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Research article

Regular sets in Cayley sum graphs on generalized dicyclic groups

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Abstract: For a graph $\Gamma = (V(\Gamma), E(\Gamma))$, a subset C of $V(\Gamma)$ is called an (α,β)-regular set in Γ, if every vertex of C is adjacent to exactly α vertices of C and every vertex of $V(\Gamma) \setminus C$ is adjacent to exactly β vertices of C. In particular, if C is an (α,β)-regular set in some Cayley sum graph of a finite group G with connection set S, then C is called an (α,β)-regular set of G. In this paper, we considered a generalized dicyclic group G and for each subgroup H of G, by giving an appropriate connection set S, we determined each possibility for (α,β) such that H is an (α,β)-regular set of G.

Keywords: regular set; Cayley graph; Cayley sum graph; generalized dicyclic group; dicyclic group
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1. Introduction

Let $\mathbb{F}_q$ be the finite field of order q. Then, $C \subseteq \mathbb{F}_q^n$ is called a code of length n over $\mathbb{F}_q$, and its elements are referred to as codewords. For any two vectors $x, y \in \mathbb{F}_q^n$, their Hamming distance $d(x,y)$ is the number of coordinates in which they differ. Given any vector $x \in \mathbb{F}_q^n$, its distance to the code C is $d(x,C) = \min\{d(x,y) \mid y \in C\}$. The covering radius of a code C is the smallest integer ρ such that for every vector y, there is a codeword $x \in C$ satisfying $d(x,y) \le \rho$. For a vector $x \in \mathbb{F}_q^n$, let $B_{x,i}$ be the number of codewords in C having distance i from x. A code C with covering radius ρ is called *t-regular* with $0 \le t \le \rho$ if $B_{x,i}$ depends only on i and $d(x,C)$ for $0 \le i \le \rho$ whenever $d(x,C) \le t$, see [9, 13] as examples. A code C is called completely regular if it is ρ-regular [9]. In the context of 1-regular code, two key parameters arise: $\alpha = B_{x,1}$ for all $x \in C$ and $\beta = B_{x,1}$ for all $x \in \mathbb{F}_q^n \setminus C$. For any $x \in C$, there are precisely α codewords in C having distance 1 from x; for any $x \notin C$, there are precisely β codewords in C having distance 1 from x. In particular, a 1-regular code with $B_{x,1} = 0$ for $x \in C$ and $B_{x,1} = 1$ for $x \in \mathbb{F}_q^n \setminus C$ is known as a perfect code. The combinatorial properties of completely regular codes enable their connection to various other combinatorial structures, such as association schemes and combinatorial designs. To generalize the idea of regular codes to other structures, many authors have extended the idea to perfect codes and regular sets in graphs.

A graph Γ is a pair $(V(\Gamma), E(\Gamma))$ of vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ is a subset of the set of 2-element subsets of $V(\Gamma)$. Throughout this paper, all groups are assumed to be finite, and all graphs are finite, simple, and undirected.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. For two different vertices $x, y \in V(\Gamma)$, they are said to be adjacent if the set $\{x, y\} \in E(\Gamma)$. A subset C of $V(\Gamma)$ is a perfect code in Γ if C is an independent set of Γ and every vertex of $V(\Gamma) \setminus C$ is adjacent to exactly one vertex of C. A subset C of $V(\Gamma)$ is said to be a total perfect code in Γ if every vertex of Γ has exactly one neighbor in C. In the literature, a perfect code is also called an efficient dominating set [8, 10, 11, 14, 18, 19] or an independent perfect dominating set [14, 20, 28], and a total perfect code is also called an efficient open dominating set [6, 12, 14].

As a generalization of perfect codes and total perfect codes in a graph, regular sets are defined as follows. For non-negative integers α and β, a subset C of $V(\Gamma)$ is called an (α,β)-regular set [4] in Γ, if every vertex of C is adjacent to exactly α vertices of C and every vertex of $V(\Gamma) \setminus C$ is adjacent to exactly β vertices of C. In particular, a $(0, 1)$-regular set in Γ is a perfect code; a $(1, 1)$-regular set in Γ is a total perfect code. Clearly, the definition of a regular set in a graph arises from the definition of the 1-regular code, by replacing the Hamming distance of two vectors by the distance of the vertices in the graph, as the metric. Moreover, an (α,β)-regular set in an r-regular graph Γ coincides precisely with a completely regular code C in Γ, see, for example, [24], where the corresponding distance partition consists of exactly two parts: $\{C, V(\Gamma) \setminus C\}$.

Let G be a group with the identity element e and S an inverse-closed subset of $G \setminus \{e\}$ (that is, $S^{-1} := \{s^{-1} : s \in S\} = S$). The Cayley graph $\mathrm{Cay}(G, S)$ on G with connection set S is the graph with vertex set G and edge set $\{\{g, gs\} : g \in G, s \in S\}$.

Huang, Xia, and Zhou [15] first introduced the definition of a subgroup (total) perfect code of a group G by using Cayley graphs on G. A subset of a group G is called a (total) perfect code of G if it is a (total) perfect code in some Cayley graph of G. A (total) perfect code of G is called a subgroup (total) perfect code of G if it is also a subgroup of G. Also in [15], the authors gave a necessary and sufficient condition for a normal subgroup of a group G to be a subgroup (total) perfect code of G. Chen, Wang, and Xia generalized this result on perfect codes to arbitrary subgroups [5]. Ma, Walls, Wang, and Zhou [22] proved that a group G admits every subgroup as a perfect code if and only if G has no elements of order 4. In [36, 37], Zhang and Zhou gave a few necessary and sufficient conditions for a subgroup of a group to be a subgroup perfect code, and several results on subgroup perfect codes of metabelian groups, generalized dihedral groups, nilpotent groups, and 2-groups. For further results on subgroup perfect codes in Cayley graphs, see [2, 3, 16, 31, 35].

In [29, 30], Wang, Xia, and Zhou first introduced the concept of an (α,β)-regular set of a group G by using Cayley graphs on G. A subset (resp. subgroup) of a group G is called an (α,β)-regular set (resp. a subgroup (α,β)-regular set) of G if it is an (α,β)-regular set in some Cayley graph of G. In particular, a subgroup $(0, 1)$-regular set (resp. $(1, 1)$-regular set) of G is a subgroup perfect code (resp. subgroup total perfect code) of G. In [32], Wang, Xu, and Zhou determined when a non-trivial proper normal subgroup of a group is a $(0,\beta)$-regular set of the group and determined all subgroup $(0,\beta)$-regular sets of dihedral groups and dicyclic groups. In [17], Khaefi, Akhlaghi, and Khosravi proved that if H is a subgroup of G, then H is an (α,β)-regular set of G, for each $0 \le \alpha \le |H| - 1$ such that $\gcd(2, |H| - 1)$ divides α, and for each $0 \le \beta \le |H|$ such that β is even. Also, they proved that a subgroup H of a group G is a perfect code of G if and only if it is an (α,β)-regular set of G, for each $0 \le \alpha \le |H| - 1$ such that $\gcd(2, |H| - 1)$ divides α, and for each $0 \le \beta \le |H|$. In addition, they showed that if H is a subgroup of

G, then H is a perfect code of G if and only if it is an (α,β)-regular set of G for each $0 \leq \alpha \leq |H| - 1$ such that $\gcd(2, |H| - 1)$ divides α, and for each $0 \leq \beta \leq |H|$ such that β is odd.

The Cayley sum graph is first defined for abelian groups [7] and then it is generalized to any arbitrary group in [1]. Let G be a group. An element $x \in G$ is called square if $x = y^2$ for some element $y \in G$. A subset S of G is called square-free if every element of S is not square. We say that a subset S of G is normal if $g^{-1}Sg = \{g^{-1}sg : s \in S\} = S$ for every element $g \in G$. Let S be a normal square-free subset of G. The Cayley sum graph of G with the connection set S, denoted by $\mathrm{CayS}(G,S)$, is the graph with vertex set G and two vertices x and y are adjacent whenever $xy \in S$. Note that the normality of S implies that $xy \in S$ if and only if $yx \in S$. The square-free condition of S ensures that $\mathrm{CayS}(G,S)$ has no loops, so that $\mathrm{CayS}(G,S)$ is a simple graph. Clearly, $\mathrm{CayS}(G,S)$ is an $|S|$-regular graph.

In [21], Ma, Feng, and Wang studied the perfect codes in Cayley sum graphs, and defined a subgroup perfect code of a group by using Cayley sum graphs instead of Cayley graphs. More precisely, a subgroup of a group G is said to be a subgroup perfect code of G if the subgroup is a perfect code in some Cayley sum graph of G. Also in [21], the authors reduced the problem of determining when a given subgroup of an abelian group is a perfect code to the case of abelian 2-groups, and classified the abelian groups whose all non-trivial subgroups are perfect codes. Ma, Wang, and the second author [23] characterized all subgroup perfect codes of abelian groups.

The total perfect codes in Cayley sum graphs have been studied, and a total perfect code of a group is also defined by using Cayley sum graphs instead of Cayley graphs. A subset (resp. subgroup) C of a group G is called a total perfect code (resp. subgroup total perfect code) of G if it is a total perfect code in some Cayley sum graph of G. Zhang [34] gave some necessary conditions of a subgroup of a given group being a (total) perfect code in a Cayley sum graph of the group, and classified the Cayley sum graphs of some families of groups which admit a subgroup as a (total) perfect code. Wang, Wei, Xu, and Zhou [26] gave two necessary and sufficient conditions for a subgroup of a group G to be a total perfect code of G, and obtained two necessary and sufficient conditions for a subgroup of an abelian group G to be a total perfect code of G. They also gave a classification of subgroup total perfect codes of a cyclic group, a dihedral group, and a dicyclic group.

Replacing Cayley graphs with Cayley sum graphs in the concept of a subgroup (α,β)-regular sets of a group, the authors in [27] obtained the concept of regular sets in a group. More precisely, a subset (resp. subgroup) C of a group G is called an (α,β)-regular set (resp. a subgroup (α,β)-regular set) of G if it is an (α,β)-regular set in some Cayley sum graph of G. In particular, a subgroup $(0,1)$-regular set (resp. $(1,1)$-regular set) of G is a subgroup perfect code (resp. subgroup total perfect code) of G. Also in [27], the authors obtained some necessary and sufficient conditions for a subgroup of a group G to be a $(0,\beta)$-regular set of G, and characterized all possible subgroup $(0,\beta)$-regular sets of a cyclic group, a dihedral group, and a dicyclic group. Seiedali, Khosravi, and Akhlaghi [25] gave a necessary and sufficient condition for a subgroup of an abelian group to be a subgroup (α,β)-regular set. For each subgroup H of a dihedral group G, by giving an appropriate connection set S, they determined each possibility for (α,β) such that H is an (α,β)-regular set of G.

In this paper, we study regular sets in Cayley sum graphs on a generalized dicyclic group. For each subgroup H of a generalized dicyclic group G, by giving an appropriate connection set S, we determine each possibility for (α,β) such that H is an (α,β)-regular set of G. In order to state our main results, we need additional notations and terminologies.

Let G be a group with the identity element e. For $g \in G$, let $o(g)$ denote the order of g, that is, the smallest positive integer m such that $g^m = e$. An element a is called an involution if $o(a) = 2$. For a subgroup H of G and an element $a \in G$, $Ha = \{ha : h \in H\}$ (resp. $aH = \{ah : h \in H\}$) is called a right coset (resp. left coset) of H in G. The index of a subgroup H in G, denoted by $|G : H|$, is defined as the number of distinct right (or left) cosets of H in G.

