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Riemann solitons on perfect fluid within $f(r, T^2)$-gravity

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Abstract: This work is devoted to the study of Riemann solitons in the setting of the energy-momentum squared gravity framework, expressed as $f(r, T^2)$, regarded as a deformation of Einstein's general relativity. Our attention is placed on the particular model $f(r, T^2) = r + \lambda T^2$, coupled with a perfect fluid, where the dynamics naturally admit Riemann solitons. Within the steady formulation of such solitons, we obtain the corresponding fluid relation of state under $f(r, T^2)$-gravity. Moreover, by exploiting the solitonic structure, we further analyze the admissibility of energy conditions, the emergence of black hole geometries, and the manifestation of singularities in the presence of a perfect fluid for this modified gravitational scenario.

Keywords: $f(r, T^2)$-gravity; conformal Ricci solitons; perfect fluid; concircular vector field; black holes; singularity
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1. Introduction

Consider a pseudo-Riemannian manifold (M, g) and let r denote its corresponding Riemann curvature tensor. The evolution of the metric $g(t)$ under the Riemann flow, as formulated by Udrişte [1], is governed by the following:

$$\frac{\partial}{\partial t} G(t) = -2\, r(g(t)),$$

where the tensor G is defined by $G = \frac{1}{2} g \odot g$, and for any pair of $(0,2)$-tensors ω, θ, the symbole $\odot$ is evaluated on vector fields U_1, U_2, U_3, U_4 by the following:

$$\begin{aligned}(\omega \odot \theta)(U_1, U_2, U_3, U_4) &= \omega(U_1, U_4)\theta(U_2, U_3) + \omega(U_2, U_3)\theta(U_1, U_4) \\ &\quad - \omega(U_1, U_3)\theta(U_2, U_4) - \omega(U_2, U_4)\theta(U_1, U_3).\end{aligned}$$

A smooth vector field V is said to generate a Riemann soliton (RS) if it satisfies the following:

$$2r + \mu\, g \odot g + g \odot \mathcal{L}_V g = 0, \tag{1.1}$$

where μ is a real constant, and (M^n, g, μ, V) denotes the soliton structure. The classification follows the sign of μ: expanding when $\mu > 0$, steady for $\mu = 0$, and shrinking when $\mu < 0$. If the vector field is a gradient (i.e., $V = \operatorname{grad} f$ for some smooth potential f), then (1.1) reduces to the following:

$$2r + \mu\, g \odot g + 2\, g \odot \nabla^2 f = 0. \tag{1.2}$$

This condition defines what is known as a gradient Riemann soliton (GRS). If the function μ is permitted to change smoothly across the manifold M, the concept naturally extends to encompass almost Riemann solitons (ARS) and their gradient counterparts, referred to as almost gradient Riemann solitons (AGRS).

The literature on geometric solitons and the distinction between different soliton types is determined by the curvature tensor on which the defining condition is imposed. Ricci solitons solely involve the Ricci tensor and therefore merely constrain the trace part of the full curvature, while Riemann solitons represent a strictly stronger notion because they are defined in terms of the entire Riemann curvature tensor and thus encode sectional curvature information that Ricci solitons do not capture.

The ARS further generalizes this framework by allowing the soliton function to vary, thereby providing an intermediate concept between strict Riemann solitons and more flexible geometric flows. Under additional assumptions, such as a constant scalar curvature or the presence of special vector fields, the Riemann-level condition may reduce to Ricci-type equations, thus leading to results analogous to those known for Ricci solitons (see, e.g., [2–5]). The present work adopts this broader ARS perspective and thereby complements the existing Ricci-soliton literature by examining curvature constraints at the full Riemann-tensor level within the context of energy-momentum-squared gravity (EMSG).

A wide range of investigations have been devoted to the study of RSs on smooth manifolds. For example, Biswas et al. [6] analyzed RS structures on almost co-Kähler manifolds, while Venkatesha and collaborators [7, 8] considered their realization in contact geometry and in almost Kenmotsu settings. In another direction, De et al. [9] addressed the theory of almost RSs. In 2025, Azami et al. studied RSs on spacetimes with pure radiation metrics [10, 11], and Riemann solitons on Egorov and Cahen-Wallach symmetric spaces [12].

On the other hand, a growing body of literature has dealt with geometric solitons in relativistic spacetimes. Among these, Azami et al. studied multiple soliton families in the context of pseudo-Finsler spacetimes (PFSs), thereby encompassing Riemann solitons [2], gradient Ricci-Bourguignon solitons [3], further generalizations of RSs [13, 14], and hyperbolic Ricci solitons [15]. Additionally, they investigated almost gradient $\mathcal{Z}$-solitons (AGZS) in magneto-fluid spacetimes formulated within a $f(r)$-gravity [16].

The study of large-scale cosmic evolution has classically been formulated through Einstein's field equations [17], which provide a remarkably precise description consistent with astrophysical observations. Within this framework, an additional non-luminous component of matter, termed dark matter (DM), is usually postulated to account for unexplained gravitational effects [18].

Alongside DM, the universe hosts another exotic entity called dark energy (DE), which alters the global energy budget, accelerates the cosmic expansion, and emerges as the prevailing factor on cosmological scales. These empirical facts have inspired the development of generalized theories of gravitation, such as $f(r)$-gravity [19]. Such frameworks extend beyond standard general relativity and are often regarded as plausible low-energy manifestations of a prospective quantum theory of gravity [20].

A natural generalization of general relativity (GR) is the $f(r)$-gravity theory, where the Einstein-Hilbert action is extended by substituting the scalar curvature r with a function $f(r)$ [21]. The presence of higher-order curvature terms within this setting offers potential resolutions to challenges in describing compact astrophysical objects, such as massive neutron stars. However, it has been shown that certain equilibrium stellar models fail to sustain $f(r)$-gravity solutions [22, 23]. Since the theory introduces higher-order differential operators in r, the specification of the stellar equation of state (EoS) becomes essential in determining viable models. Frequently, polytropic forms of the EoS are employed to manage complications associated with realistic stellar matter. In addition, perturbative consistency tests and order-reduction techniques are widely used to examine $f(r)$-gravity in strong-field regimes [24–26]. Altogether, such frameworks provide refined descriptions of neutron stars, including predictions of maximal mass thresholds and their dependence on sound speed variations across different EoS choices [23].

Further developments include the proposal by Harko et al. [27] of $f(r, T)$-gravity, which generalizes the action by allowing the Ricci scalar r and the trace of the energy-momentum tensor (EMT) T to enter the Lagrangian as arbitrary functions. In this formulation, T is the trace of the EMT, and the resulting theory has been shown to effectively account for the late-time acceleration of cosmic expansion.

Following this development, Katirci and Kavuk [28] proposed a covariant extension of GR by introducing an additional term in the action proportional to $T_{ab}T^{ab}$. Various specific realizations that emerged from this framework have been investigated in later works [29, 30]. Moreover, Roshan et al. [31] examined the possibility of a cosmological bounce in the early universe within the energy-momentum squared gravity (EMSG) context, thereby adopting the particular functional form

$$f(r, T^2) = r + \lambda T^2,$$

where λ denotes a constant parameter.

