AIMS Mathematics

AIMS Mathematics, 10(12): 30162–30185.
DOI: 10.3934/math.20251325
Received: 05 September 2025
Revised: 28 November 2025
Accepted: 09 December 2025
Published: 24 December 2025

https://www.aimspress.com/journal/Math

Research article

Actuarial pricing of European options under mixed sub-fractional Brownian motion with the Vasicek interest rate

Fangling Ren* **and Hui Feng***

College of Mathematics and Computer Science, Yan'an University, Shaanxi 716000, China

* **Correspondence:** Email: yadxrfl@yau.edu.cn; 13892150268@163.com.

Abstract: To better capture the long-term memory, non-stationary increments, self-similarity, and stochastic nature of interest rates in financial markets, we introduce the Vasicek stochastic interest rate model within a mixed sub-fractional Brownian motion framework to study European option pricing. Using actuarial pricing methods, we derived a closed-form solution for European options under this model. Through numerical simulations and empirical analysis, we examined how variables such as the underlying asset's initial price, strike price, maturity time, volatility, Hurst index, and correlation coefficients influence option prices.

Keywords: mixed sub-fractional Brownian motion; Vasicek stochastic interest rate model; option pricing; actuarial method
Mathematics Subject Classification: 60G22, 60J70, 91G20

1. Introduction

Option pricing theory has remained a central topic in financial engineering since the development of the Black–Scholes model, continuously advancing stochastic finance. The classical assumption of Brownian motion has been considered inadequate for capturing complex market fluctuations, particularly long-memory and non-Markovian characteristics. To address these limitations, alternative stochastic processes have been introduced. Fractional Brownian motion was proposed as a generalization of Brownian motion, while sub-fractional Brownian motion later gained importance for its ability to characterize dependence between standard and fractional Brownian motions. In 2014, Zili [1] introduced the mixed sub-fractional Brownian motion, which was shown to effectively represent shortterm volatility together with long-term dependence. This process has since demonstrated significant advantages in financial modeling and has been applied extensively in financial engineering.

The methodological framework of option pricing has evolved with increasing market complexity and advances in mathematical techniques, forming three major approaches: The first is the classical analytical model, in which closed-form solutions are derived through strict mathematical deduction. These models are characterized by high efficiency and theoretical rigor, making them suitable for options with simple structures under assumptions, such as complete markets and constant volatility. The second is numerical pricing, applied when analytical solutions are intractable, particularly for American options, path-dependent options (e.g., Asian options), and multi-asset options. In such cases, approximate values are obtained through methods, including Monte Carlo simulation, finite difference schemes, and extensions of binomial or trinomial trees. The third is the actuarial approach, which departs from traditional no-arbitrage assumptions. Based on the principle of fair premium, options are regarded as special insurance contracts, and pricing is conducted using the value of expected losses under the real-world probability measure. This method offers greater adaptability in settings characterized by incomplete markets, stochastic interest rates, or non-normal return distributions.

The development of actuarial pricing methods has been driven by the integration of financial mathematics and actuarial science. In 1977, Merton [2] applied option pricing techniques to deposit insurance, beginning the use of actuarial principles in finance. During the 1980s, Gerber et al. [3] introduced risk measurement concepts from insurance actuarial science into option pricing. By 1996, the research team [4] employed martingale measures and transformation techniques to establish a bridge between actuarial pricing and the valuation of options. In 1998, Bladt et al. [5] formally presented the actuarial approach to option pricing in a systematic framework. In 2010, L. N. Girard [6] demonstrated the fundamental equivalence between option pricing and actuarial valuation methods through their research aimed at eliminating their differences. In 2013, Jian et al. [7] applied actuarial methods to the pricing of European options and convertible bonds. In 2015, Shokrollahi et al. [8] examined the pricing of currency options using the actuarial fair premium approach under a mixed fractional Brownian motion with jumps. In 2025, Wu et al. [9] adopted a mixed fractional jump-diffusion model to capture the long-term memory of assets and the impact of sudden events, using actuarial pricing to study the pricing of reload options and demonstrate how parameters affect the relationship between options and underlying asset prices.

The stochastic nature of interest rates represents a central issue in derivative pricing theory. In 2010, Deakin et al. [10] obtained an analytical solution to the partial differential equation for convertible bond valuation under the assumptions that interest rates follow the Vasicek model and stock prices evolve according to geometric Brownian motion. In the same year, Li et al. [11] developed a financial market model within the framework of fractional Brownian motion, assuming that stock prices and interest rates satisfy stochastic differential equations driven by this process. Through risk hedging techniques, fractional stochastic analysis, and partial differential equation methods, a general pricing formula for European options with fractional stochastic interest rates was derived. In 2016, Wang [12] proposed a financial market model in which stock prices follow a stochastic differential equation driven by double fractional Brownian motion, while interest rates adhered to the Vasicek model. Using stochastic analysis and actuarial methods, a pricing formula for backward options under the double fractional Vasicek interest rate framework was derived. In 2019, Kim et al. [13] established analytical pricing formulas for European currency options and exchange options, assuming that the spot exchange rate evolves according to a generalized mixed fractional Brownian motion with jumps.

Numerical experiments demonstrated that the generalized mixed fractional Brownian motion with jumps model exhibits significant differences compared to other models. In 2023, Tao et al. [14] addressed the pricing of geometric Asian options, where the underlying asset follows sub-fractional Brownian motion and the interest rate is modeled by the sub-fractional Vasicek process. They derived a Black–Scholes-type partial differential equation, reformulated it as a Cauchy problem, and obtained an explicit pricing formula. In the same year, Yao, et al. [15] extended the Vasicek model to multi-asset scenarios, conducting pricing research on Asian rainbow options. In 2025, Djeutcha et al. [16] introduced a mixed modified fractional Vasicek interest rate model in their study, employing Kalman filtering for parameter estimation and state reconstruction to empirically validate its effectiveness in capturing interest rate dynamics in non-tradable economies. That same year, Fullerton et al. [17] addressed the vulnerability of traditional multi-Vasicek models to outliers in parameter estimation, proposing maximum likelihood estimation to enhance model robustness.

In 2024, Oldouz et al. [18] developed an improved interest rate model based on the Levy process for bond option pricing. That same year, Wang et al. [19] pioneered the integration of mixed sub-fractional Brownian motion with the Vasicek interest rate model, deriving the explicit formula for geometric average Asian options. In 2025, Zhi et al. [20,21] pioneered the application of mixed subfractional Brownian motion to an actuarial pricing framework for geometric Asian option valuation. They subsequently extended this theoretical framework to the more complex foreign exchange market, conducting pricing research on forex options and providing closed-form solutions and numerical analyses. Moreover, Li et al. [22] applied the same model to gap option pricing, utilizing the delta-hedging principle to derive the partial differential equation governing option prices. Through variable substitution, they established a pricing formula for gap options, demonstrating the model's effectiveness in complex derivative pricing.

Despite significant progress in related studies, particularly the exploration of mixed sub-Brownian motion with stochastic interest rates, the integration of three key elements: The long-term memory of mixed sub-fractal Brownian motion, the mean-reversion property of the Vasicek model, and the flexibility of actuarial pricing methods, for systematic pricing research targeting European options remains an underexplored field. Building on this foundation, we derive the pricing formula for European options using actuarial principles, conduct empirical analysis and numerical simulations with real market data, and investigate the factors influencing option price fluctuations.

2. Preliminary knowledge

2.1. Mixed sub-fractional Brownian motion

Definition 1 [23]. *A mixed sub-fractional Brownian motion is defined as a stochastic process represented by a linear combination of a standard Brownian motion and a sub-fractional Brownian motion, where the first component denotes the standard Brownian motion and the second denotes the sub-fractional Brownian motion.*

Properties [24]:

$$E(M_t^H M_s^H) = \beta^2 \left\{ s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right] \right\} + \alpha^2 \min(s,t). \tag{2.1}$$

When $\alpha = 0, \beta = 1$, M_t^H it is a sub-fractional Brownian motion. When $\alpha = 1$, $\beta = 0$ and

$\alpha=0, \beta=1, H=\frac{1}{2}$, M_t^H is a standard Brownian motion.

$$D(M_t^H)=\alpha^2 D(B_t)+\beta^2 D(\xi_t^H)=\alpha^2 t+\beta^2(2-2^{2H-1})t^{2H}, \tag{2.2}$$

$$M_t^H \sim N(0,\alpha^2 t+\beta^2(2-2^{2H-1})t^{2H}). \tag{2.3}$$

2.2. Actuarial option pricing method

Definition 2 [25]**.** *The expected rate of return of a risky asset price over a time period is defined as:*

$$\exp\left\{\int_0^t \beta_u \mathrm{d}u\right\}=\frac{E(S_t)}{S_0}. \tag{2.4}$$

Alternatively, the expected rate of return of the asset at t is the ratio of the expected value of the risky asset price at maturity to its initial price.

