcal{N}_r^*(x)$ and $N_1 \subseteq N_2$, then $N_2 \in \mathcal{N}_r^*(x)$.

iii. If $U \in N(x)$, then $N_r(B)^*(U) \in \mathcal{N}_r^*(x)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space $N, N_1, N_2, U \subseteq X$, and $x \in X$.

i. Assume that $N \in \mathcal{N}_r^*(x)$. From Definition 11, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq N$. Hence, $x \in N$.

ii. Assume that $N_1 \in \mathcal{N}_r^*(x)$ and $N_1 \subseteq N_2$. From Definition 11, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq N_1$. Since $N_1 \subseteq N_2$, then there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq N_2$. Hence, $N_2 \in \mathcal{N}_r^*(x)$.

iii. Assume that $U \in N(x)$. Then, there exists a $G \in \tau$ such that $x \in G$ and $G \subseteq U$. Moreover,

$$\begin{aligned} G_r^* &= N_r(B)^*(G) \\ &= \bigcup_{[x]_{B_r} \cap G \neq \emptyset} [x]_{B_r} \\ &\subseteq \bigcup_{[x]_{B_r} \cap U \neq \emptyset} [x]_{B_r}, \text{ as } G \subseteq U \\ &= N_r(B)^*(U). \end{aligned}$$

From Theorem 3 *iii*, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq N_r(B)^*(U)$. Hence, $N_r(B)^*(U) \in \mathcal{N}_r^*(x)$.

□

Example 13. *In Example 8,* $\{a,b,e\} \cap \{b,c,d,e,f\} = \{b,e\} \notin \mathcal{N}_2^*(b)$*, for* $\{a,b,e\}, \{b,c,d,e,f\} \in \mathcal{N}_2^*(b)$*. Hence, in the 2-near topology* τ_2^**, the intersection of two 2-near neighborhoods of the point b is not a 2-near neighborhood of the point b.*

Corollary 6. *In an r-near topological space, the finite intersection of r-near neighborhoods of a point may not be an r-near neighborhood of the point.*

Example 14. *In Example 8,* $\{b,c,d,e,f\} \in \mathcal{N}_2^*(b)$*, but* $\{b,c,d,e,f\} \notin \mathcal{N}_3^*(b)$*. Hence, although the set* $\{b,c,d,e,f\}$ *is a* 2*-near neighborhood of point b, it is not a* 3*-near neighborhood of point b. Consequently, for all r and s such that* $r \leq s \leq |B|$*, although* $N \in \mathcal{N}_r^*(x)$*, it may not be* $N \in \mathcal{N}_s^*(x)$*.*

Theorem 10. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space,* $A \subseteq X$*, and* $x \in X$*. Then, the set A is an r-near open set if and only if, for all* $x \in A$*, there exists a* $G_r^* \in \tau_r^*$ *such that* $x \in G_r^*$ *and* $G_r^* \subseteq A$*.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $A \subseteq X$, and $x \in X$.
($\Rightarrow$): Assume that the set A is an r-near open set and $x \in A$. Then, let $G_r^* := A$. Thus, for all $x \in A$, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq A$.
($\Leftarrow$): Assume that, for all $x \in A$, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq A$. Therefore, for all $x \in A$, $A = \bigcup_{\substack{x \in G_r^* \\ G_r^* \subseteq A}} G_r^*$. Thus, the set A is an r-near open set from Theorem 2. □

Corollary 7. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space,* $A \subseteq X$*, and* $x \in X$*. Then, the set A is an r-near open set if and only if* $A \in \mathcal{T}_r^*(x)$*, for all* $x \in A$*.*

Secondly, this section defines the concepts of r-near accumulation points, r-near isolated points, r-near exterior points, and r-near boundary points in accordance with Definition 11 and Theorem 6 in Section 3. The redefined definition and the revised theorem ensure that the characterizations and properties of these novel concepts are logically consistent with the foundational structure of the r-near topology. This refinement also enables these concepts to parallel their classical topological counterparts coherently. In addition, this section provides further clarification of the theoretical contributions through illustrative examples.

Definition 13. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space,* $A \subseteq X$*, and* $x \in X$*. For all* $N \in \mathcal{T}_r^*(x)$*, if* $(N \setminus \{x\}) \cap A \neq \emptyset$*, then the point x is called an r-near accumulation point of the set A. Moreover, the set of all r-near accumulation points of the set A is denoted by* $\mathrm{acc}_r^*(A)$*.*

Theorem 11. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A_1, A_2 \subseteq X$*. If* $A_1 \subseteq A_2$*, then* $\mathrm{acc}_r^*(A_1) \subseteq \mathrm{acc}_r^*(A_2)$*.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A_1, A_2 \subseteq X$. Assume that $A_1 \subseteq A_2$ and $x \in \mathrm{acc}_r^*(A_1)$. From Definition 13, $(N \setminus \{x\}) \cap A_1 \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$. Since $A_1 \subseteq A_2$, then $(N \setminus \{x\}) \cap A_2 \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$. Therefore, $x \in \mathrm{acc}_r^*(A_2)$. Hence, $\mathrm{acc}_r^*(A_1) \subseteq \mathrm{acc}_r^*(A_2)$. □

Example 15. *Consider Examples 1 and 8 and the set* $A = \{c,d,f\}$*. Then,*

$$\mathrm{acc}(A) = \{c,e,f\}, \quad \mathrm{acc}_1^*(A) = X, \quad \mathrm{acc}_2^*(A) = A, \quad \textit{and} \quad \mathrm{acc}_3^*(A) = \{c,d,e,f\}.$$

Therefore, $\operatorname{acc}(A) \subseteq \operatorname{acc}_1^*(A)$ *and* $\operatorname{acc}(A) \subseteq \operatorname{acc}_3^*(A)$*, but* $\operatorname{acc}(A) \nsubseteq \operatorname{acc}_2^*(A)$*. Hence, every accumulation point of the set A is both a* 1*-near and a* 3*-near accumulation point of it. However, every accumulation point of the set A is not a* 2*-near accumulation point of it. Consequently, for all r, the inclusion* $\operatorname{acc}(A) \subseteq \operatorname{acc}_r^*(A)$ *is not always valid. To exemplify, from Example 8, the set* $\operatorname{acc}_3^*(A)$ *is as follows:*