In [31], the late-time acceleration of the universe was also studied in the setting of EMSG, thereby focusing on a pressureless fluid configuration. Within $f(r, T^2)$-gravity, the gravitational Lagrangian explicitly depends on the Ricci scalar r and the quadratic invariant of the EMT, $T^2 = T_{ab}T^{ab}$ [29, 30]. Importantly, departures from standard GR only manifest in the presence of matter fields. The term $f(T_{ab}T^{ab})$ can be specified in various forms, thus producing different realizations of the theory; for instance, if $f(T_{ab}T^{ab}) \propto (T_{ab}T^{ab})^{\lambda}$ with the constant λ, then the resulting framework corresponds to EMSG.

The concept of EMSG has garnered significant interest and has been studied in a variety of contexts. In particular, Akarsu et al. [32] explored the use of energy-momentum powered gravity as a framework to describe the accelerated expansion of the universe. Within the EMSG paradigm, numerous cosmological constructions have been proposed, such as brane-world models and bulk viscous cosmologies [30]. The influence of EMSG on the internal structure of neutron stars and its associated cosmological implications were analyzed in [29], while spherically symmetric compact objects were examined in [33]. Furthermore, the phase space behavior and its physical significance for different functional forms of $f(r, T^2)$ have been studied using dynamical systems methods [34].

In the framework of GR, a cosmological or spacetime model is typically represented by a connected four-dimensional Lorentzian manifold equipped with a time orientation. Such manifolds form a subclass of pseudo-Riemannian manifolds distinguished by the Lorentzian signature $(-, +, +, +)$, which underpins the mathematical formulation of GR. Within Lorentzian geometry, the analysis of vector field properties and their evolution on the manifold is of central importance, thus making Lorentzian manifolds the natural and most effective setting for rigorous investigations in GR [35–38].

Definition 1.1 (Quasi-Einstein–Lorentzian manifold [38])**.** *A Lorentzian manifold is referred to as a quasi-Einstein manifold, or equivalently a perfect fluid spacetime, if its Ricci tensor S can be decomposed as follows:*

$$S = A_1 g + A_2 \eta \otimes \eta,$$

where A_1 and A_2 are scalar functions on the manifold, and η is a 1-form.

At its core, the EMT [39] provides a compact representation to express the relations of motion within GR. This tensor can be written as follows [38, 40]:

$$T_{ab} = (\sigma + p)\,\eta_a \eta_b - p\,g_{ab},$$

where σ denotes the energy density, and p corresponds to the isotropic pressure of a perfect fluid [40].

Within the framework of GR, the distribution and symmetries of matter are intrinsically connected to the geometric characteristics of spacetime. In particular, the symmetries of the metric tensor play a central role in classifying solutions to Einstein's field equations. Among these geometric features, solitonic structures emerge as important manifestations associated with the flow of the manifold's geometry.

Extensive research has been conducted on Ricci solitons in various spacetime models. For example, Ahsan et al. [41] and Li et al. [42] investigated spacetime configurations under the Ricci soliton and η-Ricci soliton formalism, respectively; alternatively, Venkatesha et al. [43] examined Ricci solitons in perfect fluid spacetimes. Recently, the solitonic aspect of relativistic magneto-fluid spacetime if its metric is a Ricci Bourguignon soliton was studied in [44]. Siddiqi et al. [45] analyzed $f(r, T^2)$-gravity, all filled with perfect fluid matter that supports a variety of solitonic and gradient solitonic structures. Inspired by these recent advances, the present work focuses on a particular perfect fluid model in $f(r, T^2)$-gravity that supports ARSs.

In the context of geometric solitons, our work positions itself relative to earlier studies by extending the analysis of Riemann solitons to the setting of energy–momentum–squared gravity with a perfect fluid source. Unlike previous works that mainly focused on pure geometric structures, we consider the coupling between the soliton condition and physically motivated matter content. This allows us to derive explicit relations between geometric quantities and fluid variables, classify solutions under various symmetry assumptions, and examine the resulting equations of state. In this way, the present study complements the existing Riemann soliton literature by combining geometric constraints with matter effects and exploring their implications for energy conditions and fluid properties.

This paper is organized in the following manner: Section 2 presents the field equations for the specific choice $f(r, T^2) = r + \lambda T^2$ in the presence of a perfect fluid, where within this modified gravitational framework, we compute essential geometric quantities such as the Ricci tensor, the scalar curvature, and the quadratic invariant of the EMT, and we analyze the corresponding solutions; in Section 3, we examine ARSs within the context of $f(r, T^2)$ EMSG, assuming a perfect fluid whose concircular vector field ζ coincides with the fluid's timelike velocity; and finally, the study addresses the related energy conditions for conformal Ricci solitons under this EMSG setup.

2. Field equations on a perfect fluid within $f(r, T^2)$-gravity

In this section, we focus on the EMSG model $f(r, T^2)$ interacting with a perfect fluid, where the quadratic energy-momentum term is defined as $T^2 = T_{ab}T^{ab}$. Since the model is explicitly related to the physical properties, one can generate a variety of theoretical frameworks by selecting different functional dependencies on r and T [27]. For the purposes of this study, we consider the specific form

$$f(r, T^2) = r + \lambda T^2,$$

where $f(r)$ and $f(T)$ represent general functions of the Ricci scalar r and the trace T, respectively, and λ is treated as a constant parameter.

Accordingly, the Einstein-Hilbert action for the $f(r, T^2)$-gravity model, with the cosmological constant Λ, can be expressed as follows:

$$\mho_E = \frac{1}{16\pi} \int \left[2\Lambda + L_n + f(r, T_{ab}T^{ab}) \right] \sqrt{-g}\, d^4x.$$

Here, L_n denotes the matter Lagrangian. The associated EMT of the matter content is defined by the following:

$$T_{ab} = g_{ab}L_n - \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_n)}{\delta g^{ab}}. \tag{2.1}$$

In this formulation, we assume that L_n exclusively depends on the metric components g_{ab} and does not involve derivatives of the metric.

We can write the following:

$$\delta\mho_E = \frac{1}{16\pi} \int \left[f_r\, \delta r + f_{T^2}\, \delta T^2 - \frac{1}{2} g_{ab} f\, \delta g^{ab} - g_{ab}\Lambda\, \delta g^{ab} + \frac{1}{\sqrt{-g}} \delta(\sqrt{-g}L_n) \right] \sqrt{-g}\, d^4x. \tag{2.2}$$

For clarity, we adopt the following definitions:

$$f = f(r, T_{ab}T^{ab}), \quad f_r = \frac{\partial f}{\partial r}, \quad f_{T^2} = \frac{\partial f}{\partial T^2}.$$

It is well-known that the variation of the Ricci scalar takes the following form:

$$\delta r = S^{ab}\delta g_{ab} + g_{ab}\Box\delta g^{ab} - \nabla_a\nabla_b\delta g^{ab},$$

where $\Box = \nabla_i \nabla^i$. Next, the variation of the quadratic energy-momentum term T^2 with respect to the metric can be expressed as follows:

$$\Theta_{ab} \equiv \frac{\delta(T_{ij}T^{ij})}{\delta g^{ab}} = T_{ij}\frac{\delta(T^{ij})}{\delta g^{ab}} + \frac{\delta(T_{ij})}{\delta g^{ab}}T^{ij},$$

which may alternatively be rewritten in the following form:

$$\Theta_{ab} = \frac{\delta(T_{ij})}{\delta g^{ab}}T^{ij} + 2T_a^i T_{bi} + \frac{\delta(T_{ij})}{\delta g^{ab}}T^{ij}. \tag{2.3}$$

