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Research article

Analytic normalization and geometric behavior of generalized Bessel functions

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Abstract: This work introduced normalized representations of generalized Bessel functions, denoted by ${}_kH_{\beta,g}(\wp)$ and ${}_kM_{\beta,g}(\wp; s)$ in terms of k and (s, k), where $s > k > 0$ and $s < 1$. The motivation stemed from the increasing relevance of such functions in diverse fields such as mathematical physics, wave dynamics and fractional calculus, where classical Bessel functions may be limited. This study also gave detailed geometric investigation of these generalized Bessel functions. We derived sufficient conditions for these functions to be starlike and convex of order ζ in the open unit disk $\mathbb{U}$. The results of our analysis revealed enhanced structural flexibility over classical counterparts, making them well-suited for advanced applications in complex modeling theory. Several illustrative examples and plots were provided to validate and visualize the behavior of the proposed functions, highlighting their potential in mathematical modeling and theoretical studies. These findings enhanced the geometric function theory framework for special functions and provided a solid foundation for future analytical and applied investigations.

Keywords: starlikeness; convexity; Gamma function; generalized Bessel function
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1. Introduction

The functions which can be defined as mathematical functions appeared in many problems of different fields. They play a crucial role in engineering, physics, statistics, quantum mechanics, electro-optics, and applied and pure mathematics. They are often defined as the power series, integral, and differential equations etc. Many complex mathematical or physical problems are resolved by special functions. They are used in various approximations and numerical methods. Elementary special functions include polynomials, trigonometric functions and exponential functions. High transcendental functions are not algebraic, and include Gamma function, Beta function and Bessel functions etc. Some other special functions are hypergeometric functions, Wright functions, Weierstrass functions, Fox H-functions, Meijer G-functions etc.

Bessel functions were introduced by Friedrich Bessel in the 18th century as a solution of Bessel differential equation [10], which is given as

$$\wp^2 w''(\wp) + \wp w'(\wp) + (\wp^2 - \eta^2)w = 0,$$

with order η which can be integer or non-integer.

Several kinds of Bessel functions have been studied [10], among which the first kind is defined by

$$J_\eta(\wp) = \sum_{\eta=0}^{\infty} \frac{(-1)^\eta}{\Gamma(\eta + k + 1)\Gamma(k + 1)} \left(\frac{\wp}{2}\right)^{\eta+2k}.$$

The second kind, known as Neuman function, are given as

$$Y_\eta(\wp) = \frac{J_{-\eta}(\wp)cos(\eta\pi) - J_{-\eta}(\wp)}{sin(\eta\pi)}.$$

Hankel functions are third kind, which are given as

$$H_\eta^{(1)}(\wp) = J_\eta(\wp) + iY_\eta(\wp), \;\; H_\eta^{(2)}(\wp) = J_\eta(\wp) - iY_\eta(\wp).$$

Khosravian et al. [18] worked on generalized Bessel functions defined as

$${}^\tau J_\eta^1(\lambda\wp) = e^{-\tau\wp} x^{\frac{\eta}{2}} J_\eta(\lambda\sqrt{\wp}), \;\; {}^\tau J_\eta^2(\lambda\wp) = e^{\tau\wp} x^{\frac{-\eta}{2}} J_\eta(\lambda\sqrt{\wp}),$$

where $\lambda, \tau \in \mathbb{R}^+$.

The two indices cylindrical extended Bessel function was proposed by Cesarano and Assante [8], which is given as

$$J_{n,\eta}(\wp) = \sum_{s=-\infty}^{\infty} J_{n-s}(\wp)J_{\eta-s}(\wp)J_s(\wp).$$

Bessel functions are used in the study of cylindrical geometries of fluid flow, analysis of cylindrical symmetry electrical networks, modeling heat transfer problems, describing the radial part of vibrations of cell membrane and optical fibres etc. Reynolds and Stauffer [22] established a double integral, in which the kernel consists of a product of two Bessel functions, 1st kind and modified Bessel functions, in terms of the Hurwitz-Lerch zeta function. In a related study, Reynolds and Stauffer [23] introduced a triple integral involving the Bessel-integral function $Ji_v(z)$ in terms of product of Gamma and Hurwitz-Lerch zeta functions. Schrodinger equation in cylindrical function is solved by Faisal et al. [17] using Bessel function. Parand and Nikarya [21] investigated solutions of fractional differential equations using Bessel functions. In contemporary wave propagation and inverse scattering analysis, Bessel functions are essential. In recent works like Colton and Kress [9] and Diao et al. [11], where Bessel-type structures naturally arise in acoustic, electromagnetic and coupled-physics transmission models, their significance is emphasized. Incorporating these advancements reflects the wider applicability of Bessel functions in advanced scattering theory and enhances the motivation of the current work. Recent contributions related to Bessel functions in the form of k, (s,k), (p,k), and q along with their properties are given in [1, 25]. Furthermore, modified Bessel functions [4], and some other generalizations such as the Wright function and Meijer G-function are discussed in [3, 15, 24].

The Bessel k-function [28] is given as

$$ {}_kW_{p,g}(\wp) = \sum_{p=0}^{\infty} \frac{(-g)^p}{p!\Gamma_k(pk+\eta+k)} \left(\frac{\wp}{2}\right)^{2p+\frac{\eta}{k}}, \quad k \in \mathbb{R}^+, \quad \eta > -k, \quad g \in \mathbb{R}, \tag{1.1} $$

where Γ_k stands for k-Gamma function which is given as

$$ \Gamma_k(\ell) = \int_0^{\infty} t^{\ell-1} e^{\frac{-t^k}{k}} dt, \quad \Re(\ell) > 0, $$

with some basic properties [12] given as

$$ \begin{aligned} \Gamma_k(\ell) &= k^{\frac{\ell}{k}-1}\Gamma\left(\frac{\ell}{k}\right), \ \Gamma_k(\ell+k) = \ell\Gamma_k(\ell), \\ (\ell)_{n,k} &= \frac{\Gamma_k(\ell+nk)}{\Gamma_k(\ell)}, \ \Gamma_k(k) = 1. \end{aligned} $$

The generalized Bessel (s,k)-function [20] is expressed as follows:

$$ {}_kJ_{p,g}(\wp; s) = \sum_{p=0}^{\infty} \frac{(-g)^p}{p!\Gamma_k(\frac{ps}{k}+\eta+\frac{s}{k})} \left(\frac{\wp}{2}\right)^{2p+\frac{\eta k}{s}}, \tag{1.2} $$

where s and k are any positive $\mathbb{R}$ with $0 < s < 1$ and $s > k$, $\eta > -k$, $g \in \mathbb{R}$ and $\Gamma_{s,k}$ denotes (s,k)-Gamma function defined by

$$ \Gamma_{s,k}(\ell) = \int_0^{\infty} t^{\ell-1} e^{\frac{-t^{s/k}}{s}} dt, \quad \Re(\ell) > 0, $$

having properties [20]

$$ \begin{aligned} \Gamma_{s,k}(\ell) &= \left(\frac{s}{k}\right)^{\frac{\ell k}{s}-1} \Gamma\left(\frac{\ell k}{s}\right), \ \Gamma_{s,k}\left(\ell+\frac{s}{k}\right) = \ell\Gamma_{s,k}(\ell), \\ {}^s(\ell)_{n,k} &= \frac{\Gamma_{s,k}(\ell+\frac{ns}{k})}{\Gamma_{s,k}(\ell)}, \qquad \Gamma_{s,k}\left(\frac{s}{k}\right) = 1. \end{aligned} $$

When choosing $s = k = g = 1$, the generalized (s,k)-Bessel function coincides with the classical Bessel function J_η as

$$ J_\eta(\wp) = {}_1W_{\eta,1}(\wp) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(p+\eta+1)} \left(\frac{\wp}{2}\right)^{2p+\eta}. $$

If we choose $g = -1$, the function in Eq (1.2) simplifies to modified Bessel function I_η as

$$ I_\eta(\wp) = {}_1W_{\eta,-1}(\wp) = \sum_{p=0}^{\infty} \frac{1}{p!\Gamma(p+\eta+1)} \left(\frac{\wp}{2}\right)^{2p+\eta}. $$

In domains like laser physics, electromagnetics and signal processing, generalized Bessel functions are useful for modelling radiation phenomena, wave propagation, and filter designs. They are especially helpful for examining systems with complex interactions and cylindrical or spherical symmetry, like multi-photon ionization processes or the analysis of charged radiating particles in magnetic devices. Huo et al. [16] achieved quick analytical evaluation of relative-phase contrasts in two-color heavy-field ionization using generalized Bessel functions, matching fitting-based results with significantly less computation. Dattoli et al. [13] demonstrated how generalized Bessel functions can support intricate analyses of multi-harmonic undulators and nonlinear Compton effects and elegantly model spectral properties of radiation from relativistic electrons. The Laguerre-Bessel function derived from generalized Bessel functions allows for concise analytical descriptions of radiation processes and paraxial wave dynamics, which is demonstrated by Torre et al. [14].