- *For* $\{a, b\} \in \mathcal{T}_3^*(a)$*, since* $(\{a, b\} \setminus \{a\}) \cap A = \emptyset$*, then* $a \notin \operatorname{acc}_3^*(A)$.
- *For* $\{a, b\} \in \mathcal{T}_3^*(b)$*, since* $(\{a, b\} \setminus \{b\}) \cap A = \emptyset$*, then* $b \notin \operatorname{acc}_3^*(A)$.
- *For* $\{c, d, e, f\}, X \in \mathcal{T}_3^*(c)$*, since*

$$(\{c, d, e, f\} \setminus \{c\}) \cap A = \{d, e, f\} \cap \{c, d, f\} = \{d, f\} \neq \emptyset,$$

  and

$$(X \setminus \{c\}) \cap A = \{a, b, d, e, f\} \cap \{c, d, f\} = \{d, f\} \neq \emptyset,$$

  then $c \in \operatorname{acc}_3^*(A)$.
- *For* $\{d, f\}, \{a, b, d, f\}, \{c, d, e, f\}, X \in \mathcal{T}_3^*(d)$*, since*

$$(\{d, f\} \setminus \{d\}) \cap A = \{f\} \cap \{c, d, f\} = \{f\} \neq \emptyset,$$

$$(\{a, b, d, f\} \setminus \{d\}) \cap A = \{a, b, f\} \cap \{c, d, f\} = \{f\} \neq \emptyset,$$

$$(\{c, d, e, f\} \setminus \{d\}) \cap A = \{c, e, f\} \cap \{c, d, f\} = \{c, f\} \neq \emptyset,$$

  and

$$(X \setminus \{d\}) \cap A = \{a, b, c, e, f\} \cap \{c, d, f\} = \{c, f\} \neq \emptyset,$$

  then $d \in \operatorname{acc}_3^*(A)$.
- *For* $\{c, d, e, f\}, X \in \mathcal{T}_3^*(e)$*, since*

$$(\{c, d, e, f\} \setminus \{e\}) \cap A = \{c, d, f\} \cap \{c, d, f\} = \{c, d, f\} \neq \emptyset,$$

  and

$$(X \setminus \{e\}) \cap A = \{a, b, c, d, f\} \cap \{c, d, f\} = \{c, d, f\} \neq \emptyset,$$

  then $e \in \operatorname{acc}_3^*(A)$.
- *For* $\{d, f\}, \{a, b, d, f\}, \{c, d, e, f\}, X \in \mathcal{T}_3^*(f)$*, since*

$$(\{d, f\} \setminus \{f\}) \cap A = \{d\} \cap \{c, d, f\} = \{d\} \neq \emptyset,$$

$$(\{a, b, d, f\} \setminus \{f\}) \cap A = \{a, b, d\} \cap \{c, d, f\} = \{d\} \neq \emptyset,$$

$$(\{c, d, e, f\} \setminus \{f\}) \cap A = \{c, d, e\} \cap \{c, d, f\} = \{c, d\} \neq \emptyset,$$

  and

$$(X \setminus \{f\}) \cap A = \{a, b, c, d, e\} \cap \{c, d, f\} = \{c, d\} \neq \emptyset,$$

  then $f \in \operatorname{acc}_3^*(A)$.

Corollary 8. *Every accumulation point of a set may not be an r-near accumulation point of the set.*

Example 16 manifests that for all r and s such that $r \le s \le |B|$, the inclusions $\mathrm{acc}_r^*(A) \subseteq \mathrm{acc}_s^*(A)$ and $\mathrm{acc}_s^*(A) \subseteq \mathrm{acc}_r^*(A)$ may not be valid.

Example 16. *Consider Example 15. Then,* $\mathrm{acc}_1^*(A) \nsubseteq \mathrm{acc}_3^*(A)$ *and* $\mathrm{acc}_3^*(A) \nsubseteq \mathrm{acc}_2^*(A)$*. Hence, every* 1*-near accumulation point of the set A is not a* 3*-near accumulation point of it, and every* 3*-near accumulation point of the set A is not a* 2*-near accumulation point of it.*

Proposition 7. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* A, A_1*, and* $A_2 \subseteq X$*. Then, the following properties are valid:*

i. $\mathrm{acc}_r^*(A) \subseteq \mathrm{cl}_r^*(A)$.

ii. $\mathrm{acc}_r^*(A_1 \cap A_2) \subseteq \mathrm{acc}_r^*(A_1) \cap \mathrm{acc}_r^*(A_2)$.

iii. $\mathrm{acc}_r^*(A_1) \cup \mathrm{acc}_r^*(A_2) \subseteq \mathrm{acc}_r^*(A_1 \cup A_2)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and A, A_1, and $A_2 \subseteq X$.

i. Assume that $x \in \mathrm{acc}_r^*(A)$. From Definition 13, $(N \setminus \{x\}) \cap A \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$. Thus, $N \cap A \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$. Therefore, $x \in \mathrm{cl}_r^*(A)$ from Theorem 6. Hence, $\mathrm{acc}_r^*(A) \subseteq \mathrm{cl}_r^*(A)$.

ii. Since $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$, then $\mathrm{acc}_r^*(A_1 \cap A_2) \subseteq \mathrm{acc}_r^*(A_1)$ and $\mathrm{acc}_r^*(A_1 \cap A_2) \subseteq \mathrm{acc}_r^*(A_2)$ from Theorem 11. Hence, $\mathrm{acc}_r^*(A_1 \cap A_2) \subseteq \mathrm{acc}_r^*(A_1) \cap \mathrm{acc}_r^*(A_2)$.

iii. Since $A_1 \subseteq A_1 \cup A_2$ and $A_2 \subseteq A_1 \cup A_2$, then $\mathrm{acc}_r^*(A_1) \subseteq \mathrm{acc}_r^*(A_1 \cup A_2)$ and $\mathrm{acc}_r^*(A_2) \subseteq \mathrm{acc}_r^*(A_1 \cup A_2)$ from Theorem 11. Hence, $\mathrm{acc}_r^*(A_1) \cup \mathrm{acc}_r^*(A_2) \subseteq \mathrm{acc}_r^*(A_1 \cup A_2)$.