In the remainder of this paper, A always denotes an abelian group of even order with an involution b^2, and G denotes the generalized dicyclic group generated by A and b, where $bab^{-1} = a^{-1}$ for all $a \in A$, that is, $G = \langle A, b : b^2 \in A, b^4 = e, bab^{-1} = a^{-1}, a \in A\rangle$. Since A is an abelian group, from the fundamental theorem of finitely generated abelian groups, we may assume

$$A = \langle a_1\rangle \times \cdots \times \langle a_\lambda\rangle \times \langle a_{\lambda+1}\rangle \times \cdots \times \langle a_{\lambda+\mu}\rangle,$$

with $o(a_i) = p_i^{e_i}$ for all $1 \leq i \leq \lambda+\mu$, where p_i is a prime, $e_1 \leq \cdots \leq e_\lambda$, and $p_j = 2$ if and only if $1 \leq j \leq \lambda$. Let $a_0 = b^2$ and k be the maximum nonnegative integer such that a_k is an involution. Let φ_i be the projection from A to $\langle a_i\rangle$ for $1 \leq i \leq \lambda+\mu$. Denote

$$B = \langle a_1^2\rangle \times \cdots \times \langle a_\lambda^2\rangle \times \langle a_{\lambda+1}\rangle \times \cdots \times \langle a_{\lambda+\mu}\rangle. \tag{1.1}$$

Throughout this paper, we always assume that H is a subgroup of A and $z \in A$. Denote

$$L_H = \{a_i : \varphi_i(H) = \langle a_i\rangle,\ 1 \leq i \leq \lambda\},\ r_H = |\{i : \varphi_i(H) = \langle a_i\rangle,\ 1 \leq i \leq k\}|,$$
$$m_H = |\{i : \varphi_i(H) \neq \{e\},\ 1 \leq i \leq \lambda\}|.$$

If no confusion occurs, we write t instead of t_H for all symbols $t \in \{L, r, m\}$.

Recall that A is an abelian group, $G = \langle A, b : b^2 \in A, b^4 = e, bab^{-1} = a^{-1}, a \in A\rangle$, and $H \leqslant A$. All subgroups of G are of the form H or $\langle H, zb\rangle$ with $|\langle H, zb\rangle : H| = 2$, see Lemma 2.2. In the following theorem, by giving an appropriate connection set S, we determine each possibility for (α,β) such that H is an (α,β)-regular set of G.

Theorem 1.1. *The subgroup H is an (α,β)-regular set of G for $(\alpha,\beta) \neq (0,0)$, if and only if $0 \leq \alpha \leq (2^{|L|}-1)|H|/2^{|L|} - \epsilon$, $\beta = t|H|/2^{|L|}$ with $0 \leq t \leq 2^{|L|} - \varepsilon$, and one of the following occurs:*

(1) $\epsilon = 1$ and α is even, when $m = |L| = 1$ and $b^2 \in H \setminus B$;

(2) $\epsilon = 1$, when $m \geq |L| \geq 1$, $m \neq 1$ and $b^2 \in H \setminus B$;

(3) $\epsilon = 0$ and α is even, when $r = 0$ and $b^2 \notin H \setminus B$;

(4) $\epsilon = 0$, when $r > 0$ and $b^2 \notin H \setminus B$.

Here, $\varepsilon = 0$ if $B \cup \{b^2\} \subseteq H$, $\varepsilon = 2$ if $B \nleq H$, $b^2 \notin H \cup B$, and $Hb^2 \cap B \neq \emptyset$, and $\varepsilon = 1$, otherwise.

In the following theorem, by giving an appropriate connection set S, we determine each possibility for (α,β) such that $\langle H, zb\rangle$ is an (α,β)-regular set of G.

Theorem 1.2. *Without loss of generality we assume $|\langle H, zb\rangle : H| = 2$. Then $\langle H, zb\rangle$ is an (α,β)-regular set of G for $(\alpha,\beta) \neq (0,0)$, if and only if $(\alpha,\beta) = (\eta + t'|H|/2^{|L|},\ \zeta + t|H|/2^{|L|})$ with $0 \leq \eta \leq (2^{|L|}-1)|H|/2^{|L|} - \epsilon$, $0 \leq \zeta \leq (2^{|L|} - \varepsilon)|H|/2^{|L|}$, and $0 \leq t', t \leq 2^{|L|}$, and one of the following occurs:*

(1) $\epsilon = 1$ and β is even, when $m > |L|$ and $b^2 \in H \setminus B$;

(2) $\epsilon = 1$, when $m = |L|$ and $b^2 \in H \setminus B$;

(3) $\epsilon = 0$, α and β are even, when $m > |L|$, $r = 0$, and $b^2 \notin H \setminus B$;

(4) $\epsilon = 0$ and β is even, when $m > |L|$, $r > 0$, and $b^2 \notin H \setminus B$;

(5) $\epsilon = 0$, when $m = |L|$ and $b^2 \notin H \setminus B$.

Here, $\varepsilon = 0$ if $B \leqslant H$, and $(\varepsilon, t') = (1, t)$ if $B \nleqslant H$.

The paper is organized as follows. In Section 2, we give some preliminary results which will be used in subsequent sections. In Section 3, by giving an appropriate connection set S, we determine each possibility for β such that H or $\langle H, zb\rangle$ is a $(0,\beta)$-regular set of G. In Section 4, we prove Theorems 1.1 and 1.2.

2. Preliminary

In this section, we give some basic results which will be used frequently in this paper.

Let $\mathrm{Sq}(G)$ and $\mathrm{Nsq}(G)$ be the sets of all square and non-square elements of G, respectively. For a subgroup K of G, denote $\mathcal{L}(K) = \min\{|\mathrm{Nsq}(G) \cap Kx| : x \in G \setminus K\}$.

Lemma 2.1. *Let K be a subgroup of G and S be a normal and square-free subset of G. The following hold:*

(1) The subgroup K is an (α,β)-regular set in $\mathrm{CayS}(G, S)$ if and only if $\alpha = |S \cap K|$ and $\beta = |S \cap Kx|$ for each $x \in G \setminus K$;

(2) If K is an (α,β)-regular set of $\mathrm{CayS}(G, S)$, then $\alpha \leq |\mathrm{Nsq}(G) \cap K|$ and $\beta \leq \mathcal{L}(K)$.

Proof. (1) The necessity is immediate from [25, Lemma 2.4 and Corollary 2.5]. Next, we prove the sufficiency. Since $\alpha = |S \cap K|$ and $\beta = |S \cap Kx|$ for each $x \in G \setminus K$, S contains exactly α elements of K and β elements of Kx for each $x \in G \setminus K$. Let $a \in G \setminus K$. Without loss of generality, we may assume $S \cap K = \{k_1, k_2, \dots, k_\alpha\}$ and $S \cap Ka = \{a_1a, a_2a, \dots, a_\beta a\}$ with $a_j \in K$ for $1 \leq j \leq \beta$. For each $k \in K$, there are exactly α elements $k^{-1}k_1, k^{-1}k_2, \dots, k^{-1}k_\alpha \in K$ such that $kk^{-1}k_i = k_i \in S$ for $1 \leq i \leq \alpha$. For $a \in G \setminus K$, there are exactly β elements $a_1, a_2, \dots, a_\beta \in K$ such that $a_ja \in S$ for $1 \leq j \leq \beta$. Since $k \in K$ and $a \in G \setminus K$ are arbitrary, every element of K is adjacent to exactly α elements of K in $\mathrm{CayS}(G, S)$, and every element of $G \setminus K$ is adjacent to exactly β elements of K in $\mathrm{CayS}(G, S)$. Since S is a normal and square-free subset of G, K is an (α,β)-regular set in $\mathrm{CayS}(G, S)$.

(2) This is immediate from [25, Theorem 2.9]. □

The following result determines all subgroups of a generalized dicyclic group.

Lemma 2.2. *All subgroups of G are as follows:*

(1) A_1, where $A_1 \leqslant A$;

(2) $\langle A_1, xb\rangle$, where $x \in A$ and $|\langle A_1, xb\rangle : A_1| = 2$.

Proof. Let M be a subgroup of G. Since $|G : A| = 2$, we have $G = A \cup Ab$, which implies $M = A_1 \cup (M \cap Ab)$, where $A_1 = M \cap A$. If $M \cap Ab = \emptyset$, then $A_1 = M \leq A$.

We only need to consider the case $M \cap Ab \neq \emptyset$. Let $xb \in M \cap Ab$ with $x \in A$. It follows that $b^2 = (xb)^2 \in M \cap A = A_1$, and so $|\langle A_1, xb\rangle : A_1| = 2$.

Pick $w \in M$. If $w \in A$, then $w \in A_1$, and so $w \in \langle A_1, xb\rangle$. Now, suppose $w \in Ab$. Then, there exists $a \in A$ such that $w = ab$. It follows that $ab = ax^{-1}xb$. Since $ab, xb, b^2 \in M$, one gets $ax^{-1} = (ab)(xb)b^2 \in M \cap A = A_1$, which implies $w = ab \in \langle A_1, xb\rangle$. Since w is arbitrary, we get $M \leqslant \langle A_1, xb\rangle$.

Let $a'(xb)^i \in \langle A_1, xb\rangle$ with $a' \in A_1$ and $i \in \{0,1,2,3\}$. If $i \in \{0,2\}$, then $a'(xb)^i \in A_1 \subseteq M$ since $b^2 \in A_1$. If $i = \{1,3\}$, then $a'(xb)^i = a'xb$ or $a'b^2xb$, which implies $a'(xb)^i \in M$ since $a', xb, b^2 \in M$. Therefore, $\langle A_1, xb\rangle \leqslant M$, and so $M = \langle A_1, xb\rangle$. □

Recall that H is a subgroup of A.

Lemma 2.3. *The subgroup H is an (α,β)-regular set of G if and only if H is an $(\alpha,0)$-regular set of G and a $(0,\beta)$-regular set of G.*

Proof. This is immediate from the fact that H is a normal subgroup of G and [25, Lemma 2.6]. □

Recall $B = \langle a_1^2\rangle \times \cdots \times \langle a_\lambda^2\rangle \times \langle a_{\lambda+1}\rangle \times \cdots \times \langle a_{\lambda+\mu}\rangle$. The following fact gives all square elements in G.

Fact 2.4. *The set $B \cup \{b^2\}$ consists of all square elements in G, that is,* $\mathrm{Sq}(G) = B \cup \{b^2\}$.

Denote

$$A' = \langle a_1^{2^{e_1-1}}\rangle \times \cdots \times \langle a_\lambda^{2^{e_\lambda-1}}\rangle. \tag{2.1}$$

The following fact determines all involutions of G.

Fact 2.5. *The set $A' \setminus \{e\}$ consists of all involutions of G.*

Two elements a and h of G are said to be *conjugate* if there exists an element $g \in G$ such that $h = g^{-1}ag$. The *conjugacy class* of an element $a \in G$ is the set of conjugates of $a \in G$, which is denote by a^G. The following result classifies the conjugacy classes of G.

Lemma 2.6. ([33, Lemma 2.4]) *The following hold:*

(1) $a^G = \{a\}$, for each $a \in A'$;

(2) $a^G = \{a, a^{-1}\}$, for each $a \in A \setminus A'$;

(3) $(ab)^G = Bab$, for each $a \in A$.

