AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 30109–30133.
DOI: 10.3934/math.20251323
Received: 24 October 2025
Revised: 05 December 2025
Accepted: 11 December 2025
Published: 23 December 2025

Research article

Hermite-Hadamard, Fejér and Jensen-type inequalities via p-harmonic convex functions with numerical validation and graphical insights

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Abstract: This paper deals with the generalized version of Hermite-Hadamard and Fejér-type inequalities. Some classical results are extended to a newly introduced category of p-harmonic convex functions for which explicit bounds are also established. Additionally, novel discrete versions of existing inequalities for univariate p-harmonic convex functions over linear spaces are also provided. For validation, the classical results are retrieved by letting $p \to 1$, which clearly indicates the accuracy of the achieved results. Finally, tabular and graphical analysis is presented to show the improved results with p-harmonic convex functions over existing literature.

Keywords: convex functions; harmonic convex functions; p-harmonic convex functions; Jensen's inequality; Fejér type inequality; Hermite-Hadamard inequality
Mathematics Subject Classification: 26A15, 26A51, 26D10, 26D15

1. Introduction

Convexity is a fundamental notion in mathematical analysis, and plays a crucial role in diverse fields including geometry, optimization, control systems, information theory, operations research, and functional analysis. Its impact carries over to economics, finance, engineering, and management, so it is an important concept in both theory and practice. Owing to this wide applicability, scholars have dedicated significant interest to researching generalized convexity and establishing weaker types of convexity in order to solve practical problems more effectively [1, 2]. A function is called convex if for any two points within its domain, the value of the function at any point on the line segment joining them does not exceed the linear combination of its values at the points. Geometrically, this means that a chord connecting any two points on the graph of the function is above or on the curve itself. While this property appears simple, it has deep implications and is the basis of a large number of classical inequalities and approximation results that are of profound importance in optimization theory, computational mathematics, and the applied sciences. In recent years, numerous mathematicians have focused on generalizing the notion of convex functions, introducing notions like h-convexity, p-convexity, and s-convexity. In this framework, various extensions of the Hermite-Hadamard inequality have been developed to correspond to these generalized convexities. Anderson et al. [3] developed a comprehensive framework for generalized convexity and established fundamental inequalities that underpin later Hermite-Hadamard type generalizations. Chen et al. [4] extended these inequalities to generalized p-convex functions using Raina's fractional integral operators. Dragomir [5, 6] derived Hermite-Hadamard type inequalities for harmonic-convex and harmonic harmonically-convex functions, respectively, enriching the theory of harmonic convexity. Eken et al. [7] established Hermite-Hadamard inequalities for p-convex functions, while Mehreen and Anwar [8] obtained Hadamard and Fejér-type inequalities via Caputo fractional derivatives. Noor et al. [9] investigated Hermite-Hadamard inequalities for harmonic preinvex functions.

Integral inequalities are basic tools in many fields of mathematics with wide-ranging use in approximation theory, spectral analysis, statistics, and distribution theory. They are also important in various branches of science and engineering. One of the most famous implications based on convexity is Jensen's inequality, which is an extension of convexity to weighted averages. Its integral version is a very useful tool for bounding expected values and finds wide application in probability theory and statistics [10]. Among the most significant of these is the Hermite-Hadamard inequality, which offers sharp bounds for the average value of a convex function on a closed interval in terms of its values at the endpoints and the midpoint. This inequality is an important tool of approximation theory and integral analysis since it enables the function's average value to be approximated from weighted endpoint evaluations [11]. Fejér-type inequalities also refine the Hermite-Hadamard estimates using weighted integrals with symmetric weights or by using fractional integral operators. These inequalities detect a finer relationship between what happens to a function and its integral, resulting in better estimates that are best for particular types of functions. Latif et al. [12] investigated Fejér-type inequalities to harmonically convex functions. Jensen's inequality [13] has been established in a harmonic context, which gives more accurate integral inequalities. Similarly, Fejér-type inequalities with fractional integrals and weighted means with harmonic structures provide more accurate integral bounds and better approximation outcomes.

Harmonic convexity is a generalization of the classical concept of convexity, modifying the traditional framework by applying the inequality defining property with respect to the harmonic mean, as opposed to the arithmetic mean. This approach gives rise to new families of inequalities in which harmonic means replace arithmetic means, leading to redescribed versions of well-known results such as the Jensen, Hermite-Hadamard, and Fejér-type inequalities. These advancements play an important role in various areas, including the study of special means, integral estimation, and fractional analysis. Iscan et al. [14], Dragomir et al. [15], and Baloch and his co-authors have played a pivotal role in advancing harmonic convexity. Petrović-type inequalities for harmonic h-convex functions were established in [16], while structural characterizations of different classes of harmonic convex functions and their applications were investigated in [17]. Further developments include variants of Jensen-type inequalities and related results for harmonic convex functions [18], as well as detailed analysis of the properties and bounds of Jensen-type functionals within the harmonic convex framework [19]. Building on this foundation, p-harmonic convexity adds a parameter $p \neq 0$ to characterize convexity in terms of the p-harmonic mean. This extension not only includes harmonic convexity as a special instance when $p = 1$, but also establishes connections with other generalized convexity notions for other choices of p. In [20, 21], Noor et al. established Jensen, Hermite-Hadamard, and Fejér-type inequalities for p-harmonic convex functions with improved bounds, and extended their range to applications in fractional calculus, stochastic control, and optimization. These generalizations provide effective tools for both theoretical development and practical applications in mathematics and engineering. The subtle interaction between mean values and convexity types enhances the understanding of integral inequalities and stimulates future research on related functional inequalities and their multidimensional analogues.

This study is driven by the increasing necessity to generalize traditional convexity notions to advanced structures, such as interval-valued functions and fractional calculus. Classical convexity often proves inadequate for capturing the complex behavior of such functions, especially with respect to nonlinear averaging and weighted means. To overcome these limitations, the notion of p-harmonic convex functions has been developed, extending harmonic convexity by employing the p-harmonic mean. This framework offers a flexible and strong basis for obtaining sharper inequalities, sharpening current results, and investigating their uses in different branches of mathematical analysis.

The primary objective of the present work is to establish Hermite-Hadamard and Fejér-type inequalities for p-harmonic convex functions, extend these inequalities to a broader class of such functions, and find sharp estimates. Furthermore, we provide improved estimates of previously known results and introduce novel discrete inequalities related to one of the recent Jensen-type modifications by Baloch et al. and Dragomir. In the applications section, our theoretical findings are further depicted and validated through numerical tables and graphical illustrations.

The principal contribution of this work is that it systematically defines the notion of p-harmonic convexity as a generalized framework that extends the conventional harmonic convexity concept. By refining the Hermite-Hadamard and Fejér-type inequalities in this generalized framework, we obtain sharper bounds and better estimates of existing results. Furthermore, the development of new Jensen-type discrete inequalities constitutes a significant contribution to convex analysis. These findings not only unify and enhance several known inequalities, but also provide versatile tools for analysis with applications in real analysis, fractional calculus, and optimization theory.

2. Preliminaries

Definition 2.1. *[21] Let $\mathscr{J}$ be a set that is p-harmonically convex. A function $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ is called p-harmonically convex, if*

$$\mathscr{G}\left(\left[\frac{\mathrm{w}_1^p \mathrm{w}_2^p}{\zeta \mathrm{w}_1^p + (1-\zeta)\mathrm{w}_2^p}\right]^{\frac{1}{p}}\right) \leq (1-\zeta)\mathscr{G}(\mathrm{w}_1) + \zeta\mathscr{G}(\mathrm{w}_2), \quad \forall \mathrm{w}_1, \mathrm{w}_2 \in \mathscr{J}, \zeta \in [0,1]. \tag{2.1}$$

A function $\mathscr{G}$ is defined to be p-harmonic concave if and only if the function -$\mathscr{G}$ is p-harmonically convex.

For $p = 1$, the concept of a p-harmonic convex set aligns exactly with that of a harmonic convex set. Conversely, when $p = -1$, the p-harmonic convex set coincides with the traditional notion of a convex set. This demonstrates that the p-harmonic convex set framework generalizes and unifies these concepts.

Next, we present several examples of functions that are p-harmonic convex.

Example 2.1. Below, we provide several non-trivial examples.

- The functions $\Phi_1(\mathrm{w}) = \ln \mathrm{w}$, $\Phi_2(\mathrm{w}) = e^{-c\mathrm{w}^{-p}}$, defined for all $\mathrm{c}, p > 0$ and $\mathrm{w} \in (0, \infty)$, serve as examples of functions that are both concave and p-harmonically convex.
- Consider $\Phi_3(\mathrm{w}) = \mathrm{w}^{\alpha} + \frac{1}{\mathrm{w}^p}$ and $\Phi_4(\mathrm{w}) = \frac{1}{\mathrm{w}^p}$, defined for $p > 0$, $\alpha \geq 0$, and $\mathrm{w} \in (0, \infty)$. These provide examples of p-harmonic convex functions that are also convex functions.
- The function

$$\Psi_p(\mathrm{w}) = \begin{cases} \mathrm{w}^{p-2}, & 0 < \mathrm{w} < 2^{\frac{1}{p}}, \\ 0, & 2^{-\frac{1}{p}} \leq \mathrm{w} \leq 1, \\ 1 - \mathrm{w}^{-p}, & \mathrm{w} > 1 \end{cases}$$

provides an example of a p-harmonic convex function that is neither convex nor concave.