□

Example 17. *Consider* τ_1^* *and* τ_2^* *in Example 1 and the sets* $A_1 = \{a\}$ *and* $A_2 = \{d, f\}$*. From Example 8, since*

$$\mathrm{acc}_1^*(A_1) = \{b, c, d, e, f\},\ \ \mathrm{acc}_1^*(A_2) = X,\ \ \mathrm{acc}_1^*(A_1) \cup \mathrm{acc}_1^*(A_2) = X,\ \ \textit{and}\ \ \mathrm{acc}_1^*(A_1 \cup A_2) = X,$$

then $\mathrm{acc}_1^*(A_1 \cup A_2) = \mathrm{acc}_1^*(A_1) \cup \mathrm{acc}_1^*(A_2)$*. However, since*

$$\mathrm{acc}_2^*(A_1) = \emptyset,\ \ \mathrm{acc}_2^*(A_2) = \{c, d, f\},\ \ \mathrm{acc}_2^*(A_1) \cup \mathrm{acc}_2^*(A_2) = \{c, d, f\},\ \ \textit{and}\ \ \mathrm{acc}_2^*(A_1 \cup A_2) = \{b, c, d, e, f\},$$

then $\mathrm{acc}_2^*(A_1) \cup \mathrm{acc}_2^*(A_2) \subsetneq \mathrm{acc}_2^*(A_1 \cup A_2)$*. Consequently, for all r,* $\mathrm{acc}_r^*(A_1 \cup A_2) = \mathrm{acc}_r^*(A_1) \cup \mathrm{acc}_r^*(A_2)$ *is not always valid.*

Theorem 12 proposes a characterization of r-near closed sets via r-near accumulation points.

Theorem 12. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A \subseteq X$*. Then, the set A is an r-near closed set if and only if* $\mathrm{acc}_r^*(A) \subseteq A$*.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.
$(\Rightarrow)$: Assume that the set A is an r-near closed set, $x \in \mathrm{acc}_r^*(A)$, and $x \notin A$. Then, the set $X \setminus A$ is an r-near open set and $x \in X \setminus A$. From Corollary 7, $X \setminus A \in \mathcal{T}_r^*(x)$. Thus, there exists an $X \setminus A \in \mathcal{T}_r^*(x)$ such that $[(X \setminus A) \setminus \{x\}] \cap A = \emptyset$. Thereby, $x \notin \mathrm{acc}_r^*(A)$ from Definition 13. This is a contradiction. Therefore, $x \in A$. Hence, $\mathrm{acc}_r^*(A) \subseteq A$.

($\Leftarrow$): Assume that $\mathrm{acc}_r^*(A) \subseteq A$ and $x \notin A$. Therefore, $x \notin \mathrm{acc}_r^*(A)$. Thereby, there exists an $N \in \mathcal{T}_r^*(x)$ such that $(N \setminus \{x\}) \cap A = \emptyset$. Since $x \notin A$, then there exists an $N \in \mathcal{T}_r^*(x)$ such that $N \cap A = \emptyset$. Thus, $N \subseteq X \setminus A$. Hence, for all $x \in X \setminus A$, there exists an $N \in \tau_r^*$ such that $x \in N$ and $N \subseteq X \setminus A$. Then, the set $X \setminus A$ is an r-near open set from Theorem 10. Consequently, the set A is an r-near closed set. □

Theorem 13. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then,*

$$\mathrm{cl}_r^*(A) = A \cup \mathrm{acc}_r^*(A).$$

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Assume that $x \in \mathrm{cl}_r^*(A)$. Then, there are two cases.
Case 1. Assume that $x \in A$. Then, $x \in A \cup \mathrm{acc}_r^*(A)$. Hence, $\mathrm{cl}_r^*(A) \subseteq A \cup \mathrm{acc}_r^*(A)$.
Case 2. Assume that $x \notin A$. From Theorem 6, since $x \in \mathrm{cl}_r^*(A)$, then $N \cap A \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$. Moreover, since $x \notin A$, then $(N \setminus \{x\}) \cap A \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$. Therefore, $x \in \mathrm{acc}_r^*(A)$ from Definition 13. Thus, $x \in A \cup \mathrm{acc}_r^*(A)$. Hence, $\mathrm{cl}_r^*(A) \subseteq A \cup \mathrm{acc}_r^*(A)$.
Conversely, assume that $x \in A \cup \mathrm{acc}_r^*(A)$. Then, $x \in A$ or $x \in \mathrm{acc}_r^*(A)$. If $x \in A$, then $x \in \mathrm{cl}_r^*(A)$ from Proposition 6 (i). If $x \in \mathrm{acc}_r^*(A)$, then $x \in \mathrm{cl}_r^*(A)$ from Proposition 7 (i). Hence, $A \cup \mathrm{acc}_r^*(A) \subseteq \mathrm{cl}_r^*(A)$. Consequently, $\mathrm{cl}_r^*(A) = A \cup \mathrm{acc}_r^*(A)$. □

Definition 14. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $A \subseteq X$, and $x \in A$. If there exists an $N \in \mathcal{T}_r^*(x)$ such that $(N \setminus \{x\}) \cap A = \emptyset$, then the point x is called an r-near isolated point of the set A. Moreover, the set of all r-near isolated points of the set A is denoted by* $\mathrm{iso}_r^*(A)$.

Example 18. *Consider Examples 1 and 8 and the set $A = \{c, e, f\}$. Then,*

$$\mathrm{iso}(A) = \emptyset \quad \textit{and} \quad \mathrm{iso}_2^*(A) = \{e, f\}.$$

Therefore, $\mathrm{iso}_2^*(A) \not\subseteq \mathrm{iso}(A)$. *Hence, every 2-near isolated point of the set A is not an isolated point of it. Consequently, for all r, the inclusion* $\mathrm{iso}_r^*(A) \subseteq \mathrm{iso}(A)$ *is not always valid. To exemplify, from Example 8, the set* $\mathrm{iso}_2^*(A)$ *is as follows:*

- *Since there exist $\{a, b, e\} \in \mathcal{T}_2^*(e)$ and $\{d, f\} \in \mathcal{T}_2^*(f)$ such that*

$$(\{a, b, e\} \setminus \{e\}) \cap A = \{a, b\} \cap \{c, e, f\} = \emptyset,$$

and

$$(\{d, f\} \setminus \{f\}) \cap A = \{d\} \cap \{c, e, f\} = \emptyset,$$

then $e, f \in$ $\mathrm{iso}_2^*(A)$.

Corollary 9. *Every r-near isolated point of a set may not be an isolated point of the set.*

Example 19 signifies that for all r and s such that $r \leq s \leq |B|$, the inclusions $\mathrm{iso}_r^*(A) \subseteq \mathrm{iso}_s^*(A)$ and $\mathrm{iso}_s^*(A) \subseteq \mathrm{iso}_r^*(A)$ may not be valid.