Corollary 2.7. *Each inverse-closed subset of A is normal.*

Recall $L = \{a_i : \varphi_i(H) = \langle a_i\rangle,\ 1 \leq i \leq \lambda\}$ and $m = |\{i : \varphi_i(H) \neq \{e\},\ 1 \leq i \leq \lambda\}|$. In the following lemmas, we give some results on the subgroup H.

Lemma 2.8. *The following hold:*

(1) $|H \cap B| = |H|/2^{|L|}$, $|H \cap A'| = 2^m$, and $|H \cap A' \cap B| = 2^{m-r}$;

(2) If $m > |L|$, then $|H|/2^{|L|}$ is even.

Proof. (1) Since $m \geq |L|$, we may assume

$$H = \langle a_{i_1} \rangle \times \cdots \times \langle a_{i_{|L|}} \rangle \times \langle a'_{i_{|L|+1}} \rangle \times \cdots \times \langle a'_{i_m} \rangle \times \langle a'_{\lambda+1} \rangle \times \cdots \times \langle a'_{\lambda+\mu} \rangle,$$

where $1 \leq i_1 < i_2 < \cdots < i_m \leq \lambda$, $\{e\} \neq \langle a'_h \rangle \subsetneq \langle a_h \rangle$ with $i_{|L|+1} \leq h \leq i_m$, and $a'_j \in \langle a_j \rangle$ with $\lambda+1 \leq j \leq \lambda+\mu$. In view of (1.1), we get $H \cap B = \langle a_{i_1}^2 \rangle \times \cdots \times \langle a_{i_{|L|}}^2 \rangle \times \langle a'_{i_{|L|+1}} \rangle \times \cdots \times \langle a'_{i_m} \rangle \times \langle a'_{\lambda+1} \rangle \times \cdots \times \langle a'_{\lambda+\mu} \rangle$. It follows that $|H \cap B| = |H|/2^{|L|}$. Since $r = |\{i : \varphi_i(H) = \langle a_i \rangle,\ 1 \leq i \leq k\}|$, from (2.1), one gets $H \cap A' = \langle a_{i_1} \rangle \times \cdots \times \langle a_{i_r} \rangle \times \langle a_{i_{r+1}}^{2^{e_{i_{r+1}}-1}} \rangle \times \cdots \times \langle a_{i_m}^{2^{e_{i_m}-1}} \rangle$ with $1 \leq i_1 < i_2 < \cdots < i_r \leq k$, which implies $H \cap A' \cap B = \langle a_{i_{r+1}}^{2^{e_{i_{r+1}}-1}} \rangle \times \cdots \times \langle a_{i_m}^{2^{e_{i_m}-1}} \rangle$ by (1.1). Then, $|H \cap A'| = 2^m$ and $|H \cap A' \cap B| = 2^{m-r}$.

(2) By (1), we have $2^m \mid |H|$. The fact that $m > |L|$ implies that $|H|/2^{|L|}$ is even. □

Let C be a subset of A. Denote $a_C = e$ if $C = \emptyset$, and $a_C = \prod_{a \in C} a$ if $C \neq \emptyset$. Let T be the set consisting of a_i with $\varphi_i(H) \neq \langle a_i \rangle$ for $1 \leq i \leq \lambda$. Throughout the paper, $\sqcup$ denotes the disjoint union. Then $L \sqcup T = \{a_1, \ldots, a_\lambda\}$.

Lemma 2.9. *If $L' \subseteq L$ and $T' \subseteq T$, then $|Ba_{L'}a_{T'}b \cap Ha_{T'}b| = |H|/2^{|L|}$.*

Proof. Since $a_{L'} \in H$, we have $|Ba_{L'}a_{T'}b \cap Ha_{T'}b| = |Ba_{L'} \cap H| = |B \cap H|$. This together with Lemma 2.8 (1) leads to $|Ba_{L'}a_{T'}b \cap Ha_{T'}b| = |H|/2^{|L|}$. □

Lemma 2.10. *There are $2^{|L|}$ distinct $a_{L'}$ for $L' \subseteq L$ and $2^{\lambda-|L|}$ distinct $a_{T'}$ for $T' \subseteq T$.*

Proof. Let $L' \subseteq L$ and $T' \subseteq T$. For each element $a_i \in L$ and each symbol $K \in \{L', T'\}$, if $a_i \in K$, then $\varphi_i(a_K) = a_i$; if $a_i \notin K$, then $\varphi_i(a_K) = e$. It follows that $\varphi_i(a_{L'}) \in \{a_i, e\}$ for each $a_i \in L$ and $\varphi_i(a_{T'}) \in \{a_i, e\}$ for each $a_i \in T$. Note that $|T| = \lambda - |L|$. Since $L' \subseteq L$ and $T' \subseteq T$ are arbitrary, there are exactly $2^{|L|}$ distinct $a_{L'}$ for $L' \subseteq L$ and $2^{\lambda-|L|}$ distinct $a_{T'}$ for $T' \subseteq T$. □

Lemma 2.11. *We have $A = \sqcup_{L' \subseteq L, T' \subseteq T} Ba_{L'}a_{T'}$ and $H = \sqcup_{L' \subseteq L} (H \cap B)a_{L'}$.*

Proof. In view of Lemma 2.10, there are $2^{|L|}$ distinct $a_{L'}$ for $L' \subseteq L$ and $2^{\lambda-|L|}$ distinct $a_{T'}$ for $T' \subseteq T$. By (1.1), Ba_K and $Ba_{K'}$ are pairwise disjoint for distinct subsets K and K' of $\{a_1, \ldots, a_\lambda\}$. In view of (1.1), we get $|A : B| = 2^\lambda$. The fact that $L \sqcup T = \{a_1, \ldots, a_\lambda\}$ implies $A = \sqcup_{L' \subseteq L, T' \subseteq T} Ba_{L'}a_{T'}$. Since $L \subseteq H$, from Lemma 2.8 (1), we get $H = \sqcup_{L' \subseteq L} (H \cap B)a_{L'}$. □

Lemma 2.12. *Let $a \in A \setminus H$ and $m = |L|$. If $Ha = Ha^{-1}$, then there exists $h \in H$ such that $a^2 = h^2$.*

Proof. Since $m = |L|$, we may assume

$$H = \langle a_{i_1} \rangle \times \cdots \times \langle a_{i_{|L|}} \rangle \times \langle a'_{\lambda+1} \rangle \times \cdots \times \langle a'_{\lambda+\mu} \rangle,$$

where $1 \leq i_1 < i_2 < \cdots < i_{|L|} \leq \lambda$ and $a'_j \in \langle a_j \rangle$ with $\lambda+1 \leq j \leq \lambda+\mu$. Since $Ha = Ha^{-1}$, we have $a^2 \in H$. Since $a \in A$, from (1.1) and Fact 2.4, we obtain $a^2 \in B$, and so $a^2 \in H \cap B = \langle a_{i_1}^2 \rangle \times \cdots \times \langle a_{i_{|L|}}^2 \rangle \times \langle a'_{\lambda+1} \rangle \times \cdots \times \langle a'_{\lambda+\mu} \rangle$. Since $\cup_{x \in H} x^2 = H \cap B$, there exists $h \in H$ such that $a^2 = h^2$. □

Lemma 2.13. *Let $a \in A \setminus H$. If $m = |L|$, then $Ha = Ha^{-1}$ if and only if $Ha \cap A' \neq \emptyset$.*

Proof. We first prove the necessity. Assume $Ha = Ha^{-1}$. In view of Lemma 2.12, we have $a^2 = h^2$ for some $h \in H$. It follows that $(h^{-1}a)^2 = e$. By Fact 2.5, we get $h^{-1}a \in Ha \cap A'$, which implies $Ha \cap A' \neq \emptyset$.

We next prove the sufficiency. Let $a' \in Ha \cap A'$. Then, there exists $h \in H$ such that $a' = ha$. It follows that $a = h^{-1}a'$. Since $a' \in A'$, from Fact 2.5, one gets $a^2 = (h^{-1}a')^2 = h^{-2}a'^2 = h^{-2} \in H$. Then $Ha = Ha^{-1}$. □

Lemma 2.14. *Let $a \in A \setminus H$. Suppose $Ha = Ha^{-1}$. If $Ha \cap A' \neq \emptyset$ and $Ha \cap \mathrm{Sq}(G) \neq \emptyset$, then $Ha \cap A' \cap \mathrm{Sq}(G) \neq \emptyset$.*

Proof. In view of Fact 2.4, we obtain $\mathrm{Sq}(G) = B \cup \{b^2\}$, and so $Ha \cap \mathrm{Sq}(G) = Ha \cap (B \cup \{b^2\}) \neq \emptyset$. If $b^2 \in Ha$, then $Hb^2 = Ha$, which implies $b^2 \in Ha \cap A' \cap \mathrm{Sq}(G)$ from Fact 2.5.

Now, we only need to consider the case $b^2 \notin Ha$. It follows that $Ha \cap B \neq \emptyset$. Since $a \notin H$, there exists $c \in B \setminus H$ such that $Hc = Ha$. Since $Ha = Ha^{-1}$, one gets $(Hc)^2 = (Ha)^2 = H$, which implies $Hc = Hc^{-1}$. Since $Ha = Hc$, we have $Hc \cap A' \neq \emptyset$, and so $(hc)^2 = e$ for some $h \in H$ from Fact 2.5. It follows that $h^2 = c^{-2}$. Since $h \in H$, from Lemma 2.11, we have $h = c' a_{L'}$ for some $c' \in H \cap B$ and $L' \subseteq L$. It follows that $h^2 = c'^2 a_{L'}^2 = c^{-2}$, and so $(c'c)^2 = a_{L'}^{-2}$. Since $c \in B \setminus H$ and $c' \in H \cap B$, we get $c'c \in B \setminus H$. It follows from Fact 2.4 that $c'c \in Hc \cap \mathrm{Sq}(G)$. Since $Hc = Ha$, we obtain $c'c \in Ha \cap \mathrm{Sq}(G)$.

By Fact 2.5, it suffices to show that $(c'c)^2 = a_{L'}^{-2} = e$. The case $a_{L'} = e$ is trivial. Now suppose $a_{L'} \neq e$. Since $c'c \in (B \setminus H) \cap \mathrm{Sq}(G)$, from (1.1), there exists $a' \in A \setminus H$ such that $c'c = a'^2$. Pick $a_i \in L'$. Let $\varphi_i(a') = a_i^{t_i}$. Since $a'^4 = (c'c)^2 = a_{L'}^{-2}$ and $\varphi_i(a_{L'}) = a_i$, we have ${a_i}^{4t_i} = {a_i}^{-2}$, and so $4t_i \equiv -2 \pmod{2^{e_i}}$ with $e_i \geq 1$, which implies $2^{e_i - 1} \mid 2t_i + 1$. It follows that $e_i = 1$, and so a_i is an involution. Since $a_i \in L'$ is arbitrary, one gets $(c'c)^2 = a_{L'}^{-2} = e$. □

3. (0,β)-Regular set

In this section, we give some results concerning (0,β)-regular sets in Cayley sum graphs on generalized dicyclic groups.