Based on these examples, in [17], the authors asserted that harmonic convex functions are distinct from, yet closely related to, convex functions. In this study, we generalize these observations to p-harmonic convex functions and examine the relationships between both frameworks under specific analytical conditions. Specifically, we explore significant connections between these classes that can be established under specific conditions, which are outlined below.

Lemma 2.1. *[17] Let $p > 0$ and $\mathscr{J} \subseteq \mathbb{R}\backslash\{0\}$ be an interval such that $\mathscr{J}^{-1} = \{\mathrm{w}_2 \in \bar{\mathbb{R}}, \mathrm{w}_2 = \frac{1}{\mathrm{w}_1{}^p}, \mathrm{w}_1 \in \mathscr{J}\}$. The function $\Phi : \mathscr{J} \to \mathbb{R}$ is p-harmonically convex if the associated function $\Psi : \mathscr{J}^{-1} \to \mathbb{R}$, given by $\Psi(\mathrm{w}_2) = \Phi(\mathrm{w}_1)$, is p-harmonic convex on $\mathscr{J}^{-1}$.*

Lemma 2.2. *[17] Let $\mathscr{J} \subseteq (0, \infty)$ and $\mathscr{J}^{-1}$ be defined as above in Lemma 2.1. Then, $\Phi : \mathscr{J} \to \mathbb{R}$ is p-harmonically convex iff the function $\Psi : \mathscr{J} \to \mathbb{R}$, $\Psi(\mathrm{w}) = \mathrm{w}^p\Phi(\mathrm{w})$, is convex on $\mathscr{J}$.*

In [22], the authors established important inequalities of Hermite-Hadamard- and Fejér-type that hold for p-harmonic convexity.

Theorem 2.1. *[22] Consider $\mathscr{J} \subset \mathbb{R}\backslash\{0\}$ to be a real interval and $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ a harmonically p-convex. Let $\mathrm{a}, \mathrm{b} \in I$ with $\mathrm{a} < \mathrm{b}$. If the function $\mathscr{G}$ is integrable on the interval $[\mathrm{a}, \mathrm{b}]$, then the following inequalities are satisfied:*

$$\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{[\mathrm{w}_1{}^p + \mathrm{w}_2{}^p]^{\frac{1}{p}}}\right) \leq \frac{p(\mathrm{w}_1\mathrm{w}_2)^p}{\mathrm{w}_2{}^p - \mathrm{w}_1{}^p}\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Phi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \leq \frac{\Phi(\mathrm{w}_1) + \Phi(\mathrm{w}_2)}{2}. \tag{2.2}$$

Theorem 2.2. *[22] Assume $\Phi : \mathscr{J} \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is a p-harmonic convex function, and suppose $\mathrm{w}_1, \mathrm{w}_2 \in \mathscr{J}$ with $\mathrm{w}_1 < \mathrm{w}_2$. If Φ is integrable over the interval $[\mathrm{w}_1, \mathrm{w}_2]$, then the following inequalities* *are valid:*

$$\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{[\mathrm{w}_1{}^p+\mathrm{w}_2{}^p]^{\frac{1}{p}}}\right)\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \le \int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Phi(\mathrm{w})\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \le \frac{\Phi(\mathrm{w}_1)+\Phi(\mathrm{w}_2)}{2}\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w}, \tag{2.3}$$

where $\Psi(\mathrm{w}):[\mathrm{w}_1,\mathrm{w}_2]\to\mathbb{R}$ *is a non-negative that is integrable and satisfies*

$$\Psi\left(\frac{\mathrm{w}_1\mathrm{w}_2}{\mathrm{w}}\right)=\Psi\left(\frac{\mathrm{w}_1\mathrm{w}_2}{[\mathrm{w}_1{}^p+\mathrm{w}_2{}^p-\mathrm{w}^p]^{\frac{1}{p}}}\right). \tag{2.4}$$

Dragomir et al. established a result commonly recognized as the Jensen's inequality applicable to harmonic convex functions.

Theorem 2.3. *[15] Consider the interval* $\mathscr{J}\subseteq(0,\infty)$*, and let* $\Phi:\mathscr{J}\to\mathbb{R}$ *be a harmonic convex function. Consequently, the given inequality is satisfied:*

$$\Phi\left(\frac{1}{\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k}}\right) \le \sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w}_k), \tag{2.5}$$

$\forall\mathrm{w}_1,\ldots,\mathrm{w}_n\in\mathscr{J}$ *and weights* $\mathrm{a_k}\in[0,1]$ *with* $\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}=1$.

Theorem 2.4. *[18] Consider the interval* $[\mathrm{w}_1,\mathrm{w}_2]\subseteq(0,\infty)$*, and let* $\Phi:[\mathrm{w}_1,\mathrm{w}_2]$*, a mapping into* $\mathbb{R}$*, be a harmonic convex function. For every finite sequence of positive terms* $(\mathrm{w}_k)_{\mathrm{k}=1}^{\mathrm{n}}\in[\mathrm{w}_1,\mathrm{w}_2]$ *with weights* $\mathrm{a_k}\in[0,1]$ *satisfying* $\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}=1$*, the given inequality is satisfied:*

$$\Phi\left(\frac{1}{\frac{1}{\mathrm{w}_1}+\frac{1}{\mathrm{w}_2}-\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k}}\right)\le\Phi(\mathrm{w}_1)+\Phi(\mathrm{w}_2)-\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w}_k). \tag{2.6}$$

Baloch et al. further developed this concept by introducing a Jensen's inequality adapted for *p*-harmonic convexity.

Theorem 2.5. *[23] Suppose* $\mathscr{J}$ *is an interval contained in* $(0,\infty)$ *and* $\Phi:\mathscr{J}\to\mathbb{R}$ *is a p-harmonic convex. Under these conditions, Jensen's inequality can be expressed as follows:*

$$\Phi\left(\frac{1}{\left(\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w_k}^p}\right)^{\frac{1}{p}}}\right)\le\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w}_k), \tag{2.7}$$

where $\mathrm{w}_1,\ldots,\mathrm{w}_n\in\mathscr{J}$*, and the weights* $\mathrm{a}_1,\ldots,\mathrm{a_n}\ge0$ *with* $\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}=1$.

Theorem 2.6. *[23] Let* $\Phi\subseteq\mathbb{R}\backslash\{0\}$ *be an interval, and consider a function* $\Phi:\mathscr{J}\to\mathbb{R}$ *that is p-harmonic convex. Under these conditions, the following inequality*

$$\Phi\left(\frac{1}{\left(\frac{1}{\mathrm{w}_1{}^p}+\frac{1}{\mathrm{w}_2{}^p}-\sum_{k=1}^{n}\frac{\mathrm{a}_k}{\mathrm{w}_k^p}\right)^{\frac{1}{p}}}\right)\le\Phi(\mathrm{w}_1)+\Phi(\mathrm{w}_2)-\sum_{k=1}^{n}\mathrm{a}_k\Phi(\mathrm{w}_k) \tag{2.8}$$

holds for any finite positive sequence $\{\mathrm{w}_k\}_{k=1}^{n}\in\mathscr{J}$*, and* $\mathrm{a}_1,.....,\mathrm{a}_n\ge0$ *with* $\sum_{k=1}^{n}\mathrm{a}_k=1$.

Theorem 2.7. *[19] Let $\Phi : \mathscr{J} \to \mathbb{R}$ be a harmonic convex function defined on $\mathscr{J} \subseteq \mathbb{R}\backslash\{0\}$. Then for every finite sequence of positive terms $\{\mathrm{w}_k\}_{\mathrm{k}=1}^{\mathrm{n}} \in \mathscr{J}$ and weights $\mathrm{a_k}$ with $\mathrm{A_n} := \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a_k} > 0$, the given inequality is satisfied:*

$$\begin{aligned} \mathrm{n} \min_{1\le \mathrm{k}\le \mathrm{n}}\{\mathrm{a_k}\}\left[\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\Phi(\mathrm{w}_k) - \Phi\left(\frac{1}{\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{1}{\mathrm{w}_k}}\right)\right] &\le \frac{1}{\mathrm{A_n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w}_k) - \Phi\left(\frac{1}{\frac{1}{\mathrm{A_n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k}}\right) \\ &\le \mathrm{n}\max_{1\le \mathrm{k}\le \mathrm{n}}\{\mathrm{a_k}\}\left[\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\Phi(\mathrm{w}_k) - \Phi\left(\frac{1}{\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{1}{\mathrm{w}_k}}\right)\right]. \end{aligned} \tag{2.9}$$

3. Main results

This section begins by presenting several significant inequalities of the Hermite-Hadamard, Jensen and Fejér-type for p-harmonic convex functions.