Example 19. *Consider Examples 1 and 8 and the set $A = \{c, e, f\}$. Then,* $\mathrm{iso}_1^*(A) = \emptyset$, $\mathrm{iso}_2^*(A) = \{e, f\}$, *and* $\mathrm{iso}_3^*(A) = \{f\}$. *Then,* $\mathrm{iso}_3^*(A) \not\subseteq \mathrm{iso}_1^*(A)$ *and* $\mathrm{iso}_2^*(A) \not\subseteq \mathrm{iso}_3^*(A)$. *Hence, every 3-near isolated point of the set A is not a 1-near isolated point of it, and every 2-near isolated point of the set A is not a 3-near isolated point of it.*

Theorem 14. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A_1, A_2 \subseteq X$. If $A_1 \subseteq A_2$, then* $\mathrm{iso}_r^*(A_2) \subseteq \mathrm{iso}_r^*(A_1)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A_1, A_2 \subseteq X$. Assume that $A_1 \subseteq A_2$ and $x \in \mathrm{iso}_r^*(A_2)$. From Definition 14, there exists an $N \in \mathcal{T}_r^*(x)$ such that $(N \setminus \{x\}) \cap A_2 = \emptyset$. Since $A_1 \subseteq A_2$, then there exists an $N \in \mathcal{T}_r^*(x)$ such that $(N \setminus \{x\}) \cap A_1 = \emptyset$. Therefore, $x \in \mathrm{iso}_r^*(A_1)$. Hence, $\mathrm{iso}_r^*(A_2) \subseteq \mathrm{iso}_r^*(A_1)$. □

Definition 15. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then, an r-near interior point of the set $X \setminus A$ is called an r-near exterior point of the set A. In other words, if there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq X \setminus A$, then the point x is called an r-near exterior point of the set A. Moreover, the set of all r-near exterior points of the set A is called r-near exterior of the set A and denoted by* $\mathrm{ext}_r^*(A)$. *That is, the r-near exterior of the set A is as follows:*

$$\mathrm{ext}_r^*(A) := \bigcup \{G_r^* \subseteq X : G_r^* \in \tau_r^* \text{ and } G_r^* \subseteq X \setminus A\} = \mathrm{int}_r^*(X \setminus A).$$

Theorem 15. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A_1, A_2 \subseteq X$. If $A_1 \subseteq A_2$, then* $\mathrm{ext}_r^*(A_2) \subseteq \mathrm{ext}_r^*(A_1)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A_1, A_2 \subseteq X$. Assume that $A_1 \subseteq A_2$ and $x \in \mathrm{ext}_r^*(A_2)$. From Definition 15, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq X \setminus A_2$. Since $A_1 \subseteq A_2$, then there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq X \setminus A_1$. Thus, $x \in \mathrm{ext}_r^*(A_1)$. Hence, $\mathrm{ext}_r^*(A_2) \subseteq \mathrm{ext}_r^*(A_1)$. □

Example 20. *Consider Example 1 and the set $A = \{c, f\}$. Then,*

$$\mathrm{ext}(A) = \{a, b, d\} \quad \text{and} \quad \mathrm{ext}_2^*(A) = \{a, b, e\}.$$

Therefore, $\mathrm{ext}_2^*(A) \nsubseteq \mathrm{ext}(A)$. *Hence, every* 2*-near exterior point of the set A is not an exterior point of it. Consequently, for all r, the inclusion* $\mathrm{ext}_r^*(A) \subseteq \mathrm{ext}(A)$ *is not always valid.*

Corollary 10. *Every r-near exterior point of a set may not be an exterior point of the set.*

Example 21 indicates that for all r and s such that $r \leq s \leq |B|$, the inclusions $\mathrm{ext}_r^*(A) \subseteq \mathrm{ext}_s^*(A)$ and $\mathrm{ext}_s^*(A) \subseteq \mathrm{ext}_r^*(A)$ may not be valid.

Example 21. *Consider Example 1 and the set $A = \{c, f\}$. Then,* $\mathrm{ext}_1^*(A) = \emptyset$, $\mathrm{ext}_2^*(A) = \{a, b, e\}$, *and* $\mathrm{ext}_3^*(A) = \{a, b\}$. *Therefore,* $\mathrm{ext}_3^*(A) \nsubseteq \mathrm{ext}_1^*(A)$ *and* $\mathrm{ext}_2^*(A) \nsubseteq \mathrm{ext}_3^*(A)$. *Hence, every* 3*-near exterior point of the set A is not a* 1*-near exterior point of it, and every* 2*-near exterior point of the set A is not a* 3*-near exterior point of it.*

Proposition 8. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then,* $\mathrm{ext}_r^*(A) \subseteq X \setminus A$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Assume that, $x \in \mathrm{ext}_r^*(A)$. From Definition 15, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq X \setminus A$. Thus, $x \in X \setminus A$. Hence, $\mathrm{ext}_r^*(A) \subseteq X \setminus A$. □

Theorem 16. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then, the set A is an r-near closed set if and only if* $\mathrm{ext}_r^*(A) = X \setminus A$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.
($\Rightarrow$): Assume that the set A is an r-near closed set. From Definition 8, the set $X \setminus A$ is an r-near open set. From Proposition 4 (*ii*), $\mathrm{int}_r^*(X \setminus A) = X \setminus A$. Hence, $\mathrm{ext}_r^*(A) = X \setminus A$ from Definition 15.
($\Leftarrow$): Assume that $\mathrm{ext}_r^*(A) = X \setminus A$. From Definition 15, $\mathrm{int}_r^*(X \setminus A) = X \setminus A$. Therefore, the set $X \setminus A$ is an r-near open set from Proposition 4 (*ii*). Hence, the set A is an r-near closed set from Definition 8. □

Definition 16. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $A \subseteq X$, and $x \in X$. If a point x is neither an r-near interior point nor an r-near exterior point of the set A, then the point x is called an r-near boundary point of the set A. Moreover, the set of all r-near boundary points of the set A is called r-near boundary of the set A and denoted by* $\mathrm{bnd}_r^*(A)$.

Example 22. *Consider Example 1 and the set $A = \{c, d, f\}$. Since* $\mathrm{int}_1^*(A) = \emptyset$ *and* $\mathrm{ext}_1^*(A) = \emptyset$*, then* $\mathrm{bnd}_1^*(A) = X$. *Similarly, since* $\mathrm{int}_2^*(A) = \{d, f\}$ *and* $\mathrm{ext}_2^*(A) = \{a, b, e\}$, *then* $\mathrm{bnd}_2^*(A) = \{c\}$. *Finally, since* $\mathrm{int}_3^*(A) = \{d, f\}$ *and* $\mathrm{ext}_3^*(A) = \{a, b\}$, *then* $\mathrm{bnd}_3^*(A) = \{c, e\}$.