Lemma 3.1. *Let $a \in A$ and S be a normal and square-free subset of G. Suppose $a \in Ba_{L'}a_{T'}$ for some $L' \subseteq L$ and $T' \subseteq T$. If S contains exactly t sets of type $Ba_{L''}a_{T'}b$ with $L'' \subseteq L$, then $|S \cap Hab| = t|H|/2^{|L|}$.*

Proof. Let $\mathscr{A}$ be the set consisting of (L'', T'') such that $L'' \subseteq L$, $T'' \subseteq T$, and $S \cap Ba_{L''}a_{T''}b \neq \emptyset$. In view of Lemma 2.6, $Ba_{L''}a_{T''}b$ is normal for $(L'', T'') \in \mathscr{A}$. Since S is normal, $(L'', T'') \in \mathscr{A}$ if and only if $Ba_{L''}a_{T''}b \subseteq S$ for $L'' \subseteq L$ and $T'' \subseteq T$. By Lemma 2.11, we have $\cup_{(L'',T'') \in \mathscr{A}} Ba_{L''}a_{T''}b = S \cap Ab$.

Since $a \in Ba_{L'}a_{T'}$, there exists $c \in B$ such that $a = ca_{L'}a_{T'}$. Since $a_{L'} \in H$ and $c \in B$, we get

$$\begin{aligned} |S \cap Hab| &= |S \cap Ab \cap Hca_{L'}a_{T'}b| \\ &= |S \cap Ab \cap Hca_{T'}b| \\ &= |(\cup_{(L'',T'') \in \mathscr{A}} Ba_{L''}a_{T''}b) \cap Hca_{T'}b| \\ &= |(\cup_{(L'',T'') \in \mathscr{A}} Ba_{L''}a_{T''}b) \cap Ha_{T'}b|. \end{aligned} \tag{3.1}$$

By Lemma 2.11, we get $H = \sqcup_{L'' \subseteq L} (H \cap B) a_{L''} \subseteq \sqcup_{L'' \subseteq L} Ba_{L''}$, and so $Ha_{T'}b \subseteq \sqcup_{L'' \subseteq L} Ba_{L''}a_{T'}b$. In view of (1.1), Ba_K and $Ba_{K'}$ are pairwise disjoint for distinct subsets K and K' of $\{a_1, \ldots, a_\lambda\}$, which implies $(\cup_{(L'',T'') \in \mathscr{A}} Ba_{L''}a_{T''}b) \cap Ha_{T'}b = (\cup_{(L'',T') \in \mathscr{A}} Ba_{L''}a_{T'}b) \cap Ha_{T'}b$.

By (3.1), we obtain $|S \cap Hab| = |(\cup_{(L'',T') \in \mathscr{A}} Ba_{L''}a_{T'}b) \cap Ha_{T'}b|$. Since $(L'', T') \in \mathscr{A}$ if and only if $Ba_{L''}a_{T'}b \subseteq S$ for $L'' \subseteq L$, from Lemmas 2.9 and 2.11, we get $|S \cap Hab| = t|H|/2^{|L|}$, where S contains exactly t sets of type $Ba_{L''}a_{T'}b$ with $L'' \subseteq L$. □

In this section, let J be the union of Ha for $a \in A \setminus H$ satisfying $Ha = Ha^{-1}$ and $Ha \cap A' = \emptyset$.

Lemma 3.2. *If $m > |L|$, then $J \neq \emptyset$.*

Proof. Since $m > |L|$, we may assume

$$H = \langle a_{i_1}\rangle \times \cdots \times \langle a_{i_{|L|}}\rangle \times \langle a'_{i_{|L|+1}}\rangle \times \cdots \times \langle a'_{i_m}\rangle \times \langle a'_{\lambda+1}\rangle \times \cdots \times \langle a'_{\lambda+\mu}\rangle,$$

where $1 \le i_1 < i_2 < \cdots < i_m \le \lambda$, $\{e\} \ne \langle a'_h\rangle \subsetneq \langle a_h\rangle$ with $i_{|L|+1} \le h \le i_m$, and $a'_j \in \langle a_j\rangle$ with $\lambda + 1 \le j \le \lambda + \mu$. Let l be the minimal positive integer such that $a^l_{i_m} \in H$. Since $o(a_{i_m}) = 2^{e_{i_m}}$ and $\{e\} \ne \langle a'_{i_m}\rangle \subsetneq \langle a_{i_m}\rangle$, one gets $l = 2^{e'_{i_m}}$ with $1 \le e'_{i_m} < e_{i_m}$. It follows that $2 \le l < 2^{e_{i_m}}$ and $a^{l/2}_{i_m} \in A \setminus H$. Then, $Ha^{l/2}_{i_m} = Ha^{-l/2}_{i_m}$.

It suffices to show that $Ha^{l/2}_{i_m} \cap A' = \emptyset$. Since $Ha^{l/2}_{i_m} \cap H = \emptyset$ and $\varphi_i(Ha^{-l/2}_{i_m}) = \varphi_i(H)$ for $1 \le i \le \lambda + \mu$ with $i \ne i_m$, we have $\varphi_{i_m}(Ha^{l/2}_{i_m}) \cap \varphi_{i_m}(H) = \varphi_{i_m}(Ha^{l/2}_{i_m}) \cap \langle a'_{i_m}\rangle = \emptyset$. By the minimality of l, one gets $\langle a^{2^{e_{i_m}-1}}_{i_m}\rangle \leqslant \langle a'_{i_m}\rangle = \langle a^l_{i_m}\rangle$, and so $\varphi_{i_m}(Ha^{l/2}_{i_m}) \cap \langle a^{2^{e_{i_m}-1}}_{i_m}\rangle = \emptyset$, which implies $\varphi_{i_m}(Ha^{l/2}_{i_m}) \cap A' = \emptyset$ from (2.1). Then, $Ha^{l/2}_{i_m} \cap A' = \emptyset$. □

A right transversal (resp. left transversal) of H in A is defined as a subset of A which contains exactly one element in each right coset (resp. left coset) of H in A. Since A is abelian, every right coset of any subgroup is also a left coset of the subgroup. For the sake of simplicity, we use the term "transversal" to substitute for "right transversal" or "left transversal".

Lemma 3.3. *There exists a transversal I of H in A containing e such that $I \setminus J$ is inverse-closed and I contains a square element in each coset of H having nonempty intersection with* $\mathrm{Sq}(G)$.

Proof. Let I be a transversal of H in A containing e such that I contains a square element in each coset of H having nonempty intersection with $\mathrm{Sq}(G)$. Note that $Ha \ne Ha^{-1}$, $Ha = Ha^{-1}$, and $Ha \cap A' \ne \emptyset$, or $Ha = Ha^{-1}$ and $Ha \cap A' = \emptyset$ for each $Ha \in A/H \setminus \{H\}$. For each $Ha \in A/H \setminus \{H\}$ satisfying $Ha \ne Ha^{-1}$, let $ha \in Ha$ and $h^{-1}a^{-1} \in Ha^{-1}$ belong to I for some $h \in H$. For each $Ha \in A/H \setminus \{H\}$ satisfying $Ha = Ha^{-1}$ and $Ha \cap A' \ne \emptyset$, if $Ha \cap \mathrm{Sq}(G) = \emptyset$, then let $ha \in Ha \cap A'$ belong to I for some $h \in H$; if $Ha \cap \mathrm{Sq}(G) \ne \emptyset$, from Lemma 2.14, then let $ha \in Ha \cap A' \cap \mathrm{Sq}(G)$ belong to I for some $h \in H$. Since J is the union of Ha for $a \in A \setminus H$ satisfying $Ha = Ha^{-1}$ and $Ha \cap A' = \emptyset$, from Fact 2.5, $I \setminus J$ is inverse-closed. □

3.1. H is a $(0,\beta)$*-regular set of G*

In this subsection, by giving an appropriate connection set S, we determine each possibility for β such that H is a $(0,\beta)$-regular set of G. Recall $\mathcal{L}(H) = \min\{|\mathrm{Nsq}(G) \cap Hx| : x \in G \setminus H\}$.

Lemma 3.4. *The following hold:*

(1) If $B \cup \{b^2\} \subseteq H$, then $\mathcal{L}(H) = |H|$;

(2) If $B \leqslant H$ and $b^2 \notin H$, then $\mathcal{L}(H) = |H| - 1$;

(3) If $B \nleqslant H$, $b^2 \notin H \cup B$, and $Hb^2 \cap B \ne \emptyset$, then $\mathcal{L}(H) = (2^{|L|} - 1)|H|/2^{|L|} - 1$;

(4) If $B \nleqslant H$, and $b^2 \in H \cup B$ or $Hb^2 \cap B = \emptyset$, then $\mathcal{L}(H) = (2^{|L|} - 1)|H|/2^{|L|}$.

Proof. By Fact 2.4, one gets $\mathrm{Sq}(G) = B \cup \{b^2\}$. It follows that $Hx \subseteq \mathrm{Nsq}(G)$ for each $Hx \in G/H \setminus \{H\}$ with $Hx \cap (B \cup \{b^2\}) = \emptyset$. For each $Hx \in G/H \setminus \{H\}$ with $Hx \cap B \ne \emptyset$ and $b^2 \notin Hx$, there exists $c \in B$ such that $Hx = Hc$, which implies $|\mathrm{Nsq}(G) \cap Hx| = |\mathrm{Nsq}(G) \cap Hc| = |Hc \setminus (Hc \cap B)| = |Hc \setminus (H \cap B)c| = (2^{|L|} - 1)|H|/2^{|L|}$ from Lemma 2.8 (1). To determine $\mathcal{L}(H)$, it suffices to consider the coset Hb^2.

(1) Since $B \cup \{b^2\} \subseteq H$, one gets $Hx \subseteq \mathrm{Nsq}(G)$ for each $x \in G \setminus H$, and so $\mathcal{L}(H) = |H|$.

(2) Since $B \leqslant H$, we have $Hx \cap B = \emptyset$ for each $Hx \in G/H \setminus \{H\}$. Since $B \leqslant H$ and $b^2 \notin H$, one obtains $\mathrm{Nsq}(G) \cap Hb^2 = Hb^2 \setminus \{b^2\}$, and so $\mathcal{L}(H) = |H| - 1$.

(3) Since $Hb^2 \cap B \neq \emptyset$, one gets $Hb^2 = Ha$ for some $a \in B$. Since $b^2 \notin H \cup B$ and $Ba = B$, from Lemma 2.8 (1), one gets $|\mathrm{Nsq}(G) \cap Hb^2| = |Ha \setminus (Ba \cup \{b^2\})| = |Ha \setminus (Ha \cap Ba)| - 1 = (2^{|L|} - 1)|H|/2^{|L|} - 1$. Then $\mathcal{L}(H) = (2^{|L|} - 1)|H|/2^{|L|} - 1$.

(4) If $b^2 \in H$, then $\mathcal{L}(H) = (2^{|L|} - 1)|H|/2^{|L|}$, and so (4) is valid. Now suppose $b^2 \notin H$. If $b^2 \in B$, from Lemma 2.8 (1), then $|\mathrm{Nsq}(G) \cap Hb^2| = |Hb^2 \setminus B| = |Hb^2 \setminus (H \cap B)b^2| = (2^{|L|} - 1)|H|/2^{|L|}$, which implies that (4) holds. If $b^2 \notin B$, then $Hb^2 \cap B = \emptyset$, and so $|\mathrm{Nsq}(G) \cap Hb^2| = |Hb^2 \setminus \{b^2\}| = |H| - 1$. Thus, (4) holds. □

Lemma 3.5. *Suppose that I is a transversal of H in A containing e such that I contains a square element in each coset of H having nonempty intersection with* $\mathrm{Sq}(G)$*. The following hold:*

(1) If $B \cup \{b^2\} \subseteq H$, then $(I \setminus \{e\})x$ is square-free for each $x \in H$;

(2) If $B \leqslant H$ and $b^2 \notin H$, then $(I \setminus \{e\})x$ is square-free for each $x \in H \setminus B$;

(3) If $B \nleqslant H$, $b^2 \notin H \cup B$, and $Hb^2 \cap B \neq \emptyset$, then $(I \setminus \{e\})x$ is square-free for each $x \in H \setminus (B \cup Bb^2)$;

(4) If $B \nleqslant H$, and $b^2 \in H \cup B$ or $Hb^2 \cap B = \emptyset$, then $(I \setminus \{e\})x$ is square-free for each $x \in H \setminus B$.