By employing (2.1) to compute the variation of the EMT in (2.3), one obtains the following:

$$\frac{\delta(T_{ij})}{\delta g^{ab}}T^{ij} = -L_n T_{ab} + \frac{1}{2}g_{ab}L_n T - \frac{1}{2}TT_{ab} - \frac{\partial^2 L_n}{\partial g^{ab}\partial g_{ab}}T^{ab}. \tag{2.4}$$

By substituting (2.4) into (2.3), the variation Θ_{ab} can be expressed as follows:

$$\Theta_{ab} = 2T_a^i T_{bi} - TT_{ab} - 2L_n\left(T_{ab} - \frac{1}{2}g_{ab}T\right) - 4T^{ij}\frac{\partial^2 L_n}{\partial g^{ab}\partial g_{ab}}.$$

Taking the variation in (2.2) into account, the field equations for the EMSG framework $f(r, T^2)$ are given by the following:

$$f_r S_{ab} = \Lambda g_{ab} + \frac{1}{2}f g_{ab} + (g_{ab}\nabla_i\nabla^i - \nabla_a\nabla_b)f_r + \frac{1}{8\pi}\Big(T_{ab} - \frac{1}{8\pi}f_{T^2}\Theta_{ab}\Big). \tag{2.5}$$

In addition to the geometric contributions, the effects of matter must be included. Here, we describe the matter as a perfect fluid, whose EMT is written as follows [40]:

$$T_{ab} = (\sigma + p)\eta_a\eta_b - pg_{ab}, \tag{2.6}$$

where η denotes the unit timelike four-velocity, p is the isotropic pressure, and σ corresponds to the energy density of the fluid. For a perfect fluid with $L_n = -p$ satisfying $\eta_a\eta^a = 1$, and using the EMT (2.6), the tensor Θ_{ab} reduces to the following [33]:

$$\Theta_{ab} = -(\sigma^2 - p^2)\eta_a\eta_b.$$

It is important to observe that when the scalar curvature is assumed to be constant, the field equation (2.5) simplifies to

$$f_r S_{ab} = \Lambda g_{ab} + \frac{1}{2}f g_{ab} + \frac{1}{8\pi}\Big(T_{ab} - \frac{1}{8\pi}f_{T^2}\Theta_{ab}\Big).$$

Consequently, constructing the EMSG model $f(r, T^2)$ requires a careful evaluation of (2.5) using the perfect fluid EMT defined in (2.6). For simplicity, we adopt the specific form

$$f(r, T^2) = r + \lambda T^2,$$

with λ treated as a constant parameter. Under this assumption, the Ricci tensor acquires the structure of a perfect fluid, explicitly given by the following:

$$S_{ab} = \{\Lambda + \frac{r}{2} + \frac{1}{2}\lambda T^2 - 8\pi p\}g_{ab} + \{f_{T^2}(\sigma^2 - p^2) + 8\pi(\sigma + p)\}\eta_a\eta_b. \tag{2.7}$$

Thus, for a spacetime (M^4, g) filled with a perfect fluid in the context of $f(r, T^2)$-gravity, the Ricci tensor can be expressed in the following form:

$$S_{ab} = \alpha g_{ab} + \beta\eta_a\eta_b, \tag{2.8}$$

where the coefficients α and β are defined by the following:

$$\alpha = \Lambda - 8\pi p + \frac{r}{2} + \frac{1}{2}\lambda T^2, \qquad \beta = f_{T^2}(\sigma^2 - p^2) + 8\pi(\sigma + p).$$

We note that α, β depend on σ and p, so the matter EoS remains free unless symmetries fix $\beta = 0$. In this analysis, we assume that α and β are not simultaneously zero. For completeness, a brief verification is presented here, although the derivation of the Ricci tensor using a similar method has been previously discussed in [45]. Accordingly, one finds the following:

$$r = 8\pi(3p - \sigma) - f'(T^2)(\sigma^2 - p^2) - 4\Lambda - 2\lambda T^2.$$

3. ARSs in the EMSG framework $f(r, T^2)$ with perfect fluid

In the present section, we examine the characteristics of ARSs within the EMSG framework $f(r, T^2)$, thereby considering a perfect fluid whose timelike velocity is described by ζ, which simultaneously functions as a concircular vector field.

For the sake of clarity and conciseness, we introduce the notation Ξ_f to represent "a spacetime (M^4, g) in the EMSG model $f(r, T^2)$ endowed with a perfect fluid".

Theorem 3.1. *Let a Ξ_f spacetime admit an ARS structure (M^4, g, V, μ). Then, the divergence of the vector field V is determined by the following:*

$$\mathrm{div}V = -\frac{r}{6} - 2\mu,$$

where div *denotes the divergence operator.*

Proof. Starting from the defining relation of a RS (1.1), we can rewrite it in components as follows:

$$\begin{aligned} 2r(U_1, U_2, U_3, U_4) \;=\; & -2\mu\left[g(U_1, U_4)g(U_2, U_3) - g(U_1, U_3)g(U_2, U_4)\right] \\ & - \left[g(U_1, U_4)(\mathcal{L}_V g)(U_2, U_3) + g(U_2, U_3)(\mathcal{L}_V g)(U_1, U_4)\right] \\ & + \left[g(U_1, U_3)(\mathcal{L}_V g)(U_2, U_4) + g(U_2, U_4)(\mathcal{L}_V g)(U_1, U_3)\right]. \end{aligned} \tag{3.1}$$

Contracting the indices U_1 and U_4 in (3.1) leads to the following:

$$S(U_2, U_3) = -(3\mu + \mathrm{div}V)g(U_2, U_3) - (\mathcal{L}_V g)(U_2, U_3). \tag{3.2}$$

A subsequent full contraction of (3.2) with the metric g gives the following:

$$r + 12\mu + 6\,\mathrm{div}V = 0,$$

which establishes the claimed relation for the divergence of V and thus concludes the proof.

If the vector field V satisfies the Killing condition (i.e., $\mathcal{L}_V g = 0$), then it immediately follows that $\mathrm{div}V = 0$ and the scalar curvature satisfies $r = -12\mu$. Combining Eqs (2.8) and (3.2) under this assumption yields the following:

$$3\mu = -\alpha, \qquad 0 = \beta,$$

which leads to the explicit relations

$$-3\mu = \Lambda - 8\pi p + \frac{r}{2} + \frac{1}{2}\lambda T^2, \qquad f_{T^2}(\sigma^2 - p^2) + 8\pi(\sigma + p) = 0.$$

Corollary 3.1. *Let a Ξ_f admit an ARS (M^4, g, V, μ) with V being a Killing vector field. Then, the following identities hold:*

$$-3\mu = \Lambda - 8\pi p + \frac{r}{2} + \frac{1}{2}\lambda T^2, \qquad f_{T^2}(\sigma^2 - p^2) + 8\pi(\sigma + p) = 0.$$

Alternatively, if the vector field V is taken to be a conformal Killing vector (i.e., $\mathcal{L}_V g = \psi g$ for a smooth function ψ), then the ARS Eq (3.1) can be rewritten as follows:

$$\begin{aligned}
&-2\mu[g(U_1, U_4)g(U_2, U_3) - g(U_1, U_3)g(U_2, U_4)]\\
&-[g(U_1, U_4)(\mathcal{L}_V g)(U_2, U_3) + g(U_2, U_3)(\mathcal{L}_V g)(U_1, U_4)]\\
&+[g(U_1, U_3)(\mathcal{L}_V g)(U_2, U_4) + g(U_2, U_4)(\mathcal{L}_V g)(U_1, U_3)]\\
&= -2(\mu + \psi)[g(U_1, U_4)g(U_2, U_3) - g(U_1, U_3)g(U_2, U_4)].
\end{aligned}$$

Moreover, in a Ξ_f spacetime where the scalar curvature is constant and equal to k, the Riemann tensor can be expressed as follows:

$$R(U_1, U_2, U_3, U_4) = k[g(U_1, U_4)g(U_2, U_3) - g(U_1, U_3)g(U_2, U_4)].$$

Hence, in a Ξ_f with constant scalar curvature k admitting $\mathcal{L}_V g = \psi g$, then the following corollary holds.