Theorem 17. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $A \subseteq X$, and $x \in X$. Then, $x \in \mathrm{bnd}_r^*(A)$ if and only if $N \cap A \neq \emptyset$ and $N \cap (X \setminus A) \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$.*

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space, $A \subseteq X$, and $x \in X$.
($\Rightarrow$): Assume that $x \in \mathrm{bnd}_r^*(A)$, and $N \cap A = \emptyset$ or $N \cap (X \setminus A) = \emptyset$, for an $N \in \mathcal{T}_r^*(x)$. Suppose that $N \cap A = \emptyset$, for an $N \in \mathcal{T}_r^*(x)$. Then, there exists an $N \in \tau_r^*$ such that $x \in N$ and $N \subseteq X \setminus A$. Therefore, $x \in \mathrm{ext}_r^*(A)$. Because $x \in \mathrm{bnd}_r^*(A)$, then $x \notin \mathrm{ext}_r^*(A)$. This is a contradiction. Similarly, suppose that $N \cap (X \setminus A) = \emptyset$, for an $N \in \mathcal{T}_r^*(x)$. Then, there exists an $N \in \tau_r^*$ such that $x \in N$ and $N \subseteq A$. Therefore, $x \in \mathrm{int}_r^*(A)$. Because $x \in \mathrm{bnd}_r^*(A)$, then $x \notin \mathrm{int}_r^*(A)$. This is a contradiction. Consequently, $N \cap A \neq \emptyset$ and $N \cap (X \setminus A) \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$.
($\Leftarrow$): Assume that $N \cap A \neq \emptyset$ and $N \cap (X \setminus A) \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$, and $x \notin \mathrm{bnd}_r^*(A)$. Then, $x \in \mathrm{int}_r^*(A)$ or $x \in \mathrm{ext}_r^*(A)$ from Definition 16. Suppose that $x \in \mathrm{int}_r^*(A)$. From Definition 12, there exists a $G_r^* \in \tau_r^*$ such that $x \in G_r^*$ and $G_r^* \subseteq A$. Therefore, there exists a $G_r^* \in \mathcal{T}_r^*(x)$ such that $G_r^* \cap (X \setminus A) = \emptyset$. This contradicts the condition that $N \cap (X \setminus A) \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$. Similarly, suppose that $x \in \mathrm{ext}_r^*(A)$. From Definition 15, there exists an $H_r^* \in \tau_r^*$ such that $x \in H_r^*$ and $H_r^* \subseteq X \setminus A$. Therefore, there exists an $H_r^* \in \mathcal{T}_r^*(x)$ such that $H_r^* \cap A = \emptyset$. This contradicts the condition that $N \cap A \neq \emptyset$, for all $N \in \mathcal{T}_r^*(x)$. Consequently, $x \in \mathrm{bnd}_r^*(A)$. □

Theorem 18. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then,* $\mathrm{bnd}_r^*(A) = \mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. From Theorems 6 and 17, since

$$\begin{aligned} x \in \mathrm{bnd}_r^*(A) &\Leftrightarrow N \cap A \neq \emptyset \ \wedge \ N \cap (X \setminus A) \neq \emptyset, \ \forall N \in \mathcal{T}_r^*(x) \\ &\Leftrightarrow x \in \mathrm{cl}_r^*(A) \ \wedge \ x \in \mathrm{cl}_r^*(X \setminus A) \\ &\Leftrightarrow x \in \mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A), \end{aligned}$$

then $\mathrm{bnd}_r^*(A) = \mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A)$. □

Theorem 19. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then,*

$$\mathrm{cl}_r^*(A) = \mathrm{int}_r^*(A) \cup \mathrm{bnd}_r^*(A).$$

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. From Propositions 1 (i), 4 (i), and 6 (i) and Theorem 18, since

$$\begin{aligned}
\mathrm{int}_r^*(A) \cup \mathrm{bnd}_r^*(A) &= \mathrm{int}_r^*(A) \cup (\mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A)) \\
&= (\mathrm{int}_r^*(A) \cup \mathrm{cl}_r^*(A)) \cap (\mathrm{int}_r^*(A) \cup \mathrm{cl}_r^*(X \setminus A)) \\
&= \mathrm{cl}_r^*(A) \cap (\mathrm{int}_r^*(A) \cup \mathrm{cl}_r^*(X \setminus A)) \\
&= \mathrm{cl}_r^*(A) \cap (\mathrm{int}_r^*(A) \cup (X \setminus \mathrm{int}_r^*(A))) \\
&= \mathrm{cl}_r^*(A) \cap X \\
&= \mathrm{cl}_r^*(A),
\end{aligned}$$

then $\mathrm{cl}_r^*(A) = \mathrm{int}_r^*(A) \cup \mathrm{bnd}_r^*(A)$. □

Proposition 9. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then, the following properties are valid:*

i. $\mathrm{bnd}_r^*(A) \subseteq \mathrm{cl}_r^*(A)$.

ii. $\mathrm{bnd}_r^*(A) = \mathrm{bnd}_r^*(X \setminus A)$.

iii. $\mathrm{bnd}_r^*(A) = \mathrm{cl}_r^*(A) \setminus \mathrm{int}_r^*(A)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.

i. From Theorem 18, since $\mathrm{bnd}_r^*(A) = \mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A)$, then $\mathrm{bnd}_r^*(A) \subseteq \mathrm{cl}_r^*(A)$.

ii. From Theorem 18, since

$$\begin{aligned}
\mathrm{bnd}_r^*(A) &= \mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A) \\
&= \mathrm{cl}_r^*(X \setminus (X \setminus A)) \cap \mathrm{cl}_r^*(X \setminus A) \\
&= \mathrm{cl}_r^*(X \setminus A) \cap \mathrm{cl}_r^*(X \setminus (X \setminus A)) \\
&= \mathrm{bnd}_r^*(X \setminus A),
\end{aligned}$$

then $\mathrm{bnd}_r^*(A) = \mathrm{bnd}_r^*(X \setminus A)$.

iii. From Proposition 1 (i) and Theorem 18,

$$\mathrm{bnd}_r^*(A) = \mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A) = \mathrm{cl}_r^*(A) \cap (X \setminus \mathrm{int}_r^*(A)) = \mathrm{cl}_r^*(A) \setminus \mathrm{int}_r^*(A).$$

□

Theorem 20. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then, the following properties are valid:*

i. The set $\mathrm{bnd}_r^*(A)$ *is an r-near closed set.*

ii. The set A is an r-near open set if and only if $\mathrm{bnd}_r^*(A) \cap A = \emptyset$.

iii. The set A is an r-near closed set if and only if $\mathrm{bnd}_r^*(A) \subseteq A$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.