Proof. Note that $I \setminus \{e\} = ((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha)) \cup (I \setminus (\cup_{a \in B \cup \{b^2\}} Ha))$. By Fact 2.4, one gets $\mathrm{Sq}(G) = B \cup \{b^2\} \subseteq \cup_{a \in B \cup \{b^2\}} Ha$. Since $Hah = Ha$ for each $h \in H$ with $a \in A$, one gets $(I \setminus (\cup_{a \in B \cup \{b^2\}} Ha))h \subseteq A \setminus (B \cup \{b^2\})$, which implies that $(I \setminus (\cup_{a \in B \cup \{b^2\}} Ha))h$ is square-free. It suffices to show that $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x$ is square-free for each case.

(1) Since $B \cup \{b^2\} \subseteq H$, we get $\cup_{a \in B \cup \{b^2\}} Ha = H$, and so $(I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha) = (I \setminus \{e\}) \cap H = \emptyset$, which implies $(I \setminus \{e\})x = (I \setminus (\cup_{a \in B \cup \{b^2\}} Ha))x$ for each $x \in H$. Thus, (1) is valid.

(2) Since $b^2 \notin H$, one gets $Hb^2 \cap H = \emptyset$. Since $B \leqslant H$, we have $Hb^2 \cap B = \emptyset$ and $\cup_{a \in B \cup \{b^2\}} Ha = H \cup Hb^2$. Since I contains a square element in each coset of H having nonempty intersection with $\mathrm{Sq}(G)$ and $Hb^2 \cap B = \emptyset$, from Fact 2.4, we have $b^2 \in I$, which implies $(I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha) = (I \setminus \{e\}) \cap (H \cup Hb^2) = (I \setminus \{e\}) \cap Hb^2 = \{b^2\}$. Then $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x = \{b^2 x\} \subseteq Hb^2 \setminus (B \cup Bb^2) \subseteq A \setminus (B \cup Bb^2)$ for each $x \in H \setminus B$, and so $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x$ is square-free. Thus, (2) holds.

(3) Note that I contains a square element in each coset of H having nonempty intersection with $\mathrm{Sq}(G)$. Since $B \nleqslant H$, $b^2 \notin H \cup B$, and $Hb^2 \cap B \neq \emptyset$, from Fact 2.4, one obtains $(I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha) \subseteq B \cup \{b^2\} \subseteq B \cup Bb^2$, which implies $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x \subseteq (B \cup Bb^2)x \subseteq A \setminus (B \cup Bb^2)$ for each $x \in H \setminus (B \cup Bb^2)$. Then $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x$ is square-free for each $x \in H \setminus (B \cup Bb^2)$. Thus, (3) is valid.

(4) Note that I contains a square element in each coset of H having nonempty intersection with $\mathrm{Sq}(G)$. If $b^2 \in H$, from Fact 2.4, then $(I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha) = (I \setminus \{e\}) \cap (\cup_{a \in B} Ha) \subseteq B \setminus \{e\}$, and so $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x \subseteq A \setminus (H \cup B)$ for each $x \in H \setminus B$, which implies that $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x$ is square-free. If $b^2 \in B$, then $B \cup \{b^2\} = B$, and so $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x \subseteq Bx \subseteq A \setminus B$ for each $x \in H \setminus B$ from Fact 2.4, which implies that $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x$ is square-free.

Now, suppose $b^2 \notin H \cup B$. It follows that $Hb^2 \cap B = \emptyset$. Since I contains a square element in each coset of H having nonempty intersection with $\mathrm{Sq}(G)$, one has $I \cap Hb^2 = \{b^2\}$. For each $x \in H \setminus B$, since $Hb^2 \cap B = \emptyset$, we obtain $(I \cap Hb^2)x = \{b^2 x\} \subseteq Hb^2 \setminus Bb^2 \subseteq A \setminus (B \cup \{b^2\})$ and $((I \setminus \{e\}) \cap (\cup_{a \in B} Ha))x \subseteq Bx \subseteq A \setminus (B \cup Bb^2)$, which implies that $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x \subseteq A \setminus (B \cup \{b^2\})$. Then, $((I \setminus \{e\}) \cap (\cup_{a \in B \cup \{b^2\}} Ha))x$ is square-free for each $x \in H \setminus B$. Thus, (4) is valid. □

Lemma 3.6. *The subgroup H is a $(0,\beta)$-regular set in* $\mathrm{CayS}(G,S)$ *if and only if* $\beta = t|H|/2^{|L|}$ *and one of the following holds:*

(1) $0 \leq t \leq 2^{|L|}$, $B \cup \{b^2\} \subseteq H$;

(2) $0 \leq t \leq 2^{|L|} - 1$, $B \leqslant H$, and $b^2 \notin H$;

(3) $0 \leq t \leq 2^{|L|} - 2$, $B \nleqslant H$, $b^2 \notin H \cup B$, and $Hb^2 \cap B \neq \emptyset$;

(4) $0 \leq t \leq 2^{|L|} - 1$, $B \nleqslant H$, and $b^2 \in H \cup B$ or $Hb^2 \cap B = \emptyset$.

Proof. We first prove the necessity. Pick an element $a \in A$. In view of Lemma 2.11, one gets $a \in Ba_{L'}a_{T'}$ for some $L' \subseteq L$ and $T' \subseteq T$. Since S is normal, from Lemma 2.6, we may assume that S contains exactly t sets of type $Ba_{L''}a_{T'}b$ with $L'' \subseteq L$. In view of Lemmas 2.1 and 3.1, we get $\beta = |S \cap Hab| = t|H|/2^{|L|}$ with $0 \leq t \leq \frac{\mathcal{L}(H)\cdot 2^{|L|}}{|H|}$.

If $B \cup \{b^2\} \subseteq H$, from Lemma 3.4 (1), then $\mathcal{L}(H) = |H|$, and so $0 \leq t \leq 2^{|L|}$, which imply that (1) is valid. If $B \leqslant H$ and $b^2 \notin H$, from Lemma 3.4 (2), one gets $\mathcal{L}(H) = |H| - 1$, and so $0 \leq t \leq 2^{|L|} - 1$, which implies that (2) holds. If $B \nleqslant H$, $b^2 \notin H \cup B$, and $Hb^2 \cap B \neq \emptyset$, from Lemma 3.4 (3), then $\mathcal{L}(H) = (2^{|L|} - 1)|H|/2^{|L|} - 1$, and so, $0 \leq t \leq 2^{|L|} - 2$, which implies that (3) is valid. If $B \nleqslant H$, and $b^2 \in H \cup B$ or $Hb^2 \cap B \neq \emptyset$, from Lemma 3.4 (4), then $\mathcal{L}(H) = (2^{|L|} - 1)|H|/2^{|L|}$, and so $0 \leq t \leq 2^{|L|} - 1$, which implies that (4) holds.

Next, we prove the sufficiency. If $t = 2^{|L|}$, then $\beta = |H|$, and so $S = G \setminus H$. Now, we consider $0 \leq t \leq 2^{|L|} - 1$. By Lemma 3.3, let I be a transversal of H in A containing e such that $I \setminus J$ is inverse-closed and I contains a square element in each coset of H having nonempty intersection with $\mathrm{Sq}(G)$.

In view of Lemma 2.10, there are $2^{|L|} - 1$ distinct $a_{L'}$ for $\emptyset \neq L' \subseteq L \subseteq H$. By (1.1), Ba_K and $Ba_{K'}$ are pairwise disjoint for distinct subsets K and K' of $\{a_1, \ldots, a_\lambda\}$. If $b^2 \notin H \cup B$ and $Hb^2 \cap B \neq \emptyset$, then $H \cap Bb^2 \neq \emptyset$, and there are $2^{|L|} - 2$ distinct $a_{L'}$ for $\emptyset \neq L' \subseteq L$ and $a_{L'} \notin Bb^2$ from Lemma 2.11. Since $0 \leq t|H|/2^{|L|} \leq \mathcal{L}(H)$ from Lemma 3.4, each coset Hx with $x \in G \setminus H$ has at least $t|H|/2^{|L|}$ non-square elements. Since $a_{L'} \in H$ with $L' \subseteq L$, from Lemma 2.8 (1), one has $|H \cap Ba_{L'}| = |H|/2^{|L|}$.

If $B \nleqslant H$, $b^2 \notin H \cup B$, and $Hb^2 \cap B \neq \emptyset$, then let S be a union of $t|H|/2^{|L|+1}$ sets of type $\{ha', h^{-1}a'^{-1}\}$ with $ha' \in \mathrm{Nsq}(G)$ and $h \in H$ for each $Ha' \subseteq J$, t sets of type $\cup_{x \in H \cap Ba_{L'}} (I \setminus (J \cup \{e\}))x$, and t sets of type $\cup_{T' \subseteq T} Ba_{L'}a_{T'}b$ with $\emptyset \neq L' \subseteq L$ and $a_{L'} \notin Bb^2$. If $B \leqslant H$, or $b^2 \in H \cup B$, or $Hb^2 \cap B = \emptyset$, then let S be a union of $t|H|/2^{|L|+1}$ sets of type $\{ha', h^{-1}a'^{-1}\}$ with $ha' \in \mathrm{Nsq}(G)$ and $h \in H$ for each $Ha' \subseteq J$, t sets of type $\cup_{x \in H \cap Ba_{L'}} (I \setminus (J \cup \{e\}))x$, and t sets of type $\cup_{T' \subseteq T} Ba_{L'}a_{T'}b$ with $\emptyset \neq L' \subseteq L$.

We now claim that H is a $(0, t|H|/2^{|L|})$-regular set in $\mathrm{CayS}(G,S)$. The proof proceeds in the following steps.

Step 1. *S is normal and square-free.*

In view of Fact 2.4 and Lemma 2.6, $\cup_{T' \subseteq T} Ba_{L'}a_{T'}b$ with $\emptyset \neq L' \subseteq L$ is normal and square-free. By Lemma 3.5, S is square-free.

Let L' be a nonempty subset of L and $d \in H \cap Ba_{L'}$. Since $d \in Ba_{L'}$, there exists $c \in B$ such that $d = ca_{L'}$. By (1.1), we have $a_{L'}^2 \in B$, and so $d^{-1} = c^{-1}a_{L'}^{-1} \in H \cap Ba_{L'}^{-1} = H \cap Ba_{L'}$. Since L' and d are arbitrary, $H \cap Ba_{L'}$ is inverse-closed for each nonempty subset of L.