Corollary 3.2. *Let (M, g) be a Ξ_f spacetime that possess a constant scalar curvature k and $\mathcal{L}_V g = \psi g$. Under these conditions, the manifold M naturally supports an ARS configuration described by $(M, g, V, -\mu - \psi)$.*

Next, let us assume that the velocity vector field V of the RS is of a gradient type (i.e., $V = \nabla\Phi$, where Φ is a smooth function on (M^4, g)). Then, from (1.2) and using the fact that $\mathrm{trace}(\mathcal{L}_{\nabla\Phi} g) = 2\Delta\Phi$, we deduce the following result.

Theorem 3.2. *Let (M^4, g) be a Ξ_f spacetime admitting a AGRS $(M^4, g, \nabla\Phi, \mu)$. Then, the scalar function Φ satisfies the modified Poisson equation in the context of $f(r, T)$-gravity:*

$$\Delta\Phi = -\frac{1}{6}r - 2\mu = -\frac{2}{3}\alpha + \frac{1}{6}\beta - 2\mu.$$

Applying the divergence theorem yields the following corollary.

Corollary 3.3. *Let (M^4, g) be a closed Ξ_f that admits a GRS $(M^4, g, \nabla\Phi, \mu)$. Then, the soliton constant μ is given by the following:*

$$\mu = -\frac{1}{12}\int_M r\, dV_g.$$

In particular, for a smooth function $\Phi \in C^\infty(M)$, the divergence of the scaled vector field ΦV satisfies the following:

$$\mathrm{div}(\Phi V) = \zeta(d\Phi) + \Phi\,\mathrm{div}(V).$$

The function Φ is called a last multiplier of the vector field V with respect to the metric g if

$$\mathrm{div}(\Phi V) = 0.$$

In this case, one obtains the following Liouville-type equation:

$$V(d\ln\Phi) = -\mathrm{div}(V),$$

which describes how the function Φ compensates for the divergence of V (see [46]).

Using this framework, we can further state the following result.

Theorem 3.3. *Consider a Ξ_f spacetime (M^4, g) which supports an AGRS (M^4, g, V, μ) with $V = \nabla\Phi$. Then, the vector field Φ and V satisfy the generalized Liouville-type equation in the context of $f(r, T)$-gravity:*

$$V(d\ln\Phi) = \frac{1}{6} + 2\mu.$$

Let $\Phi : M \to \mathbb{R}$ be a smooth scalar function. We say that Φ is harmonic if $\Delta\Phi = 0$. Under the assumption $V = \mathrm{grad}(\Phi)$, Theorem 3.2 implies the following result.

Theorem 3.4. *Let (M^4, g) possess a gradient ARS $(M^4, g, \nabla\Phi, \mu)$ in which Φ is harmonic. Then, the constant μ satisfies the following:*

$$\mu = -\frac{r}{12}.$$

Consequently, the soliton is classified as expanding, steady, or shrinking according to whether the scalar curvature r is negative, zero, or positive, respectively.

Next, consider a concircular vector field (CVF) ζ on (M^4, g) [47], defined by the following:

$$\nabla_X\zeta = \Omega X, \tag{3.3}$$

where Ω is a smooth function on M^4.

If we take $V = \zeta$, then from (3.2), it follows that

$$(\mathcal{L}_\zeta g)(U_1, U_2) + S(U_1, U_2) + (3\mu + \mathrm{div}\,\zeta)g(U_1, U_2) = 0,$$

for any pair of timelike vector fields $U_1, U_2 \in \chi(M_4)$. Using (3.3), this simplifies to the following:

$$S(U_1, U_2) + (3\mu + 6\Omega)g(U_1, U_2) = 0.$$

Combining with the Ricci tensor decomposition (2.8), we deduce the following:

$$(\alpha + 3\mu + 6\Omega)g(U_1, U_2) + \beta\,\eta(U_1)\eta(U_2) = 0,$$

which implies

$$\alpha + 3\mu + 6\Omega = 0, \quad \beta = 0.$$

This leads to the following conclusion.

Theorem 3.5. *Consider a Ξ_f spacetime that admits an ARS (M^4, g, ζ, μ) where ζ is a CVF. Then, the soliton is classified as expanding, steady, or shrinking according to the sign of $2\alpha + 3\Omega$:*

1) *Expanding if $2\alpha + 3\Omega < 0$;*
2) *Steady if $2\alpha + 3\Omega = 0$; and*
3) *Shrinking if $2\alpha + 3\Omega > 0$.*

Remark 3.1. *Srivastava [48] studied an EoS of the form $\sigma + p = h(r)$, where $h(r)$ depends on the scale factor $r(t)$ and cosmic time t. In this framework, the choices $\sigma = -p$, $p = \sigma/3$, $\sigma = p$, and $p = 0$ correspond to the dark energy, radiation, stiff matter, and dust-dominated eras, respectively.*

For a Ξ_f spacetime that supports a steady ARS (M^4, g, ζ, μ) with CVF ζ, Theorem 3.5 implies

$$(\sigma + p)((\sigma - p) f_{T^2} + 8\pi) = 0, \tag{3.4}$$

together with

$$\Lambda - 8\pi p + \frac{r}{2} - \frac{1}{2}\lambda T^2 = -\frac{3}{2}\Omega.$$

In the case of stiff matter, $\sigma = p$, Eq (3.4) enforces $p = 0$, thus indicating that the corresponding Ξ_f spacetime behaves like a dust-dominated era.

Corollary 3.4. *If a Ξ_f spacetime is sourced by stiff matter and supports a steady ARS, then it effectively represents a dust-like era.*

When a phantom barrier occurs, σ and p satisfy the following:

$$\sigma = -p = \frac{1}{8\pi}\Big(\Lambda + \frac{r}{2} - \frac{1}{2}\lambda T^2 + \frac{3}{2}\Omega\Big).$$

Corollary 3.5. *For a Ξ_f spacetime with a phantom barrier source that admits a steady ARS, σ and p are determined as follows:*

$$\sigma = -p = \frac{1}{8\pi}\Big(\Lambda + \frac{r}{2} - \frac{1}{2}\lambda T^2 + \frac{3}{2}\Omega\Big).$$

A unit timelike vector field ζ is called a torse-forming vector field if

$$g(\zeta, \zeta) = -1 \quad \text{and} \quad \nabla_W \zeta = \phi(W + \eta(W)\zeta),$$

for any vector field W, where ϕ is a smooth scalar function, and η is the 1-form associated with ζ, defined as $\eta(W) = g(W, \zeta)$.