i. From Theorem 18, $\mathrm{bnd}_r^*(A) = \mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A)$. Moreover, the sets $\mathrm{cl}_r^*(A)$ and $\mathrm{cl}_r^*(X \setminus A)$ are r-near closed sets from Proposition 5 (*i*). Thus, the set $\mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A)$ is an r-near closed set from Corollary 4. Hence, the set $\mathrm{bnd}_r^*(A)$ is an r-near closed set.

ii. ($\Rightarrow$): Assume that the set A is an r-near open set. From Propositions 1 (*i*), 4 (*ii*), and 6 (*i*) and Theorem 18, since

$$\begin{aligned}
\mathrm{bnd}_r^*(A) \cap A &= (\mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A)) \cap A \\
&= (A \cap \mathrm{cl}_r^*(A)) \cap \mathrm{cl}_r^*(X \setminus A) \\
&= A \cap \mathrm{cl}_r^*(X \setminus A) \\
&= A \cap (X \setminus \mathrm{int}_r^*(A)) \\
&= A \cap (X \setminus A) \\
&= \emptyset,
\end{aligned}$$

then $\mathrm{bnd}_r^*(A) \cap A = \emptyset$.

($\Leftarrow$): Assume that $\mathrm{bnd}_r^*(A) \cap A = \emptyset$. Then, $A \cap (X \setminus \mathrm{int}_r^*(A)) = \emptyset$ from Propositions 1 (*i*) and 6 (*i*) and Theorem 18. Therefore, $A \subseteq \mathrm{int}_r^*(A)$. From Proposition 4 (*i*), $\mathrm{int}_r^*(A) = A$. Hence, the set A is an r-near open set from Proposition 4 (*ii*).

iii. ($\Rightarrow$): Assume that the set A is an r-near closed set. From Proposition 6 (*ii*) and Theorem 18, since

$$\begin{aligned}
\mathrm{bnd}_r^*(A) &= \mathrm{cl}_r^*(A) \cap \mathrm{cl}_r^*(X \setminus A) \\
&= A \cap \mathrm{cl}_r^*(X \setminus A) \\
&\subseteq A,
\end{aligned}$$

then $\mathrm{bnd}_r^*(A) \subseteq A$.

($\Leftarrow$): Assume that $\mathrm{bnd}_r^*(A) \subseteq A$. From Theorem 19, since $\mathrm{cl}_r^*(A) = \mathrm{int}_r^*(A) \cup \mathrm{bnd}_r^*(A)$, then $\mathrm{cl}_r^*(A) \subseteq \mathrm{int}_r^*(A) \cup A$. Besides, $\mathrm{cl}_r^*(A) \subseteq A \cup A$ from Proposition 4 (*i*). Therefore, $\mathrm{cl}_r^*(A) \subseteq A$. From Proposition 6 (*i*), $\mathrm{cl}_r^*(A) = A$. Hence, the set A is an r-near closed set from Proposition 6 (*ii*).

□

Finally, Table 3 provides a comparison of classical topological properties and their r-near topological counterparts, together with the corresponding hold/fail conditions.

Table 3. Comparison of some properties in the classical and r-near topologies.

Classical Topology	**r-Near Topology**	**Hold/Fail Conditions**
The finite union of closed sets is a closed set.	The finite union of r-near closed sets may not be an r-near closed set.	Hold: If an r near topology is classical topology, then this property holds. Fail: For $r \leq \lvert B \rvert$, this property generally fails (see Example 10).
$\mathrm{int}(A_1 \cap A_2) = \mathrm{int}(A_1) \cap \mathrm{int}(A_2)$	$\mathrm{int}_r^*(A_1 \cap A_2) \subseteq \mathrm{int}_r^*(A_1) \cap \mathrm{int}_r^*(A_2)$	Hold: The inclusion always holds. Fail: The Inclusion $\mathrm{int}_r^*(A_1) \cap \mathrm{int}_r^*(A_2) \subseteq \mathrm{int}_r^*(A_1 \cap A_2)$ (and thus equality) generally fails (see Example 11).
$\mathrm{cl}(A_1) \cup \mathrm{cl}(A_2) = \mathrm{cl}(A_1 \cup A_2)$	$\mathrm{cl}_r^*(A_1) \cup \mathrm{cl}_r^*(A_2) \subseteq \mathrm{cl}_r^*(A_1 \cup A_2)$	Hold: The inclusion always holds. Fail: The inclusion $\mathrm{cl}_r^*(A_1 \cup A_2) \subseteq \mathrm{cl}_r^*(A_1) \cup \mathrm{cl}_r^*(A_2)$ (and thus equality) generally fails (see Example 12).
The finite intersection of neighborhoods of a point is a neighborhood of the point.	The finite intersection of r-near neighborhoods of a point may not be an r-near neighborhood of the point.	Hold: If an r near topology is classical topology, then this property holds. Fail: For $r \leq \lvert B \rvert$, this property generally fails (see Example 13).
$\mathrm{acc}(A_1) \cup \mathrm{acc}(A_2) = \mathrm{acc}(A_1 \cup A_2)$	$\mathrm{acc}_r^*(A_1) \cup \mathrm{acc}_r^*(A_2) \subseteq \mathrm{acc}_r^*(A_1 \cup A_2)$	Hold: The inclusion always holds. Fail: The inclusion $\mathrm{acc}_r^*(A_1 \cup A_2) \subseteq \mathrm{acc}_r^*(A_1) \cup \mathrm{acc}_r^*(A_2)$ (and thus equality) generally fails (see Example 17).

$\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ is an r-near topological space and $A \subseteq X$.

5. Comparison of concepts in the r-near topological spaces and the classical topological spaces

Section 3 demonstrated by means of counterexamples that the inclusion relations $\mathrm{int}_r^*(A) \subseteq \mathrm{int}(A)$ and $\mathrm{cl}(A) \subseteq \mathrm{cl}_r^*(A)$ do not hold in general. This failure is not due to an issue with the definitions themselves but rather to the absence of additional assumptions needed for these relations to be valid. The aim of this section is to determine the conditions under which the inclusion relations $\mathrm{int}_r^*(A) \subseteq \mathrm{int}(A)$, $\mathrm{int}(A) \subseteq \mathrm{int}_r^*(A)$, $\mathrm{cl}(A) \subseteq \mathrm{cl}_r^*(A)$, and $\mathrm{cl}_r^*(A) \subseteq \mathrm{cl}(A)$ hold. By doing so, the section clarifies the relationships between the r-near interior/r-near closure/r-near exterior of a set and the interior/closure/exterior of a set, respectively. Moreover, it provides several examples related to those.