If the mapping of a function is starlike from unit disk $\mathbb{U}$ onto domain D regarding origin, then the function is starlike. It can be of any order ζ which describes how starlike the image is. Similarly if the image is convex set, then the function is convex and it can also have an order ζ. A function f analytic in the unit disk $\mathbb{D} = z : |z| < 1$ is said to be starlike of order ζ $(0 \leq \zeta < 1)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \zeta \quad \text{for all } z \in \mathbb{D}.$$

And it is convex if it satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \zeta \quad \text{for all } z \in \mathbb{D}.$$

In this paper, we consider starlikeness and convexity of the k and (s, k)-Bessel function in this sense, where the function is normalized and analytic in the unit disk. Moreover, Salah [26] worked on uniformly starlike functions involving Mittag-Leffler function and Lambert series with respect to symmetrical points. Bulboacua and Zayed [5] discussed the starlikeness and convexity of Bessel functions, which is extended by Shah et al. [27] in terms of generalized Bessel k-functions. Recent researchers worked on the starlikeness and convexity of generalized Bessel functions [6, 30], Wright function [7], Sturve function [29] and Hardy class of q-Bessel functions [2]. Khan et al. [19] established analytic inequalities, Fekete-Szegý estimates, and sharp coefficient bounds by applying the theory of starlikeness and convexity to quantum-difference-based multivalent subclasses in the cardioid domain.

Normalization is a scaling or transformation applied to a function to make sure $f(0) = 0, f'(0) = 1$. This makes the functions analytic in the unit disc, maintains geometric properties like convexity and starlikeness, and permits standard inequalities, such as $\Re\left(\frac{zf'(z)}{f(z)}\right) > \zeta$ and $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \zeta$, to hold. Unlike classical functions normalized directly, the Bessel functions are normalized via parameter-dependent transformations to satisfy analyticity.

Our current approach is to define normalized forms of generalized Bessel function different from the normalized form defined by Shah et al. [27] in terms of k and (s, k)-functions where $k > 0$, $0 < s < 1$, $s > k$. These conditions are taken to guarantee the validity of the results. In particular, $s > k > 0$ ensures that the (s, k)-Bessel function is analytic and well-defined in the region of interest. Another parameter β is used in the following results, and appropriate conditions on β are necessary for the inequalities employed to establish starlikeness and convexity properties. Moreover, the condition $s < 1$ guarantees the convergence of the corresponding series representation. These assumptions are essential for the theoretical framework and cannot be relaxed without affecting the validity of the main theorems. Sufficient conditions for starlikesness and convexity of defined normalized forms will also be discussed. Some illustrative examples will also be provided to authenticate our results.

2. Main results

In the following section, we modify the functions ${}_kW_{\eta,g}(\wp)$ and ${}_kJ_{\eta,g}(\wp; s)$ to ensure analyticity and subsequently introduce the normalized forms of the k-Bessel and (s, k)-Bessel functions, which serve as the basis for further analysis in the following sections.

The function ${}_kW_{\eta,g}(\wp)$ defined in (1.1) is not analytic. The non-analytic behavior results from the presence of the exponent $2p + \frac{\eta}{k}$. If $\frac{\eta}{k}$ is non-integer, the function contains fractional powers of $\wp$, producing a branch point at the origin and preventing expansion in integer powers. Analyticity at origin occurs only when $\frac{\eta}{k}$ is a non-negative integer. Thus, we define a new function based on ${}_kW_{\eta,g}(\wp)$ with parameters k be any positive real number, $\eta > -k$ and $g \in \mathbb{R}$, which is given as

$$\begin{aligned} {}_kh_{\eta,g}(\wp) &= \left(2\sqrt{k}\right)^{\frac{\eta}{k}} \Gamma_k(\eta + k)\, \wp^{1-\frac{\eta}{2k}}\ {}_kW_{\eta,g}\Big(\sqrt{\frac{\wp}{k}}\Big) \\ &= \left(2\sqrt{k}\right)^{\frac{\eta}{k}} \Gamma_k(\eta + k)\, \wp^{1-\frac{\eta}{2k}} \left[\sum_{p=0}^{\infty} \frac{(-g)^p \left(\frac{\sqrt{\wp}}{2\sqrt{k}}\right)^{2p+\frac{\eta}{k}}}{p!\, \Gamma_k(pk + \eta + k)}\right] \\ &= \sum_{p=0}^{\infty} \frac{(-g)^p\, \wp^{p+1}}{p!\, 4^p\, k^p\, (\eta + k)_{p,k}} = \sum_{p=0}^{\infty} \frac{(-g)^p\, \wp^{p+1}}{p!\, 4^p\, k^p\, (\beta)_{p,k}}, \end{aligned} \tag{2.1}$$

where $\beta = \eta + k$ and $\beta \in \mathbb{Z}^+$.

The normalization of ${}_kh_{\beta,g}(\wp)$ can be described as:

Definition 2.1. *For k as any positive real number, $\eta > -k$ and g be a real number, the normalized version of ${}_kh_{\beta,g}(\wp)$ is given by*

$${}_kH_{\beta,g}(\wp) = \wp \,.\, {}_kh_{\beta,g}(\wp) = \wp^2 + \sum_{p=1}^{\infty} \frac{(-g)^p\, \wp^{p+2}}{p!\, 4^p\, k^p\, (\beta)_{p,k}}, \tag{2.2}$$

where $\beta = \eta + k \in (0, +\infty)$.

Similarly, to make the function ${}_kJ_{\eta,g}(\wp; s)$, defined in (1.2) analytic, we introduced a function that originates from ${}_kJ_{\eta,g}(\wp; s)$ with parameters $s > k > 0$, $s < 1$, $\eta > -k$, $g \in \mathbb{R}$, where k and s be any positive real numbers, given as

$$\begin{aligned} {}_km_{\eta,g}(\wp; s) &= \left(2\sqrt{\frac{s}{k}}\right)^{\frac{\eta k}{s}} \Gamma_{s,k}\Big(\eta + \frac{s}{k}\Big)\, \wp^{1-\frac{\eta k}{2s}}\ {}_kJ_{\eta,g}\Big(\sqrt{\frac{\wp k}{s}}; s\Big) \\ &= \left(2\sqrt{\frac{s}{k}}\right)^{\frac{\eta k}{s}} \Gamma_{s,k}\Big(\eta + \frac{s}{k}\Big)\, \wp^{1-\frac{\eta k}{2s}} \left[\sum_{p=0}^{\infty} \frac{(-g)^p \left(\frac{\sqrt{\wp}}{2\sqrt{\frac{s}{k}}}\right)^{2p+\frac{\eta k}{s}}}{p!\, \Gamma_{s,k}\big(\frac{ps}{k} + \eta + \frac{s}{k}\big)}\right] \end{aligned}$$

$$= \sum_{p=0}^{\infty} \frac{(-g)^p \ \wp^{p+1}}{p! \ 4^p \ (\frac{s}{k})^p \ (\eta + \frac{s}{k})_{p,k}}$$

$$= \sum_{p=0}^{\infty} \frac{(-g)^p \ \wp^{p+1}}{p! \ (\frac{4s}{k})^p \ {}^s(\beta)_{p,k}}, \tag{2.3}$$

where ${}^s(\beta) = \eta + \frac{s}{k}$ and $\beta \in \mathbb{Z}^+$.

The normalization of ${}_k m_{\beta,g}(\wp; s)$ can be defined as:

Definition 2.2. *For any positive real numbers k and s with $s > k > 0$, $s < 1$, $\eta > -k$, $g \in \mathbb{R}$, the normalized form of ${}_k m_{\beta,g}(\wp; s)$ is given by*

$${}_k M_{\beta,g}(\wp; s) = \wp \ . \ {}_k m_{\beta,g}(\wp) \ = \wp^2 + \sum_{p=1}^{\infty} \frac{(-g)^p \ \wp^{p+2}}{p!(\frac{4s}{k})^p \ {}^s(\beta)_{p,k}}, \tag{2.4}$$

where ${}^s(\beta) = \eta + \frac{s}{k} \in (0, +\infty)$.

3. Starlike and convex behavior of generalized k-Bessel function

This segment states theorems that offer enhanced results concerning the starlikeness and convexity of some order for Bessel function ${}_k H_{\beta,g}$ defined in Eq (2.2):

Theorem 3.1. *Consider $\beta > 2$, $g \in \mathbb{C}^*$ and $k \in \mathbb{R}^+$ with*

$$0 < |g| < \frac{4k\beta}{\beta - 2} =: g_*, \tag{3.1}$$

and if

$$\zeta \leq 1 - \frac{\beta(4k - |g|) + 2|g|}{\beta(4k - |g|) - |g|} =: \zeta_*, \tag{3.2}$$

then ${}_k H_{\beta,g}$ is starlike of order ζ.