Assume that ζ is a unit timelike torse-forming vector field (UTTFF) in a spacetime (M^4, g) governed by an EMSG model $f(r, T^2)$. Then, it follows that $\nabla_\zeta \zeta = 0$, and the derivative of η satisfies the following:

$$(\nabla_{W_1}\eta)(W_2) = \phi(g(W_1, W_2) + \eta(W_1)\eta(W_2)).$$

Lemma 3.1. *In a Ξ_f spacetime, the following identities hold for a UTTFF ζ:*

$$R(W_1, W_2)\zeta = (\zeta\phi + \phi^2)(\eta(W_2)W_1 - \eta(W_1)W_2), \tag{3.5}$$

$$S(W_1, \zeta) = 3(\zeta\phi + \phi^2)\eta(W_1), \tag{3.6}$$

where R and S denote the curvature and Ricci tensors, respectively.

Proof. Let ζ be a UTTFF. From its defining relation, we have the following:

$$\nabla_{W_1}\zeta = \phi(W_1 + \eta(W_1)\zeta). \tag{3.7}$$

Assume the Ricci tensor along ζ takes the form

$$S(W_1, \zeta) = G\,\eta(W_1),$$

for some nonzero function G. Applying the covariant derivative ∇_{W_2} to both sides of (3.7) yields the following:

$$\nabla_{W_2}\nabla_{W_1}\zeta = (W_2\phi)(W_1 + \eta(W_1)\zeta) + \phi\Big[\nabla_{W_2}W_1 + (\nabla_{W_2}\eta)(W_1)\zeta + \phi(W_2 + \eta(W_2)\zeta)\eta(W_1)\Big]. \tag{3.8}$$

This expression forms the basis to derive (3.5) and (3.6).

By swapping W_1 and W_2 in (3.8), we obtain the following:

$$\nabla_{W_1}\nabla_{W_2}\zeta = (W_1\phi)(W_2 + \eta(W_2)\zeta) + \phi\Big[\nabla_{W_1}W_2 + (\nabla_{W_1}\eta)(W_2)\zeta + \phi(W_1 + \eta(W_1)\zeta)\eta(W_2)\Big]. \tag{3.9}$$

Replacing W_1 with the Lie bracket $[W_1, W_2]$ in (3.7) gives the following:

$$\nabla_{[W_1,W_2]}\zeta = \phi([W_1, W_2] + \eta([W_1, W_2])\zeta). \tag{3.10}$$

Combining (3.9) and (3.10) leads to the following curvature relation:

$$R(W_1, W_2)\zeta = (W_1\phi)(W_2 + \eta(W_2)\zeta) - (W_2\phi)(W_1 + \eta(W_1)\zeta) + \phi^2(\eta(W_2)W_1 - \eta(W_1)W_2). \tag{3.11}$$

Contracting (3.11) with respect to W_2 yields the following Ricci tensor component along ζ:

$$S(W_1, \zeta) = -2(W_1\phi) + (\zeta\phi)\eta(W_1) - 3\phi^2\eta(W_1). \tag{3.12}$$

Comparing (3.9) and (3.12) gives the following:

$$G\eta(W_1) = -2(W_1\phi) + (\zeta\phi)\eta(W_1) - 3\phi^2\eta(W_1), \tag{3.13}$$

where setting $W_1 = \zeta$ results in

$$G = 3(\zeta\phi + \phi^2). \tag{3.14}$$

Substituting (3.14) back into (3.13) gives the following:

$$W_1\phi = -(\zeta\phi)\eta(W_1). \tag{3.15}$$

Finally, directly inserting (3.15) into (3.11) leads to the identities (3.5) and (3.6).

Theorem 3.6. *Consider a Ξ_f spacetime with a UTTFF ζ. If it supports an ARS of the form $(M^4, g, \vartheta\zeta, \mu)$, where ϑ is a smooth function, then the gradient of ϑ satisfies*

$$D\vartheta = -(\zeta\vartheta)\zeta,$$

and the scalar curvature relation

$$4\alpha - \beta = -6(\phi\vartheta + 2\mu + 2\phi) - 2\zeta\vartheta$$

must hold.

Proof. For a UTTFF ζ, the Lie derivative of the metric is given by the following:

$$(\mathcal{L}_\zeta g)(W_2, W_3) = 2\phi(g(W_2, W_3) + \eta(W_2)\eta(W_3)),$$

so that $\operatorname{div}\zeta = 3\phi$.

Considering the scaled vector field $\vartheta\zeta$, one obtains the following:

$$(\mathcal{L}_{\vartheta\zeta} g)(W_2, W_3) = (W_2\vartheta)\eta(W_3) + (W_3\vartheta)\eta(W_2) + 2\phi\vartheta(g(W_2, W_3) + \eta(W_2)\eta(W_3)).$$

Using (3.2) together with the above, the Ricci tensor reads as follows:

$$\begin{aligned} S(W_2, W_3) = &-(2\phi\vartheta + 3\phi + 3\mu)g(W_2, W_3) \\ &- \Big[(W_2\vartheta)\eta(W_3) + (W_3\vartheta)\eta(W_2) + (2\phi\vartheta)\eta(W_2)\eta(W_3)\Big]. \end{aligned} \tag{3.16}$$

Substituting $W_3 = \zeta$ in (3.16) and simplifying yields the following:

$$3(\zeta\phi + \phi^2)\eta(W_2) = -(2\phi\vartheta + 3\phi + 3\mu)\eta(W_2) - [-(W_2\vartheta) + (\zeta\vartheta - 2\phi\vartheta)\eta(W_2)]. \tag{3.17}$$

Setting $W_2 = \zeta$ in (3.17) gives the following:

$$3(\zeta\phi + \phi^2) = -2\zeta\vartheta - 3\phi - 3\mu. \tag{3.18}$$

From (3.17) and (3.18), it follows that

$$W_2\vartheta = -(\zeta\vartheta)\eta(W_2),$$

which establishes

$$D\vartheta = -(\zeta\vartheta)\zeta.$$

Plugging this back into (3.16) simplifies the Ricci tensor to the following:

$$S(W_2, W_3) = -(2\phi\vartheta + 3\phi + 3\mu)g(W_2, W_3) - (-2\zeta\vartheta + 2\phi\vartheta)\eta(W_2)\eta(W_3). \tag{3.19}$$

Finally, contracting (3.19) gives the scalar curvature

$$r = -6(\phi\vartheta + 2\mu + 2\phi) - 2\zeta\vartheta,$$

and using the relation $r = 4\alpha - \beta$ completes the proof.

Corollary 3.6. *Consider a Ξ_f spacetime that possesses a UTTFF ζ. If it admits an ARS of the form (M^4, g, ζ, μ), then the scalar curvature satisfies the following:*

$$r = -6\,(2\mu + 3\phi).$$

Theorem 3.7. *Let Ξ_f be a spacetime that admits a gradient ARS $(M^4, g, \nabla\psi, \mu)$. Then, the following relation holds:*

$$\zeta\alpha + \beta\phi + \zeta(\Delta\psi + 3\mu) = \frac{2}{3}(\alpha - \beta)\zeta\psi.$$

Moreover, if $3\phi + 3\mu$ is constant and $\zeta\phi + \phi^2 \neq 0$, then

$$D\psi = -(\zeta\psi)\zeta, \quad \phi(\zeta\psi) = \frac{1}{2}(\Delta\psi + 3\mu) + \frac{1}{2}\alpha, \quad -3(\zeta\phi + \phi^2) - 3\mu - \Delta\psi + 2\phi(\zeta\psi) = \beta.$$