Theorem 3.1. *Assume $\Phi : \mathscr{J} \subset \mathbb{R}\backslash\{0\}$ maps into $\mathbb{R}$ such that Φ is a p-harmonic convex function, and suppose $\mathrm{w}_1, \mathrm{w}_2 \in \mathscr{J}$ with $\mathrm{w}_1 < \mathrm{w}_2$. If Φ is integrable on $[\mathrm{w}_1, \mathrm{w}_2]$ and $p > 0$, then for every $\lambda \in [0,1]$, the given inequalities are satisfied:*

$$\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p + \mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) \le l_p(\lambda) \le \frac{p(\mathrm{w}_1\mathrm{w}_2)^p}{\mathrm{w}_2{}^p - \mathrm{w}_1{}^p}\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Phi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \le L_p(\lambda) \le \frac{\Phi(\mathrm{w}_1)+\Phi(\mathrm{w}_2)}{2}, \tag{3.1}$$

where

$$l_p(\lambda) := \lambda\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\lambda\mathrm{w}_1{}^p + (2-\lambda)\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) + (1-\lambda)\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{((1-\lambda)\mathrm{w}_2{}^p + (1-\lambda)\mathrm{w}_1{}^p)^{\frac{1}{p}}}\right)$$

and

$$L_p(\lambda) := \frac{\lambda\Phi(\mathrm{w}_1) + \Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\lambda\mathrm{w}_1{}^p+(1-\lambda)\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) + (1-\lambda)\Phi(\mathrm{w}_2)}{2}.$$

Proof. Assume Φ is a p-harmonic convex function defined on $\mathscr{J}$. Since, $\left[\frac{w_1w_2}{(\lambda w_1^p+(1-\lambda)w_2^p)^{\frac{1}{p}}}\right]$ is a p-harmonic mean between w_1 and w_2, clearly $\left[\frac{w_1w_2}{(\lambda w_1^p+(1-\lambda)w_2^p)^{\frac{1}{p}}}\right] \in [w_1, w_2]$. By applying inequality (2.2) on the sub-interval $\left[\mathrm{w}_1, \frac{\mathrm{w}_1\mathrm{w}_2}{(\lambda\mathrm{w}_1{}^p+(1-\lambda)\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right]$, with $\lambda \ne 0$, the following result is obtained:

$$\begin{aligned} \Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\lambda\mathrm{w}_1{}^p + (2-\lambda)\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) &\le \frac{p(\mathrm{w}_1\mathrm{w}_2)^p}{\lambda(\mathrm{w}_2{}^p - \mathrm{w}_1{}^p)}\int_{\mathrm{w}_1}^{\frac{\mathrm{w}_1\mathrm{w}_2}{(\lambda\mathrm{w}_1{}^p+(1-\lambda)\mathrm{w}_2{}^p)^{\frac{1}{p}}}}\frac{\Phi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \\ &\le \frac{\Phi(\mathrm{w}_1) + \Phi\left(\frac{\mathrm{w}_1\mathrm{w}_2}{(\lambda\mathrm{w}_1{}^p+(1-\lambda)\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right)}{2}. \end{aligned} \tag{3.2}$$

Applying inequality (2.2) again on the subinterval $\left[\frac{w_1 w_2}{(\lambda w_1{}^p+(1-\lambda)w_2{}^p)^{\frac{1}{p}}}, w_2\right]$ for $\lambda \neq 0$ yields

$$\Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{((1-\lambda)w_2{}^p+(1+\lambda)w_1{}^p)^{\frac{1}{p}}}\right) \leq \frac{p(w_1w_2)^p}{(1-\lambda)(w_2{}^p-w_1{}^p)}\int_{\frac{w_1w_2}{(\lambda w_1{}^p+(1-\lambda)w_2{}^p)^{\frac{1}{p}}}}^{w_2}\frac{\Phi(w)}{w^{p+1}}dw \leq \frac{\Phi\left(\frac{w_1w_2}{(\lambda w_1{}^p+(1-\lambda)w_2{}^p)^{\frac{1}{p}}}\right)+\Phi(w_2)}{2}. \tag{3.3}$$

By multiplying inequality (3.2) by λ and (3.3) by $(1-\lambda)$, then summing the results together, the following combined inequality is obtained:

$$l_p(\lambda) \leq \frac{p(w_1w_2)^p}{(w_2{}^p-w_1{}^p)}\int_{w_1}^{w_2}\frac{\Phi(w)}{w^{p+1}}dw \leq L_p(\lambda). \tag{3.4}$$

Here, $l_p(\lambda)$ and $L_p(\lambda)$ are defined as above. This result follows from the p-harmonic convexity of Φ.

$$\begin{aligned}
\Phi\left(\frac{2^{\frac{1}{p}} w_1w_2}{w_1{}^p+w_2{}^p}\right) &= \Phi\left(\frac{1}{\lambda\frac{[\lambda w_1{}^p+(2-\lambda)w_2{}^p]^{\frac{1}{p}}}{2^{\frac{1}{p}}w_1w_2}+(1-\lambda)\frac{[(1-\lambda)w_2{}^p+(1+\lambda)w_1{}^p]^{\frac{1}{p}}}{2^{\frac{1}{p}}w_1w_2}}\right) \\
&\leq \lambda\Phi\left(\frac{2^{\frac{1}{p}}w_1w_2}{[\lambda w_1{}^p+(2-\lambda)w_2{}^p]^{\frac{1}{p}}}\right)+(1-\lambda)\Phi\left(\frac{2^{\frac{1}{p}}w_1w_2}{[(1-\lambda)w_2{}^p+(1+\lambda)w_1{}^p]^{\frac{1}{p}}}\right) \\
&\leq \lambda\left[\frac{\lambda}{2}\Phi(w_2)+\left(\frac{2-\lambda}{2}\right)\Phi(w_1)\right]+(1-\lambda)\left[\left(\frac{1-\lambda}{2}\right)\Phi(w_1)+\left(\frac{1+\lambda}{2}\right)\Phi(w_2)\right] \\
&= \frac{1}{2}\left[\lambda^2\Phi(w_2)+\lambda(2-\lambda)\Phi(w_1)+(1-\lambda)^2\Phi(w_1)+(1-\lambda^2)\Phi(w_2)\right] \\
&= \frac{\Phi(w_1)+\Phi(w_2)}{2}.
\end{aligned} \tag{3.5}$$

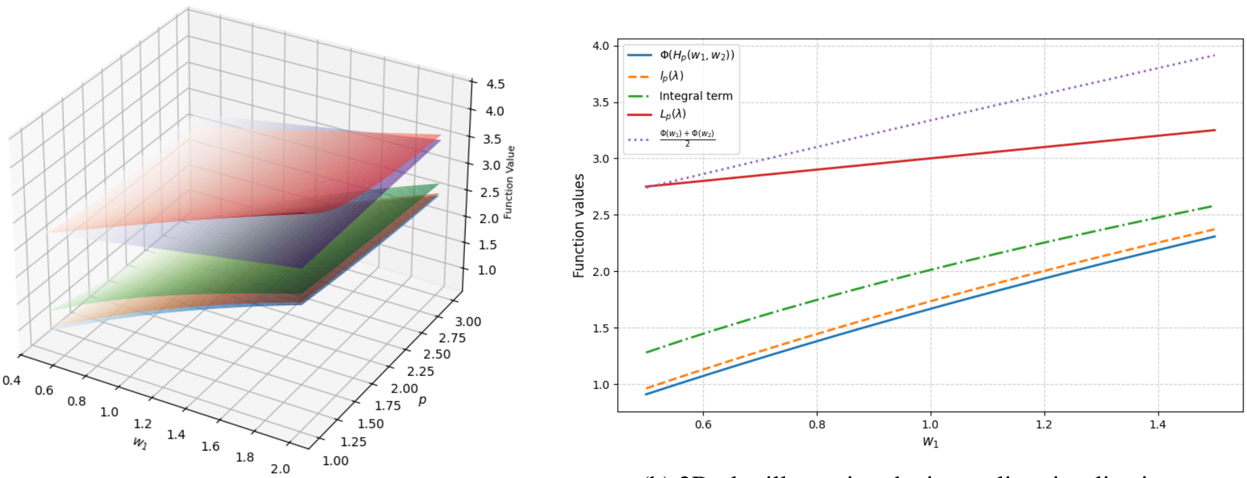
Then, by inequalities (3.4) and (3.5), we get (3.1).

The numerical validity of Theorem 3.1 is illustrated in Table 1.

Table 1. Numerical verification of the p-harmonic Hermite-Hadamard type inequality for $\Phi(w) = w^p$, $w_1 \in [0.5, 1.5]$, $w_2 = 5$, $\lambda = 0.15$, and $\Phi(w) = w^p$.

p	w_1	λ	$\Phi(H_p)$	$l_p(\lambda)$	Integral term	$L_p(\lambda)$	$\frac{\Phi(w_1)+\Phi(w_2)}{2}$
1.0	0.5	0.15	0.909091	0.961258	1.279214	2.740535	2.750000
1.5	0.5	0.10	0.777397	0.798861	1.059152	2.699733	2.750000
2.0	0.5	0.10	0.703598	0.718044	0.909091	2.647471	2.750000
3.0	0.5	0.05	0.629751	0.633770	0.743243	2.707906	2.750000

The graphical representation of Theorem 3.1 is shown as Figure 1.



(a) 3D plot illustrating the inequality visualization

(b) 2D plot illustrating the inequality visualization

Figure 1. Inequality chain for $p = 1$.

Corollary 3.1. *For the conditions specified in Theorem 3.1, the following result follows:*

$$\Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{({w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right) \le l_p(\lambda) \le \frac{p(w_1 w_2)^p}{{w_2}^p - {w_1}^p} \int_{w_1}^{w_2} \frac{\Phi(w)}{w^{p+1}} dw \le L_p(\lambda) \le \frac{\Phi(w_1) + \Phi(w_2)}{2}, \tag{3.6}$$

where

$$l_p := \frac{1}{2}\left[\Phi\left(\frac{4^{\frac{1}{p}} w_1 w_2}{(3{w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right) + \Phi\left(\frac{4^{\frac{1}{p}} w_1 w_2}{{w_1}^p + 3{w_2}^p}\right)\right]$$

and

$$L_p := \frac{\Phi(w_1) + 2\Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{({w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right) + \Phi(w_2)}{4}.$$

The numerical validity of Corollary 3.1 is illustrated in Table 2.