Proof. To prove the statement, we will use the condition given in [27] for starlikeness, which is

$$\left| \frac{\wp({}_k H_{\beta,g}(\wp))'}{{}_k H_{\beta,g}(\wp)} - 1 \right| < 1 - \zeta, \qquad \zeta \leq 1. \tag{3.3}$$

Taking

$$\left| ({}_k H_{\beta,g}(\wp))' - \frac{{}_k H_{\beta,g}(\wp)}{\wp} \right| = \left| \left(\wp^2 + \sum_{p=1}^{\infty} \frac{(-g)^p \ \wp^{p+2}}{p! \ 4^p \ k^p \ (\beta)_{p,k}} \right)' - \frac{\wp^2 + \sum_{p=1}^{\infty} \frac{(-g)^p \ \wp^{p+2}}{p! \ 4^p \ k^p \ (\beta)_{p,k}}}{\wp} \right|$$

$$= \left| 2\wp + \sum_{p=1}^{\infty} \frac{(p+2)(-g)^p \ \wp^{p+1}}{p! \ 4^p \ k^p \ (\beta)_{p,k}} - \frac{\wp \left[\wp + \sum_{p=1}^{\infty} \frac{(-g)^p \ \wp^{p+1}}{4^p \ k^p \ p! \ (\beta)_{p,k}} \right]}{\wp} \right|$$

$$
\begin{aligned}
&= \left| \wp + \sum_{p=1}^{\infty} \frac{(p+2)\,(-g)^p\,\wp^{p+1}}{4^p\,k^p\,p!\,(\beta)_{p,k}} - \sum_{p=1}^{\infty} \frac{(-g)^p\,\wp^{p+1}}{4^p\,k^p\,p!\,(\beta)_{p,k}} \right| \\
&= \left| \wp + \sum_{p=1}^{\infty} \frac{(p+1)\,(-g)^p\,\wp^{p+1}}{4^p\,k^p\,p!\,(\beta)_{p,k}} \right| < \sup_{\theta\in 2\pi} \left| e^{i\theta p} + \sum_{p=1}^{\infty} \frac{(p+1)\,(-g)^p\,e^{i\theta p}}{4^p\,k^p\,p!\,(\beta)_{p,k}} \right| \\
&\leq 1 + \sum_{p=1}^{\infty} \frac{(p+1)\,|g|^p}{4^p\,k^p\,p!\,\frac{\Gamma_k(\beta+pk)}{\Gamma_k(\beta)}}, \qquad |\wp| \leq 1 \\
&\leq 1 + \frac{\Gamma_k(\beta+k)}{\beta} \sum_{p=1}^{\infty} \frac{|g|^p\,(p+1)}{4^p\,k^p\,\Gamma_k(p+k)\Gamma_k(\beta+pk)}.
\end{aligned} \tag{3.4}
$$

Letting $\sigma_k(p)$ be the function defined as

$$
\sigma_k(p) = \frac{p+1}{\Gamma_k(p+k)\Gamma_k(\beta+pk)}, \qquad p \in \mathbb{N}. \tag{3.5}
$$

Taking the difference of two consecutive terms to analyze the behavior of the function $\sigma_k(p+1)$, we get

$$
\begin{aligned}
\sigma_k(p+1) - \sigma_k(p) &= \frac{p+2}{\Gamma_k(p+1+k)\Gamma_k(\beta+(p+1)k)} - \frac{p+1}{\Gamma_k(p+k)\Gamma_k(\beta+pk)} \\
&= \frac{p+2-(p+1)(p+k)(\beta+pk)}{(p+k)(\beta+pk)\Gamma_k(p+k)\Gamma_k(\beta+pk)} \\
&= -\frac{p^2\beta + p^3k + pk\beta + p^2k^2 + p\beta + p^2k - p - 2}{\Gamma_k(p+k+1)\Gamma_k(\beta+pk+k)}.
\end{aligned} \tag{3.6}
$$

Since the numerator is positive for all $p \in \mathbb{N}$ because the highest-order term $p^3k > 0$ dominates for all $p \geq 1$, and the remaining terms are either positive or grow slower than p^3k and the denominator is also positive as $\Gamma_k(x)$ for $x > 0$, which immediately follows that

$$
\sigma_k(p+1) - \sigma_k(p) < 0, \; p \in \mathbb{N}. \tag{3.7}
$$

Inequality (3.7) shows that the function is decreasing strictly. Therefore,

$$
\frac{p+1}{\Gamma_k(p+k)\Gamma_k(\beta+pk)} \leq \sigma_k(1) = \frac{2}{\Gamma_k(1+k)\Gamma_k(\beta+k)}.
$$

By the inequality (3.4), we get

$$
\begin{aligned}
\left| ({}_kH_{\beta,g}(\wp))' - \frac{{}_kH_{\beta,g}(\wp)}{\wp} \right| &< 1 + \frac{\Gamma_k(\beta+k)}{\beta} \sum_{p=1}^{\infty} \frac{2\,|g|^p}{4^p\,k^p\,\Gamma_k(1+k)\Gamma_k(\beta+k)} \\
&= 1 + \frac{2}{\beta} \sum_{p=1}^{\infty} \left(\frac{|g|}{4k} \right)^p = 1 + \frac{2|g|}{\beta(4k-|g|)} \\
&= \frac{2|g| + \beta(4k-|g|)}{\beta(4k-|g|)}, \qquad |\wp| < 1, \; k \geq 0.
\end{aligned} \tag{3.8}
$$

Provided that the inequality (3.8) satisfies with a value greater than zero, we obtain

$$|g| < \frac{4k\beta}{\beta - 2}, \tag{3.9}$$

equivalent to $0 < |g| < 4k$. If $|g| = 0$ the function reduces to identity function which is trivial case. As

$$\left|\frac{{}_kH_{\beta,g}(\wp)}{\wp}\right| = \left|\wp + \sum_{p=1}^{\infty} \frac{(-g)^p\ \wp^{p+1}}{p!\ 4^p\ k^p\ (\beta)_{p,k}}\right|.$$

From the triangular inequality and the maximum modulus theorem, it follows that

$$\begin{aligned}
\left|\frac{{}_kH_{\beta,g}(\wp)}{\wp}\right| &> |\wp| - \sup\left|\sum_{p=1}^{\infty} \frac{(-g)^p\ e^{i\theta p}}{4^p\ k^p\ p!\ (\beta)_{p,k}}\right| \\
&\geq 1 - \sum_{p=1}^{\infty} \frac{|g|^p}{4^p\ k^p\ \ p!\ (\beta)_{p,k}} \\
&= 1 - \sum_{p=1}^{\infty} \frac{|g|^p\ \Gamma_k(\beta)}{4^p\ k^p\ \Gamma_k(\beta + pk)\ \Gamma_k(p + k)} \\
&= 1 - \frac{\Gamma_k(\beta + k)}{\beta} \sum_{p=1}^{\infty} \frac{|g|^p}{4^p\ k^p\ \Gamma_k(\beta + pk)\ \Gamma_k(p + k)}.
\end{aligned}$$

As we know that $\frac{1}{\Gamma_k(\beta+pk)\ \Gamma_k(p+k)}$ is strictly decreasing, so

$$\begin{aligned}
\left|\frac{{}_kH_{\beta,g}(\wp)}{\wp}\right| &> 1 - \frac{\Gamma_k(\beta + k)}{\beta} \sum_{p=1}^{\infty} \frac{|g|^p}{4^p\ k^p\ \Gamma_k(\beta + k)\ \Gamma_k(1 + k)} \\
&= 1 - \frac{1}{\beta} \sum_{p=1}^{\infty} \left(\frac{|g|}{4k}\right)^p = 1 - \frac{|g|}{\beta(4k - |g|)} \\
&= \frac{\beta(4k - |g|) - |g|}{\beta(4k - |g|)},
\end{aligned} \tag{3.10}$$

where

$$\frac{\beta(4k - |g|) - |g|}{\beta(4k - |g|)} > 0. \tag{3.11}$$

Equation (3.11) holds true provided that $\beta > 2$ and

$$|g| < \min\left\{4; \frac{4k\beta}{\beta - 2}\right\} = \frac{4k\beta}{\beta - 2}. \tag{3.12}$$

Since

$$\left|\frac{\wp({}_kH_{\beta,g}(\wp))'}{{}_kH_{\beta,g}(\wp)} - 1\right| = \left|({}_kH_{\beta,g}(\wp))' - \frac{{}_kH_{\beta,g}(\wp)}{\wp}\right| \left|\frac{\wp}{{}_kH_{\beta,g}(\wp)}\right|$$

$$< \frac{2|g| + \beta(4k - |g|)}{\beta(4k - |g|)} \times \frac{\beta(4k - |g|)}{\beta(4k - |g|) - |g|}$$
$$< \frac{2|g| + \beta(4k - |g|)}{\beta(4k - |g|) - |g|} \leq 1 - \zeta,$$

which gives

$$\zeta \leq 1 - \frac{2|g| + \beta(4k - |g|)}{\beta(4k - |g|) - |g|}. \tag{3.13}$$

Finally from inequality (3.2), it is proved that ${}_kH_{\beta,g}$ is a starlike function of order ζ.