Proof. Let Ξ_f be a gradient ARS $(M^4, g, \nabla\psi, \mu)$. From Eq (3.7), we obtain

$$S(W_1, W_2) = -(\Delta\psi + 3\mu)g(W_1, W_2) - 2g(\nabla_{W_1} D\psi, W_2),$$

which leads to

$$\nabla_{W_1} D\psi = -\frac{1}{2}\left[QW_1 + (\Delta\psi + 3\mu)W_1\right], \tag{3.20}$$

where Q denotes the Ricci operator. Taking the covariant derivative of (3.20) along W_2 yields the following:

$$\nabla_{W_1}\nabla_{W_2} D\psi = -\frac{1}{2}\Big[\nabla_{W_1}(QW_2) + W_1(\Delta\psi + 3\mu)W_2 + (\Delta\psi + 3\mu)\nabla_{W_1} W_2\Big]. \tag{3.21}$$

Exchanging W_1 and W_2 gives the following:

$$\nabla_{W_2}\nabla_{W_1} D\psi = -\frac{1}{2}\Big[\nabla_{W_2}(QW_1) + W_2(\Delta\psi + 3\mu)W_1 + (\Delta\psi + 3\mu)\nabla_{W_2} W_1\Big].$$

Additionally, (3.20) implies the following

$$\nabla_{[W_1, W_2]} D\psi = -\frac{1}{2}\Big[Q[W_1, W_2] + (\Delta\psi + 3\mu)[W_1, W_2]\Big]. \tag{3.22}$$

Substituting (3.21) and (3.22) into the definition of the Riemann curvature operator

$$r(W_1, W_2)D\psi = \nabla_{W_1}\nabla_{W_2} D\psi - \nabla_{W_2}\nabla_{W_1} D\psi - \nabla_{[W_1, W_2]} D\psi,$$

and using $(\nabla_{W_1}\theta)W_2 = \phi(g(W_1, W_2) + \eta(W_1)\eta(W_2))$, we obtain

$$r(W_1, W_2)D\psi = -\frac{1}{2}\left[(\nabla_{W_1} Q)W_2 - (\nabla_{W_2} Q)W_1\right] - \frac{1}{2}\left[W_1(\Delta\psi + 3\mu)W_2 - W_2(\Delta\psi + 3\mu)W_1\right]. \tag{3.23}$$

From Eq (2.8), the Ricci operator can be written as $QX = \alpha X + \beta\eta(X)\zeta$. Differentiating this with respect to V gives the following:

$$(\nabla_V Q)(X) = V(\alpha)X + V(\beta)\eta(X)\zeta + \beta(\nabla_V \eta)(X)\zeta + \beta\omega(X)\nabla_V \zeta. \tag{3.24}$$

Substituting (3.24) into (3.23) leads to the following:

$$\begin{aligned} r(W_1, W_2)D\psi &= -\frac{1}{2}\Big[W_1(\alpha)W_2 - W_2(\alpha)W_1 + W_1(\beta)\eta(W_2)\zeta - W_2(\beta)\eta(W_1)\zeta\Big] \\ &\quad -\frac{1}{2}\Big[\beta(\nabla_{W_1}\eta)(W_2)\zeta - \beta(\nabla_{W_2}\eta)(W_1)\zeta + \beta\omega(W_2)\nabla_{W_1}\zeta - \beta\omega(W_1)\nabla_{W_2}\zeta\Big] \\ &\quad -\frac{1}{2}\Big[W_1(\Delta\psi + 3\mu)W_2 - W_2(\Delta\psi + 3\mu)W_1\Big]. \end{aligned}$$

Taking the inner product with W_1 and contracting over W_1, we find the following:

$$\begin{aligned} S(W_2, D\psi) = &-\frac{1}{2}\Big[-3W_2(\alpha) + (\zeta\beta)\eta(W_2) + W_2(\beta) + \beta(\nabla_\zeta\eta)(W_2) - \beta(\nabla_{W_2}\eta)(\zeta)\Big] \\ &-\frac{1}{2}\Big[3\beta\phi\omega(W_2) - 3W_2(\Delta\psi + 3\mu)\Big]. \end{aligned} \tag{3.25}$$

On the other hand, setting $W_1 = D\psi$ in (2.8) gives the following:

$$S(W_2, D\psi) = \alpha(W_2\psi) + \beta\omega(W_2)(\zeta\psi). \tag{3.26}$$

Finally, replacing W_2 with ζ in (3.25) and (3.26) results in the following:

$$\zeta\alpha + \beta\phi + \zeta(\Delta\psi + 3\mu) = \frac{2}{3}(\alpha - \beta)\zeta\psi.$$

Using the relation $Q\zeta = 3(\zeta\phi + \phi^2)\zeta$, we also deduce the following:

$$\begin{aligned} g((\nabla_{W_1}Q)W_2, \zeta) = &g((\nabla_{W_1}Q)\zeta, W_2) = 3\phi(\zeta\phi + \phi^2)g(W_1, W_2) \\ &+ 3W_1(\zeta\phi + \phi^2)\eta(W_2) - \phi S(W_1, W_2). \end{aligned} \tag{3.27}$$

By applying the inner product of Eq (3.23) with the vector field ζ and using (3.27), the following is obtained:

$$\begin{aligned} g(r(W_1, W_2)D\psi, \zeta) &= -\frac{3}{2}\Big[W_1(\zeta\phi + \phi^2)\eta(W_2) - W_2(\zeta\phi + \phi^2)\eta(W_1)\Big] \\ &\quad -\frac{1}{2}\left[W_1(\Delta\psi + 3\mu)\eta(W_2) - W_2(\Delta\psi + 3\mu)\eta(W_1)\right]. \end{aligned} \tag{3.28}$$

Using (3.5) in (3.28) leads to the following:

$$\begin{aligned} &(\zeta\phi + \phi^2)((W_2\psi)\eta(W_1) - (W_1\psi)\eta(W_2)) \\ = &-\frac{3}{2}\Big[W_1(\zeta\phi + \phi^2)\eta(W_2) - W_2(\zeta\phi + \phi^2)\eta(W_1)\Big] \\ &-\frac{1}{2}\left[W_1(\Delta\psi + 3\mu)\eta(W_2) - W_2(\Delta\psi + 3\mu)\eta(W_1)\right]. \end{aligned} \tag{3.29}$$

Substituting $W_2 = \zeta$ into (3.29) yields the following:

$$(\zeta\phi + \phi^2)(W_1\psi + (\zeta\psi)\eta(W_1)) = \frac{3}{2}\Big[W_1(\zeta\phi + \phi^2) + \zeta(\zeta\phi + \phi^2)\eta(W_1)\Big]$$

$$-\frac{1}{2}\left[-W_1(\Delta\psi + 3\mu) - \zeta(\Delta\psi + 3\mu)\eta(W_1)\right].$$

From [49, Proposition 2.2], we have the following:

$$W_1(\zeta\phi + \phi^2) + \zeta(\zeta\phi + \phi^2)\eta(W_1) = 0.$$

Assuming $\Delta\psi + 3\mu$ is constant, it follows that

$$W_1\psi + (\zeta\psi)\eta(W_1) = 0,$$

which implies

$$D\psi = -(\zeta\psi)\zeta.$$

Taking the covariant derivative with respect to W_1, we obtain the following:

$$\nabla_{W_1} D\psi = -(W_1(\zeta\psi))\zeta - \phi(\zeta\psi)[W_1 + \eta(W_1)\zeta]. \tag{3.30}$$

Taking the inner product of (3.30) with ζ and using (3.6) and (3.20), we find the following:

$$W_1(\zeta\psi) = -\frac{1}{2}(3(\zeta\phi + \phi^2) + \Delta\psi + 3\mu)\eta(W_1). \tag{3.31}$$