Table 2. Numerical verification of Corollary 3.1 for $w_1 = 0.5,\ w_2 = 5$, and $\Phi(w) = w$.

p	$\Phi(H_p)$	l_p	$\frac{p(w_1 w_2)^p}{{w_2}^p - {w_1}^p} \int_{w_1}^{w_2} \frac{\Phi(w)}{w^{p+1}} dw$	L_p	$\frac{\Phi(w_1) + \Phi(w_2)}{2}$
1.0	0.9091	1.0918	1.2792	1.8295	2.7500
1.5	0.7774	0.8938	1.0592	1.7637	2.7500
2.0	0.7036	0.7809	0.9091	1.7268	2.7500
3.0	0.6298	0.6716	0.7432	1.6899	2.7500

Corollary 3.2. *For the conditions specified in Theorem 3.1, we have the following result:*

$$\begin{aligned}\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p+\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) &\leq \sup_{\lambda\in[0,1]} l_p(\lambda) \leq \frac{p(\mathrm{w}_1\mathrm{w}_2)^p}{\mathrm{w}_2{}^p-\mathrm{w}_1{}^p}\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Phi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w}\\ &\leq \inf_{\lambda\in[0,1]} L_p(\lambda) \leq \frac{\Phi(\mathrm{w}_1)+\Phi(\mathrm{w}_2)}{2},\end{aligned} \tag{3.7}$$

where l_p and L_p are same the as defined in Corollary 3.1.

Theorem 3.2. *Assume $\Phi : \mathscr{J} \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ such that Φ is a p-harmonic convex function and let* $\mathrm{w}_1, \mathrm{w}_2 \in \mathscr{J}$ *with* $\mathrm{w}_1 < \mathrm{w}_2$. *If $\Phi, \Psi \in L[\mathrm{w}_1, \mathrm{w}_2]$, and Ψ is non-negative, satisfying condition (2.3). Then, for every $\lambda \in [0, 1]$, $p > 0$, the given inequalities are satisfied:*

$$\begin{aligned}\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p+\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right)\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} &\leq l_p(\lambda)\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w}\\ &\leq \frac{p(\mathrm{w}_1\mathrm{w}_2)^p}{\mathrm{w}_2{}^p-\mathrm{w}_1{}^p}\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Phi(\mathrm{w})\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w}\\ &\leq L_p(\lambda)\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w}\\ &\leq \frac{\Phi(\mathrm{w}_1)+\Phi(\mathrm{w}_2)}{2}\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w},\end{aligned} \tag{3.8}$$

where $l_p(\lambda)$ and $L_p(\lambda)$ are the same as those defined in Theorem 3.1.

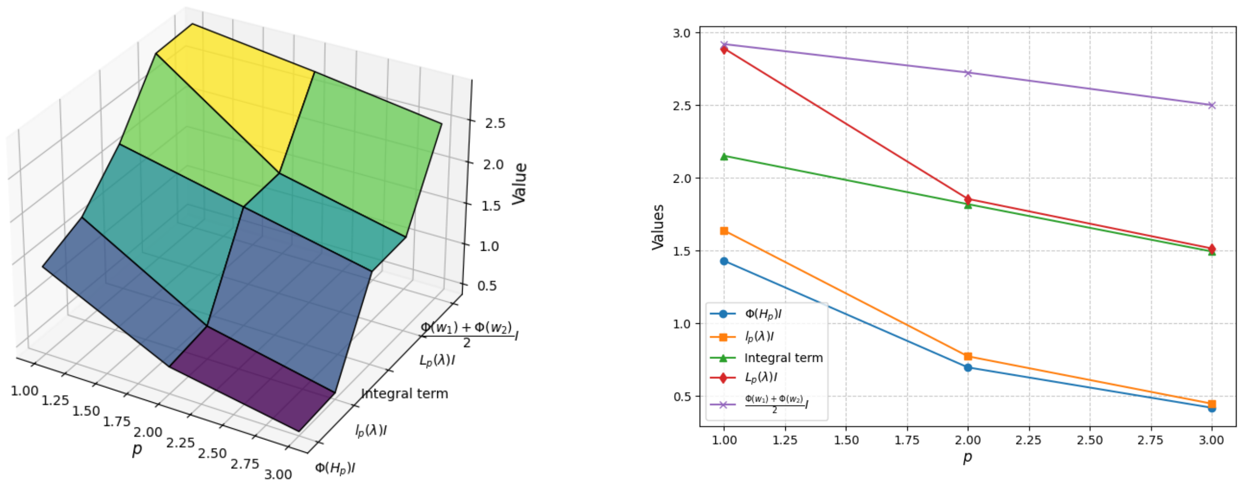
Proof. The proof proceeds in a manner similar to that of Theorem 3.1.

The numerical validity of Theorem 3.2 is illustrated in Table 3.

Table 3. Numerical verification of the inequality chain of Theorem 3.2 for $\Phi(\mathrm{w}) = \mathrm{w}$ and $\Psi(\mathrm{w}) = 1$ (values rounded).

p	(w_1, w_2)	λ	$\Phi\big(H_p\big)I$	$l_p(\lambda)I$	$\frac{p(w_1w_2)^p}{w_2^p-w_1^p}\int_{w_1}^{w_2}\frac{\Phi(\mathrm{w})\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}\,d\mathrm{w}$	$L_p(\lambda)I$	$\frac{\Phi(w_1)+\Phi(w_2)}{2}I$
1	(1,6)	0.50	1.429	1.637	2.150	2.887	2.917
2	(1,10)	0.50	0.697	0.773	1.818	1.854	2.722
3	(1,14)	0.50	0.420	0.448	1.493	1.514	2.499

The graphical comparison of the Theorem 3.2 as shown in Figure 2.



(a) 3D plot illustrating the inequality visualization

(b) 2D plot illustrating the inequality visualization

Figure 2. Inequality chain according to the above table values.

Corollary 3.3. *Given the conditions stated in Theorem 3.2, we get*

$$\begin{aligned}
\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p+\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right)\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} &\leq \sup_{\lambda\in[0,1]} l_p(\lambda)\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \\
&\leq \frac{p(\mathrm{w}_1\mathrm{w}_2)^p}{\mathrm{w}_2{}^p-\mathrm{w}_1{}^p}\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Phi(\mathrm{w})\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \\
&\leq \inf_{\lambda\in[0,1]} L_p(\lambda)\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \\
&\leq \frac{\Phi(\mathrm{w}_1)+\Phi(\mathrm{w}_2)}{2}\int_{\mathrm{w}_1}^{\mathrm{w}_2}\frac{\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w},
\end{aligned} \tag{3.9}$$

where l_p and L_p are the same as defined in Corollary 3.2.

The numerical validity of Corollary 3.3 is illustrated in Table 4.

Table 4. Examples where the Corollary 3.3 chain holds strictly. Here, $\Phi(\mathrm{w}) = \mathrm{w}^{\alpha}$ and $\Psi(\mathrm{w}) = \mathrm{w}^{\beta}$. Values rounded to 8 decimals.

p	(w_1, w_2)	α	β	$\Phi(H_p)I$	$\sup_{\lambda\in[0,1]} l_p(\lambda)\,I$	*2nd Inequality*	$\inf_{\lambda\in[0,1]} L_p(\lambda)\,I$	$\frac{\Phi(w_1)+\Phi(w_2)}{2}\,I$
1.5	(1,4)	1.5	-0.5	0.83333333	1.66592621	1.71428571	1.87500001	2.10937500
2.0	(1,4)	2.0	-1.0	0.72941176	1.23471306	1.60000000	1.85615531	2.78906250
2.0	(1,4)	2.0	-0.5	0.72941176	1.45813733	2.13333333	2.19203104	3.29375000

Symmetrical and anti-symmetrical transform

Let $\Phi : \mathscr{J} \subset \mathbb{R}\backslash\{0\} \to \mathbb{C}$ be a given function. The symmetrical transform of Φ is defined as

$$\breve{\Phi}(\zeta) \;:=\; \frac{1}{2}\left[\Phi(\zeta) + \Phi\left(\left(\frac{\mathrm{w}_1{}^p\mathrm{w}_2{}^p\zeta^p}{(\mathrm{w}_1{}^p + \mathrm{w}_2{}^p)\zeta^p - \mathrm{w}_1{}^p\mathrm{w}_2{}^p}\right)^{\frac{1}{p}}\right)\right], \;\; \zeta \in \mathscr{J}, \;\; p \geq 1.$$

Similarly, the anti-symmetrical transform is expressed as

$$\tilde{\Phi}(\zeta) \;:=\; \frac{1}{2}\left[\Phi(\zeta) - \Phi\left(\left(\frac{\mathrm{w}_1{}^p\mathrm{w}_2{}^p\zeta^p}{(\mathrm{w}_1{}^p + \mathrm{w}_2{}^p)\zeta^p - \mathrm{w}_1{}^p\mathrm{w}_2{}^p}\right)^{\frac{1}{p}}\right)\right], \;\; \zeta \in \mathscr{J}, \;\; p \geq 1.$$

It is straightforward to verify that for any function Φ, the following identity holds $\Phi = \breve{\Phi} + \tilde{\Phi}$. Moreover, if Φ is p-harmonic convex on $\mathscr{J}$, then its symmetrical transform $\breve{\Phi}$ is also possesses the property of p-harmonic convexity. However, the converse is generally not true.