Definition 3 [26]**.** *The actuarial value of a European option is defined as the expectation, under the actual probability distribution of the stock price, of the difference between the discounted stock price at maturity and the discounted strike price when exercised. The risk-free asset is discounted at the risk-free interest rate, while the risky asset is discounted at its expected rate of return. The exercise of a European option at maturity depends on the following conditions:*

$$\exp\left\{-\int_0^t \beta(u)\,\mathrm{d}u\right\}S_t>\exp\left\{-\int_0^t r(u)\mathrm{d}u\right\}K.$$

Condition for exercising a European put option:

$$\exp\left\{-\int_0^t \beta(u)du\right\}S_t<\exp\left\{-\int_0^t r(u)du\right\}K.$$

The actuarial prices of European call and put options are represented by $C(K,t)$ *and* $P(K,t)$*, respectively, with the underlying asset price* S_t*, strike price* K*, and maturity date* t*.*

Based on the above definitions, the following expressions are obtained:

$$C(K,t)=E\left[\left(\exp\left\{-\int_0^t \beta(u)\mathrm{d}u\right\}S_t-\exp\left\{-\int_0^t r(u)\mathrm{d}u\right\}K\right)I_{\left\{\exp\left\{-\int_0^t \beta(u)\mathrm{d}u\right\}S_t>\exp\left\{-\int_0^t r(u)\mathrm{d}u\right\}K\right\}}\right],$$

$$P(K,t)=E\left[\left(\exp\left\{-\int_0^t r(u)\mathrm{d}u\right\}K-\exp\left\{-\int_0^t \beta(u)\mathrm{d}u\right\}S_t\right)I_{\left\{\exp\left\{-\int_0^t r(u)\mathrm{d}u\right\}K>\exp\left\{-\int_0^t \beta(u)\mathrm{d}u\right\}S_t\right\}}\right].$$

Where r represents the risk-free interest rate, β *represents the expected return, and* I_{A_1} *represents the indicator function of event* A_1*.*

Lemma 1 [27]**.** *Two random variables satisfy* $Y_1 \sim N(0,1)$ *and* $Y_2 \sim N(0,1)$ *with* $Cov(Y_1,Y_2)=\rho$*. For any real number* a、b、c、d、k*, the following equation holds:*

$$E\left[\mathrm{e}^{cY_1+dY_2}I_{\{aY_1+bY_2\ge k\}}\right]=\mathrm{e}^{\frac{1}{2}(c^2+d^2+2\rho cd)}N\left(\frac{ac+bd+\rho(ad+bc)-k}{\sqrt{a^2+b^2+2\rho ab}}\right), \tag{2.5}$$

where N_x denotes the standard normal distribution function.

2.3. The Vasicek interest rate model

Definition 4. *The spot interest rate r satisfies the following stochastic differential equation under the risk-neutral measure Q:*

$$\mathrm{d}r_t = a(\theta - r_t)\mathrm{d}t + \sigma \mathrm{d}B_t, \tag{2.6}$$

*where a、θ、σ are positive constants, $\{B_t; t>0\}$ represents a standard Brownian motion, a denotes the speed of interest rate adjustment, θ signifies the long-term interest rate, and σ refers to the coefficient of the interest rate. This stochastic differential equation is referred to as the Vasicek mode*l [28].

If $\{B_t; t>0\}$ in Eq (2.6) is replaced by a sub-fractional Brownian motion $\{\xi_t^H; t>0\}$, i.e.,

$$\mathrm{d}r_t = a(\theta - r_t)\mathrm{d}t + \sigma \mathrm{d}\xi_t^H, \tag{2.7}$$

then this stochastic differential equation is referred to as the sub-fractional Vasicek model.

Definition 5. *If $\{B_t; t>0\}$ in Eq (2.6) is replaced by a mixed sub-fractional Brownian motion $\{M_t^H; t>0\}$, i.e.,*

$$\mathrm{d}r_t = a(\theta - r_t)\mathrm{d}t + \sigma \mathrm{d}M_t^H,\ M_t^H = \alpha B_t + \beta \xi_t^H, \tag{2.8}$$

where α and β represent the volatilities of the underlying asset price under the standard Brownian motion and the sub-fractional Brownian motion, respectively, with α and β being constants. This stochastic differential equation is termed the mixed sub-fractional Vasicek model.

3. Option pricing model under a mixed sub-fractional Brownian motion with Vasicek interest rate

Model assumptions:

(1) The financial market is frictionless, with no transaction or tax costs.

(2) All securities are perfectly divisible and can be freely short-sold.

(3) The price S of risky assets follows a mixed sub-fractional Brownian motion, satisfying the differential equation below [29]:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t(\rho \mathrm{d}B_t + \sqrt{1-\rho^2}\mathrm{d}\xi_t^H), 0 \le t \le T, \tag{3.1}$$

where S_t represents the price of the risky asset at time t, μ denotes the expected return rate, σ indicates the volatility of the asset price, and ρ signifies the coefficient of the linear combination of the Brownian motion B_t and sub-fractional Brownian motion ξ_t^H, and $|\rho| \le 1$. ξ_t^H and B_t are Gaussian processes and more general than B_t. In addition to the primary properties of B_t, parameter H in ξ_t^H captures fractal features of financial asset prices, such as long-memory and self-similarity.

(4) When $M_t^H = \rho B_t + \sqrt{1-\rho^2}\xi_t^H$, the stochastic interest rate r_t is assumed to satisfy the following stochastic differential equation:

$$\mathrm{d}r_t = (a - br_t)\mathrm{d}t + c\mathrm{d}\widetilde{M}_{1t}^H, \tag{3.2}$$

where a, b, and c are constants. The first one is the adjustment parameter, the second is the average rate of adjustment, and the third is the volatility of interest rate.

Furthermore,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}\widetilde{M}_{2t}^H. \tag{3.3}$$

It is assumed that $\{M_{1t}^H\}$ and $\{M_{2t}^H\}$ are mixed sub-fractional Brownian motions on the complete probability space (Ω, F, P) with correlation coefficient δ, for

$$|\delta| \le 1,\ \widetilde{M}_{1t}^H = M_{1t}^H,\ \widetilde{M}_{2t}^H = \delta M_{1t}^H + \sqrt{1-\delta^2} M_{2t}^H,$$

where

$$M_{1t}^H = \rho_1 B_{1t} + \sqrt{1-\rho_1^2}\,\xi_{1t}^H,\ M_{2t}^H = \rho_2 B_{2t} + \sqrt{1-\rho_2^2}\,\xi_{2t}^H,$$

and ρ_1 denote the coefficient of the linear combination of the standard Brownian motion B_{1t} and the sub-fractional Brownian motion ξ_{1t}^H, ρ_2 represents the coefficient of the linear combination of the standard Brownian motion B_{2t} and the sub-fractional Brownian motion ξ_{2t}^H, with $|\rho_1| \le 1$, $|\rho_2| \le 1$.

Therefore,

$$\mathrm{d}r_t = (a - br_t)\mathrm{d}t + c\mathrm{d}M_{1t}^H, \tag{3.4}$$

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma\delta \mathrm{d}M_{1t}^H + \sigma\sqrt{1-\delta^2}\mathrm{d}M_{2t}^H). \tag{3.5}$$

Theorem 1. Under the mixed sub-fractional Vasicek stochastic interest rate model, the solution to the stochastic differential Eq (3.5) for the price of the risky asset is given by:

$$S_t = S_0 \exp\Big\{\mu t - \frac{1}{2}\sigma^2\big[\delta^2\rho_1^2 + (1-\delta^2)\rho_2^2\big]t$$

$$-\frac{1}{2}\sigma^2 \cdot [\delta^2(1-\rho_1^2) + (1-\delta^2)(1-\rho_2^2)] \cdot (2 - 2^{2H-1})t^{2H} + \sigma\delta M_{1t}^H + \sigma\sqrt{1-\delta^2} M_{2t}^H\Big\}.$$

Proof. Based on the definition $M_{1t}^H = \rho_1 B_{1t} + \sqrt{1-\rho_1^2}\,\xi_{1t}^H,\ M_{2t}^H = \rho_2 B_{2t} + \sqrt{1-\rho_2^2}\,\xi_{2t}^H$, we can get

$$(\mathrm{d}M_{1t}^H)^2 = \big[\rho_1^2 + (1-\rho_1^2)(2-2^{2H-1})2Ht^{2H-1}\big]\mathrm{d}t,\ (\mathrm{d}M_{2t}^H)^2 = \big[\rho_2^2 + (1-\rho_2^2)(2-2^{2H-1})2Ht^{2H-1}\big]\mathrm{d}t, \tag{3.6}$$

$$(\mathrm{d}S_t)^2 = S_t^2[\mu^2(\mathrm{d}t)^2 + \sigma^2\delta^2(\mathrm{d}M_{1t}^H)^2 + \sigma^2(1-\delta^2)(\mathrm{d}M_{2t}^H)^2]$$

$$= S_t^2\sigma^2\{[\delta^2\rho_1^2 + (1-\delta^2)\rho_2^2] + [\delta^2(1-\rho_1^2) + (1-\delta^2)(1-\rho_2^2)](2-2^{2H-1})2Ht^{2H-1}\}\mathrm{d}t. \tag{3.7}$$

Let

$$A = \delta^2\rho_1^2 + (1-\delta^2)\rho_2^2,\ B = \delta^2(1-\rho_1^2) + (1-\delta^2)(1-\rho_2^2)](2-2^{2H-1})2Ht^{2H-1}. \tag{3.8}$$

Then, $(\mathrm{d}S_t)^2 = S_t^2\sigma^2(A+B)\mathrm{d}t$, and let $f(x) = \ln x$, then $f(S_t) = \ln S_t$.

According to Itô′s lemma

$$\mathrm{d}(\ln S_t) = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial S_t}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(\mathrm{d}S_t)^2 + o(\mathrm{d}t), \tag{3.9}$$

substituting Eqs (3.5) and (3.7) into Eq (3.9) yields

$$\mathrm{d}(\ln S_t) = \left\{\mu - \frac{1}{2}\sigma^2 A - \frac{1}{2}\sigma^2 \cdot B \cdot 2H(2-2^{2H-1})t^{2H-1}\right\}\mathrm{d}t + \sigma\left(\delta \mathrm{d}M_{1t}^H + \sqrt{1-\delta^2}\mathrm{d}M_{2t}^H\right).c.$$

Integrate both sides of the equation

$$\int_0^t \mathrm{d}(\ln S_u) = \int_0^t \{\mu - \frac{1}{2}\sigma^2 A - \frac{1}{2}\sigma^2 \cdot B \cdot 2H(2-2^{2H-1})u^{2H-1}\}\mathrm{d}u + \sigma\left(\delta\int_0^t \mathrm{d}M_{1u}^H + \sqrt{1-\delta^2}\int_0^t \mathrm{d}M_{2u}^H\right),$$

then

$$S_t = S_0 \exp\{\mu t - \frac{1}{2}\sigma^2 \cdot A \cdot t - \frac{1}{2}\sigma^2 \cdot B \cdot (2-2^{2H-1})t^{2H} + \sigma\delta M_{1t}^H + \sigma\sqrt{1-\delta^2}M_{2t}^H\}. \tag{3.10}$$

Substituting (3.8) into (3.10) gives

$$S_t = S_0 \exp\{\mu t - \frac{1}{2}\sigma^2[\delta^2\rho_1^2 + (1-\delta^2)\rho_2^2]t$$

$$-\frac{1}{2}\sigma^2 \cdot [\delta^2(1-\rho_1^2) + (1-\delta^2)(1-\rho_2^2)] \cdot (2-2^{2H-1})t^{2H} + \sigma\delta M_{1t}^H + \sigma\sqrt{1-\delta^2}M_{2t}^H\}. \tag{3.11}$$

Remark 1.
(1) When $\rho_1 = 0$ and $\rho_2 = 0$, the formula reduces to the asset price under the sub-fractional Vasicek stochastic interest rate.
(2) When $\rho_1 = 1$ and $\rho_2 = 1$, it corresponds to the asset price under the Vasicek stochastic interest rate in the classical Black–Scholes model.
(3) When $\rho_1 = 0$, $\rho_2 = 0$ and $H = \frac{1}{2}$, it represents the asset price under the Vasicek stochastic interest rate from the fractional Brownian motion perspective.