Theorem 21. *Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$. Then, the following properties are valid:*

i. If the set A is an open set in τ, then $\mathrm{int}_r^*(A) \subseteq \mathrm{int}(A)$, for all r.

ii. If the set A is an r-near open set, then $\mathrm{int}(A) \subseteq \mathrm{int}_r^*(A)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.

i. Assume that the set A is an open set in τ. Then, $\mathrm{int}(A) = A$. From Proposition 4 (*i*), $\mathrm{int}_r^*(A) \subseteq \mathrm{int}(A)$, for all r.

ii. Assume that the set A is an r-near open set. From Proposition 4 *ii*, $\mathrm{int}_r^*(A) = A$. Moreover, $\mathrm{int}(A) \subseteq A$. Hence, $\mathrm{int}(A) \subseteq \mathrm{int}_r^*(A)$.

□

The converse of Theorem 21 *i* is not always correct. In other words, a set whose every r-near interior point is an interior point may not be an open set in the classical topology (see Example 23).

Example 23. *Consider Example 1 and the set* $A = \{a, c, d, e, f\}$*. Since* $\mathrm{int}(A) = \{c, d, e, f\}$, $\mathrm{int}_1^*(A) = \emptyset$, $\mathrm{int}_2^*(A) = \{d, f\}$, *and* $\mathrm{int}_3^*(A) = \{c, d, e, f\}$, *then* $\mathrm{int}_1^*(A) \subseteq \mathrm{int}(A)$, $\mathrm{int}_2^*(A) \subseteq \mathrm{int}(A)$, *and* $\mathrm{int}_3^*(A) \subseteq \mathrm{int}(A)$. *However, the set A is not an open set in* τ.

The converse of Theorem 21 *ii* is not always correct. In other words, a set whose every interior point is an r-near interior point may not be an r-near open set (see Example 24).

Example 24. *Consider Example 1 and the set* $A = \{a, b, c, e\}$*. Since*

$$\mathrm{int}(A) = \{a, b\} \quad \textit{and} \quad \mathrm{int}_2^*(A) = \{a, b, e\},$$

then $\mathrm{int}(A) \subseteq \mathrm{int}_2^*(A)$*. However, the set A is not a* 2*-near open set. Similarly, since* $\mathrm{int}(A) = \{a, b\}$ *and* $\mathrm{int}_3^*(A) = \{a, b\}$, *then* $\mathrm{int}(A) \subseteq \mathrm{int}_3^*(A)$*. However, the set A is not a* 3*-near open set.*

Theorem 22. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A \subseteq X$*. Then, the following properties are valid:*

i. If the set A is a closed set in τ*, then* $\mathrm{cl}(A) \subseteq \mathrm{cl}_r^*(A)$*, for all r.*

ii. If the set A is an r-near closed set, then $\mathrm{cl}_r^*(A) \subseteq \mathrm{cl}(A)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.

i. Assume that the set A is a closed set in τ. Then, $\mathrm{cl}(A) = A$. From Proposition 6 (*i*), $\mathrm{cl}(A) \subseteq \mathrm{cl}_r^*(A)$, for all r.

ii. Assume that the set A is an r-near closed set. From Proposition 6 *ii*, $\mathrm{cl}_r^*(A) = A$. Moreover, $A \subseteq \mathrm{cl}(A)$. Hence, $\mathrm{cl}_r^*(A) \subseteq \mathrm{cl}(A)$.

□

The converse of Theorem 22 *i* is not always correct. In other words, a set whose every closure point is an r-near closure point may not be a closed set in the classical topology (see Example 25).

Example 25. *Consider Example 1 and the set* $A = \{b\}$*. Since*

$$\mathrm{cl}(A) = \{a, b\}, \quad \mathrm{cl}_1^*(A) = X, \quad \mathrm{cl}_2^*(A) = \{a, b, c, e\}, \quad \textit{and} \quad \mathrm{cl}_3^*(A) = \{a, b\},$$

then $\mathrm{cl}(A) \subseteq \mathrm{cl}_1^*(A)$, $\mathrm{cl}(A) \subseteq \mathrm{cl}_2^*(A)$, *and* $\mathrm{cl}(A) \subseteq \mathrm{cl}_3^*(A)$*. However, the set A is not a closed set in* τ.

The converse of Theorem 22 *ii* is not always correct. In other words, a set whose every r-near closure point is a closure point may not be an r-near closed set (see Example 26).

Example 26. *Consider Example 1 and the set* $A = \{c, d\}$*. Since*

$$\mathrm{cl}(A) = \{c, d, e, f\} \quad \textit{and} \quad \mathrm{cl}_2^*(A) = \{c, d, f\},$$

then $\mathrm{cl}_2^*(A) \subseteq \mathrm{cl}(A)$*. However, the set A is not a* 2*-near closed set. Similarly, since*

$$\mathrm{cl}(A) = \{c, d, e, f\} \quad \textit{and} \quad \mathrm{cl}_3^*(A) = \{c, d, e, f\},$$

then $\mathrm{cl}_3^*(A) \subseteq \mathrm{cl}(A)$*. However, the set A is not a* 3*-near closed set.*

Theorem 23. *Let* $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ *be an r-near topological space and* $A \subseteq X$. *Then, the following properties are valid:*

i. If the set A is a closed set in τ, *then* $\mathrm{ext}_r^*(A) \subseteq \mathrm{ext}(A)$, *for all r.*

ii. If the set A is an r-near closed set, then $\mathrm{ext}(A) \subseteq \mathrm{ext}_r^*(A)$.

Proof. Let $\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ be an r-near topological space and $A \subseteq X$.

i. Assume that the set A is a closed set in τ. Then, the set $X \setminus A$ is an open set in τ. Therefore, $\mathrm{int}(X \setminus A) = X \setminus A$ and thus $\mathrm{ext}(A) = X \setminus A$. From Proposition 8, $\mathrm{ext}_r^*(A) \subseteq \mathrm{ext}(A)$, for all r.

ii. Assume that the set A is an r-near closed set. Then, the set $X \setminus A$ is an r-near open set. From Theorem 21 *ii*, $\mathrm{int}(X \setminus A) \subseteq \mathrm{int}_r^*(X \setminus A)$. Hence, $\mathrm{ext}(A) \subseteq \mathrm{ext}_r^*(A)$ from Definition 15.

□

The converse of Theorem 23 *i* is not always correct. In other words, a set whose every r-near exterior point is an exterior point may not be a closed set in the classical topology (see Example 27).