Definition 3.1. *Assume $\Phi : \mathscr{J} \subset \mathbb{R}\backslash\{0\} \to \mathbb{C}$ is called symmetrized p-harmonic convex (concave) on $\mathscr{J}$ if $\breve{\Phi}$ satisfies the property of p-harmonic convexity (concavity).*

Theorem 3.3. *Let $\Phi : \mathscr{J} \subset \mathbb{R}\backslash\{0\} \to \mathbb{C}$ be a symmetrized p-harmonic convex function that is integrable on $\mathscr{J}$, and let Ψ be a non-negative, integrable function satisfying the required condition*

$$\int_{\mathrm{w}_1}^{\mathrm{w}_2} \frac{\Phi(\mathrm{w})\Psi\left(\left(\frac{\mathrm{w}_1{}^p\mathrm{w}_2{}^p\mathrm{w}^p}{(\mathrm{w}_1{}^p+\mathrm{w}_2{}^p)\mathrm{w}^p-\mathrm{w}_1{}^p\mathrm{w}_2{}^p}\right)^{\frac{1}{p}}\right)}{\mathrm{w}^{p+1}}d\mathrm{w} \;=\; \int_{\mathrm{w}_1}^{\mathrm{w}_2} \frac{\Phi(\mathrm{w})\Psi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w}. \tag{3.10}$$

Then, we have inequalities (2.2) and (2.3) for p-harmonic convex functions.

Proof. Suppose that $\Phi : \mathscr{J} \subset \mathbb{R}\backslash\{0\} \to \mathbb{C}$ is both a symmetrized p-harmonic and integrable function. By employing the Hermite-Hadamard-type inequality (2.2) for $\breve{\Phi}$, we get

$$\breve{\Phi}\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p + \mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) \leq \frac{p\mathrm{w}_1{}^p\mathrm{w}_2{}^p}{\mathrm{w}_2{}^p - \mathrm{w}_1{}^p}\int_{\mathrm{w}_1}^{\mathrm{w}_2} \frac{\breve{\Phi}(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} \leq \frac{\breve{\Phi}(\mathrm{w}_1) + \breve{\Phi}(\mathrm{w}_2)}{2}. \tag{3.11}$$

By some calculations, it follows that

$$\breve{\Phi}\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p + \mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) = \Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p + \mathrm{w}_2{}^p)^{\frac{1}{p}}}\right), \qquad \frac{\breve{\Phi}(\mathrm{w}_1) + \breve{\Phi}(\mathrm{w}_2)}{2} = \frac{\Phi(\mathrm{w}_1) + \Phi(\mathrm{w}_2)}{2},$$

and

$$\int_{\mathrm{w}_1}^{\mathrm{w}_2} \frac{\breve{\Phi}(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w} = \int_{\mathrm{w}_1}^{\mathrm{w}_2} \frac{\Phi(\mathrm{w})}{\mathrm{w}^{p+1}}d\mathrm{w}.$$

Therefore, by substituting these values into inequality (3.11), we get the result (2.2).

In a similar way, one can prove inequality (2.3) for the symmetrized p-harmonic convex function on Φ.

Theorem 3.4. *Assume* $\Phi : [w_1, w_2] \subset \mathbb{R}\backslash\{0\} \to \mathbb{C}$ *is a symmetrized p-harmonic convex function on* $[w_1, w_2]$, $p > 0$. *Then, for any* $w \in [w_1, w_2]$, *we have the following bounds:*

$$\Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{({w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right) \leq \breve{\Phi}(w) \leq \frac{\Phi(w_1) + \Phi(w_2)}{2}. \tag{3.12}$$

Proof. Since Φ is symmetrized p-harmonic convex on $[w_1, w_2]$, then for any $w \in [w_1, w_2]$ we have

$$\breve{\Phi}\left(\frac{2^{\frac{1}{p}} w_1 w_2}{({w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right) \leq \frac{\breve{\Phi}(w) + \breve{\Phi}\left(\left(\frac{{w_1}^p {w_2}^p w^p}{({w_1}^p + {w_2}^p) w^p - {w_1}^p {w_2}^p}\right)^{\frac{1}{p}}\right)}{2}.$$

With a few straightforward algebraic steps, we find that $\breve{\Phi}\left(\frac{2^{\frac{1}{p}} w_1 w_2}{({w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right) = \Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{({w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right)$, $\breve{\Phi}(w) = \breve{\Phi}(w)$, and

$$\breve{\Phi}\left(\left(\frac{{w_1}^p {w_2}^p w^p}{({w_1}^p + {w_2}^p) w^p - {w_1}^p {w_2}^p}\right)^{\frac{1}{p}}\right) = \breve{\Phi}(w).$$

Hence, inequality (3.12) is established.

Moreover, using the p-harmonic convexity of $\breve{\Phi}$ on $[w_1, w_2]$, we can also derive

$$\begin{aligned} \breve{\Phi}(w) &\leq \frac{{w_2}^p({w_1}^p - w^p)}{w^p({w_1}^p - {w_2}^p)} \breve{\Phi}(w_1) + \frac{{w_1}^p(w^p - {w_2}^p)}{w^p({w_1}^p - {w_2}^p)} \breve{\Phi}(w_2) \\ &= \frac{{w_2}^p({w_1}^p - w^p)}{w^p({w_1}^p - {w_2}^p)} \frac{\Phi(w_1) + \Phi(w_2)}{2} + \frac{{w_1}^p(w^p - {w_2}^p)}{w^p({w_1}^p - {w_2}^p)} \frac{\Phi(w_1) + \Phi(w_2)}{2} \\ &= \frac{\Phi(w_1) + \Phi(w_2)}{2}. \end{aligned}$$

This leads directly to the second inequality in (3.12).

We have the following corollary.

Corollary 3.4. *Let the conditions specified in Theorem 3.4 be satisfied. Then,*

$$\inf_{w \in [w_1, w_2]} \breve{\Phi}(w) = \breve{\Phi}\left(\frac{2^{\frac{1}{p}} w_1 w_2}{({w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right) = \Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{({w_1}^p + {w_2}^p)^{\frac{1}{p}}}\right)$$

and

$$\sup_{w \in [w_1, w_2]} \breve{\Phi}(w) = \breve{\Phi}(w_1) = \breve{\Phi}(w_2) = \frac{\Phi(w_1 + \Phi(w_2))}{2}.$$

Next, we derive a set of novel discrete inequalities for univariate p-harmonic convex functions, which serve as analogues of the result presented in Theorem 2.7.

Theorem 3.5. *Let* $[w_1, w_2] \subseteq (0, \infty)$ *be a interval, and let* $\Phi : [w_1, w_2] \to \mathbb{R}$ *be a p-harmonic convex functions. Then, for every finite sequence* $(w_k)_{k=1}^{n} \in [w_1, w_2]$ *and non-negative real numbers* $a_k \in [0, 1]$, *satisfying* $A_n := \sum_{k=1}^{n} a_k = 1$, *we get*

$$\begin{aligned}
&\Phi\left(\frac{1}{\left(\frac{1}{w_1{}^p} + \frac{1}{w_2{}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right) \qquad (3.13)\\
\geq\ &\Phi\left(\frac{1}{\left(\frac{1}{w_1{}^p} + \frac{1}{w_2{}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right) - \min_{1\leq k\leq n} a_k \left[\sum_{k=1}^{n} \Phi(w_k) - n\Phi\left(\frac{1}{\frac{1}{n}\sum_{k=1}^{n} \frac{1}{w_k^p}}\right)^{\frac{1}{p}}\right]\\
\geq\ &\Phi\left(\frac{1}{\left(\frac{1}{w_1{}^p} + \frac{1}{w_2{}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right) - \left[\sum_{k=1}^{n} a_k\Phi(w_k) - \Phi\left(\frac{1}{\sum_{k=1}^{n} \frac{a_k}{w_k^p}}\right)^{\frac{1}{p}}\right]\\
\geq\ &2\Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{(w_1{}^p + w_2{}^p)^{\frac{1}{p}}}\right) - \sum_{k=1}^{n} a_k\Phi(w_k).
\end{aligned}$$

Proof. Since $\frac{1}{\left(\frac{1}{w_1{}^p} + \frac{1}{w_2{}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}, \frac{2^{\frac{1}{p}} w_1 w_2}{(w_1{}^p + w_2{}^p)^{\frac{1}{p}}}$, by applying the p-harmonic convexity of Φ over the interval $[w_1, w_2]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left[\Phi\left(\frac{1}{\left(\frac{1}{w_1{}^p} + \frac{1}{w_2{}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right) + \Phi\left(\frac{1}{\sum_{k=1}^{n} \frac{a_k}{w_k^p}}\right)^{\frac{1}{p}}\right] \qquad (3.14)\\
\geq\ &\Phi\left(\frac{1}{\frac{1}{2}\left[\left(\frac{1}{w_1{}^p} + \frac{1}{w_2{}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p} + \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}\right]}\right)\\
=\ &\Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{(w_1{}^p + w_2{}^p)^{\frac{1}{p}}}\right).
\end{aligned}$$

Equivalently,

$$\Phi\left(\frac{1}{\left(\frac{1}{w_1{}^p} + \frac{1}{w_2{}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right) + \Phi\left(\frac{1}{\sum_{k=1}^{n} \frac{a_k}{w_k^p}}\right)^{\frac{1}{p}} \geq 2\Phi\left(\frac{2^{\frac{1}{p}} w_1 w_2}{(w_1{}^p + w_2{}^p)^{\frac{1}{p}}}\right). \qquad (3.15)$$