Substituting (3.31) into (3.30) gives the following:

$$\nabla_{W_1} D\psi = \frac{1}{2}(3(\zeta\phi + \phi^2) + \Delta\psi + 3\mu)\eta(W_1)\zeta - \phi(\zeta\psi)[W_1 + \eta(W_1)\zeta]. \tag{3.32}$$

Finally, substituting (3.32) into (3.20) results in the following:

$$S(W_1, W_2) = -[3(\zeta\phi + \phi^2) + \Delta\psi + 3\mu - 2\phi(\zeta\psi)]\eta(W_1)\eta(W_2) + [2\phi(\zeta\psi) - 3\mu - \Delta\psi]g(W_1, W_2). \tag{3.33}$$

Comparing (3.33) with (2.8) gives the following:

$$\phi(\zeta\psi) = \frac{1}{2}(\Delta\psi + 3\mu) + \frac{1}{2}\alpha, \quad -3(\zeta\phi + \phi^2) - 3\mu - \Delta\psi + 2\phi(\zeta\psi) = \beta.$$

Within a Lorentzian manifold M, a vector field v is called a v(Ric)-vector when

$$\nabla_X v = \psi Q(X)$$

with ψ being a constant.

Theorem 3.8. *Let Ξ_f admit a v(Ric)-vector field v. If the system allows an ARS (M^4, g, v, μ), then*

$$3\mu = -(1 + 6\psi)\alpha.$$

Furthermore, if $\psi \neq -\frac{1}{2}$, the spacetime Ξ_f is Einstein, which implies that $\beta = 0$.

Proof. For a ν(Ric)-vector field, the Lie derivative of the metric satisfies

$$(\mathcal{L}_\nu g)(W_1, W_2) = 2\psi S(W_1, W_2), \tag{3.34}$$

which leads to $\operatorname{div} \nu = \psi r$. Substituting (3.34) into (3.2) gives the following:

$$S(W_1, W_2) + (3\mu + \operatorname{div} \nu) g(W_1, W_2) + 2\psi S(W_1, W_2) = 0. \tag{3.35}$$

Contracting (3.35) allows us to solve for μ:

$$\mu = -\frac{(1 + 6\psi) r}{12}.$$

If $\psi \neq -\frac{1}{2}$, Eq (3.35) further implies

$$S(U_1, U_2) = -\frac{3\mu + \psi r}{1 + 2\psi} g(U_1, U_2),$$

so that the spacetime Ξ_f is Einstein. Then, from Eq (2.8), it follows that $\beta = 0$.

An n-dimensional spacetime is said to be $\mathcal{W}_2$-flat if its $\mathcal{W}_2$-curvature tensor, as defined in [50], satisfies

$$\begin{aligned} \mathcal{W}_2(W_1, W_2, W_3, W_4) \;=\;& r(W_1, W_2, W_3, W_4) \\ & + \frac{1}{n-1}\Big[g(W_1, W_3) S(W_2, W_4) - g(W_2, W_3) S(W_1, W_4)\Big] = 0, \end{aligned}$$

for all vector fields W_1, W_2, W_3, W_4.

Theorem 3.9. *Let Ξ_f be a $\mathcal{W}_2$-flat spacetime that admits an ARS (M^4, g, ζ, μ). Then, the following results hold:*

$$\mu = -\frac{r}{12}, \quad \phi = 0,$$

so that Ξ_f is an Einstein spacetime. Moreover, Ξ_f corresponds to a dust matter era with

$$f_{T^2} = -\frac{8\pi}{\sigma - p}.$$

Proof. Consider an EMSG model $f(r, T^2)$ spacetime (M^4, g) that is $\mathcal{W}_2$-flat. By definition, this implies the following:

$$R(W_1, W_2, W_3, W_4) = -\frac{1}{3}\Big[g(W_1, W_3) S(W_2, W_4) - g(W_2, W_3) S(W_1, W_4)\Big]. \tag{3.36}$$

Contracting (3.36) with respect to W_1 and W_4 gives the following:

$$S(W_2, W_3) = \frac{1}{4} r\, g(W_2, W_3). \tag{3.37}$$

On the other hand, from Eq (1.1), we have the following:

$$S(W_2, W_3) = -(5\phi + 3\mu) g(W_2, W_3) - 2\phi\, \eta(W_2)\eta(W_3). \tag{3.38}$$

Comparing (3.37) and (3.38), we obtain the following:

$$\Big(5\phi + 3\mu + \frac{1}{4}r\Big)g(W_2, W_3) + 2\phi\,\eta(W_2)\eta(W_3) = 0. \tag{3.39}$$

Equation (3.39) implies that $\phi = 0$ and $\mu = -\frac{r}{12}$. Moreover, from (3.37) it follows that $\beta = 0$, thus confirming that the EMSG model $f(r, T^2)$ spacetime is an Einstein spacetime.

Theorem 3.10. *Let Ξ_f be a pseudo-projectively flat and admit an AGZS (M^4, g, ζ, μ) with $c_1 + 3c_2 \neq 0$. Then we have $\mu = -\frac{r}{12}$ and $\phi = 0$, implying that Ξ_f is an Einstein spacetime.*

Proof. Consider the EMSG model spacetime (M^4, g) whose pseudo-projective curvature tensor $\bar{P}$ identically vanishes. Taking the inner product of $\bar{P}$ with a vector field W_4, one finds the following:

$$\begin{aligned} c_1 R(W_1, W_2, W_3, W_4) = &-c_2\Big[S(W_2, W_3)g(W_1, W_4) - S(W_1, W_3)g(W_2, W_4)\Big] \\ &+ \frac{r}{4}\Big(\frac{c_1}{3} + c_2\Big)\Big[g(W_2, W_3)g(W_1, W_4) - g(W_1, W_3)g(W_2, W_4)\Big]. \end{aligned} \tag{3.40}$$

Contracting (3.40) over W_1 and W_4 and using the assumption $c_1 + 3c_2 \neq 0$ leads to the following:

$$S(W_2, W_3) = \frac{r}{4}g(W_2, W_3). \tag{3.41}$$

Substituting (3.41) into the previous expression for the Ricci tensor (3.38) gives the following:

$$(5\phi + 3\mu + \frac{1}{4}r)g(W_2, W_3) + 2\phi\,\eta(W_2)\eta(W_3) = 0. \tag{3.42}$$

From (3.42), it directly follows that $\phi = 0$ and $\mu = -\frac{r}{12}$. Consequently, the EMSG spacetime satisfies $p + \sigma = 0$ or equivalently $f_{T^2} = -\frac{8\pi}{\sigma - p}$, thus confirming that Ξ_f is an Einstein spacetime.

The dust-like behavior ($p = 0$) and the phantom-barrier relation $p+\sigma = 0$ derived in Corollaries 3.10 and 3.11 (and in Theorems 3.17 and 3.18) are consequences of the additional geometric assumptions imposed (e.g., ζ concircular or torse-forming, steady ARS, W_2-flatness or pseudo-projective flatness). In the absence of those symmetry constraints, the ARS ansatz together with the EMSG field Eq (2.5) does not force these particular equations of state, and more general matter relations remain admissible.

According to [5], the Ricci tensor S in a spacetime satisfies

$$S(\zeta, \zeta) > 0, \tag{3.43}$$

for all timelike vector fields $\zeta \in \chi(M^4)$. Equation (3.43) represents the timelike convergence condition (TCC).