Theorem 2. For the risk asset price process $\{S_t, t \geq 0\}$, its expected return is given by the following formula: $\beta_u = \mu,\ u \in [0,t]$.

Proof. According to the definition of M_{1t}^H, M_{2t}^H, it is known that it satisfies the normal distribution, then $E(M_{1t}^H) = 0$, Therefore,

$$D(M_{1t}^H) = \rho_1^2 D(B_{1t}) + (\sqrt{1-\rho_1^2})^2 D(\xi_{1t}^H) = \rho_1^2 t + (1-\rho_1^2)(2-2^{2H-1})t^{2H},$$

$$M_{1t}^H \sim N\left(0,\ \rho_1^2 t + (1-\rho_1^2)(2-2^{2H-1})t^{2H}\right).$$

Similarly, for $M_{2t}^H \sim N\left(0,\ \rho_2^2 t + (1-\rho_2^2)(2-2^{2H-1})t^{2H}\right)$.

According to Theorem 1 and the properties of expectation, it follows that

$$E[S_t] = S_0 E[\exp\{\mu t - \frac{1}{2}\sigma^2[\delta^2\rho_1^2 + (1-\delta^2)\rho_2^2]t$$

$$-\frac{1}{2}\sigma^2 \cdot [\delta^2(1-\rho_1^2) + (1-\delta^2)(1-\rho_2^2)] \cdot (2-2^{2H-1})t^{2H} + \sigma\delta M_{1t}^H + \sigma\sqrt{1-\delta^2}M_{2t}^H\}],$$

$$= S_0 \exp\{\mu t - \frac{1}{2}\sigma^2[\delta^2\rho_1^2 + (1-\delta^2)\rho_2^2]t - \frac{1}{2}\sigma^2 \cdot [\delta^2(1-\rho_1^2) \tag{3.12}$$

$$+(1-\delta^2)(1-\rho_2^2)] \cdot (2-2^{2H-1})t^{2H} + \frac{1}{2}\sigma^2\delta^2[\rho_1^2 T + (1-\rho_1^2) \cdot (2-2^{2H-1})t^{2H}]$$

$$+\frac{1}{2}\sigma^2(1-\delta^2)[\rho_2^2 t + (1-\rho_2^2) \cdot (2-2^{2H-1})t^{2H}]\}$$

$$= S_0 \exp\{\mu t\}.$$

Thus,

$$\frac{E[S_t]}{S_0} = \exp\{\mu t\}, \exp\left\{\int_0^t \beta_u \mathrm{d}u\right\} = \exp\{\beta_u t\}. \tag{3.13}$$

From Definition 2,

$$\exp\{\beta_u t\} = \exp\{\mu t\}.$$

It follows that,

$$\beta_u = \mu, \quad u \in [0, t]. \tag{3.14}$$

Theorem 3. If the interest rate r_t satisfies the stochastic differential equation:

$$\mathrm{d}r_t = (a - br_t)\mathrm{d}t + c\mathrm{d}M_{1t}^H = (a - br_t)\mathrm{d}t + c\rho_1 \mathrm{d}B_{1t} + c\sqrt{1-\rho_1^2}\mathrm{d}\xi_{1t}^H, \tag{3.15}$$

then the solution to the equation is

$$r_t = r_0 \mathrm{e}^{-bt} + \frac{a}{b}(1-\mathrm{e}^{-bt}) + c\rho_1 \int_0^t \mathrm{e}^{b(u-t)}\, \mathrm{c}B_{1u} + c\sqrt{1-\rho_1^2}\int_0^t \mathrm{e}^{b(u-t)}\, \mathrm{d}\xi_{1u}^H. \tag{3.16}$$

Where, B_{1u} and ξ_{1u}^H are standard Brownian motion and sub-fractional Brownian motion, respectively.

Proof. Given that

$$\mathrm{d}r_t = a\mathrm{d}t - br_t\mathrm{d}t + c\rho_1\mathrm{d}B_{1t} + c\sqrt{1-\rho_1^2}\mathrm{d}\xi_{1t}^H,$$

$$\mathrm{d}r_t + br_t\mathrm{d}t = a\mathrm{d}t + c\rho_1\mathrm{d}B_{1t} + c\sqrt{1-\rho_1^2}\mathrm{d}\xi_{1t}^H. \tag{3.17}$$

Multiplying both sides of Eq (3.17) by e^{bt}, then

$$\mathrm{e}^{\mathrm{bt}}\mathrm{d}r_t + b\mathrm{e}^{\mathrm{bt}}r_t\mathrm{d}t = a\mathrm{e}^{\mathrm{bt}}\mathrm{d}t + c\rho_1\mathrm{e}^{\mathrm{bt}}\mathrm{d}B_{1t} + c\sqrt{1-\rho_1^2}\mathrm{e}^{\mathrm{bt}}\mathrm{d}\xi_{1t}^H. \tag{3.18}$$

Therefore,

$$\mathrm{d}(r_t\mathrm{e}^{\mathrm{bt}}) = a\mathrm{e}^{\mathrm{bt}}\mathrm{d}t + c\rho_1\mathrm{e}^{\mathrm{bt}}\mathrm{d}B_{1t} + c\sqrt{1-\rho_1^2}\mathrm{e}^{\mathrm{bt}}\mathrm{d}\xi_{1t}^H.$$

Incorporating both sides gives,

$$\int_0^t (r_u \mathrm{e}^{\mathrm{bu}})\mathrm{d}u = a\int_0^t \mathrm{e}^{\mathrm{bu}}\mathrm{d}u + c\rho_1\int_0^t \mathrm{e}^{\mathrm{bu}}\mathrm{d}B_{1u} + c\sqrt{1-\rho_1^2}\int_0^t \mathrm{e}^{\mathrm{bu}}\mathrm{d}\xi_{1u}^H .$$

Accordingly,

$$r_t = r_0\,\mathrm{e}^{-bt} + \frac{a}{b}(1-\mathrm{e}^{-bt}) + c\rho_1\int_0^t \mathrm{e}^{b(u-t)}\,\mathrm{d}B_{1u} + c\sqrt{1-\rho_1^2}\int_0^t \mathrm{e}^{b(u-t)}\,\mathrm{d}\xi_{1u}^H . \tag{3.19}$$

4. Actuarial pricing formulas for European options under the mixed sub-fractional Brownian motion with Vasicek interest rate

Theorem 4. Under the mixed sub-fractional Brownian motion with the Vasicek interest rate, the actuarial pricing formulas for European call and put options with maturity time T are given, respectively, by

$$\begin{aligned} C(K,T) &= S_0 N\left(\frac{D_{11}+\rho'\sqrt{D_{11}D_{12}}-k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}}\right) \\ &\quad - K\exp\left\{\left(\frac{r_0}{b}-\frac{a}{b^2}\right)(\mathrm{e}^{-bT}-1)-\frac{a}{b}T\right\}\mathrm{e}^{\frac{1}{2}D_{12}}\,N\left(-\frac{D_{12}+\rho'\sqrt{D_{11}D_{12}}-k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}}\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} P(K,T) &= K\exp\left\{\left(\frac{r_0}{b}-\frac{a}{b^2}\right)(\mathrm{e}^{-bT}-1)-\frac{a}{b}T\right\}\mathrm{e}^{\frac{1}{2}D_{12}}\,N\left(-\frac{D_{12}+\rho'\sqrt{D_{11}D_{12}}-k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}}\right) \\ &\quad - S_0 N\left(\frac{D_{11}+\rho'\sqrt{D_{11}D_{12}}-k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}}\right). \end{aligned} \tag{4.2}$$

Among them

$$D_{11} = \sigma^2\delta^2\left[\rho_1^2 T + (1-\rho_1^2)(2-2^{2H-1})T^{2H}\right] + \sigma^2(1-\delta^2)\left[\rho_2^2 T + (1-\rho_2^2)(2-2^{2H-1})T^{2H}\right],$$

$$\begin{aligned} D_{12} &= Var\{c\rho_1\int_0^T\int_0^t \mathrm{e}^{b(u-t)}\,\mathrm{d}B_{1u}\mathrm{d}t + c\sqrt{1-\rho_1^2}\int_0^T\int_0^t \mathrm{e}^{b(u-t)}\,\mathrm{d}\xi_{1u}^H\mathrm{d}t\}, \\ &= \left(\frac{c\rho_1}{b}\right)^2\left(T-\frac{3}{2b}+\frac{2}{b}e^{-bT}-\frac{1}{2b}e^{-2bT}\right) + \frac{c^2(1-\rho^2)}{b^2}H(2H-1)\cdot(V_1-V_2), \end{aligned}$$

where

$$V_1 = \frac{2}{2H-1}A - \frac{2}{b^{2H-1}}B,\quad V_2 = \frac{2}{2H-1}(C-A) - \frac{1}{2H-1}(D-E),$$

$$A = \frac{T^{2H}}{2H}[1 - e^{-bT}\,{}_1F_1(2H;2H+1;bT)],$$

$$B = \frac{b^{2H-1}T^{2H}}{2H(2H-1)}[e^{-bT}\phi_2(2H,2H-1,2H;bT,-bT) - e^{-2bT}\phi_2(2H,2H-1,2H;2bT,-bT)],$$

$$C = \frac{(2T)^{2H}-T^{2H}}{2H} - e^{-2bT}\left[\frac{(2T)^{2H}}{2H}{}_1F_1(2H;2H+1;2bT) - \frac{T^{2H}}{2H}{}_1F_1(2H;2H+1;bT)\right],$$

$$D=\left[\frac{(2T)^{2H}}{2H}\phi_2(2H,2H-1,2H;-b(2T),b(2T))-\frac{T^{2H}}{2H}\phi_2(2H,2H-1,2H;-bT,bT\right]$$
$$-e^{-2bT}\left[\frac{(2T)^{2H}}{2H}{}_2F_2(2H-1,2H;2H,2H+1;b(2T))-\frac{T^{2H}}{2H}{}_2F_2(2H-1,2H;2H,2H+1;bT\right],$$

$$E=e^{-bT}\frac{T^{2H}}{2H}\phi_2(2H,2H-1,2H;-bT,bT)-e^{-2bT}\frac{T^{2H}}{2H}{}_2F_2(2H-1,2H;2H,2H+1;bT).$$