By subtracting from both sides of inequality (3.15) the same quantity $\sum_{k=1}^{n} a_k\Phi(w_k)$, we obtain

$$\Phi\left(\frac{1}{\left(\frac{1}{w_1{}^p} + \frac{1}{w_2{}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right) - \left[\sum_{k=1}^{n} a_k\Phi(w_k) - \Phi\left(\frac{1}{\sum_{k=1}^{n} \frac{a_k}{w_k^p}}\right)^{\frac{1}{p}}\right] \qquad (3.16)$$

$$\geq \quad 2\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p+\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right)-\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w_k}).$$

Applying the first inequality from (2.9), after multiplying by (-1), we obtain

$$-\left[\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w_k})-\Phi\left(\frac{1}{\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k^p}}\right)^{\frac{1}{p}}\right]\leq-\min_{1\leq\mathrm{k}\leq\mathrm{n}}\{\mathrm{a_k}\}\left[\sum_{\mathrm{k}=1}^{\mathrm{n}}\Phi(\mathrm{w}_k)-\mathrm{n}\Phi\left(\frac{1}{\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{1}{\mathrm{w}_k^p}}\right)^{\frac{1}{p}}\right].$$

By adding $\left(\frac{1}{\left(\frac{1}{\mathrm{w}_1{}^p}+\frac{1}{\mathrm{w}_2{}^p}-\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k^p}\right)^{\frac{1}{p}}}\right)$ on the both sides of above inequality, we get

$$\left(\frac{1}{\left(\frac{1}{\mathrm{w}_1{}^p}+\frac{1}{\mathrm{w}_2{}^p}-\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k^p}\right)^{\frac{1}{p}}}\right)-\left[\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w}_k)-\Phi\left(\frac{1}{\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k^p}}\right)^{\frac{1}{p}}\right] \tag{3.17}$$

$$\leq \left(\frac{1}{\left(\frac{1}{\mathrm{w}_1{}^p}+\frac{1}{\mathrm{w}_2{}^p}-\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k^p}\right)^{\frac{1}{p}}}\right)-\min_{1\leq\mathrm{k}\leq\mathrm{n}}\{\mathrm{a_k}\}\left[\sum_{\mathrm{k}=1}^{\mathrm{n}}\Phi(\mathrm{w}_k)-\mathrm{n}\Phi\left(\frac{1}{\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{1}{\mathrm{w}_k^p}}\right)^{\frac{1}{p}}\right].$$

By applying inequalities (3.16) and (3.17), we obtain the second and third inequalities presented in (3.13).

Corollary 3.5. *With the conditions specified in Theorem 3.1, we obtain*

$$\frac{1}{2}\left[\Phi\left(\frac{1}{\left(\frac{1}{\mathrm{w}_1{}^p}+\frac{1}{\mathrm{w}_2{}^p}-\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k^p}\right)^{\frac{1}{p}}}\right)+\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w}_k)\right]-\Phi\left(\frac{2^{\frac{1}{p}}\mathrm{w}_1\mathrm{w}_2}{(\mathrm{w}_1{}^p+\mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) \tag{3.18}$$

$$\geq \frac{1}{2}\left[\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}\Phi(\mathrm{w_k})-\Phi\left(\frac{1}{\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{\mathrm{a_k}}{\mathrm{w}_k^p}}\right)^{\frac{1}{p}}\right]-\frac{1}{2}\min_{1\leq\mathrm{k}\leq\mathrm{n}}\{\mathrm{a_k}\}\left[\sum_{\mathrm{k}=1}^{\mathrm{n}}\Phi(\mathrm{w}_k)-\mathrm{n}\Phi\left(\frac{1}{\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\frac{1}{\mathrm{w}_k^p}}\right)^{\frac{1}{p}}\right] \tag{3.19}$$

$$\geq 0,$$

$\forall(\mathrm{w}_k)_{\mathrm{k}=1}^{\mathrm{n}}\in[\mathrm{w}_1,\mathrm{w}_2]$ *and* $\mathrm{a_k}\in[0,1],\ p>0$ *for* $\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{a_k}=1.$

Remark 3.1.
- *Classical and Harmonic convexities are retrieved from p-harmonic convexity as a limiting case.*
- *Upon letting* $p\to1$*, Theorems 3.1 and 3.2 refine and coincide with Theorems 2.1 and 2.4 of [24].*
- *Upon letting* $p\to1$*, our main results of subsection of symmetric and anti-symmetric inequalities refine as well as coincide with Baloch's symmetric and anti-symmetric inequalities [24].*

4. Applications with graphical representation

In this section, $l_p(\lambda)$ and $L_p(\lambda)$ retain the same definitions as those introduced in Theorem 3.1.

Let $\Phi : [\mathrm{w}_1, \mathrm{w}_2] \subset (0, \infty)$ map into $\mathbb{R}$ such that Φ is a p-harmonic convex function defined by $\Phi(\mathrm{w}) = \mathrm{w}^p$, $p > 0$. Then by using inequality (3.1) we get

$$\frac{2(\mathrm{w}_1\mathrm{w}_2)^p}{\mathrm{w}_1{}^p + \mathrm{w}_2{}^p} \le l_p(\lambda) \le \frac{p\mathrm{w}_1{}^p\mathrm{w}_2{}^p}{\mathrm{w}_2{}^p - \mathrm{w}_1{}^p}(\ln \mathrm{w}_2 - \ln \mathrm{w}_1) \le L_p(\lambda) \le \frac{\mathrm{w}_1{}^p + \mathrm{w}_2{}^p}{2}. \tag{4.1}$$

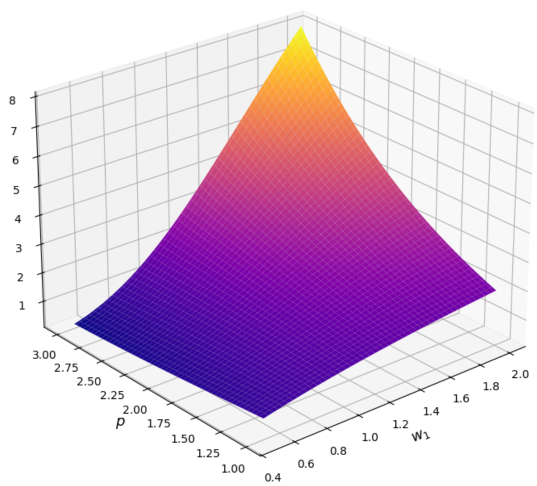
Inequality (4.1) provides a refinement of the result established in [17].

The numerical validity of inequality (4.1) is illustrated in Table 5.

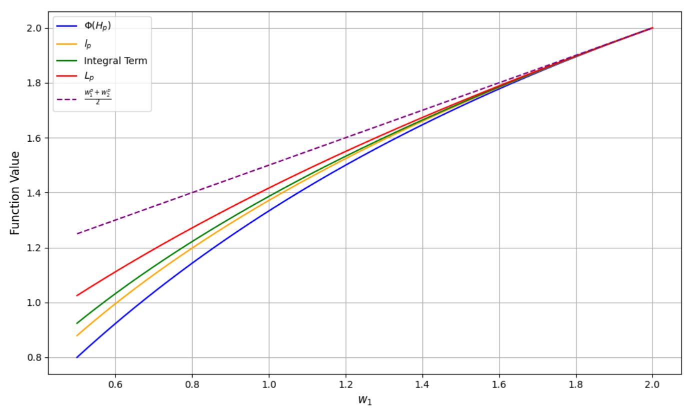
Table 5. Numerical verification of the inequality chain for $\Phi(\mathrm{w}) = \mathrm{w}^p$ with $\mathrm{w}_1 = 1$ and $\mathrm{w}_2 = 2$.

p	$\Phi(H_p)$	l_p	Integral term	L_p	$\frac{\mathrm{w}_1{}^p + \mathrm{w}_2{}^p}{2}$
1.0	1.3333	1.3714	1.3863	1.4167	1.5000
1.5	1.4780	1.5660	1.6090	1.6956	1.9142
2.0	1.6000	1.7582	1.8484	2.0500	2.5000
3.0	1.7778	2.0925	2.3760	3.1380	4.5000

The graphical representation of inequality (4.1) is shown in Figure 3.



(a) 3D plot illustrating the inequality visualization



(b) 2D plot illustrating the inequality visualization

Figure 3. Inequality visualization for $p = 1$.