Theorem 3.2. *Consider $\beta > 3$ and $g \in \mathbb{C}^*$ and for any positive real number k with*

$$0 < |g| < \frac{8k\beta}{2\beta - 6} =: g_c, \tag{3.14}$$

and if

$$\zeta \leq 1 - \frac{2\beta(4k - |g|) + 6|g|}{2\beta(4k - |g|) - 3|g|} =: \zeta_c, \tag{3.15}$$

then ${}_kH_{\beta,g}(\wp)$ is a convex function of order ζ.

Proof. To establish the statement, we make use of the convexity condition discussed in [27], modifying it appropriately for the (s, k)-function, which is

$$\left|\frac{\wp({}_kH_{\beta,g}(\wp))''}{({}_kH_{\beta,g}(\wp))'}\right| < 1 - \zeta, \qquad \zeta \leq 1. \tag{3.16}$$

Consider $\left|\wp({}_kH_{\beta,g}(\wp))''\right|$,

$$\begin{aligned}
\left|\wp({}_kH_{\beta,g}(\wp))''\right| &= \left|\wp\left(\wp^2 + \sum_{p=1}^{\infty} \frac{(-g)^p\ \wp^{p+2}}{p!\ 4^p\ k^p\ (\beta)_{p,k}}\right)''\right| \\
&= \left|\wp\left(2\wp + \sum_{p=1}^{\infty} \frac{(p+2)(-g)^p\ \wp^{p+1}}{p!\ 4^p\ k^p\ (\beta)_{p,k}}\right)'\right| \\
&= \left|\wp\left(2 + \sum_{p=1}^{\infty} \frac{(p+2)\ (p+1)\ (-g)^p\ \wp^p}{4^p\ k^p\ p!\ (\beta)_{p,k}}\right)\right| \\
&= \left|2\wp + \sum_{p=1}^{\infty} \frac{(p+2)\ (p+1)\ (-g)^p\ \wp^{p+1}}{4^p\ k^p\ p!\ (\beta)_{p,k}}\right|.
\end{aligned}$$

Employing both the triangle inequality and the maximum modulus theorem, we find

$$\begin{aligned}
\left|\wp({}_kH_{\beta,g}(\wp))''\right| &\leq |2\wp| + \left|\sum_{p=1}^{\infty} \frac{(p+1)\ (p+2)\ (-g)^p\ \wp^{p+1}}{4^p\ k^p\ p!\ (\beta)_{p,k}}\right| \\
&< |2e^{i\theta p}| + \sup_{\theta \in 2\pi} \left|\sum_{p=1}^{\infty} \frac{(p+1)\ (p+2)\ (-g)^p\ e^{i\theta p}}{4^p\ k^p\ p!\ (\beta)_{p,k}}\right|
\end{aligned}$$

$$\leq 2+\sum_{p=1}^{\infty}\frac{(p+1)\,(p+2)\,|g|^{p}}{4^{p}\,k^{p}\,p!\,\frac{\Gamma_k(\beta+pk)}{\Gamma_k(\beta)}}, \qquad |\wp|\leq 1$$

$$=2+\frac{\Gamma_k(\beta+k)}{\beta}\sum_{p=1}^{\infty}\frac{|g|^{p}\,(p+1)\,(p+2)}{4^{p}\,k^{p}\,\Gamma_k(p+k)\Gamma_k(\beta+pk)}. \tag{3.17}$$

Now let a function $\varrho_k(p)$ which is defined as

$$\varrho_k(p)=\frac{(p+1)(p+2)}{\Gamma_k(p+k)\Gamma_k(\beta+pk)}, \qquad p\in\mathbb{N}. \tag{3.18}$$

We find $\varrho_k(p+1\,)-\varrho_k(p\,)$ to check the behavior of the function.

$$\begin{aligned}
\varrho_k(p+1\,)-\varrho_k(p\,) &= \frac{(p+2)(p+3)}{\Gamma_k(p+1+k)\Gamma_k(\beta+(p+1)k)}-\frac{(p+1)\,(p+2)}{\Gamma_k(p+k)\Gamma_k(\beta+pk)}\\
&=\frac{p+2}{\Gamma_k(p+k)\Gamma_k(\beta+pk)}\left[\frac{p+3}{(p+k)(\beta+pk)}-(p+1)\right]\\
&=\frac{p+2}{\Gamma_k(p+k)\Gamma_k(\beta+pk)}\left[\frac{p+3-(p+1)(p+k)(\beta+pk)}{(p+k)(\beta+pk)}\right]\\
&=-\frac{(p^4+3p^3+3\beta p+\beta p^2+2p^2+2\beta)k+(p^3+2p+3p^2)k^2}{\Gamma_k(p+1+k)\Gamma_k(\beta+pk+k)}\\
&\quad-\frac{(p^3+3p^2+2p)\beta-p^2-5p-6}{\Gamma_k(p+1+k)\Gamma_k(\beta+pk+k)}.
\end{aligned} \tag{3.19}$$

Observing that the numerator is positive for all $p\in\mathbb{N}$ since the leading term $p^4k>0$ dominates and the remaining terms are either positive or of smaller order, and noting that the denominator is positive due to $\Gamma_k(x)$ for $x>0$, we conclude that

$$\varrho_k(p+1)-\varrho_k(p)<0,\ \ p\in\mathbb{N}. \tag{3.20}$$

Inequality (3.20) relies on the recurrence relation of the k-Gamma function and its positivity, which ensures that the difference of consecutive terms is negative, confirming the monotonic decreasing behavior of $\varrho_k(p)$, so

$$\frac{(a\ +1)(a+2)}{\Gamma_k\,(a+k)\Gamma_k\,(\beta+ak)}\leq\varrho_k\,(1)=\frac{2(3)}{\Gamma_k\,(1+k)\Gamma_k\,(\beta+k)}=\frac{6}{\Gamma_k\,(1+k)\Gamma_k\,(\beta+k)}.$$

Based on inequality (3.17), we have

$$\begin{aligned}
\left|\wp({}_kH_{\beta,g}(\wp))''\right| &< 2+\frac{\Gamma_k(\beta+k)}{\beta}\sum_{p=1}^{\infty}\frac{6\,|g|^{p}}{4^{p}\,k^{p}\,\Gamma_k(1+k)\Gamma_k(\beta+k)}\\
&=2+\frac{6}{\beta}\sum_{p=1}^{\infty}\left(\frac{|g|}{4k}\right)^{p}=2+\frac{6|g|}{\beta(4k-|g|)}\\
&=\frac{6|g|+2\beta(4k-|g|)}{\beta(4k-|g|)}, \qquad \wp\in\mathbb{U},\ k\in\mathbb{R}^{+}.
\end{aligned} \tag{3.21}$$

Consider $\frac{6|g|+2\beta(4k-|g|)}{\beta(4k-|g|)} > 0$, we derive

$$|g| < \frac{8k\beta}{2\beta-6}, \tag{3.22}$$

equivalent to $0 < |g| < 4k$, which holds because of our assumptions.