Here, ϕ_2 is the bivariate confluent hypergeometric function,

$\phi_2(b,b',c;w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b)_m(b')_n}{(c)_{m+n}m!n!}w^m z^n$, ${}_1F_1$ and ${}_2F_2$ are defined as generalized hypergeometric functions.

$$\rho'=Cov(X_1,X_2),$$

$$k=\ln\frac{K}{S_0}+\frac{1}{2}\sigma^2\left[\delta^2\rho_1^2+(1-\delta^2)\rho_2^2\right]T+\frac{1}{2}\sigma^2\left[\delta^2(1-\rho_1^2)+(1-\delta^2)(1-\rho_2^2)\right](2-2^{2H-1})T^{2H}$$
$$+(\frac{r_0}{b}-\frac{a}{b^2})(\mathrm{e}^{-bT}-1)-\frac{a}{b}T.$$

Proof. According to the pricing principle of European option insurance, the following conditions must be satisfied:

$$A_1\text{: }\exp\{-\int_0^T\beta(u)\mathrm{d}u\}S_T>\exp\{-\int_0^T r(t)\mathrm{d}t\}K. \tag{4.3}$$

From Theorems 1–3, it follows that

$$\mathrm{e}^{-\mu T}S_0\exp\{\mu T-\frac{1}{2}\sigma^2\left[\delta^2\rho_1^2+(1-\delta^2)\rho_2^2\right]T-\frac{1}{2}\sigma^2\left[\delta^2(1-\rho_1^2)+(1-\delta^2)(1-\rho_2^2)\right](2-2^{2H-1})T^{2H}$$
$$+\sigma\delta M_{1T}^H+\sigma\sqrt{1-\delta^2}M_{2T}^H\}$$
$$>K\exp\{(\frac{r_0}{b}-\frac{a}{b^2})(\mathrm{e}^{-bT}-1)-\frac{a}{b}T-c\rho_1\int_0^T\int_0^t\mathrm{e}^{b(u-t)}\,\mathrm{d}B_{1u}\mathrm{d}t-c\sqrt{1-\rho_1^2}\int_0^T\int_0^t\mathrm{e}^{b(u-t)}\,\mathrm{d}\xi_{1u}^H\mathrm{d}t\}.$$

Taking the logarithm of both sides gives

$$\begin{aligned}&\ln\frac{S_0}{K}-\frac{1}{2}\sigma^2\left[\delta^2\rho_1^2+(1-\delta^2)\rho_2^2\right]T-\frac{1}{2}\sigma^2\left[\delta^2(1-\rho_1^2)+(1-\delta^2)(1-\rho_2^2)\right](2-2^{2H-1})T^{2H}\\&+\sigma\delta M_{1T}^H+\sigma\sqrt{1-\delta^2}M_{2T}^H\\&>(\frac{r_0}{b}-\frac{a}{b^2})(\mathrm{e}^{-bT}-1)-\frac{a}{b}T-c\rho_1\int_0^T\int_0^t\mathrm{e}^{b(u-t)}\,\mathrm{d}B_{1u}\mathrm{d}t-c\sqrt{1-\rho_1^2}\int_0^T\int_0^t\mathrm{e}^{b(u-t)}\,\mathrm{d}\xi_{1u}^H\mathrm{d}t,\end{aligned} \tag{4.4}$$

and is deformed into

$$\begin{aligned}&\sigma\delta M_{1T}^{H}+\sigma\sqrt{1-\delta^{2}}M_{2T}^{H}+c\rho_{1}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\,\mathrm{d}B_{1u}\mathrm{d}t+c\sqrt{1-\rho_{1}^{2}}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\,\mathrm{d}\xi_{1u}^{H}\mathrm{d}t\\&>\ln\frac{K}{S_{0}}+\frac{1}{2}\sigma^{2}\left[\delta^{2}\rho_{1}^{2}+(1-\delta^{2})\rho_{2}^{2}\right]T+\frac{1}{2}\sigma^{2}\left[\delta^{2}(1-\rho_{1}^{2})+(1-\delta^{2})(1-\rho_{2}^{2})\right]\\&\cdot(2-2^{2H-1})T^{2H}+(\frac{r_{0}}{b}-\frac{a}{b^{2}})(\mathrm{e}^{-bT}-1)-\frac{a}{b}T,\end{aligned}\tag{4.5}$$

it is assumed that

$$\begin{aligned}X&=\sigma\delta M_{1T}^{H}+\sigma\sqrt{1-\delta^{2}}M_{2T}^{H}+c\rho_{1}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\,\mathrm{d}B_{1u}\mathrm{d}t+c\sqrt{1-\rho_{1}^{2}}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\,\mathrm{d}\xi_{1u}^{H}\mathrm{d}t\\&=\sqrt{D_{11}}X_{1}+\sqrt{D_{12}}X_{2},\end{aligned}\tag{4.6}$$

where

$$D_{11}=\sigma^{2}\delta^{2}\left[\rho_{1}^{2}T+(1-\rho_{1}^{2})(2-2^{2H-1})T^{2H}\right]+\sigma^{2}(1-\delta^{2})\left[\rho_{2}^{2}T+(1-\rho_{2}^{2})(2-2^{2H-1})T^{2H}\right],$$

$$D_{12}=Var\{c\rho_{1}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\,\mathrm{d}B_{1u}\mathrm{d}t+c\sqrt{1-\rho_{1}^{2}}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\,\mathrm{d}\xi_{1u}^{H}\mathrm{d}t\},$$

$$\begin{aligned}k&=\ln\frac{K}{S_{0}}+\frac{1}{2}\sigma^{2}\left[\delta^{2}\rho_{1}^{2}+(1-\delta^{2})\rho_{2}^{2}\right]T+\frac{1}{2}\sigma^{2}\left[\delta^{2}(1-\rho_{1}^{2})+(1-\delta^{2})(1-\rho_{2}^{2})\right](2-2^{2H-1})T^{2H}\\&\quad+(\frac{r_{0}}{b}-\frac{a}{b^{2}})(\mathrm{e}^{-bT}-1)-\frac{a}{b}T,\end{aligned}$$

$$X_{1}\sim N(0,1),\quad X_{2}\sim N(0,1),\quad \rho'=Cov(X_{1},X_{2}).$$

The following is the solving process of D_{12}:

$$D_{12}=Var\{c\rho_{1}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\,\mathrm{d}B_{1u}\mathrm{d}t+c\sqrt{1-\rho_{1}^{2}}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\,\mathrm{d}\xi_{1u}^{H}\mathrm{d}t\}.$$

Let $X=c\rho_{1}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\mathrm{d}B_{1u}\mathrm{d}t, Y=c\sqrt{1-\rho_{1}^{2}}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\mathrm{d}\xi_{1u}^{H}\mathrm{d}t,$

and $Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)$.

Since B_{1u} and ξ_{1u}^{H} are independent, then $Cov(X,Y)=0$, thus $D_{12}=Var(X)+Var(Y)$.

Calculate first X:

$$X=c\rho_{1}\int_{0}^{T}\int_{0}^{t}\mathrm{e}^{b(u-t)}\mathrm{d}B_{1u}\mathrm{d}t=c\rho_{1}\int_{0}^{T}\left[\int_{u}^{T}\mathrm{e}^{b(u-t)}\mathrm{d}t\right]\mathrm{d}B_{1u}=c\rho_{1}\int_{0}^{T}\frac{1}{b}(1-\mathrm{e}^{b(u-T)})\mathrm{d}B_{1u},$$

this is an *Itô* integral with the variance

$$Var(X)=\left(\frac{c\rho_{1}}{b}\right)^{2}\int_{0}^{T}\left(1-\mathrm{e}^{b(u-T)}\right)^{2}\mathrm{d}u=\left(\frac{c\rho_{1}}{b}\right)^{2}(T-\frac{3}{2b}+\frac{2}{b}\mathrm{e}^{-bT}-\frac{1}{2b}\mathrm{e}^{-2bT}).$$

Recalculation Y:

$$Y = c\sqrt{1-\rho_1^2}\int_0^T\int_0^t e^{b(u-t)} d\xi_{1u}^H dt = c\sqrt{1-\rho_1^2}\int_0^T\left[\int_u^T e^{b(u-t)} dt\right] d\xi_{1u}^H$$
$$= \frac{c\sqrt{1-\rho_1^2}}{b}\int_0^T \left[1-e^{-b(T-u)}\right] d\xi_{1u}^H .$$

Let $K = \dfrac{c\sqrt{1-\rho_1^2}}{b}, g(u) = 1 - e^{-b(T-u)}$, then

$$Y = K\int_0^T g(u) d\xi_{1u}^H ,\quad D_{12} = Var\{c\rho_1\int_0^T\int_0^t e^{b(u-t)}\, dB_{1u} dt + c\sqrt{1-\rho_1^2}\int_0^T\int_0^t e^{b(u-t)}\, d\xi_{1u}^H dt\}.$$