Since $\Phi(\mathrm{w}) = \mathrm{w}, \forall \mathrm{w} \in (0, \infty)$ is a p-harmonic convex function, applying inequality (2.7) yields

$$\frac{1}{\left(\sum_{k=1}^{n} \frac{a_k}{(\mathrm{w}_k)^p}\right)^{\frac{1}{p}}} \le \sum_{k=1}^{n} a_k \mathrm{w}_k. \tag{4.2}$$

Taking

$$w_k = k \;\; with \;\; a_k = \frac{1}{n} \, (1 \le k \le n) \tag{4.3}$$

in inequality (4.2), we get

$$\frac{2^p n}{(n+1)^p} \le \sum_{w_1=1}^{n} \frac{1}{{w_1}^p}. \tag{4.4}$$

Now, using inequality (2.8) for $\Phi(w) = w^p, \forall w \in (0, \infty)$, we have

$$\frac{1}{\left(\frac{1}{{w_1}^p} + \frac{1}{{w_2}^p} - \sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}} \le w_1 + w_2 - \sum_{k=1}^{n} a_k w_k, \tag{4.5}$$

and under condition (4.3), inequality (4.5) becomes

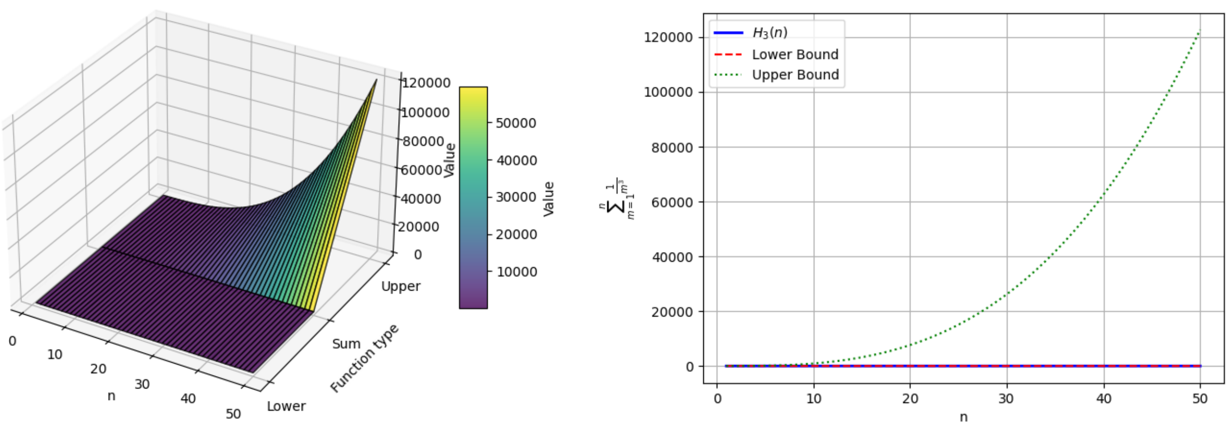
$$\sum_{w_1=1}^{n} \frac{1}{{w_1}^p} \le \frac{n^{p+1} + 1}{n+1}. \tag{4.6}$$

Hence, by combining inequalities (4.2) and (4.6), we get

$$\frac{2^p n}{(n+1)^p} \le \sum_{w_1=1}^{n} \frac{1}{{w_1}^p} \le \frac{n^{p+1} + 1}{n+1}. \tag{4.7}$$

The numerical validity of inequality (4.7) is illustrated in Table 6.

The graphical representation of inequality (4.7) is shown in Figure 4.



(a) 3D plot illustrating the inequality visualization

(b) 2D plot illustrating the inequality visualization

Figure 4. Inequality visualization for $p = 3$.

Table 6. Numerical verification of $\frac{2^p n}{(n+1)^p} \leq H_p(n) \leq \frac{n^{p+1}+1}{n+1}$ for $p = 1, 2, 3$.

n	p	Lower Bound $\dfrac{2^p n}{(n+1)^p}$	$H_p(n) = \sum_{w_1=1}^{n} \frac{1}{w_1{}^p}$	Upper Bound $\dfrac{n^{p+1}+1}{n+1}$	Holds
2	1	1.333333	1.500000	1.666667	Yes
3	1	1.500000	1.833333	2.500000	Yes
5	1	1.666667	2.283333	4.333333	Yes
10	1	1.818182	2.928968	9.181818	Yes
20	1	1.904762	3.597740	19.095238	Yes
50	1	1.960784	4.499205	49.039216	Yes
100	1	1.980198	5.187378	99.019802	Yes
2	2	0.888889	1.250000	3.000000	Yes
3	2	1.125000	1.361111	7.000000	Yes
5	2	1.111111	1.463611	21.000000	Yes
10	2	0.991736	1.549768	91.000000	Yes
20	2	0.907029	1.596163	381.000000	Yes
50	2	0.784314	1.625133	2451.000000	Yes
100	2	0.635604	1.634984	9901.000000	Yes
2	3	0.592593	1.125000	5.666667	Yes
3	3	0.750000	1.162037	20.500000	Yes
5	3	0.790123	1.185662	104.333333	Yes
10	3	0.726744	1.197532	909.181818	Yes
20	3	0.629737	1.200868	7619.095238	Yes
50	3	0.490197	1.201861	122549.039216	Yes
100	3	0.374532	1.202007	990099.019802	Yes

Let $\Phi : [w_1, w_2] \subset (0, \infty)$ map into $\mathbb{R}$, such that Φ is p-harmonic convex function defined by $\Phi(w) = e^{w^p},\ p \geq 1$. Then, the application of inequality (3.1) yields

$$e^{\frac{2^{\frac{1}{p}} w_1 w_2}{[w_1{}^p + w_2{}^p]^{\frac{1}{p}}}} \leq l_p(\lambda) \leq \frac{p(w_1 w_2)^p}{w_2{}^p - w_1{}^p} \int_{w_1}^{w_2} \frac{e^{w^p}}{w^{p+1}} dw \leq L_p(\lambda) \leq \frac{e^{w_1{}^p} + e^{w_2{}^p}}{2}. \tag{4.8}$$

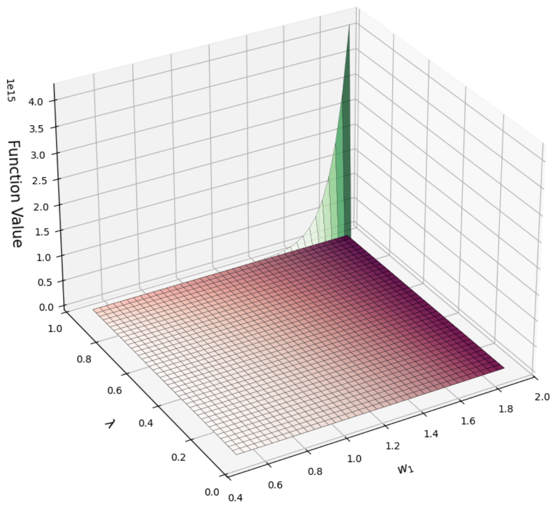
Inequality (4.8) is a refinement of the inequality presented in [17].

The numerical validity of inequality (4.8) is illustrated in Table 7.

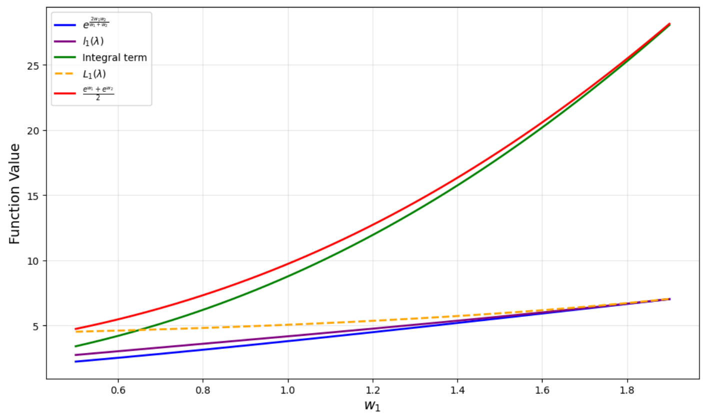
Table 7. Numerical verification of the inequality chain for $w_1 = 0.5$, $w_2 = 2.0$, and $\lambda = 0.5$.

p	$\exp\left(\dfrac{2^{1/p} w_1 w_2}{(w_1{}^p + w_2{}^p)^{1/p}}\right)$	$l_p(\lambda)$	$\dfrac{p(w_1 w_2)^p}{w_2{}^p - w_1{}^p} \displaystyle\int_{w_1}^{w_2} \frac{e^{w^p}}{w^{p+1}}\, dw$	$L_p(\lambda)$	$\dfrac{e^{w_1{}^p} + e^{w_2{}^p}}{2}$
1	2.225541	3.401700	2.735286	4.735961	4.518889
2	1.985745	2.204558	3.109329	15.289704	27.941088
3	1.871449	1.967348	9.929974	746.624052	1491.045568

The graphical representation of inequality (4.8) is shown in Figure 5.



(a) 3D plot illustrating the inequality visualization



(b) 2D plot illustrating the inequality visualization

Figure 5. Inequality visualization for $p = 1$.

Assume $\Phi : [w_1, w_2] \subset (0, \infty)$ maps into $\mathbb{R}$ such that Φ is p-harmonic convex function defined by $\Phi(w) = w^{p+1} e^{w^{p+1}},\ p > 0$. Then, using inequality (3.1) we get

$$\left(\frac{2^{\frac{1}{p}} w_1 w_2}{[w_1{}^p + w_2{}^p]^{\frac{1}{p}}}\right)^{p+1} e^{\left(\frac{2^{\frac{1}{p}} w_1 w_2}{[w_1{}^p + w_2{}^p]^{\frac{1}{p}}}\right)^{p+1}} \leq l_p(\lambda) \leq \frac{p(w_1 w_2)^p}{w_2{}^p - w_1{}^p} \int_{w_1}^{w_2} e^{w^{p+1}} dw$$
$$\leq L_p(\lambda) \leq \frac{w_1{}^{p+1} e^{w_1{}^{p+1}} w_2{}^{p+1} e^{w_2{}^{p+1}}}{2}. \tag{4.9}$$

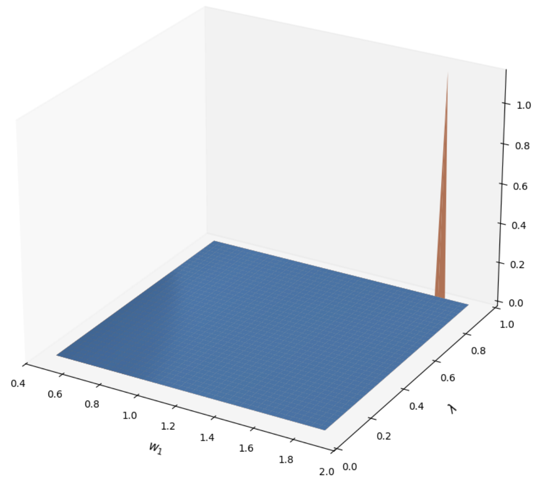
Thus, inequality (4.9) is a refinement of inequality presented in [17].