Example 27. *Consider Example 1 and the set* $A = \{b\}$. *Since*

$$\mathrm{ext}(A) = \{c, d, e, f\}, \quad \mathrm{ext}_1^*(A) = \emptyset, \quad \mathrm{ext}_2^*(A) = \{d, f\}, \quad \textit{and} \quad \mathrm{ext}_3^*(A) = \{c, d, e, f\},$$

then $\mathrm{ext}_1^*(A) \subseteq \mathrm{ext}(A)$, $\mathrm{ext}_2^*(A) \subseteq \mathrm{ext}(A)$, *and* $\mathrm{ext}_3^*(A) \subseteq \mathrm{ext}(A)$. *However, the set A is not a closed set in* τ.

The converse of Theorem 23 *ii* is not always correct. In other words, a set whose every exterior point is an r-near exterior point may not be an r-near closed set (see Example 28).

Example 28. *Consider Example 1 and the set* $A = \{d, f\}$. *Since* $\mathrm{ext}(A) = \{a, b\}$ *and* $\mathrm{ext}_2^*(A) = \{a, b, e\}$, *then* $\mathrm{ext}(A) \subseteq \mathrm{ext}_2^*(A)$. *However, the set A is not a* 2*-near closed set. Similarly, since* $\mathrm{ext}(A) = \{a, b\}$ *and* $\mathrm{ext}_3^*(A) = \{a, b\}$, *then* $\mathrm{ext}(A) \subseteq \mathrm{ext}_3^*(A)$. *However, the set A is not a* 3*-near closed set.*

Table 4 summarizes the inclusion relations between some r-near topological concepts and their classical counterparts.

Table 4. Summary of inclusion relations provided in this section.

Concepts	**Hypotheses (Conditions on the set A)**	**Conclusions (Inclusion relation)**
Interior and r-near interior	$A \in \tau$	$\mathrm{int}_r^*(A) \subseteq \mathrm{int}(A)$, for all r
	$A \in \tau_r^*$	$\mathrm{int}(A) \subseteq \mathrm{int}_r^*(A)$
Closure and r-near closure	$A \in \tau^c$	$\mathrm{cl}(A) \subseteq \mathrm{cl}_r^*(A)$, for all r
	$A \in \tau_r^{*c}$	$\mathrm{cl}_r^*(A) \subseteq \mathrm{cl}(A)$
Exterior and r-near exterior	$A \in \tau^c$	$\mathrm{ext}_r^*(A) \subseteq \mathrm{ext}(A)$, for all r
	$A \in \tau_r^{*c}$	$\mathrm{ext}(A) \subseteq \mathrm{ext}_r^*(A)$

$\left(O, F, \sim_{B_r}, N_r(B), X, \tau, \tau_r^*\right)$ is an r-near topological space and $A \subseteq X$.

6. Conclusions

The current study handles a grounding study concerning the basic concepts of r-near topology. The study first redefines the concept of r-near neighborhoods and then introduces the relationship between r-near open neighborhoods and r-near closure. Besides, it explores a number of the fundamental properties of r-near closed sets, r-near interior, r-near closure, and r-near neighborhoods. Afterward, this study defines the concepts of r-near accumulation points, r-near isolated points, r-near exterior points, and r-near boundary points. Moreover, it studies some of their basic properties and exemplifies them. Thus, the main properties attained related to the aforesaid concepts can be listed as follows:

- The arbitrary intersection of r-near closed sets is also an r-near closed set.
- The r-near interior of a set is the biggest r-near open set contained by the set.
- The r-near closure of a set is the smallest r-near closed set containing the set.
- A set, being an r-near open neighborhood of every element of the set, is an r-near open set.
- A set, where every r-near accumulation point belongs to the set, is an r-near closed set.

Additionally, this study puts forward a few relationships between the concepts in the r-near topological spaces and those in classical topological spaces. These relationships can be summarized as follows:

- Every r-near interior point of an open set is also an interior point of the set.
- Every interior point of an r-near open set is also an r-near interior point of the set.
- Every closure point of a closed set is also an r-near closure point of the set.
- Every r-near closure point of an r-near closed set is also a closure point of the set.
- Every r-near exterior point of a closed set is also an exterior point of the set.
- Every exterior point of an r-near closed set is also an r-near exterior point of the set.

Furthermore, it is observed that some inclusion relations among r-near topological operators, as well as between r-near and classical topological operators, are not valid in general, as shown by counterexamples in Examples 2–5, 15, 16, and 18–21. These examples indicate that the structural behavior of r-near topology depends on r, and that different choices of r may lead to distinct topological results. Therefore, the validity of such inclusion relations can require additional assumptions or specific constraints on the relevant parameters.

In addition, Definition 11 and Theorem 6 provide a consistent foundation for the investigation of subsequent r-near topological concepts herein. For instance, Theorems 10, 12, and 17 offer fundamental characterizations of r-near open sets, r-near closed sets, and r-near boundary points, paralleling their counterparts in the classical topology. Besides, the primary theoretical significance of the concepts of r-near accumulation points, r-near isolated points, r-near exterior points, and r-near boundary points is that they provide novel characterizations of the r-near closure; for example, $\mathrm{cl}_r^*(A) = A \cup \mathrm{acc}_r^*(A)$ and $\mathrm{cl}_r^*(A) = \mathrm{int}_r^*(A) \cup \mathrm{bnd}_r^*(A)$. Moreover, although the present study focuses on the theoretical foundation, these concepts have clear potential in subsequent applications.

For instance, in pattern recognition, r-near accumulation points can characterize patterns of dense relational similarity, whereas r-near isolated points can help identify outliers or anomalous patterns. Another significant aspect is that the refined and extended framework introduced in this paper establishes the necessary mathematical foundation for defining and characterizing further topological concepts in future theoretical studies, such as r-near continuity, r-near dense sets, nowhere r-near dense sets, r-near dense-in-itself sets, near separation axioms, r-near compactness, and r-near connectedness. Furthermore, as a continuation of this study, it is planned to define the concept of r-near continuity using Definition 11. Thus, this will ensure a consistent formulation of the relationships between the concept of r-near continuity and its characterization via r-near closure.

Beyond theoretical studies, a potential direction for future research is the integration of r-near topological concepts into applied frameworks for pattern recognition, decision-making, medical diagnosis, and face recognition. For example, [19] provides decision-making methodologies based on soft rough sets constructed via near-open sets, whereas [21] demonstrates that even small modifications to initial-neighborhood structures can substantially affect diagnostic accuracy. Building on these insights, the r-near neighborhoods introduced in this paper offer a flexible approach to modeling graded nearness in practical settings and may lead to more interpretable models.

Use of Generative-AI tools declaration

The author declares she has used Artificial Intelligence (AI) tools only to improve the article's English quality.

Conflict of interest

The author declares no conflicts of interest.

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