For $H > \dfrac{1}{2}$, the stochastic integral variance is

$$Var(Y) = K^2\int_0^T\int_0^T g(u)g(v)\frac{\partial^2 R(u,v)}{\partial u\partial v} du dv$$
$$= K^2 H(2H-1)\int_0^T\int_0^T g(u)g(v)[\left|u-v\right|^{2H-2} - (u+v)^{2H-2}] du dv$$
$$= \frac{c^2(1-\rho_1^2)}{b^2} H(2H-1)\cdot V,$$

which $V = \int_0^T\int_0^T g(u)g(v)[\left|u-v\right|^{2H-2} - (u+v)^{2H-2}] du dv.$

Reorder $V = V_1 - V_2$,

among $V_1 = \int_0^T\int_0^T g(u)g(v)\left|u-v\right|^{2H-2} du dv,\quad V_2 = \int_0^T\int_0^T g(u)g(v)(u+v)^{2H-2} du dv.$

(1) Calculate V_1

symmetry by use

$$V_1 = 2\int_0^T g(u)[\int_0^u g(v)(u-v)^{2H-2} dv] du = 2\int_0^T g(u)[\int_0^u (1-e^{-b(T-v)})(u-v)^{2H-2} dv] du.$$

Let

$$J(u) = \int_0^u (u-v)^{2H-2} dv - \int_0^u e^{-b(T-v)}(u-v)^{2H-2} dv.$$

The first item:

$$\int_0^u (u-v)^{2H-2} dv = \frac{u^{2H-1}}{2H-1}.$$

The second item: Let $w = u - v$.

$$\int_0^u e^{-b(T-v)}(u-v)^{2H-2} dv = e^{-b(T-u)}\int_0^u e^{-bw} w^{2H-2} dw = \frac{e^{-b(T-u)}}{b^{2H-1}}\gamma(2H-1, bu),$$

where $\gamma(a,z)$ is the incomplete gamma function.

Therefore,

$$J(u) = \frac{u^{2H-1}}{2H-1} - \frac{e^{-b(T-u)}}{b^{2H-1}}\gamma(2H-1, bu).$$

Hence,

$$V_1 = 2\int_0^T g(u)\left[\frac{u^{2H-1}}{2H-1} - \frac{\mathrm{e}^{-b(T-u)}}{b^{2H-1}}\gamma(2H-1,bu)\right]\mathrm{d}u = \frac{2}{2H-1}A - \frac{2}{b^{2H-1}}B,$$

where

$$A = \int_0^T (1-\mathrm{e}^{-b(T-u)})u^{2H-1}\mathrm{d}u = \frac{T^{2H}}{2H}[1-\mathrm{e}^{-bT}{}_1F_1(2H;2H+1;bT)],$$

$$B = \int_0^T (1-\mathrm{e}^{-b(T-u)})\mathrm{e}^{-b(T-u)}\gamma(2H-1,bu)\mathrm{d}u.$$

Let

$$t = bu, \mathrm{d}u = \mathrm{d}(\frac{t}{b}).$$

$$\begin{aligned} B &= \frac{b^{2H-1}}{2H-1}[\mathrm{e}^{-bT}\int_0^T \mathrm{e}^{bu}u^{2H-1}{}_1F_1(2H-1;2H;-bu)\mathrm{d}u - \mathrm{e}^{-2bT}\int_0^T \mathrm{e}^{2bu}u^{2H-1}{}_1F_1(2H-1;2H;-bu)\mathrm{d}u] \\ &= \frac{b^{2H-1}T^{2H}}{2H(2H-1)}[\mathrm{e}^{-bT}\phi_2(2H,2H-1,2H;bT,-bT) - \mathrm{e}^{-2bT}\phi_2(2H,2H-1,2H;2bT,-bT)], \end{aligned}$$

where ϕ_2 is the bivariate confluent hypergeometric function.

$$\phi_2(b,b',c;w,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b)_m(b')_n}{(c)_{m+n}m!n!}w^m z^n.$$

(2) Calculate V_2

$$V_2 = \int_0^T g(u)\left[\int_0^T g(v)(u+v)^{2H-2}\mathrm{d}v\right]\mathrm{d}u,$$

$$L(u) = \int_0^T g(v)(u+v)^{2H-2}\mathrm{d}v = \int_0^T (u+v)^{2H-2}\mathrm{d}v - \int_0^T e^{-b(T-v)}(u+v)^{2H-2}\mathrm{d}v.$$

Let $w = u+v$, then

$$\begin{aligned} L(u) &= \frac{(u+T)^{2H-1} - u^{2H-1}}{2H-1} - \mathrm{e}^{-b(T+u)}\int_u^{u+T} w^{2H-2}\mathrm{e}^{bw}\mathrm{d}w \\ &= \frac{(u+T)^{2H-1} - u^{2H-1}}{2H-1} - \frac{\mathrm{e}^{-b(T+u)}}{2H-1}[(u+T)^{2H-1}{}_1F_1(2H-1;2H;b(u+T)) - u^{2H-1}{}_1F_1(2H-1;2H;bu)], \end{aligned}$$

$$L(u) = L_1(u) + L_2(u),$$

$$L_1(u) = \frac{(u+T)^{2H-1} - u^{2H-1}}{2H-1},$$

$$L_2(u) = \frac{\mathrm{e}^{-b(T+u)}}{2H-1}[(u+T)^{2H-1}{}_1F_1(2H-1;2H;b(u+T)) - u^{2H-1}{}_1F_1(2H-1;2H;bu)].$$

Therefore,

$$V_2=\frac{1}{2H-1}[\int_0^T g(u)(u+T)^{2H-1}\mathrm{d}u-\int_0^T g(u)u^{2H-1}\mathrm{d}u]$$

$$-\frac{1}{2H-1}\int_0^T g(u)\mathrm{e}^{-b(T+u)}[(u+T)^{2H-1}{}_1F_1(2H-1;2H;b(u+T))-u^{2H-1}{}_1F_1(2H-1;2H;bu)]\mathrm{d}u,$$

where $\int_0^T g(u)u^{2H-1}\mathrm{d}u=A$ has been calculated.

Step 1: Calculate $C=\int_0^T g(u)(u+T)^{2H-1}\mathrm{d}u$

$$C=\int_0^T g(u)(u+T)^{2H-1}\mathrm{d}u=\int_0^T(1-\mathrm{e}^{-b(T-u)})(u+T)^{2H-1}\mathrm{d}u$$

$$=(u+T)^{2H-1}\mathrm{d}u-\mathrm{e}^{-bT}\int_0^T \mathrm{e}^{bu}(u+T)^{2H-1}\mathrm{d}u$$

$$=\frac{(2T)^{2H}-T^{2H}}{2H}-\mathrm{e}^{-2bT}[\frac{(2T)^{2H}}{2H}{}_1F_1(2H;2H+1;2bT)-\frac{T^{2H}}{2H}{}_1F_1(2H;2H+1;bT)].$$

Step2: Calculate $D=\int_0^T g(u)\mathrm{e}^{-b(T+u)}(u+T)^{2H-1}{}_1F_1(2H-1;2H;b(u+T))\mathrm{d}u$

$$D=\int_0^T g(u)\mathrm{e}^{-b(T+u)}(u+T)^{2H-1}{}_1F_1(2H-1;2H;b(u+T))\mathrm{d}u$$

$$\underline{\underline{w=u+T}}\int_T^{2T}[1-\mathrm{e}^{-b(2T-w)}]\mathrm{e}^{-bw}w^{2H-1}{}_1F_1(2H-1;2H;bw)\mathrm{d}w$$

$$=\int_T^{2T}\mathrm{e}^{-bw}w^{2H-1}{}_1F_1(2H-1;2H;bw)\mathrm{d}w-\mathrm{e}^{-2bT}\int_T^{2T}w^{2H-1}{}_1F_1(2H-1;2H;bw)\mathrm{d}w.$$

Step 3: Calculate $E=\int_0^T g(u)\mathrm{e}^{-b(T+u)}u^{2H-1}{}_1F_1(2H-1;2H;bu)\mathrm{d}u$

$$E=\int_0^T g(u)\mathrm{e}^{-b(T+u)}u^{2H-1}{}_1F_1(2H-1;2H;bu)\mathrm{d}u$$

$$=\mathrm{e}^{-bT}\int_0^T \mathrm{e}^{-bu}u^{2H-1}{}_1F_1(2H-1;2H;bu)\mathrm{d}u-\mathrm{e}^{-2bT}\int_0^T u^{2H-1}{}_1F_1(2H-1;2H;bu)\mathrm{d}u.$$

Therefore, $Y=\frac{c\sqrt{1-\rho_1^2}}{b}\int_0^T[1-\mathrm{e}^{-b(T-u)}]\mathrm{d}\xi_{1u}^H$,

$$Var(Y)=\frac{c^2(1-\rho_1^2)}{b^2}H(2H-1)\cdot(V_1-V_2).$$

To sum up,

$$D_{12}=Var(X)+Var(Y)$$

$$=(\frac{c\rho_1}{b})^2(T-\frac{3}{2b}+\frac{2}{b}\mathrm{e}^{-bT}-\frac{1}{2b}\mathrm{e}^{-2bT})+\frac{c^2(1-\rho^2)}{b^2}H(2H-1)\cdot(V_1-V_2),$$

where $V_1=\frac{2}{2H-1}A-\frac{2}{b^{2H-1}}B$, $V_2=\frac{2}{2H-1}(C-A)-\frac{1}{2H-1}(D-E)$,

$A=\frac{T^{2H}}{2H}[1-\mathrm{e}^{-bT}{}_1F_1(2H;2H+1;bT)]$,

$$B=\frac{b^{2H-1}T^{2H}}{2H(2H-1)}[\mathrm{e}^{-bT}\phi_2(2H,2H-1,2H;bT,-bT)-\mathrm{e}^{-2bT}\phi_2(2H,2H-1,2H;2bT,-bT)],$$

$$C=\frac{(2T)^{2H}-T^{2H}}{2H}-\mathrm{e}^{-2bT}[\frac{(2T)^{2H}}{2H}{}_1F_1(2H;2H+1;2bT)-\frac{T^{2H}}{2H}{}_1F_1(2H;2H+1;bT)],$$

$$D=\left[\frac{(2T)^{2H}}{2H}\phi_2(2H,2H-1,2H;-b(2T),b(2T))-\frac{T^{2H}}{2H}\phi_2(2H,2H-1,2H;-bT,bT\right]$$
$$-\mathrm{e}^{-2bT}\left[\frac{(2T)^{2H}}{2H}{}_2F_2(2H-1,2H;2H,2H+1;b(2T))-\frac{T^{2H}}{2H}{}_2F_2(2H-1,2H;2H,2H+1;bT\right],$$

$$E=\mathrm{e}^{-bT}\frac{T^{2H}}{2H}\phi_2(2H,2H-1,2H;-bT,bT)-\mathrm{e}^{-2bT}\frac{T^{2H}}{2H}{}_2F_2(2H-1,2H;2H,2H+1;bT).$$

Here, ϕ_2 is the bivariate confluent hypergeometric function，

$\phi_2(b,b',c;w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b)_m(b')_n}{(c)_{m+n}m!n!}w^m z^n$, ${}_1F_1$ and ${}_2F_2$ are defined as generalized hypergeometric functions.