From (2.8), one has

$$S(\zeta, \zeta) = \alpha + \beta,$$

and therefore the TCC holds if $S(\zeta, \zeta) > 0$, which can be expressed as follows:

$$8\pi + (\sigma - p)f_{T^2} > \frac{1}{p + \sigma}\left(\frac{r}{2} + \frac{1}{2}\lambda T^2 + \Lambda - 8\pi p\right). \tag{3.44}$$

Equation (3.44) implies that M satisfies the cosmological strong energy condition (SEC) [51]. Based on this, we can state the following.

Theorem 3.11. *Let Ξ_f be a spacetime that admits an ARS (M^4, g, ζ, μ) with a timelike CVF ζ, and assume $p + \sigma \neq 0$. Then, the ARS is shrinking if $\Omega > 0$.*

Remark 3.2. *As shown by Hawking and Ellis [52], the following implications hold: TCC ⇒ SEC, TCC ⇒ NCC (Null Convergence Condition), and SEC ⇒ NEC (Null Energy Condition). Consequently, TCC also implies NCC.*

Using Remark 3.2 together with Theorem 3.11, we obtain the following results.

Theorem 3.12. *If a Ξ_f admits an expanding ARS (M^4, g, ζ, μ) with a CVF ζ, and if* (3.44) *is satisfied, then the spacetime obeys the SEC.*

Corollary 3.7. *Under the same conditions as above, the spacetime also satisfies the NCC.*

Corollary 3.8. *Suppose a Ξ_f spacetime admits an ARS structure (M^4, g, ζ, μ) with a CVF ζ and satisfies the SEC. Then, for an expanding ARS, the Ricci tensor corresponds to the second Segre type [52].*

Example 3.1. *The metric for a conformally Ricci-flat pure radiation spacetime in the coordinates $(u; v; x; y)$, where $x > 0$, is expressed as follows:*

$$g = (x - p^2\frac{v^2}{x^2})du^2 + 2dudv - \frac{4v}{x}dudx - \frac{1}{p^2}(dx^2 + dy^2), \tag{3.45}$$

where p is nonzero constant. This spacetime admits a steady Riemann soliton with the potential vector field $V = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial u}$ for some constants a, b [10].

Example 3.2. *The Vaidya solution represents a non-vacuum configuration in GR, specifically describing a spacetime geometry generated by radially moving photons with spherical symmetry. This solution is expressed in a coordinate system (u, r, θ, ϕ) defined in the vicinity of any point, where the metric tensor takes the following form:*

$$g = -\left(1 - \frac{2m(u)}{r}\right)du^2 - 2drdu + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.46}$$

In this expression, the function $m(u)$ represents a time-dependent mass parameter, while u denotes the retarded time coordinate, particularly relevant in scenarios involving matter accretion. An important limiting case occurs when the mass function becomes constant. Under this condition, the Vaidya metric reduces to: the Schwarzschild solution in advanced Eddington-Finkelstein coordinates, thereby describing the well-known non-rotating black hole geometry. Recent investigations have revealed that the Vaidya spacetime exhibits interesting geometric characteristics. When the mass function vanishes ($m = 0$), the spacetime described by Eq (3.46) constitutes a steady Riemann soliton [53]. In this case, the geometry admits a potential vector field of the following form:

$$V = V^1\frac{\partial}{\partial u} + V^2\frac{\partial}{\partial r} + V^3\frac{\partial}{\partial \theta} + V^4\frac{\partial}{\partial \phi} \tag{3.47}$$

where

$$V_1 = (a_2u + a_3)\cos\theta + ((a_4\cos\phi + a_5\sin\phi)u + a_6\cos\phi + a_7\sin\phi)\sin\theta + a_1,$$

$$
\begin{aligned}
V_2 &= -(a_2 r + a_2 u + a_3)\cos\theta \\
&\quad -((a_4\cos\phi + a_5\sin\phi)(r+u) + a_6\cos\phi + a_7\sin\phi)\sin\theta, \\
V_3 &= -\frac{1}{r}\left[-(a_2 u + a_3)\sin\theta + ((a_4\cos\phi + a_5\sin\phi)u + a_6\cos\phi + a_7\sin\phi)\cos\theta\right] \\
&\quad + a_2\sin\theta - (a_4\cos\phi + a_5\sin\phi)\cos\theta + a_8\cos\phi + a_9\sin\phi, \\
V_4 &= -\frac{1}{r\sin\theta}\left[(-a_4\sin\phi + a_5\cos\phi)u - a_6\sin\phi + a_7\cos\phi\right] \\
&\quad -\frac{1}{\sin\theta}(-a_4\sin\phi + a_5\cos\phi) + \cot\theta(-a_8\sin\phi + a_9\cos\phi) + a_{10},
\end{aligned}
\tag{3.48}
$$

for some constants $a_2, \ldots, a_{10}$.

We remark that our analysis is primarily structural and several directions remain open: explicit metric constructions that realize the ARS solutions in EMSG, numerical modeling of compact objects with the induced equations of state, and extensions to anisotropic or nonperfect fluids and to non-timelike soliton vectors.

4. Conclusions

In this paper, we investigated ARS within the framework of EMSG with a perfect fluid source and the specific model $f(r, T^2) = r + \lambda T^2$. By combining the ARS ansatz with the modified field equations, we derived explicit algebraic relations between geometric quantities and matter variables, including the effective coupling coefficients (α,β) and the links between the soliton function μ and the scalar curvature r. Our results show that, under mild assumptions, steady ARS configurations often lead to rigid geometric structures; in particular, the relation $\mu = -r/12$ appears in several settings.

We further demonstrated that when additional geometric symmetries are imposed, such as timelike concircular or torse-forming fields, W_2-flatness, or pseudo-projective flatness, the matter content becomes severely restricted. In these situations, the fluid equations of state reduce either to dust-like behavior ($p = 0$) or to the phantom-barrier condition $p + \sigma = 0$. However, these conclusions are conditional and specifically arise from the extra symmetry hypotheses; without such assumptions, the ARS condition together with the EMSG field equations admits more general matter configurations. This distinction was clarified at the end of Section 3, with additional remarks following Eq (2.8).

Moreover, we discussed the relation between ARS and other soliton concepts. Unlike Ricci solitons, which constrain only the Ricci tensor, ARS imposes conditions that involve the full Riemann tensor and therefore produces stronger geometric restrictions. Nevertheless, under special assumptions such as a constant scalar curvature, the ARS framework may reduce to Ricci-type relations, thus creating overlap with the existing soliton literature.

Finally, we acknowledge the limitations of the present study and outline potential extensions. Our analysis was primarily structural and analytic; explicit metric realizations of the obtained ARS solutions in EMSG (for instance cosmological models or static spherically symmetric spacetimes) are still lacking. Numerical investigations of compact objects within this framework, as well as generalizations to anisotropic or nonperfect fluids and to non-timelike soliton vector fields, remain important directions for future work. These developments will further test the physical relevance of the conditional dust and phantom-barrier regimes identified here, and will broaden the applicability of ARS methods in modified gravity.

Author contributions

Wedad A Alharbi: Conceptualization, investigation, methodology, writing-review & editing; Shahroud Azami: Conceptualization, investigation, methodology, writing-original draft; Mehdi Jafari: Conceptualization, methodology, writing-review & editing; Abdul Haseeb: Conceptualization, investigation, methodology, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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