Now consider $\left|({}_kH_{\beta,g}(\wp))'\right|$, which is

$$\left|({}_kH_{\beta,g}(\wp))'\right| = \left|\left(\wp^2 + \sum_{p=1}^{\infty} \frac{(-g)^p\ \wp^{p+2}\Gamma_k(\beta)}{k^p\ 4^p\ \Gamma_k(p+k)\ \Gamma_k(\beta+pk)}\right)'\right| = \left|2\wp + \sum_{p=1}^{\infty} \frac{(-g)^p\ \wp^{p+1}\ (p+2)\Gamma_k(\beta+k)}{\beta\ k^p\ 4^p\ \Gamma_k(p+k)\ \Gamma_k(\beta+pk)}\right|.$$

Employing the triangular inequality along with the maximum modulus theorem, we get

$$\begin{aligned}\left|({}_kH_{\beta,g}(\wp))'\right| &> |2e^{i\theta p}| - \sup\left|\sum_{p=1}^{\infty} \frac{(p+2)\ (-g)^p\ e^{i\theta p}\ \Gamma_k(\beta+k)}{4^p\ k^p\ \Gamma_k(p+k)\ \Gamma_k(\beta+pk)}\right| \\ &\geq 2 - \frac{\Gamma_k(\beta+k)}{\beta} \sum_{p=1}^{\infty} \frac{(p+2)\ |g|^p}{4^p\ k^p\ \Gamma_k(\beta+pk)\ \Gamma_k(p+k)}, \quad \wp \in \mathbb{U}.\end{aligned}$$

As fact $\frac{1}{\Gamma_k(\beta+pk)\ \Gamma_k(p+k)}$ is strictly decreasing, we have

$$\begin{aligned}\left|({}_kH_{\beta,g}(\wp))'\right| &> 2 - \frac{\Gamma_k(\beta+k)}{\beta} \sum_{p=1}^{\infty} \frac{3\ |g|^p}{4^p\ k^p\ \Gamma_k(\beta+k)\ \Gamma_k(1+k)} \\ &= 2 - \frac{3}{\beta}\sum_{p=1}^{\infty}\left(\frac{|g|}{4k}\right)^p = 2 - \frac{3|g|}{\beta(4k-|g|)} \\ &= \frac{2\beta(4k-|g|)-3|g|}{\beta(4k-|g|)},\end{aligned} \tag{3.23}$$

where

$$\frac{2\beta(4k-|g|)-3|g|}{\beta(4k-|g|)} > 0. \tag{3.24}$$

The preceding equation holds true due to $\beta > 3$ and

$$|g| < \min\left\{4; \frac{8k\beta}{2\beta-6}\right\} = \frac{8k\beta}{2\beta-6}. \tag{3.25}$$

Since

$$\begin{aligned}\left|\frac{\wp({}_kH_{\beta,g}(\wp))''}{({}_kH_{\beta,g}(\wp))'}\right| &= \left|\wp({}_kH_{\beta,g}(\wp))''\right|\ \left|\frac{1}{({}_kH_{\beta,g}(\wp))'}\right| \\ &< \frac{6|g|+2\beta(4k-|g|)}{\beta(4k-|g|)} \times \frac{\beta(4k-|g|)}{2\beta(4k-|g|)-3|g|} \\ &< \frac{6|g|+2\beta(4k-|g|)}{2\beta(4k-|g|)-3|g|} \leq 1-\zeta,\end{aligned}$$

which gives

$$\zeta \leq 1 - \frac{6|g|+2\beta(4k-|g|)}{2\beta(4k-|g|)-3|g|}. \tag{3.26}$$

${}_kH_{\beta,g}$ is a convex function of order ζ by the inequality (3.15).

4. Applications of starlike and convex behavior of generalized Bessel k-function

This segment presents application of theorems to examine convexity and starlikeness of different orders and the graphical representations which are generated by Mathematica 13.3.

Example 4.1. Choose $\beta = 2.2565$, $g = 1.5453$, $\zeta = 0$ and $k = 0.1$. To check if ${}_{0.1}H_{2.2565,1.5453}$ is starlike or convex, the conditions for starlikeness and convexity given in (3.1), (3.2), (3.14), and (3.15) that must be verified, which are

$$0 < |1.5453| < \frac{4 \times 0.1(2.2565)}{2.2565 - 2} = 3.51891.$$

Also,

$$0 \leq 1 - \frac{2.2565(4 \times 0.1 - |1.5453|) + 2|1.5453|}{2.2565(4 \times 0.1 - |1.5453|) - |1.5453|} = 1.12258.$$

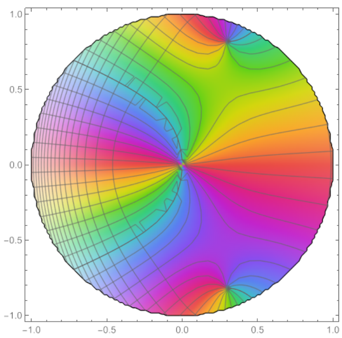
Both conditions are satisfied, hence ${}_{0.1}H_{2.2565,1.5453}$ is a starlike function of order 0.

Next, we assess the criteria for convexity, which is

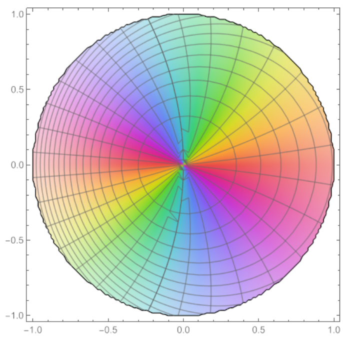
$$0 < |1.5453| \nless \frac{8 \times 0.1(2.2565)}{2 \times 2.2565 - 6}.$$

Hence, ${}_{0.1}H_{2.2565,1.5453}$ is not a convex function of order 0.

Figure 1 illustrates the contour plots of the Bessel k-function for different values of k. The symmetry about the real axis observed in Figure 1(a) supports the starlikeness property established in Theorem 3.1. As k increases in Figure 1(b), the contours gradually resemble the classical Bessel function, providing graphical evidence for the convexity behavior predicted in Theorem 3.2. This figure thus connect the numerical illustrations directly to the theoretical results.



(a) ${}_{0.1}H_{2.2565,1.5453}$



(b) ${}_{0.4}H_{2.2565,1.5453}$

Figure 1. Contours of ${}_{k}H_{2.2565,1.5453}$.

Example 4.2. Taking $\beta = 4.1258$, $g = 1.5453$, $\zeta = 0.32$, and $k = 0.2$, in order to observe if ${}_{0.2}H_{4.1258,1.5453}$ is starlike or convex, the sufficient conditions for starlikeness and convexity must be verified given in Theorems 3.1 and 3.2, which are

$$0 < |1.5453| < \frac{4 \times 0.2(4.1258)}{4.1258 - 2} = 1.55266,$$

and

$$0.32 \leq 1 - \frac{4.1258(4 \times 0.2 - |1.5453|) + 2|1.5453|}{4.1258(4 \times 0.2 - |1.5453|) - |1.5453|} = 1.01287.$$

Both conditions are satisfied, hence ${}_{0.2}H_{4.1258,1.5453}$ is starlike of order 0.32.

The above parameter values satisfy the following conditions for convexity:

$$0 < |1.5453| < \frac{8 \times 0.2(4.1258)}{2 \times 4.1258 - 6} = 2.93182,$$

$$0.32 \leq 1 - \frac{2 \times 4.1258(4 \times 0.2 - |1.5453|) + 6|1.5453|}{2 \times 4.1258(4 \times 0.2 - |1.5453|) - 3|1.5453|} = 3.24766.$$

So, ${}_{0.2}H_{4.1258,1.5453}$ is a convex function of order 0.32.

Figure 2 depicts the contour plots of the Bessel k-function for different values of k. Symmetry about the real axis in both cases supports the starlikeness property (Theorem 3.1). As k increases to 0.5, the contours resemble the classical Bessel function, providing evidence for the convexity behavior described in Theorem 3.2.

(a) ${}_{0.2}H_{4.1258,1.5453}$ (b) ${}_{0.5}H_{4.1258,1.5453}$

Figure 2. Contours of ${}_{k}H_{4.1258,1.5453}$.

5. Starlike and convex behavior of generalized (s, k)-Bessel function

In this section, we present theorems that provide refined results on the starlike and convex behavior of order ζ for the Bessel function ${}_{k}M_{\beta,g}(\wp; s)$ defined in Eq (2.4).

Theorem 5.1. *Consider $\beta > 2$, $g \in \mathbb{C}^*$ and $s, k \in \mathbb{R}^+$, where $0 < s < 1$ and $s > k$, with*

$$0 < |g| < \frac{\left(\frac{4s}{k}\right)\beta}{\beta - 2} =: g_*, \tag{5.1}$$

and

$$\zeta \leq 1 - \frac{2|g| + \beta\left(\left(\frac{4s}{k}\right) - |g|\right)}{\beta\left(\left(\frac{4s}{k}\right) - |g|\right) - |g|} =: \zeta_*, \tag{5.2}$$

then ${}_{k}M_{\beta,g}(\wp; s)$ is starlike of order ζ.

Proof. To prove the statement, we utilize the starlikeness condition from [27], suitably adapted for the (s, k)-function, which is

$$\left|\frac{\wp({}_kM_{\beta,g}(\wp; s))'}{{}_kM_{\beta,g}(\wp; s)} - 1\right| < 1 - \zeta, \tag{5.3}$$

where $\zeta \leq 1$.