Since $I \setminus (J \cup \{e\})$ is inverse-closed, $\cup_{x \in H \cap Ba_{L'}} (I \setminus (J \cup \{e\}))x$ for each nonempty subset L' of L, $\{ha', h^{-1}a'^{-1}\}$ with $ha' \in \mathrm{Nsq}(G)$, and $h \in H$ for each $Ha' \subseteq J$ are normal by Corollary 2.7, which implies that S is normal.

Thus, Step 1 holds.

Step 2. $|S \cap Hab| = t|H|/2^{|L|}$ *for each* $a \in A$.

Note that S contains t sets of type $Ba_{L'}a_{T'}b$ with $\emptyset \neq L' \subseteq L$ for each $T' \subseteq T$. For each $a \in A$, from Lemma 2.11, one gets $a \in Ba_{L''}a_{T''}$ for some $L'' \subseteq L$ and $T'' \subseteq T$. By Lemma 3.1, we get $|S \cap Hab| = t|H|/2^{|L|}$ for each $a \in A$.

Thus, Step 2 is valid.

Step 3. $|S \cap Ha| = t|H|/2^{|L|}$ *for each* $a \in A \setminus H$.

Note that $|I \cap Ha| = 1$ for each $a \in A \setminus H$. By Lemma 2.8 (1), we have $|(\cup_{x \in H \cap Ba_{L'}}(I \setminus (J \cup \{e\}))x) \cap Ha| = |H \cap Ba_{L'}| = |H|/2^{|L|}$ with $\emptyset \neq L' \subseteq L \subseteq H$ for each Ha satisfying $Ha \cap J \cap H = \emptyset$. Since S contains t sets of type $\cup_{x \in H \cap Ba_{L'}}(I \setminus (J \cup \{e\}))x$ with $\emptyset \neq L' \subseteq L$, from Lemma 2.11, we get $|S \cap Ha| = t|H|/2^{|L|}$ for each Ha satisfying $Ha \cap J \cap H = \emptyset$. If $J = \emptyset$, then the desired result follows. Now, suppose $J \neq \emptyset$. It follows from Lemma 2.13 that $m > |L|$. By Lemma 2.8 (2), $|H|/2^{|L|}$ is even. Since S contains $t|H|/2^{|L|+1}$ sets of type $\{ha', h^{-1}a'^{-1}\}$ with $ha' \in \mathrm{Nsq}(G)$ and $h \in H$ for each $Ha' \subseteq J$, one obtains $|S \cap Ha'| = t|H|/2^{|L|}$.

Thus, Step 3 holds. In view of Steps 1–3 and Lemma 2.1 (1), our claim is valid. □

3.2. $\langle H, zb \rangle$ is a $(0,\beta)$-regular set of G

In this subsection, we always assume that $\langle H, zb \rangle$ is a subgroup of G such that H has index 2 in $\langle H, zb \rangle$.

In the following lemma, by giving an appropriate connection set S with $S \cap A = \emptyset$, we determine each possibility for (α,β) such that $\langle H, zb \rangle$ is an (α,β)-regular set of G.

Lemma 3.7. *Let S be a normal and square-free subset of G with $S \cap A = \emptyset$. Then $\langle H, zb \rangle$ is an (α,β)-regular set in* $\mathrm{CayS}(G,S)$ *if and only if one of the following holds:*

(1) $B \leqslant H$, $\alpha = t'|H|/2^{|L|}$, and $\beta = t|H|/2^{|L|}$;

(2) $B \nleqslant H$ and $\alpha = \beta = t|H|/2^{|L|}$.

Here, $0 \leq t, t' \leq 2^{|L|}$.

Proof. Since H has index 2 in $\langle H, zb \rangle$, we have $\langle H, zb \rangle = H \cup Hzb$. Since $S \cap A = \emptyset$, we have

$$|S \cap \langle H, zb \rangle a| = |S \cap (H \cup Hzb)a| = |S \cap Hzba| = |S \cap Hza^{-1}b| \tag{3.2}$$

for each $a \in A$. Since $z \in A$, from Lemma 2.11, one gets $z \in Ba_{L'}a_{T'}$ for some $L' \subseteq L$ and $T' \subseteq T$.

We first prove the necessity. Let $a \in A \setminus H$. Since H has index 2 in $\langle H, zb \rangle$, we have $\langle H, zb \rangle a \neq \langle H, zb \rangle$. In view of Lemma 2.11, one gets $za^{-1} \in Ba_{L''}a_{T''}$ for some $L'' \subseteq L$ and $T'' \subseteq T$. Since S is normal, from Lemma 2.6, we may assume that S contains exactly t sets of type $Ba_{L_0}a_{T''}b$ with $L_0 \subseteq L$. By Lemma 2.1 (1), Lemma 3.1, and (3.2), one has $\beta = |S \cap \langle H, zb \rangle a| = |S \cap Hza^{-1}b| = t|H|/2^{|L|}$ with $0 \leq t \leq 2^{|L|}$.

Suppose $B \leqslant H$. Note that $z \in Ba_{L'}a_{T'}$. Since S is normal, from Lemma 2.6, we may assume that S contains exactly t' sets of type $Ba_{L''}a_{T'}b$ with $L'' \subseteq L$. By Lemma 2.1 (1), Lemma 3.1, and (3.2), we get $\alpha = |S \cap \langle H, zb \rangle| = |S \cap Hzb| = t'|H|/2^{|L|}$ with $0 \leq t' \leq 2^{|L|}$. Thus, (1) is valid.

Suppose $B \nleqslant H$. It follows that $B \setminus H \neq \emptyset$. Since $z \in Ba_{L'}a_{T'}$, we have $zc^{-1} \in Ba_{L'}a_{T'}$ with $c \in B \setminus H$. By Lemma 3.1, we have $|S \cap Hzb| = |S \cap Hzc^{-1}b|$. In view of Lemma 2.1 (1) and (3.2), one gets $\alpha = |S \cap \langle H, zb \rangle| = |S \cap Hzb| = |S \cap Hzc^{-1}b| = |S \cap \langle H, zb \rangle c| = \beta$. Thus, (2) holds.

We next prove the sufficiency. In view of Fact 2.4 and Lemma 2.6, $Ba_{L''}a_{T''}b$ is normal and square-free for each $L'' \subseteq L$ and $T'' \subseteq T$.

Suppose $B \leqslant H$. It follows from Lemma 2.11 that $H = \sqcup_{L'' \subseteq L} Ba_{L''}$. Since $z \in Ba_{L'}a_{T'} \subseteq Ha_{L'}a_{T'} = Ha_{T'}$, one gets $Hz = Ha_{T'} = \cup_{L'' \subseteq L} Ba_{L''}a_{T'}$, which implies $(\cup_{L'' \subseteq L} Ba_{L''}a_{T'}) \cap (A \setminus Hz) = \emptyset$. Let S be a union of t' sets of type $Ba_{L''}a_{T'}b$ with $L'' \subseteq L$, and t sets of type $Ba_{L''}a_{T''}b$ with $L'' \subseteq L$ for each $T'' \subseteq T$ with $(\cup_{L'' \subseteq L} Ba_{L''}a_{T''}) \cap (A \setminus Hz) \neq \emptyset$. Since $z \in Ba_{L'}a_{T'}$, from (3.2) and Lemma 3.1, we obtain $|S \cap \langle H, zb\rangle| = |S \cap Hzb| = t'|H|/2^{|L|}$. For each $x \in A \setminus H$, since $zx^{-1} \in A$, from Lemma 2.11, one gets $zx^{-1} \in Ba_{L''}a_{T''}$ for some $L'' \subseteq L$ and $T'' \subseteq T$, which implies $zx^{-1} \in Ba_{L''}a_{T''} \cap (A \setminus Hz)$. In view of (3.2) and Lemma 3.1, one gets $|S \cap \langle H, zb\rangle x| = |S \cap Hzx^{-1}b| = t|H|/2^{|L|}$ for each $x \in A \setminus H$. By Lemma 2.1 (1), $\langle H, zb\rangle$ is a $(t'|H|/2^{|L|}, t|H|/2^{|L|})$-regular set in $\mathrm{CayS}(G, S)$ with $0 \le t, t' \le 2^{|L|}$.

Suppose $B \nleqslant H$. Let S be a union of t sets of type $\cup_{T'' \subseteq T} Ba_{L''}a_{T''}b$ with $L'' \subseteq L$. For each $za^{-1} \in A$, from Lemma 2.11, one gets $za^{-1} \in Ba_{L''}a_{T''}$ for some $L'' \subseteq L$ and $T'' \subseteq T$. By (3.2) and Lemma 3.1, one gets $|S \cap \langle H, zb\rangle a| = |S \cap Hza^{-1}b| = t|H|/2^{|L|}$ for each $a \in A$. In view of Lemma 2.1 (1), $\langle H, zb\rangle$ is a $(t|H|/2^{|L|}, t|H|/2^{|L|})$-regular set in $\mathrm{CayS}(G, S)$ with $0 \le t \le 2^{|L|}$. □

In the following lemma, by giving an appropriate connection set S with $S \subseteq A$, we determine each possibility for β such that $\langle H, zb\rangle$ is a $(0,\beta)$-regular set of G.

Since $b^2 = (zb)^2$, one gets $b^2 \in H$.

Lemma 3.8. *Let S be a normal and square-free subset of G with $S \subseteq A$. Then $\langle H, zb\rangle$ is a $(0,\beta)$-regular set in* $\mathrm{CayS}(G, S)$ *if and only if one of the following holds:*

(1) $B \leqslant H$ and $0 \le \beta \le |H|$;

(2) $B \nleqslant H$ and $0 \le \beta \le (2^{|L|} - 1)|H|/2^{|L|}$.

Moreover, if $m > |L|$, then $2 \mid \beta$.

Proof. Since H has index 2 in $\langle H, zb\rangle$ and $S \subseteq A$, one gets

$$S \cap \langle H, zb\rangle x = S \cap (H \cup Hzb)x = S \cap Hx \tag{3.3}$$

for each $x \in A \setminus H$.

We first prove the necessity. Since $|G : A| = 2$, one has $G = A \cup Ab$. By Fact 2.4, we have $|\mathrm{Nsq}(G) \cap Hab| = |H|$ for each $a \in A$. It follows that $\mathcal{L}(H) = \min\{|\mathrm{Nsq}(G) \cap Hx| : x \in G \setminus H\} = \min\{|\mathrm{Nsq}(G) \cap Hx| : x \in A \setminus H\}$. Let $\mathcal{L}(H) = |\mathrm{Nsq}(G) \cap Hx_0|$ with $x_0 \in A \setminus H$. Since S is square-free, one gets $S \cap Hx_0 \subseteq \mathrm{Nsq}(G) \cap Hx_0$. By Lemma 2.1 (1) and (3.3), one obtains $\beta = |S \cap \langle H, zb\rangle x| = |S \cap Hx|$ for all $x \in A \setminus H$. It follows that $\beta = |S \cap Hx_0| \le |\mathrm{Nsq}(G) \cap Hx_0| = \mathcal{L}(H)$.

Suppose $B \leqslant H$. Since $b^2 \in H$, from Lemma 3.4 (1), we have $\beta \le \mathcal{L}(H) = |H|$ for each $x \in A \setminus H$. Thus, (1) is valid.

Suppose $B \nleqslant H$. Since $b^2 \in H$, from Lemma 3.4 (4), one gets $\beta \le \mathcal{L}(H) = (2^{|L|} - 1)|H|/2^{|L|}$ for each $x \in A \setminus H$. Thus, (2) holds.