Therefore, Eq (4.5) can be rewritten as:

$$\sqrt{D_{11}}X_1+\sqrt{D_{12}}X_2>k.$$

According to the principles of actuarial science, the price of a European call option at maturity T is expressed as:

$$C(K,T)=E\left[\exp\{-\int_0^T\beta(u)\mathrm{d}u\}S_T I_{A_1}\right]-E\left[\exp\{-\int_0^T r(t)\mathrm{d}t\}KI_{A_1}\right]=V_1-V_2, \tag{4.7}$$

where

$$\begin{aligned}V_1&=E\left[\exp\{-\int_0^T\beta_u\mathrm{d}u\}S_T I_{A_1}\right]\\&=E[S_0\exp\{-\frac{1}{2}\sigma^2\left[\delta^2\rho_1^2+(1-\delta^2)\rho_2^2\right]T-\frac{1}{2}\sigma^2[\delta^2(1-\rho_1^2)\\&\quad+(1-\delta^2)(1-\rho_2^2)(2-2^{2H-1})T^{2H}+\sigma\delta M_{1T}^H+\sigma\sqrt{1-\delta^2}M_{2T}^H\}E[I_{A_1}]\\&=S_0\exp\{-\frac{1}{2}\sigma^2\left[\delta^2\rho_1^2+(1-\delta^2)\rho_2^2\right]T-\frac{1}{2}\sigma^2[\delta^2(1-\rho_1^2)\\&\quad+(1-\delta^2)(1-\rho_2^2)(2-2^{2H-1})T^{2H}\}E[\mathrm{e}^{\sqrt{D_{11}}X_1}I_{\{\sqrt{D_{11}}X_1+\sqrt{D_{12}}X_2>k\}}]\\&=S_0N(\frac{D_{11}+\rho'\sqrt{D_{11}D_{12}}-k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}}),\end{aligned} \tag{4.8}$$

$$\begin{aligned}V_2&=E\left[\exp\{-\int_0^T r(t)\mathrm{d}t\}K\cdot I_{A_1}\right]\\&=K\exp\{(\frac{r_0}{b}-\frac{a}{b^2})(\mathrm{e}^{-bT}-1)-\frac{a}{b}T-c\rho_1\int_0^T\int_0^t\mathrm{e}^{b(u-t)}\mathrm{d}B_{1u}\mathrm{d}t\\&\quad-c\sqrt{1-\rho_1^2}\int_0^T\int_0^t\mathrm{e}^{b(u-t)}\mathrm{d}\xi_{1u}^H\mathrm{d}t\cdot I_{\{\sqrt{D_{11}}X_1+\sqrt{D_{12}}X_2>k\}}\}\end{aligned}$$

$$= K\exp\{(\frac{r_0}{b}-\frac{a}{b^2})(e^{-bT}-1)-\frac{a}{b}T\}E[e^{-\sqrt{D_{12}}X_2}]\cdot I_{\{\sqrt{D_{11}}X_1+\sqrt{D_{12}}X_2>k\}}\} \tag{4.9}$$

$$= K\exp\{(\frac{r_0}{b}-\frac{a}{b^2})(e^{-bT}-1)-\frac{a}{b}T\}e^{\frac{1}{2}D_{12}}N(-\frac{D_{12}+\rho'\sqrt{D_{11}D_{12}}+k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}}).$$

Substituting Eqs (4.8) and (4.9) into Eq (4.7) yields the price of the European call option at maturity T as:

$$C(S_T,T)=S_0N(\frac{D_{11}+\rho'\sqrt{D_{11}D_{12}}-k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}})$$

$$-K\exp\{(\frac{r_0}{b}-\frac{a}{b^2})(e^{-bT}-1)-\frac{a}{b}T\}e^{\frac{1}{2}D_{12}}N(-\frac{D_{12}+\rho'\sqrt{D_{11}D_{12}}+k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}}). \tag{4.10}$$

Similarly, the pricing formula for the European put option at maturity T is given by:

$$P(K,T)=K\exp\left\{(\frac{r_0}{b}-\frac{a}{b^2})(e^{-bT}-1)-\frac{a}{b}T\right\}e^{\frac{1}{2}D_{12}}N(-\frac{D_{12}+\rho'\sqrt{D_{11}D_{12}}+k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}})$$

$$-S_0N(\frac{D_{11}+\rho'\sqrt{D_{11}D_{12}}-k}{\sqrt{D_{11}+D_{12}+2\rho'\sqrt{D_{11}D_{12}}}}). \tag{4.11}$$

Remark 2.

(1) When $\rho_1=0$ and $\rho_2=0$, Theorem 4 reduces to the option pricing formula under the sub-fractional Vasicek stochastic interest rate.

(2) When $\rho_1=1$ and $\rho_2=1$, Theorem 4 reduces to the option pricing formula under the Vasicek stochastic interest rate within the traditional Black–Scholes framework.

(3) When $\rho_1=0$, $\rho_2=0$ and $H=\frac{1}{2}$, Theorem 4 corresponds to the option pricing formula under the Vasicek stochastic interest rate from the fractional Brownian motion perspective.

5. Empirical analysis and numerical simulation

5.1. Empirical Analysis

5.1.1. Selection and processing of data

(1) Price of the underlying asset S

The real market data in this article comes from the CSI 300 stock index options. The CSI 300 is a core broad-based index in China's A-share market, with strong representativeness and a solid foundation in its constituent stocks. The CSI 300 index is composed of the top 300 stocks ranked by comprehensive performance in the Shanghai and Shenzhen stock exchanges. Its constituent stocks are large in scale, highly liquid, and cover a wide range of industries. The trading volume and market capitalization also account for a significant proportion in the A-share market, effectively reflecting the overall market changes. The option contract under study in this article has its first trading day on August 18, 2025, with the exercise period from November 21, 2025. The underlying asset of the option contract has a total of 64 trading days during its duration. Here, we select the closing price data of the CSI 300ETF stock index options on each trading day during the duration as the underlying price for that day.

(2) The risk-free interest rate r

The Shanghai Interbank Offered Rate (Shibor) serves as the basis for determining the risk-free interest rate. In this paper, the three-month real Shanghai Interbank Offered Rate (Shibor) data during the option contract's duration (from August 18,2025 to November 21,2025) is selected, and its arithmetic average is taken as the risk-free interest rate r in the pricing formula. Data source: China Financial Exchange and the official website of the Shanghai Interbank Offered Rate.

(3) Option contract validity period T:

The term T of an option contract refers to the annualized number of remaining trading days before its expiration. The stock market typically has approximately 252 valid trading days in a year.

(4) Option market price:

For the CSI 300ETF (code IO2511), the closing price of the option contract on each trading day during its term is used as the daily market price (hereafter referred to as the true price of the option contract).

(5) Hurst index H:

We use the R/S analysis method, i.e., the re-scaled range analysis method, to estimate the Hurst index by processing and calculating the parameters through MATLAB language.

(6) Estimation of the volatility σ of the underlying assets

The historical volatility is calculated by collecting the data, calculating the daily log return, the daily log return standard deviation, and then annualizing the daily standard deviation, to estimate the volatility of the underlying asset.

(7) Parameter estimation of the Vasicek's Interest rate model

The continuous-time formulation $dr_t = (a - br_t)dt + cdM_{1t}^H$ based on the Vasicek interest rate model employs maximum likelihood estimation for parameter estimation. The continuous-time model is first discretized into a self-regressive form, with initial parameter estimates obtained through regression analysis. The maximum likelihood function is then optimized to derive the final parameters. The parameter transformation process involves mapping regression coefficients to structural parameters: The mean reversion rate, b, is derived from the logarithmic form of the self-regressive coefficients; the long-term mean reversion parameter, a, is calculated from the relationship between the intercept term and regression coefficients; and the interest rate volatility parameter, c, is estimated from the residual variance.

The parameters estimated from the real data used in this paper are as follows (Table 1):

Table 1. Parameter Estimation Table.

Parameter Name	Estimated Value	Parameter Name	Estimated Value
H	0.7512	σ	0.1523
r_0	0.0150	a	0.0346
T	0.2500	b	0.0238
δ	0.5000	c	0.0082
ρ_1	0.6000	ρ'	0.8000
ρ_2	0.4000		

5.1.2. Model comparison

The price of the option is simulated using the above parameters and the mixed sub-fractional Brownian motion Vasicek rate model (model 1) obtained from theorem 4 and the B-S option pricing model (model 2).

As evidenced by Table 2, Model 1's superiority stems from its integration of mixed sub-fractional Brownian motion with the Vasicek interest rate, which enhances its capacity to characterize long memory and volatility clustering in markets. The model employs the Hurst parameter to describe the persistent dependence of underlying asset returns, while the mixed subfractional Vasicek model endogenizes interest rate risk as a stochastic process, thereby more accurately capturing the complex dynamics of real markets. In contrast, the traditional B-S model, based on constant volatility and independent increment assumptions, neglects these crucial market microstructural features, resulting in systematic deviations in its price fitting.