Focusing on $\left|({}_kM_{\beta,g}(\wp; s))' - \frac{{}_kM_{\beta,g}(\wp;s)}{\wp}\right|$, we have

$$\begin{aligned}
\left|({}_kM_{\beta,g}(\wp; s))' - \frac{{}_kM_{\beta,g}(\wp; s)}{\wp}\right| &= \left|\left(\wp^2 + \sum_{p=1}^{\infty} \frac{(-g)^p \wp^{p+2}}{p!\,(\frac{4s}{k})^p \ {}^s(\beta)_{p,k}}\right)' - \frac{\wp^2 + \sum_{p=1}^{\infty} \frac{(-g)^p \ \wp^{p+2}}{p!\ (\frac{4s}{k})^p \ {}^s(\beta)_{p,k}}}{\wp}\right| \\
&= \left|2\wp + \sum_{p=1}^{\infty} \frac{(p+2)(-g)^p \wp^{p+1}}{p!\,(\frac{4s}{k})^p \ {}^s(\beta)_{p,k}} - \frac{\wp\left[\wp + \sum_{p=1}^{\infty} \frac{(-g)^p \wp^{p+1}}{p!\ (\frac{4s}{k})^p \ {}^s(\beta)_{p,k}}\right]}{\wp}\right| \\
&= \left|\wp + \sum_{p=1}^{\infty} \frac{(p+2)(-g)^p \wp^{p+1}}{p!\,(\frac{4s}{k})^p \ {}^s(\beta)_{p,k}} - \sum_{p=1}^{\infty} \frac{(-g)^p \wp^{p+1}}{p!\,(\frac{4s}{k})^p \ {}^s(\beta)_{p,k}}\right| \\
&= \left|\wp + \sum_{p=1}^{\infty} \frac{(p+1)(-g)^p \wp^{p+1}}{p!\,(\frac{4s}{k})^p \ {}^s(\beta)_{p,k}}\right| \\
&< \sup_{\theta \in 2\pi} \left|e^{i\theta p} + \sum_{p=1}^{\infty} \frac{(p+1)(-g)^p e^{i\theta(p+1)}}{p!\,(\frac{4s}{k})^p \ {}^s(\beta)_{p,k}}\right| \\
&\leq 1 + \sum_{p=1}^{\infty} \frac{(p+1)|g|^p}{p!\,(\frac{4s}{k})^p \ \frac{\Gamma_{s,k}\ (\beta + \frac{ps}{k})}{\Gamma_{s,k}\ (\beta)}}, \qquad |\wp| \leq 1 \\
&= 1 + \frac{\Gamma_{s,k}\ (\beta + \frac{s}{k})}{\beta} \sum_{p=1}^{\infty} \frac{|g|^p\ (p+1)}{(\frac{4s}{k})^p \Gamma_{s,k}\ (p + \frac{s}{k}) \Gamma_{s,k}\ (\beta + \frac{ps}{k})}.
\end{aligned} \tag{5.4}$$

Let us take a function ${}^s\phi_k(p)$ which is defined as

$${}^s\phi_k(p) = \frac{p+1}{\Gamma_k(p + \frac{s}{k})\Gamma_k(\beta + \frac{ps}{k})}, \quad p \in \mathbb{N}. \tag{5.5}$$

The difference between two consecutive terms of the function ${}^s\phi_k(p)$ is given as

$$\begin{aligned}
{}^s\phi_k(p+1) \ - \ {}^s\phi_k(p) &= \frac{p+2}{\Gamma_k(p + 1 + \frac{s}{k})\Gamma_k(\beta + (p+1)\frac{s}{k})} - \frac{p+1}{\Gamma_{s,k}(p + \frac{s}{k})\Gamma_{s,k}(\beta + \frac{ps}{k})} \\
&= \frac{p + 2 - (p+1)(p + \frac{s}{k})(\beta + \frac{ps}{k})}{(p + \frac{s}{k})(\beta + \frac{ps}{k})\Gamma_{s,k}(p + \frac{s}{k})\Gamma_{s,k}(\beta + \frac{ps}{k})} \\
&= -\frac{1}{\Gamma_{s,k}(p + \frac{s}{k} + 1)\Gamma_{s,k}(\beta + \frac{ps}{k} + \frac{s}{k})}\Bigg[\frac{ps^2 + p^2s^2}{k^2} \frac{p^3s + \beta ps + \beta s}{k} \\
&\quad + p\beta(p+1) - (p+2)\Bigg].
\end{aligned} \tag{5.6}$$

Examining the numerator, the leading term $\frac{p^3 s}{k}$ dominates for all $p \in \mathbb{N}$, while the other terms are either positive or of lower order. The denominator remains positive because of the properties of the (s,k)-Gamma function for positive arguments. Consequently, we have

$$^{s}\phi_k(p+1\ ) - {}^{s}\phi_k\ (p) < 0,\ p \in \mathbb{N}. \tag{5.7}$$

Inequality (5.7) confirms that consecutive differences are negative, showing that $^{s}\phi_k(p)$ is strictly decreasing. Therefore, the sequence satisfies

$$\frac{a+1}{\Gamma_{s,k}(a+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{as}{k})} \ \leq\ {}^{s}\phi_k(1) = \frac{2}{\Gamma_{s,k}(1+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{s}{k})}.$$

Applying inequality (5.4), we have

$$\begin{aligned}\left|({}_kM_{\beta,g}(\wp;s))' - \frac{{}_kM_{\beta,g}(\wp;s)}{\wp}\right| &< 1 + \frac{\Gamma_{s,k}(\beta+\frac{s}{k})}{\beta}\sum_{p=1}^{\infty}\frac{2|g|^p}{(\frac{4s}{k})^p\Gamma_{s,k}(1+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{s}{k})} \\ &= 1 + \frac{2}{\beta}\sum_{p=1}^{\infty}\left(\frac{|g|}{\frac{4s}{k}}\right)^p = 1 + \frac{2|g|}{\beta(\frac{4s}{k}-|g|)} \\ &= \frac{2|g| + \beta(\frac{4s}{k}-|g|)}{\beta(\frac{4s}{k}-|g|)}, \quad \wp \in \mathbb{U},\ k \in \mathbb{R}^+. \end{aligned} \tag{5.8}$$

Assuming $\frac{2|g|+\beta(\frac{4s}{k}-|g|)}{\beta(\frac{4s}{k}-|g|)} > 0$ implies

$$|g| < \frac{\left(\frac{4s}{k}\right)\beta}{\beta-2}, \tag{5.9}$$

which is equivalent to $0 < |g| < \frac{4s}{k}$. $|g| = 0$ corresponds to the trivial case, yielding the identity function.

Now consider $\left|\frac{{}_kM_{\beta,g}(\wp;s)}{\wp}\right|$, we have

$$\left|\frac{{}_kM_{\beta,g}(\wp;s)}{\wp}\right| = \left|\wp + \sum_{p=1}^{\infty}\frac{(-g)^p\ \wp^{p+1}}{p!\ (\frac{4s}{k})^p\ {}^{s}(\beta)_{p,k}}\right|.$$

Utilizing both the triangle inequality and the maximum modulus theorem, we arrive at

$$\begin{aligned}\left|\frac{{}_kM_{\beta,g}(\wp;s)}{\wp}\right| &> |\wp| - \sup\left|\sum_{p=1}^{\infty}\frac{(-g)^p\ e^{i\theta p}}{p!\ (\frac{4s}{k})^p\ {}^{s}(\beta)_{p,k}}\right| \\ &\geq 1 - \sum_{p=1}^{\infty}\frac{|g|^p}{(\frac{4s}{k})^p\ p!\ {}^{s}(\beta)_{p,k}}, \qquad \wp \in \mathbb{U} \\ &= 1 - \sum_{p=1}^{\infty}\frac{|g|^p\ \Gamma_{s,k}(\beta)}{(\frac{4s}{k})^p\Gamma_{s,k}(\beta+\frac{ps}{k})\ \Gamma_{s,k}(p+\frac{s}{k})} \\ &= 1 - \frac{\Gamma_{s,k}(\beta+\frac{s}{k})}{\beta}\sum_{p=1}^{\infty}\frac{|g|^p}{(\frac{4s}{k})^p\ \Gamma_{s,k}(\beta+\frac{ps}{k})\ \Gamma_{s,k}(p+\frac{s}{k})}. \end{aligned}$$

As $\frac{1}{\Gamma_{s,k}(\beta+\frac{ps}{k})\,\Gamma_{s,k}(p+\frac{s}{k})}$ decreases strictly, thus

$$\begin{aligned}\left|\frac{{}_kM_{\beta,g}(\wp;s)}{\wp}\right| &> 1-\frac{\Gamma_k(\beta+\frac{s}{k})}{\beta}\sum_{p=1}^{\infty}\frac{|g|^p}{(\frac{4s}{k})^p\,\Gamma_{s,k}(\beta+\frac{s}{k})\,\Gamma_{s,k}(1+\frac{s}{k})}\\ &= 1-\frac{1}{\beta}\sum_{p=1}^{\infty}\left(\frac{|g|}{\frac{4s}{k}}\right)^p = 1-\frac{|g|}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}\\ &= \frac{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-|g|}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)},\end{aligned} \tag{5.10}$$

where

$$\frac{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-|g|}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)} > 0. \tag{5.11}$$