Now, suppose $m > |L|$. In view of Lemma 3.2, one has $J \neq \emptyset$. By Lemma 2.1 (1) and (3.3), we get $\beta = |S \cap Ha|$ for $Ha \subseteq J$. For each $ha \in S \cap Ha$ with $Ha \subseteq J$ and $h \in H$, since S is normal and $Ha \cap A' = \emptyset$, we have $h^{-1}a^{-1} \in S \cap Ha^{-1} = S \cap Ha$ from Lemma 2.6. Since $Ha \cap A' = \emptyset$, from Fact 2.5, one obtains $2 \mid |S \cap Ha|$, and so $2 \mid \beta$. The second statement follows.

Next, we prove the sufficiency. By Lemma 3.3, let I be a transversal of H in A containing e such that $I \setminus J$ is inverse-closed and I contains a square element in each coset of H having nonempty intersection with $\mathrm{Sq}(G)$.

By Lemma 2.8 (1), we have $|H \setminus B| = (2^{|L|} - 1)|H|/2^{|L|}$, $|H \cap A'| = 2^m$, and $|(H \cap A') \setminus B| = 2^m - 2^{m-r}$. Note that $b^2 \in H$. It follows that $\langle b^2 \rangle \leqslant H$, and so $|H \cap A'| = 2^m \geq 2$, which implies $m \geq 1$. By Lemma 3.4 (1) and (4), we have $\beta \leq \mathcal{L}(H)$, which implies that each coset $\langle H, zb \rangle x$ with $x \in A \setminus H$ has at least β non-square elements in A by (3.3). Let $\sigma(n) = \frac{1-(-1)^n}{2}$ for an integer n. Now, we give an appropriate connection set S for each case respectively.

- Suppose $B \leqslant H$, $m = |L|$, and $\beta \leq 2^m$. Let S be a union of β sets of type $(I \setminus \{e\})a'$ with $a' \in H \cap A'$.
- Suppose $B \leqslant H$, $m = |L|$, and $\beta > 2^m$. Let S be a union of $2^m - \sigma(\beta)$ sets of type $(I \setminus \{e\})a'$ with $a' \in H \cap A'$ and $\frac{\beta-(2^m-\sigma(\beta))}{2}$ sets of type $(I \setminus \{e\})a \cup (I \setminus \{e\})a^{-1}$ with $a \in H \setminus A'$.
- Suppose $B \nleqslant H$, $m = |L|$, and $\beta \leq 2^m - 2^{m-r}$. Let S be a union of β sets of type $(I \setminus \{e\})c'$ with $c' \in (H \cap A') \setminus B$.
- Suppose $B \nleqslant H$, $m = |L|$, and $\beta > 2^m - 2^{m-r}$. Let S be a union of $2^m - 2^{m-r} - \sigma(\beta) + (-1)^{\beta+1} \cdot \delta_{m,r}$ sets of type $(I \setminus \{e\})c'$ with $c' \in (H \cap A') \setminus B$ and $\frac{\beta-(2^m-2^{m-r}-\sigma(\beta)+(-1)^{\beta+1}\cdot\delta_{m,r})}{2}$ sets of type $(I \setminus \{e\})c \cup (I \setminus \{e\})c^{-1}$ with $c \in H \setminus (A' \cup B)$, where δ is Kronecker's delta.
- Suppose $B \leqslant H$, $m > |L|$, and $\beta \leq 2^m$. Let S be a union of β sets of type $(I \setminus (J \cup \{e\}))a'$ with $a' \in H \cap A'$ and $\beta/2$ sets of type $\{ha'', h^{-1}a''^{-1}\}$ with $ha'' \in \mathrm{Nsq}(G)$ and $h \in H$ for each $Ha'' \subseteq J$.
- Suppose $B \leqslant H$, $m > |L|$, and $\beta > 2^m$. Let S be a union of 2^m sets of type $(I \setminus (J \cup \{e\}))a'$ with $a' \in H \cap A'$, $\frac{\beta-2^m}{2}$ sets of type $(I \setminus (J \cup \{e\}))a \cup (I \setminus (J \cup \{e\}))a^{-1}$ with $a \in H \setminus A'$, and $\beta/2$ sets of type $\{ha'', h^{-1}a''^{-1}\}$ with $ha'' \in \mathrm{Nsq}(G)$ and $h \in H$ for each $Ha'' \subseteq J$.
- Suppose $B \nleqslant H$, $m > |L|$, and $\beta \leq 2^m - 2^{m-r}$. Let S be a union of β sets of type $(I \setminus (J \cup \{e\}))c'$ with $c' \in (H \cap A') \setminus B$ and $\beta/2$ sets of type $\{ha'', h^{-1}a''^{-1}\}$ with $ha'' \in \mathrm{Nsq}(G)$ and $h \in H$ for each $Ha'' \subseteq J$.
- Suppose $B \nleqslant H$, $m > |L|$, and $\beta > 2^m - 2^{m-r}$. Let S be a union of $2^m - 2^{m-r}$ sets of type $(I \setminus (J \cup \{e\}))c'$ with $c' \in (H \cap A') \setminus B$, $\frac{\beta-(2^m-2^{m-r})}{2}$ sets of type $(I \setminus (J \cup \{e\}))c \cup (I \setminus (J \cup \{e\}))c^{-1}$ with $c \in H \setminus (A' \cup B)$, and $\beta/2$ sets of type $\{ha'', h^{-1}a''^{-1}\}$ with $ha'' \in \mathrm{Nsq}(G)$ and $h \in H$ for each $Ha'' \subseteq J$.

We now claim that H is a $(0,\beta)$-regular set in $\mathrm{CayS}(G, S)$. The proof proceeds in the following steps.

Step 1. *S is normal.*

Suppose $m = |L|$. By Lemma 2.13, we get $J = \emptyset$, and so $I \setminus \{e\}$ is inverse-closed, which implies that S is normal from Corollary 2.7.

Suppose $m > |L|$. In view of Lemma 3.2, one obtains $J \neq \emptyset$. Since $I \setminus (J \cup \{e\})$ is inverse-closed, and moreover, for each $Ha'' \subseteq J$, $\{ha'', h^{-1}a''^{-1}\}$ with $ha'' \in \mathrm{Nsq}(G)$ and $h \in H$ is also inverse-closed, it follows from Corollary 2.7 that S is normal.

Step 2. *S is square-free.*

If $B \leqslant H$, from Lemma 3.5 (1), then S is square-free; if $B \nleqslant H$, from Lemma 3.5 (4), then S is square-free.

Step 3. $|S\cap\langle H,zb\rangle x|=\beta$ *for each* $x\in A\setminus H$.

By (3.3), one gets $S\cap\langle H,zb\rangle x=S\cap Hx$ for each $x\in A\setminus H$. We only need to prove $|S\cap Hx|=\beta$ for each $x\in A\setminus H$.

Suppose $m=|L|$. Note that $J=\emptyset$. Since $|(I\setminus\{e\})\cap Hx|=1$ for each $x\in A\setminus H$, we have $|S\cap Hx|=\beta$.

Suppose $m>|L|$. By Lemma 3.2, one has $J\neq\emptyset$. Since $|(I\setminus(J\cup\{e\}))\cap Hd|=1$ for each Hd satisfying $Hd\cap J\cap H=\emptyset$, we have $|S\cap Hd|=\beta$. Since S contains $\beta/2$ sets of type $\{ha'',h^{-1}a''^{-1}\}$ with $ha''\in\mathrm{Nsq}(G)$ and $h\in H$ for each $Ha''\subseteq J$, we have $|S\cap Ha''|=\beta$.

In view of Steps 1–3 and Lemma 2.1 (1), our claim is valid. □

4. Proofs of Theorems 1.1 and 1.2

To give the proofs of Theorems 1.1 and 1.2, we need some auxiliary lemmas.

Lemma 4.1. *The subgroup H is an* $(\alpha,0)$*-regular set of G if and only if one of the following occurs:*

(1) $0\le\alpha\le|H|/2-1$ and α is even, when $m=|L|=1$ and $b^2\in H\setminus B$;

(2) $0\le\alpha\le(2^{|L|}-1)|H|/2^{|L|}-1$, when $m\ge|L|\ge1$, $m\neq1$, and $b^2\in H\setminus B$;

(3) $0\le\alpha\le(2^{|L|}-1)|H|/2^{|L|}$ and α is even, when $r=0$ and $b^2\notin H\setminus B$;

(4) $0\le\alpha\le(2^{|L|}-1)|H|/2^{|L|}$, when $r>0$ and $b^2\notin H\setminus B$.

Proof. Let H be an $(\alpha,0)$-regular set in $\mathrm{CayS}(G,S)$ for some subset S of G. By Lemma 2.1 (1), one obtains $S=S\cap H$. Since $b^2\in A'$ from Fact 2.4, we get

$$\begin{aligned}S=S\cap H&\subseteq\mathrm{Nsq}(G)\cap H\\&=(\mathrm{Nsq}(G)\cap(H\cap A'))\cup(\mathrm{Nsq}(G)\cap(H\setminus A'))\\&=((H\cap A')\setminus(B\cup\{b^2\}))\cup(H\setminus(A'\cup B)).\end{aligned}\tag{4.1}$$

We divide the proof into the following two cases.

Case 1. $b^2\in H\setminus B$.

By Fact 2.4, one gets $\mathrm{Nsq}(G)\cap H=H\setminus((H\cap B)\cup\{b^2\})$. If $|L|=0$, from Lemma 2.8 (1), then $|H\cap B|=|H|$, and so $H\leqslant B$, contrary to $b^2\in H\setminus B$. Then $|L|\ge1$. In view of Lemma 2.1 (2) and Lemma 2.8 (1), we have $0\le\alpha\le|H\setminus((H\cap B)\cup\{b^2\})|=(2^{|L|}-1)|H|/2^{|L|}-1$.

Suppose $m=1$. It follows that $m=|L|=1$. By Lemma 2.8 (1), we have $|H\cap A'|=2$. Since $\langle b^2\rangle\leqslant H\cap A'$, we get $H\cap A'=\langle b^2\rangle$, and so $H\cap A'\subseteq\mathrm{Sq}(G)$ from Fact 2.4. Since S is square-free and $S=S\cap H$, we have $S\subseteq\mathrm{Nsq}(G)\cap H=H\setminus\mathrm{Sq}(G)$. Since $H\cap A'\subseteq\mathrm{Sq}(G)$, from Fact 2.5, S has no involution. Since S is normal, from Lemma 2.6, S is a union of $\alpha/2$ sets of type $\{a,a^{-1}\}$ with $a\in H\setminus((H\cap B)\cup\{b^2\})$. It follows that α is even. Thus, (1) holds.

Suppose $m\neq1$. Then $m\ge|L|\ge1$ and $m\neq1$. By Lemma 2.8 (1), we have $|(H\cap A')\setminus B|=2^m-2^{m-r}$. Since $b^2\in H\setminus B$ and $b^2\in A'$, from Fact 2.5, one gets $|(H\cap A')\setminus(B\cup\{b^2\})|=2^m-2^{m-r}-1$, which implies that S has at most $2^m-2^{m-r}-1$ involutions from (4.1). If $\alpha\le2^m-2^{m-r}-1$, from Lemma 2.6 and (4.1), then S can be a union of α sets of type $\{a'\}$ with $a'\in(H\cap A')\setminus(B\cup\{b^2\})$ since S is normal.