Table 2. Comparison of the difference between the price of the call option and the real price under the Mixed Fractional Vasicek Interest Rate Model and the B-S Model.

Strike Price	Real Market Call Option Price	The Model 1 Fitted Call Option Price	Absolute Error	The Model 2 Fitted Call Option Price	Absolute Error
3750	462	469	7	471.6	9.6
3800	420	420.3	0.3	425.6	5.6
3850	379.8	378.2	1.6	380.9	1.1
3900	341.6	339.4	2.1	338.1	3.6
3950	305.2	308.9	3.7	297.2	8

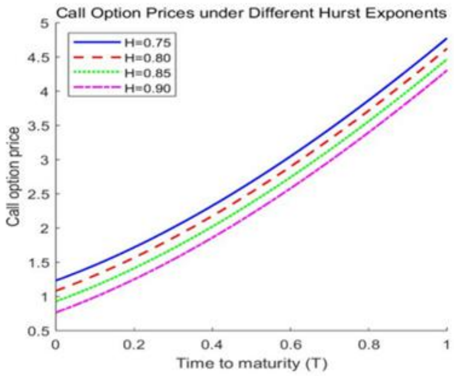
5.2. Parameter sensitivity analysis

Using the derived actuarial pricing formulas for European options with maturity T under the mixed sub-fractional Brownian motion with Vasicek interest rate, numerical simulations are conducted to examine the effects of various parameters on call option prices.

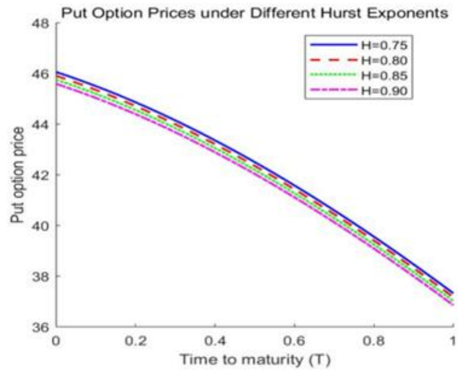
The model parameters are set as follows: $S_0 = 100,\ T = 1,\ \sigma = 0.2,\ \delta = 0.5,\ \rho' = 0.5,\ a = 0.1,$ $b = 0.3,\ \rho_1 = 0.4,\ \rho_2 = 0.6,\ r_0 = 0.05.$

5.2.1. Impact of the Hurst exponent H on the option price S

Figure 1 (a) and (b) demonstrate the nonlinear impact of the *Hurst* index (H) on the time value of options by altering the volatility term structure. At any given time (T_0), higher Hurst values correspond to lower option prices. The results indicate that higher Hurst values ($H > 0.5$) represent long-term memory, which under specific conditions can exert nonlinear or even inhibitory effects on the time value of options by either enhancing mean-reversion dynamics or modifying the cumulative efficiency of volatility (T^{2H}). This suggests that long-term memory does not always enhance option value, as its ultimate effect depends on the option's virtual or real state, highlighting the model's ability to capture complex market dynamics.



(a) Call option prices.



(b) Put option prices.

Figure 1. The option prices versus T for different Hurst exponents.

Figure 2 presents a typical Vasicek interest rate simulation path driven by a mixed sub-fractional Brownian motion in $H = 0.75$. The path shows interest rate r_t fluctuating near the long-term mean a/b, exhibiting the persistent trajectory characteristic of fractional Brownian motion. This stochastic nature forms one of the sources of randomness in the discount factor D_{12} of our option pricing formula. In other words, the option prices at each time point in Figures 1 represent the expected values derived from numerous stochastic interest rate paths similar to Figure 2. The mean-reversion property of the Vasicek model prevents interest rates from diverging, ensuring stability in the discount factor calculation and the final option pricing.

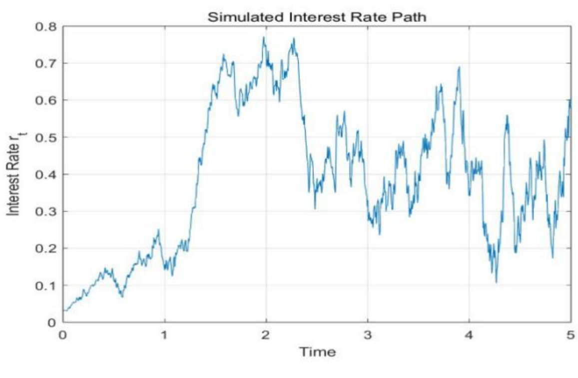
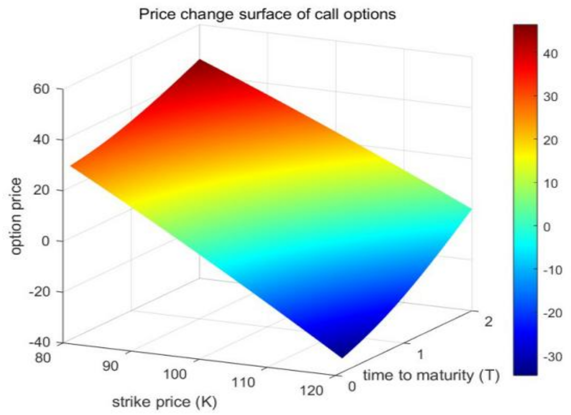


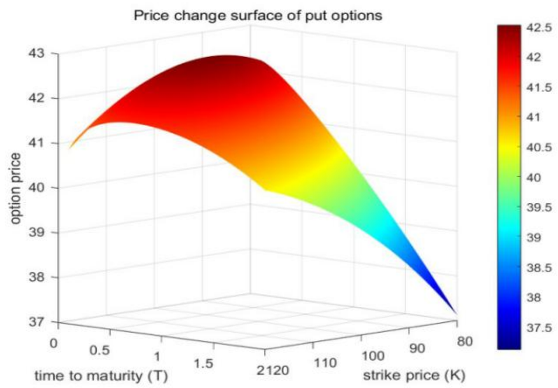
Figure 2. Stochastic interest rate trends over time t.

5.2.2. Impact of K and T on option prices

The option price curves in Figure 3 (a) and (b) vividly demonstrate the quantitative mechanism of intrinsic and time values in this model. The call option price decreases as K increases, while the put option price rises, directly reflecting the core impact of the strike price on the option's intrinsic value. More crucially, the effect of expiration time T is significant: The value of call options increases with T, as the longer duration amplifies the potential return under the $H = 0.75$ setting (driven by the T^{2H} in the variance term), resulting in positive time value. Conversely, the value of put options declines with T, indicating they are deeply in-the-money, where waiting time risks eroding intrinsic value, leading to negative time value.



(a) Call option prices.

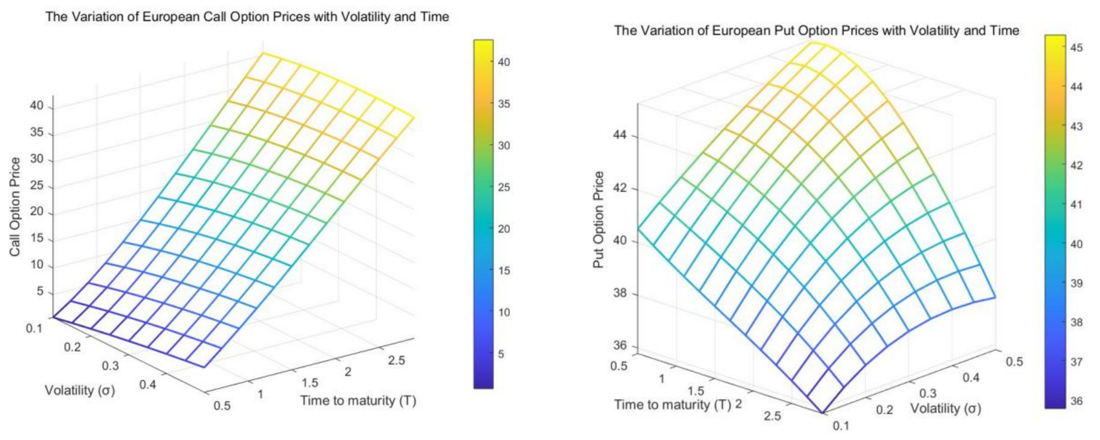


(b) Put option prices.

Figure 3. The option prices versus strike price K and time T.

5.2.3. Effects of σ and T on option prices

Figure 4 (a) and (b) demonstrate the synergistic effect of volatility (σ) and time (T) on option valuation. Both option types exhibit monotonic growth with increasing σ, stemming from Vega's fundamental positive attribute: Volatility amplifies uncertainty in underlying asset prices. The inherent asymmetry of option returns (unlimited upside potential versus limited downside risk) elevates expected returns across both options as uncertainty increases. The nonlinear effect of time T proves particularly significant: Call option values grow steadily with extended duration, as longer maturities and higher volatility collectively amplify uncertainty through a superlinear mechanism (driven by the T^{2H} term in the model), thereby enhancing positive time value. Conversely, put options demonstrate declining value with increasing T at fixed σ, confirming their deep in-the-money status. Extended maturities imply greater probability of adverse price movements for underlying assets, resulting in negative time value. The model accurately captures this complex dynamic, demonstrating its sophisticated pricing capabilities.

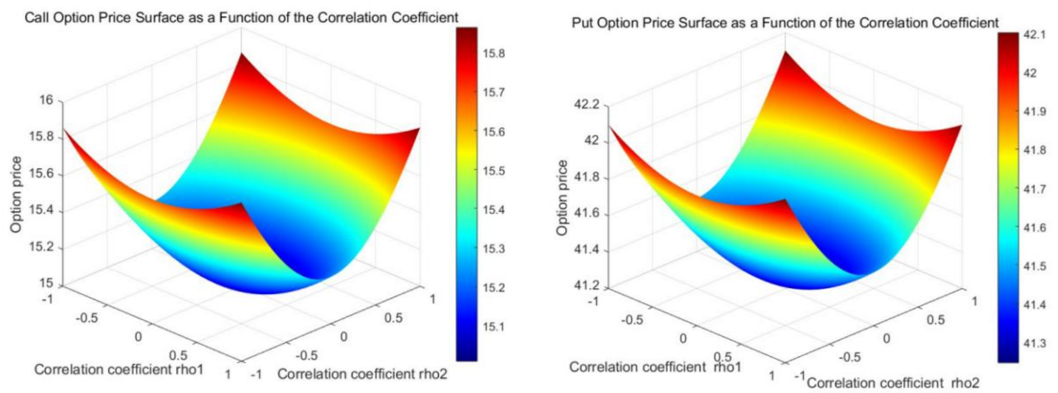


(a) Call option prices. (b) Put option prices.