The preceding equation holds true since $\beta > 2$ and

$$|g| < \min\left\{4;\frac{\left(\frac{4s}{k}\right)\beta}{\beta-2}\right\} = \frac{\left(\frac{4s}{k}\right)\beta}{\beta-2}. \tag{5.12}$$

As we know,

$$\begin{aligned}\left|\frac{\wp({}_kM_{\beta,g}(\wp;s))'}{{}_kM_{\beta,g}(\wp;s)}-1\right| &= \left|({}_kM_{\beta,g}(\wp;s))'-\frac{{}_kM_{\beta,g}(\wp;s)}{\wp}\right|\left|\frac{\wp}{{}_kM_{\beta,g}(\wp;s)}\right|\\ &< \frac{2|g|+\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)} \times \frac{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-|g|}\\ &< \frac{2|g|+\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-|g|} \le 1-\zeta\end{aligned}$$

leads to

$$\zeta \le 1-\frac{2|g|+\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-|g|}. \tag{5.13}$$

In conclusion, it follows from inequality (5.2) that ${}_kM_{\beta,g}(\wp;s)$ is a starlike function of order ζ.

Theorem 5.2. *Suppose $\beta > 3$, $g \in \mathbb{C}^*$, $s,k \in \mathbb{R}^+$, $0 < s < 1$, $s > k$ with*

$$0 < |g| < \frac{\frac{8s}{k}\beta}{2\beta-6} =: g_c \tag{5.14}$$

and

$$\zeta \le 1-\frac{6|g|+2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-3|g|} =: \zeta_c, \tag{5.15}$$

then ${}_kM_{\beta,g}(\wp;s)$ is a convex function of order ζ.

Proof. For the proof, we apply the convexity criterion given in [27], adjusted for the (s,k)-function, which is

$$\left|\frac{\wp({}_kM_{\beta,g}(\wp;s))''}{({}_kM_{\beta,g}(\wp;s))'}\right| < 1-\zeta, \quad \zeta \leq 1. \tag{5.16}$$

First, we consider $\left|\wp({}_kM_{\beta,g}(\wp;s))''\right|$, which gives

$$\begin{aligned}\left|\wp({}_kM_{\beta,g}(\wp;s))''\right| &= \left|\wp\left(\wp^2 + \sum_{p=1}^{\infty}\frac{(-g)^p \;\; \wp^{p+2}}{p\,!\left(\frac{4s}{k}\right)^p \;\; {}^s(\beta)_{p,k}}\right)''\right| \\ &= \left|\wp\left(2\wp + \sum_{p=1}^{\infty}\frac{(p+2\,)(-g)^p \;\; \wp^{p+1}}{p!\left(\frac{4s}{k}\right)^p \;\; {}^s(\beta)_{p,k}}\right)'\right| \\ &= \left|\wp\left(2 + \sum_{p=1}^{\infty}\frac{(p+2\,)\,(p+1\,)\,(-g)^p \;\; \wp^p}{\left(\frac{4s}{k}\right)^p \;\; p!\;{}^s(\beta)_{p,k}}\right)\right| \\ &= \left|2\wp + \sum_{p=1}^{\infty}\frac{(p+2)\,(p+1\,)\,(-g)^p \;\; \wp^{p+1}}{\left(\frac{4s}{k}\right)^p \;\; p!\;{}^s(\beta)_{p,k}}\right|.\end{aligned}$$

Using the triangle inequality, we have

$$\begin{aligned}\left|\wp({}_kM_{\beta,g}(\wp;s))''\right| &\leq |2\wp\,| + \left|\sum_{p=1}^{\infty}\frac{(\,p+1)\,(p+2\,)\,(-g)^p \;\; \wp^{p+1}}{\left(\frac{4s}{k}\right)^p \;\; p!\;{}^s(\beta)_{p,k}}\right| \\ &< |2e^{i\theta p}| + \sup_{\theta\in 2\pi}\left|\sum_{p=1}^{\infty}\frac{(p+1)\,(p+2)\,(-g)^p\; e^{i\theta p}}{\left(\frac{4s}{k}\right)^p \;\; p!\;{}^s(\beta)_{p,k}}\right| \\ &\leq 2 + \sum_{p=1}^{\infty}\frac{(p+1)\,(p+2)\,|g|^p}{\left(\frac{4s}{k}\right)^p \;\; p!\;\frac{\Gamma_{s,k}(\beta+\frac{ps}{k})}{\Gamma_{s,k}(\beta)}}\,, \qquad |\wp| \leq 1 \\ &= 2 + \frac{\Gamma_{s,k}(\beta+\frac{s}{k})}{\beta}\sum_{p=1}^{\infty}\frac{|g|^p\;(p+1)\,(p+2)}{\left(\frac{4s}{k}\right)^p \;\Gamma_{s,k}(p+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{ps}{k})}.\end{aligned} \tag{5.17}$$

Assume a function ${}^s\omega_k(p)$ which is defined as

$${}^s\omega_k(p) = \frac{(p+1)(p+2)}{\Gamma_{s,k}(p+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{ps}{k})}, \quad p \in \mathbb{N}. \tag{5.18}$$

To check the behavior of the function ${}^s\omega_k(p)$, we have

$$\begin{aligned}{}^s\omega_k(p+1) \;-\; {}^s\omega_k(p) &= \frac{(p+2)(p+3)}{\Gamma_{s,k}(p+1+\frac{s}{k})\Gamma_{s,k}(\beta+(p+1)\frac{s}{k})} - \frac{(p+1)(p+2)}{\Gamma_{s,k}(p+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{ps}{k})} \\ &= \frac{p+2}{\Gamma_{s,k}(p+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{ps}{k})}\left[\frac{p+3}{(p+\frac{s}{k})(\beta+\frac{ps}{k})} - (p+1)\right] \\ &= \frac{p+2}{\Gamma_{s,k}(p+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{ps}{k})}\left[\frac{p+3-(p+1)(p+\frac{s}{k})(\beta+\frac{ps}{k})}{(p+\frac{s}{k})(\beta+\frac{ps}{k})}\right]\end{aligned}$$

$$= -\frac{1}{\Gamma_{s,k}(p+1+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{ps}{k}+\frac{s}{k})}\Big[\frac{p^2s^2+ps^2}{k^2}+\frac{p^3s+p^2s+\beta ps+\beta s}{k}$$
$$+(p+1)\beta p-(p+3)\Big]. \tag{5.19}$$

Observing that numerator and denominator both are positive, thus

$$^{s}\omega_k(p+1) - {}^{s}\omega_k(p) < 0,\ p \in \mathbb{N}. \tag{5.20}$$

It follows from inequality (5.20) that consecutive differences are negative which shows that the function $^{s}\omega_k(p)$ is decreasing strictly, so

$$\begin{aligned}\frac{(a+1)(a+2)}{\Gamma_{s,k}(a+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{as}{k})} &\leq {}^{s}\omega_{s,k}(1)\\ &= \frac{2(3)}{\Gamma_{s,k}(1+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{s}{k})}\\ &= \frac{6}{\Gamma_{s,k}(1+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{s}{k})}.\end{aligned}$$

By the inequality (5.17), we get

$$\begin{aligned}\left|\wp({}_kM_{\beta,g}(\wp;s))''\right| &< 2+\frac{\Gamma_{s,k}(\beta+\frac{s}{k})}{\beta}\sum_{p=1}^{\infty}\frac{6\,|g|^p}{\left(\frac{4s}{k}\right)^p\ \Gamma_{s,k}(1+\frac{s}{k})\Gamma_{s,k}(\beta+\frac{s}{k})}\\ &= 2+\frac{6}{\beta}\sum_{p=1}^{\infty}\left(\frac{|g|}{\frac{4s}{k}}\right)^p = 2+\frac{6|g|}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}\\ &= \frac{6|g|+2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}, \qquad \wp\in\mathbb{U},\ k\in\mathbb{R}^+.\end{aligned} \tag{5.21}$$

Consider $\frac{6|g|+2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)} > 0$,

$$|g| < \frac{\left(\frac{8s}{k}\right)\beta}{2\beta-6}, \tag{5.22}$$

which is equivalent to $0 < |g| < \frac{4s}{k}$, which holds because of our assumptions.