We only need to consider the case $\alpha>2^m-2^{m-r}-1$. Note that S is normal. If α is even, from Lemma 2.6 and (4.1), then S can be a union of $2^m-2^{m-r}-2+\delta_{m,r}$ sets of type $\{a'\}$ with $a'\in(H\cap A')\setminus(B\cup\{b^2\})$

and $\frac{\alpha-(2^m-2^{m-r}-2+\delta_{m,r})}{2}$ sets of type $\{a,a^{-1}\}$ with $a\in H\setminus(A'\cup B)$. If α is odd, from Lemma 2.6 and (4.1) again, then S can be a union of $2^m-2^{m-r}-1-\delta_{m,r}$ sets of type $\{a'\}$ with $a'\in(H\cap A')\setminus(B\cup\{b^2\})$ and $\frac{\alpha-(2^m-2^{m-r}-1-\delta_{m,r})}{2}$ sets of type $\{a,a^{-1}\}$ with $a\in H\setminus(A'\cup B)$. Thus, (2) is valid.

Case 2. $b^2\notin H\setminus B$.

Since $b^2\notin H\setminus B$, from Fact 2.4, we have $\mathrm{Nsq}(G)\cap H=H\setminus(H\cap B)$. It follows from Lemma 2.1 (2) and Lemma 2.8 (1) that $0\le\alpha\le|H\setminus(H\cap B)|=(2^{|L|}-1)|H|/2^{|L|}$.

Suppose $r=0$. By Lemma 2.8 (1), one has $|H\cap A'|=|H\cap A'\cap B|=2^m$, and so $H\cap A'\subseteq B$. Since S is square-free and $S=S\cap H$, we have $S\subseteq\mathrm{Nsq}(G)\cap H=H\setminus(H\cap B)$. Since $H\cap A'\subseteq B$, from Fact 2.5, S has no involution. Since S is normal, from Lemma 2.6, S is a union of $\alpha/2$ sets of type $\{a,a^{-1}\}$ with $a\in H\setminus(H\cap B)$. It follows that α is even. Thus, (3) holds.

Suppose $r>0$. Since $b^2\notin H\setminus B$, we get $(H\cap A')\setminus(B\cup\{b^2\})=(H\cap A')\setminus B$. By Lemma 2.8 (1), we get $|(H\cap A')\setminus B|=2^m-2^{m-r}$. By (4.1) and Fact 2.5, S has at most 2^m-2^{m-r} involutions. If $\alpha\le 2^m-2^{m-r}$, from Lemma 2.6 and (4.1), then S can be a union of α sets of type $\{a'\}$ with $a'\in(H\cap A')\setminus B$ since S is normal.

We only need to consider the case $\alpha>2^m-2^{m-r}$. Note that S is normal. If α is even, from Lemma 2.6 and (4.1), then S can be a union of $2^m-2^{m-r}-\delta_{m,r}$ sets of type $\{a'\}$ with $a'\in(H\cap A')\setminus B$ and $\frac{\alpha-(2^m-2^{m-r}-\delta_{m,r})}{2}$ sets of type $\{a,a^{-1}\}$ with $a\in H\setminus(A'\cup B)$. If α is odd, from Lemma 2.6 and (4.1) again, then S can be a union of $2^m-2^{m-r}-1+\delta_{m,r}$ sets of type $\{a'\}$ with $a'\in(H\cap A')\setminus B$ and $\frac{\alpha-(2^m-2^{m-r}-1+\delta_{m,r})}{2}$ sets of type $\{a,a^{-1}\}$ with $a\in H\setminus(A'\cup B)$. Thus, (4) is valid. □

Lemma 4.2. *The subgroup $\langle H,zb\rangle$ is an (α,β)-regular set of G if and only if there exists a normal and square-free subset S of G such that $\langle H,zb\rangle$ is an (η,ζ)-regular set in* $\mathrm{CayS}(G,S\cap A)$ *and an $(\alpha-\eta,\beta-\zeta)$-regular set in* $\mathrm{CayS}(G,S\cap Ab)$ *for some $\eta\in\{0,1,\dots,\alpha\}$ and $\zeta\in\{0,1,\dots,\beta\}$.*

Proof. We first prove the necessity. Suppose that $\langle H,zb\rangle$ is an (α,β)-regular set in $\mathrm{CayS}(G,S)$. By Lemma 2.1 (1), one gets $|S\cap\langle H,zb\rangle|=\alpha$ and $|S\cap\langle H,zb\rangle x|=\beta$ for each $x\in A\setminus H$, which imply that S is the union of $S\cap\langle H,zb\rangle$ and β pairwise disjoint subsets $T_j\subseteq\mathrm{Nsq}(G)$ with $1\le j\le\beta$, where each $T_j\cup\{e\}$ is a right transversal of $\langle H,zb\rangle$ in G. Without loss of generality, we may assume that $|S\cap H|=\eta$, $\cup_{h=1}^{\zeta}T_h\subseteq A$, and $\cup_{h=\zeta+1}^{\beta}T_h\subseteq Ab$ for some $\eta\in\{0,1,\dots,\alpha\}$ and $\zeta\in\{0,1,\dots,\beta\}$. Since H has index 2 in $\langle H,zb\rangle$, we have $|S\cap Hzb|=\alpha-\eta$. Then, $|S\cap A\cap\langle H,zb\rangle|=\eta$, $|S\cap A\cap\langle H,zb\rangle x|=\zeta$ for each $x\in A\setminus H$ and $|S\cap Ab\cap\langle H,zb\rangle|=\alpha-\eta$, $|S\cap Ab\cap\langle H,zb\rangle x|=\beta-\zeta$ for each $x\in A\setminus H$. Note that $S=(S\cap A)\cup(S\cap Ab)$. Since S is a normal and square-free subset of G, $S\cap A$ and $S\cap Ab$ are both normal and square-free. By Lemma 2.1 (1), $\langle H,zb\rangle$ is an (η,ζ)-regular set in $\mathrm{CayS}(G,S\cap A)$ and an $(\alpha-\eta,\beta-\zeta)$-regular set in $\mathrm{CayS}(G,S\cap Ab)$.

Next, we prove the sufficiency. Suppose that $\langle H,zb\rangle$ is an (η,ζ)-regular set in $\mathrm{CayS}(G,S\cap A)$ and an $(\alpha-\eta,\beta-\zeta)$-regular set in $\mathrm{CayS}(G,S\cap Ab)$. By Lemma 2.1 (1), $S\cap A$ contains exactly η elements of $\langle H,zb\rangle$ and ζ elements of $\langle H,zb\rangle x$ for each $x\in A\setminus H$, and $S\cap Ab$ contains exactly $\alpha-\eta$ elements of $\langle H,zb\rangle$ and $\beta-\zeta$ elements of $\langle H,zb\rangle x$ for each $x\in A\setminus H$. Since $G=A\cup Ab$, one gets $S=(S\cap A)\cup(S\cap Ab)$, which implies that S contains exactly α elements of $\langle H,zb\rangle$ and β elements of $\langle H,zb\rangle x$ for each $x\in A\setminus H$. Since $S\cap A$ and $S\cap Ab$ are both normal and square-free, from Lemma 2.1 (1), $\langle H,zb\rangle$ is an (α,β)-regular set in $\mathrm{CayS}(G,S)$. □

Lemma 4.3. *Let S be a normal and square-free subset of G with $S\subseteq A$. Then $\langle H,zb\rangle$ is an (α,β)-regular set in* $\mathrm{CayS}(G,S)$ *if and only if H is an $(\alpha,0)$-regular set in* $\mathrm{CayS}(G,S\cap H)$ *and $\langle H,zb\rangle$ is a*

$(0,\beta)$*-regular set of* $\mathrm{CayS}(G, S \setminus H)$.

Proof. Since $S \subseteq A$ and H has index 2 in $\langle H, zb\rangle$, we have $S \cap \langle H, zb\rangle = S \cap H$ and $S \cap (G \setminus \langle H, zb\rangle) = S \cap (A \setminus H)$.

We first prove the necessity. Since H is a normal subgroup of G and S, $S \cap H$ are normal and square-free, $S \setminus H$ is normal and square-free. Since $\langle H, zb\rangle$ is an (α,β)-regular set in $\mathrm{CayS}(G, S)$, from Lemma 2.1 (1), one has $|S \cap H| = |S \cap \langle H, zb\rangle| = \alpha$ and $|(S \setminus H) \cap \langle H, zb\rangle x| = |S \cap \langle H, zb\rangle x| = \beta$ for each $x \in A \setminus H$. Since $S \cap Hb = \emptyset$, from Lemma 2.1 (1) again, H is an $(\alpha, 0)$-regular set in $\mathrm{CayS}(G, S \cap H)$ and $\langle H, zb\rangle$ is a $(0,\beta)$-regular set in $\mathrm{CayS}(G, S \setminus H)$.

We next prove the sufficiency. Since H is an $(\alpha, 0)$-regular set in $\mathrm{CayS}(G, S \cap H)$ and $\langle H, zb\rangle$ is a $(0,\beta)$-regular set in $\mathrm{CayS}(G, S \setminus H)$, from Lemma 2.1 (1), one gets $|S \cap \langle H, zb\rangle| = |S \cap H| = \alpha$ and $|S \cap \langle H, zb\rangle x| = |(S \setminus H) \cap \langle H, zb\rangle x| = \beta$ for each $x \in A \setminus H$. Since $S = (S \cap H) \cup (S \setminus H)$, from Lemma 2.1 (1), $\langle H, zb\rangle$ is an (α,β)-regular set in $\mathrm{CayS}(G, S)$. □

Now, we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.3, H is an (α,β)-regular set of G if and only if H is an $(\alpha, 0)$-regular set and a $(0,\beta)$-regular set of G. The desired result is valid from Lemmas 3.6 and 4.1. □

Next, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.2, $\langle H, zb\rangle$ is an (α,β)-regular set of G if and only if $\langle H, zb\rangle$ is an (η,ζ)-regular set in $\mathrm{CayS}(G, S \cap A)$ and an $(\alpha - \eta, \beta - \zeta)$-regular set in $\mathrm{CayS}(G, S \cap Ab)$ for some $\eta \in \{0, 1, \ldots, \alpha\}$, $\zeta \in \{0, 1, \ldots, \beta\}$, and normal, square-free subset S. In view of Lemma 4.3, $\langle H, zb\rangle$ is an (α,β)-regular set of G if and only if H is an $(\eta, 0)$-regular set in $\mathrm{CayS}(G, S \cap H)$, and $\langle H, zb\rangle$ is an $(0,\zeta)$-regular set in $\mathrm{CayS}(G, (S \cap A) \setminus H)$ and an $(\alpha - \eta, \beta - \zeta)$-regular set in $\mathrm{CayS}(G, S \cap Ab)$ for some $\eta \in \{0, 1, \ldots, \alpha\}$, $\zeta \in \{0, 1, \ldots, \beta\}$, and normal, square-free subset S. The desired result follows from Lemma 2.8 (2) and Lemmas 3.7, 3.8, and 4.1. □

5. Conclusions

In this paper, we study regular sets in Cayley sum graphs on a generalized dicyclic group. For each subgroup H of a generalized dicyclic group G, by giving an appropriate connection set S, we determine each possibility for (α,β) such that H is an (α,β)-regular set of G. Future work may investigate the regular sets in Cayley sum graphs on other non-abelian groups.

Author contributions

Meiqi Peng: Investigation, writing–original draft, writing–review and editing; Yuefeng Yang: Conceptualization, supervision, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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