Figure 4. The option price versus volatility σ and time T.

5.2.4. Effects of ρ_1, ρ_2 on option prices

Figure 5 (a) and (b) demonstrate the structural impact of "quality" from different risk sources on option pricing. The correlation coefficients ρ_1 and ρ_2 measure the mixing ratio between standard Brownian motion (incremental independence) and fractional Brownian motion (long-term memory) within each risk source. When ρ approaches 1, standard Brownian motion dominates, exhibiting irregular paths and incremental independence. This generates broader price distributions over the option's lifetime, indicating higher volatility accumulation efficiency and, consequently, greater option value. Conversely, when ρ approaches 0, fractional Brownian motion becomes dominant. Although its long-term memory effects manifest in the distant term, its relatively lower volatility accumulation efficiency in the near-term leads to lower option prices. This surface essentially reflects the market's pricing differential between the two risk sources: The model reveals that traditional, purely "stochastic walk" risks are assigned higher risk premiums, profoundly demonstrating the model's inherent ability to distinguish and quantify different risks.



(a) Call option prices. (b) Put option prices.

Figure 5. The option price versus correlation coefficient.

6. Conclusions

In this study, we establish a dynamic model capable of accurately capturing long-memory characteristics and complex volatility patterns in markets. Based on the insurance premium principle, we derive a pricing formula for European options. The theoretical model's advantages are systematically validated through empirical analysis: First, comparative analysis with the classical B-S model demonstrates significantly lower pricing errors across strike prices, confirming its practical accuracy. Second, numerical simulations reveal the classic dependence of option prices on key parameters (e.g., underlying asset prices and strike prices), while quantifying the unique impacts of novel risk factors such as long-memory (Hurst index, H) and market risk source correlation (correlation coefficient, ρ). Empirical results show differentiated pricing between traditional "stochastic walk" risks and long-memory risks, highlighting the model's robust capability in identifying and separating distinct risk sources. This research not only provides solid theoretical support for the intersection of stochastic interest rate and long-memory processes, but also offers a practical pricing framework validated through empirical testing, serving as a direct and effective tool for financial institutions to conduct derivatives pricing and risk management in complex market environments.

Author contributions

Fangling Ren derived the models and wrote the manuscript. Hui Feng framed the study and validated the results. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (Grant No. 62477029). We thank LetPub (www.letpub.com.cn) for its linguistic assistance during the preparation of this manuscript.

Conflicts of interest

The authors declare no conflict of interest in this paper.

References

1. M. Zili, Mixed sub-fractional Brownian motion, *Random Operators Sto*, **22** (2014), 163–178. http://doi.org/10.1515/rose-2014-0017
2. R. C. Merton, An analytic derivation of the cost of the deposit insurance and loan guarantees: An application of modern option pricing theory, *J Bank Financ*, **1** (1977), 3–11. http://doi.org/10.1016/0378-4266(77)90015-2

3. H. U. Gerber, E. S. W. Shiu, Option pricing by Esscher transforms, *Trans. Soc. Actuaries*, **46** (1994), 99–191.
4. H. U. Gerber, E. S. W. Shiu, Actuarial bridges to dynamic hedging and option pricing, *Insur: Math. Econ.*, **18** (1996), 183–218.
5. M. Bladt, T. H. Rydberg, An actuarial approach to option pricing under the physical measure and without market assumptions, *Insur: Math. Econ.*, **22** (1998), 65–73. http://doi.org/10.1016/s0167-6687(98)00013-4
6. L. N. Girard, Market value of insurance liabilities: Reconciling the actuarial appraisal and option pricing methods, *N. Am. Actuar. J.*, **4** (2000), 31–49. https://doi.org/10.1080/10920277.2000.10595871
7. J. Liu, L. Yan, C. Ma, Pricing options and convertible bonds based on an actuarial approach, *J. Math. Probl. Eng.*, **2013** (2013), 676148. http://doi.org/10.1155/2013/676148
8. F. Shokrollahi, A. Kılıçman, Actuarial approach in a mixed fractional Brownian motion with jumps environment for pricing currency option, *Adv. Differ. Equ.*, **2015** (2015), 257. https://doi.org/10.1186/s13662-015-0590-8
9. W. L. Wu, M. X. Shen, Mixed fractional jump-diffusion environment reinsurance pricing, *Journal of Anshan Normal University*, **27** (2025), 15–22. In Chinese. http://doi.org/10.20212/j.issn.1008-2441.2025.04.003
10. A. S. Deakin, M. Davison, An analytic solution for a Vasicek interest rate convertible bond model, *J. Appl. Math.*, **2010** (2010), 263451. http://doi.org/10.1155/2010/263451
11. W. L. Huang, G. M. Liu, S. H. Li, A. Wang, European option pricing under fractional stochastic interest rate model, *Adv. Mater. Res.*, **171-172** (2010), 787–790. http://doi.org/10.4028/www.scientific.net/AMR.171-172.787
12. Y. L. Wang, H. Xue, Pricing of backward options under double fractional Vasicek interest rate environment, *World Sci-Tech R D*, **38** (2016), 840–844. In Chinese. http://doi.org/10.16507/j.issn.1006-6055.2016.04.021
13. K. H. Kim, S. Yun, N. U. Kim, J. H. Ri, Pricing formula for European currency option and exchange option in a generalized jump mixed fractional Brownian motion with time-varying coefficients, *Physica A*, **522** (2019), 215–231. http://doi.org/10.1016/j.physa.2019.01.145
14. L. Tao, Y. Lai, Y. Ji, X. Tao, Asian option pricing under sub-fractional vasicek mode, *Quant. Financ. Econ.*, **7** (2023), 403–419. http://doi.org/10.3934/QFE.2023020
15. Y. Fu, S. Zhou, X. Li, F. Rao, Multi-assets Asian rainbow options pricing with stochastic interest rates obeying the Vasicek model, *AIMS Mathematics*, **8** (2023), 10685–10710. http://doi.org/10.3934/math.2023542
16. E. Djeutcha, J. S. Kamdem, L. A. Fono, Interest rate options in one-factor mixed modified fractional Vasicek model, *Int. J. Finan. Eng.*, **12** (2025), 2550013. http://doi.org/10.1142/S2424786325500136
17. T. M. Fullerton, M. Pokojovy, A. T. Anum, E. Nkum, Maximum trimmed likelihood estimation for discrete multivariate vasicek processes, *Economies*, **13** (2025), 68. https://doi.org/10.3390/economies13030068
18. O. Samimia, F. Mehrdoust, Vasicek interest rate model under Lévy process and pricing bond option, *Commun. Stat.-Simul. Comput.*, **53** (2024), 529–545. http://doi.org/10.1080/03610918.2022.2025837
19. X. Wang, C. Wang, Pricing geometric average Asian options in the mixed sub-fractional Brownian motion environment with Vasicek interest rate model, *AIMS Mathematics*, **9** (2024), 26579–26601. http://doi.org/10.3934/math.20241293

20. H. Zhi, Z. D. Guo, Pricing of geometric average Asian option with fixed strike price under mixed fractional Brownian motion mechanism, *Journal of Anqing Normal University*, **31** (2025), 21–26. In Chinese. http://doi.org/10.13757/j.cnki.cn34-1328/n.2025.01.004
21. H. Zhi, Z. D. Guo, Pricing model of foreign exchange options under the mixed subfractional Brownian motion mechanism, *Journal of Liaoning University of Technolog*, **45** (2025), 61–66. In Chinese. http://doi.org/10.15916/j.issn1674-3261.2025.01.010
22. L. Li, R. L. Song, Pricing of gap options under mixed fractional Vasicek stochastic interest rate model, *Journal of Shandong University*, 2025. https://link.cnki.net/urlid/37.1389.N.20251022.1553.008
23. A. A. Araneda, N. Bertschinger, The sub-fractional CEV model, *Physica A*, **573** (2021), 125974. https://doi.org/10.1016/j.physa.2021.125974
24. H. Zhou, Research on binary option pricing, *China University of Mining and Technology*, 2022. In Chinese. http://doi.org/10.27623/d.cnki.gzkyu.2022.000830
25. C. H. Cai, Q. Wang, W. L. Xiao, Mixed sub-fractional Brownian motion and drift estimation of related Orstein-Uhlenbeck process, *Commun. Math. Stat.*, **11** (2022), 229–255. http://doi.org/10.1007/S40304-021-00245-8
26. J. T. Merfeld, "Market value of insurance liabilities: Reconciling the actuarial appraisal and option pricing methods", Luke N. Girard, January 2000, *N. Am. Actuar. J.*, **4** (2013), 56–57. http://doi.org/10.1080/10920277.2000.10595874
27. P. C. Lukman, B. D. Handari, H. Tasman, Study on European put option pricing with underlying asset zero-coupon bond and interest rate following the Vasicek model with jump, *J. Phys: Conf. Ser.*, **1725** (2021), 012092. http://doi.org/10.1088/1742-6596/1725/1/012092
28. O. Vasicek, An equilibrium characterization of the term structure, *J. Financ. Econ.*, **5** (1977), 177–188. http://doi.org/10.1016/0304-405x(77)90016-2
29. M. Yu, Z. Y. Cheng, J. Deng, S. Y. Wang, A new option pricing method: A new perspective based on mixed sub-fractional Brownian motion, *Systems Engineering—Theory and Practice*, **41** (2021), 2761–2776. In Chinese. http://doi.org/10.12011/SETP2019-2880