Now consider $\left|({}_kM_{\beta,g}(\wp;s))'\right|$, we have

$$\begin{aligned}\left|({}_kM_{\beta,g}(\wp;s))'\right| &= \left|\left(\wp^2+\sum_{p=1}^{\infty}\frac{(-g)^p\ \wp^{p+2}\Gamma_{s,k}(\beta)}{\left(\frac{4s}{k}\right)^p\ \Gamma_{s,k}(p+\frac{s}{k})\ \Gamma_{s,k}(\beta+\frac{ps}{k})}\right)'\right|\\ &= \left|2\wp+\sum_{p=1}^{\infty}\frac{(-g)^p\ \wp^{p+1}\ (p+2)\Gamma_{s,k}(\beta+\frac{s}{k})}{\beta\ \left(\frac{4s}{k}\right)^p\ \Gamma_{s,k}(p+\frac{s}{k})\ \Gamma_{s,k}(\beta+\frac{ps}{k})}\right|.\end{aligned}$$

Using the maximum modulus theorem together with the triangular inequality, we find

$$\begin{aligned}\left|({}_kM_{\beta,g}(\wp;s))'\right| &> |2e^{i\theta p}| - \sup\left|\sum_{p=1}^{\infty}\frac{(p+2)\,(-g)^p\, e^{i\theta p}\,\Gamma_{s,k}(\beta+\frac{s}{k})}{\left(\frac{4s}{k}\right)^p\,\Gamma_{s,k}(p+\frac{s}{k})\,\Gamma_{s,k}(\beta+\frac{ps}{k})}\right| \\ &\geq 2 - \frac{\Gamma_{s,k}(\beta+\frac{s}{k})}{\beta}\sum_{p=1}^{\infty}\frac{(p+2)\,|g|^p}{\left(\frac{4s}{k}\right)^p\,\Gamma_{s,k}(\beta+\frac{ps}{k})\,\Gamma_{s,k}(p+\frac{s}{k})}.\end{aligned}$$

As $\frac{1}{\Gamma_{s,k}(\beta+\frac{ps}{k})\,\Gamma_{s,k}(p+\frac{s}{k})}$ decreases strictly, we have

$$\begin{aligned}\left|({}_kM_{\beta,g}(\wp;s))'\right| &> 2 - \frac{\Gamma_{s,k}(\beta+\frac{s}{k})}{\beta}\sum_{p=1}^{\infty}\frac{3\,|g|^p}{\left(\frac{4s}{k}\right)^p\,\Gamma_{s,k}(\beta+\frac{s}{k})\,\Gamma_{s,k}(1+\frac{s}{k})} \\ &= 2 - \frac{3}{\beta}\sum_{p=1}^{\infty}\left(\frac{|g|}{\frac{4s}{k}}\right)^p = 2 - \frac{3|g|}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)} \\ &= \frac{2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-3|g|}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)},\end{aligned} \tag{5.23}$$

where

$$\frac{2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-3|g|}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)} > 0. \tag{5.24}$$

The above equation holds because $\beta > 3$ and

$$|g| < \min\left\{4;\frac{\left(\frac{8s}{k}\right)\beta}{2\beta-6}\right\} = \frac{\left(\frac{8s}{k}\right)\beta}{2\beta-6}. \tag{5.25}$$

It is factual that

$$\begin{aligned}\left|\frac{\wp({}_kM_{\beta,g}(\wp;s))''}{({}_kM_{\beta,g}(\wp;s))'}\right| &= \left|\wp({}_kM_{\beta,g}(\wp;s))''\right|\,\left|\frac{1}{({}_kM_{\beta,g}(\wp;s))'}\right| \\ &< \frac{6|g|+2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)} \times \frac{\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-3|g|} \\ &< \frac{6|g|+2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-3|g|} \leq 1-\zeta,\end{aligned}$$

which implies

$$\zeta \leq 1 - \frac{6|g|+2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)}{2\beta\left(\left(\frac{4s}{k}\right)-|g|\right)-3|g|}. \tag{5.26}$$

Therefore, inequality (5.15) shows that ${}_kM_{\beta,g}(\wp;s)$ is a convex function of order ζ.

6. Application of starlike and convex behavior of generalized Bessel (s, k)-function

In this area, we provide applications of the above theorems that demonstrate the convex and starlike behaviour of order 0.32. The graphical representations are generated by Mathematica 13.3.

Example 6.1. Choose $\beta = 9.5$, $g = 3.7523$, $\zeta = 0.32$, $s = 0.76$, and $k = 0.7$ to check whether ${}_{0.7}H_{9.5,3.7523}(\wp; 0.76)$ is starlike or convex. For this, sufficient conditions for starlikeness and convexity given in Theorems 5.1 and 5.2 needs to be checked:

$$0 < |3.7523| < \frac{\left(\frac{4(0.76)}{0.7}\right) 9.5}{9.5 - 2} = 5.50095,$$

and

$$0.32 \nleq 1 - \frac{2|3.7523| + 9.5\left(\left(\frac{4(0.76)}{0.7}\right) - |3.7523|\right)}{9.5\left(\left(\frac{4(0.76)}{0.7}\right) - |3.7523|\right) - |3.7523|} = -6.05863.$$

The second condition is not satisfied, hence ${}_{0.7}H_{9.5,3.7523}(\wp; 0.76)$ is not a starlike function of order 0.32.

Subsequently, we analyze the conditions that ensure convexity:

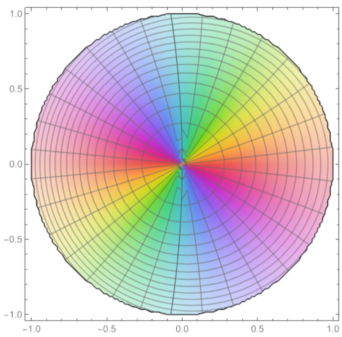
$$0 < |3.7523| < \frac{\left(\frac{8(0.76)}{0.7}\right) 9.5}{2(9.5) - 6} = 6.34725,$$

and

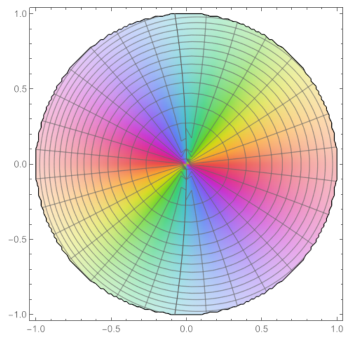
$$0.32 \leq 1 - \frac{6|3.7523| + 2(9.5)\left(\left(\frac{4(0.76)}{0.7}\right) - |3.7523|\right)}{2(9.5)\left(\left(\frac{4(0.76)}{0.7}\right) - |3.7523|\right) - 3|3.7523|} = 929.956.$$

Since the above parameter values fulfill the conditions, it follows that ${}_{0.7}H_{9.5,3.7523}(\wp; 0.76)$ is a convex function of order 0.32.

Figure 3 illustrates the contour plots of the Bessel (s, k)-function for different parameter values. Figure 3(a), with $s = 0.76$ and $k = 0.7$, shows contours symmetric about the real axis, consistent with the starlikeness property from Theorem 5.1. In Figure 3(b), for $s = 0.96$ and $k = 0.9$, the contours approach the classical Bessel function, providing graphical evidence for the convexity behavior described in Theorem 5.2. These figures explicitly connect the graphical and numerical evidence to the theoretical results.



(a) ${}_{0.7}M_{9.5,3.7523}(\wp; 0.76)$



(b) ${}_{0.9}M_{9.5,3.7523}(\wp; 0.96)$

Figure 3. Contours of ${}_{k}M_{9.5,3.7523}(\wp; s)$.

7. Conclusions

We have examined the geometric perspective for different values of k and s in our recent findings.

(1) We introduced two forms of generalized Bessel functions, ${}_kH_{\beta,g}(\wp)$ and ${}_kM_{\beta,g}(\wp;s)$, extending the classical Bessel function framework with additional flexibility in parameters.
(2) We derived the sufficient conditions under which these functions are starlike or convex of order ζ within the open unit disk $\mathbb{U}$.
(3) We proved several theorems to validate the starlike and convex nature of the proposed functions, offering theoretical support for their geometric behavior.
(4) The introduction of the additional parameter s in ${}_kM_{\beta,g}(\wp;s)$ enhanced the analytical scope, leading to a more generalized starlikeness condition involving both s and k.
(5) Specific examples with computed bounds verify the theoretical criteria, confirming starlikeness and convexity for selected values of β, g, k and s.
(6) We provided concrete numerical applications to verify the applicability of the derived findings, showing that the generalized functions meet the established criteria for various configurations of β, g, k and s.
(7) The developed criteria and function forms lay groundwork for future studies in analytic function theory and solutions to differential equations involving generalized Bessel functions.

Author contributions

Syed Ali Haider Shah and Hafsa: Conceptualization, Formal analysis, Visualization, Writing original draft, Writing–review & editing; Hajer Zaway and Salma Trabelsi: Funding acquisition, Financial support. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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