The numerical validity of inequality (4.9) is illustrated in Table 8.

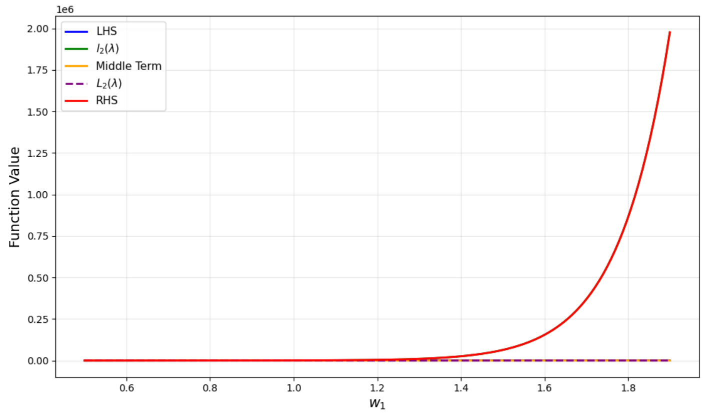
Table 8. Numerical verification of the exponential p-harmonic inequality for $\Phi(w) = e^{w^{p+1}}$. All values rounded to 6 decimals.

p	w_1	w_2	λ	LHS	$l_p(\lambda)$	Middle	$L_p(\lambda)$	RHS
1.0	0.5	2.0	0.5	1.493105	1.892300	2.002516	2.152884	2.356560
1.5	0.8	3.0	0.3	2.174352	2.951205	3.142683	3.609848	4.021700
2.0	0.8	3.0	0.3	1.951433	2.486719	2.679300	3.041177	3.485521

The graphical representation of inequality (4.9) is shown in Figure 6.



(a) 3D plot illustrating the inequality visualization



(b) 2D plot illustrating the inequality visualization

Figure 6. Inequality visualization for $p = 1$.

Let $\Phi : [\mathrm{w}_1, \mathrm{w}_2] \subset (0, \infty)$ map into $\mathbb{R}$ such that Φ is p-harmonic convex function defined by $\Phi(\mathrm{w}) = \ln \mathrm{w}$, $p > 0$. Then, using inequality (3.1) we get

$$\frac{2(\mathrm{w}_1\mathrm{w}_2)^p}{\mathrm{w}_1{}^p + \mathrm{w}_2{}^p} \leq \exp(l_p(\lambda)) \leq \exp\left(\frac{\mathrm{w}_1{}^{\mathrm{w}_2{}^p}}{\mathrm{w}_2{}^{\mathrm{w}_1{}^p}}\right)^{\frac{1}{\mathrm{w}_2{}^p - \mathrm{w}_1{}^p}} \leq \exp(L_p(\lambda)) \leq (\mathrm{w}_1\mathrm{w}_2)^{\frac{p}{2}}. \tag{4.10}$$

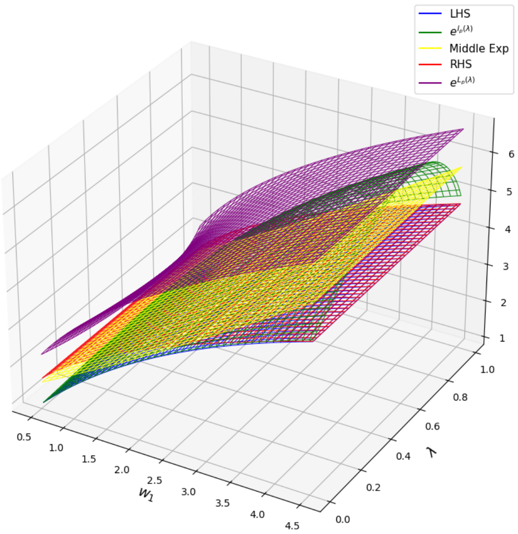
Inequality (4.10) is a refinement of inequality presented in [17].

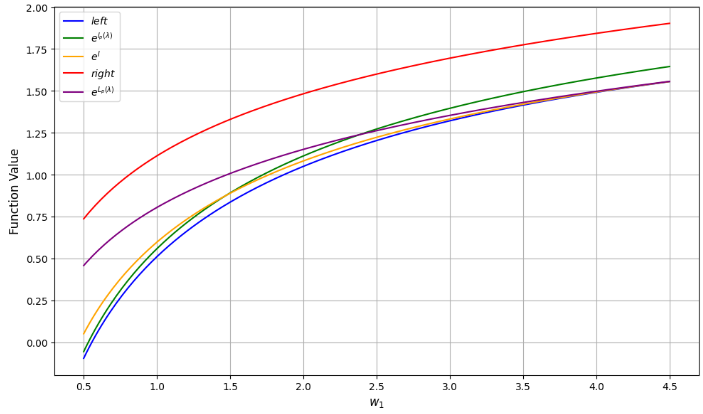
The numerical validity of inequality (4.10) is illustrated in Table 9.

Table 9. Numerical verification of the inequality chain for $\Phi(\mathrm{w}) = \ln \mathrm{w}$ (All values are rounded to 6 decimals).

p	w_1	w_2	λ	Left	$e^{l_p(\lambda)}$	e^I	$e^{L_p(\lambda)}$	Right
1.5	0.5	5.0	0.05	0.685432	0.786180	0.903325	1.912045	1.988177
1.5	0.5	5.0	0.10	0.685432	0.794999	0.903325	1.836784	1.988177
2.0	0.5	5.0	0.10	0.495050	0.757318	0.805409	1.254309	2.500000
3.0	0.5	5.0	0.05	0.249750	0.632989	0.696200	1.689859	3.952847

The graphical representation of inequality (4.10) is shown in Figure 7.





(b) 2D plot illustrating the inequality visualization

(a) 3D plot illustrating the inequality visualization

Figure 7. Inequality visualization for $p = 1$.

Next, since $\Phi(\mathrm{w}) = \ln \mathrm{w}, \forall \Phi \in (0, \infty)$, is a p-harmonically convex function and concave, the application of inequality (2.7) together with the classical Jensen's inequality and its generalized form, yields

$$\frac{1}{\left(\sum_{k=1}^{n} \frac{a_k}{w_k^p}\right)^{\frac{1}{p}}} \leq \prod_{k=1}^{n} w_k^{a_k}, \tag{4.11}$$

and

$$\prod_{k=1}^{n} w_k^{a_k} \leq \sum_{k=1}^{n} a_k w_k, \tag{4.12}$$

$$(w_1 w_2)^{\frac{1}{p}} \prod_{k=1}^{n} w_k^{-\frac{a_k}{p}} \leq \left(w_1{}^p + w_2{}^p - \sum_{k=1}^{n} a_k w_k^p\right)^{\frac{1}{p}}. \tag{4.13}$$

Now, by applying inequality (2.8) to $\mathscr{G}(x) = \ln x, \forall x \in (0, \infty)$, we obtain

$$\prod_{k=1}^{n} w_k^{a_k} \leq (w_1 + w_2) - w_1 w_2 \sum_{k=1}^{n} \frac{a_k}{w_k^p} \tag{4.14}$$

or

$$\left(w_1{}^{-1} + w_2{}^{-1} - \sum_{k=1}^{n} a_k w_k^{-p}\right)^{-1} \leq w_1 w_2 \prod_{k=1}^{n} w_k^{-a_k}. \tag{4.15}$$

Here, we propose a conjecture and leave its proof for future investigation, as stated below:

$$\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{x_k a_k} \leq (\mathrm{w}_1 + \mathrm{w}_2) - \mathrm{w}_1\mathrm{w}_2 \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w}_k^p}. \tag{4.16}$$

Hence, from inequalities (4.2), (4.11), (4.13), (4.14), and (4.16), we deduce that

$$\frac{1}{\left(\sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w}_k^p}\right)^{\frac{1}{p}}} \leq \prod_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{w}_k^{\mathrm{a_k}} \leq \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a_k w}_k \leq (\mathrm{w}_1 + \mathrm{w}_2) - \mathrm{w}_1\mathrm{w}_2 \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w}_k^p}.$$

However, from inequalities (4.2) and (4.11), we conclude that the given inequality is a weighted p-harmonic, geometric and arithmetic mean inequality:

$$\frac{1}{\left(\sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w}_k^p}\right)^{\frac{1}{p}}} \leq \prod_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{w}_k^{\mathrm{a_k}} \leq \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a_k w}_k. \tag{4.17}$$

Remark 4.1. *Inequality (4.17) establishes a connection among the harmonic mean, geometric mean and arithmetic mean inequality. Upon letting* $p \to 1$, *inequality (4.17) refines as well as concides with inequality (3.13) in [25].*

The extended version of this inequality is in the form of the p-harmonic mean ($H_p.M$), harmonic mean ($H.M$), geometric mean ($G.M$), and arithmetic mean ($A.M$) inequality as follows:

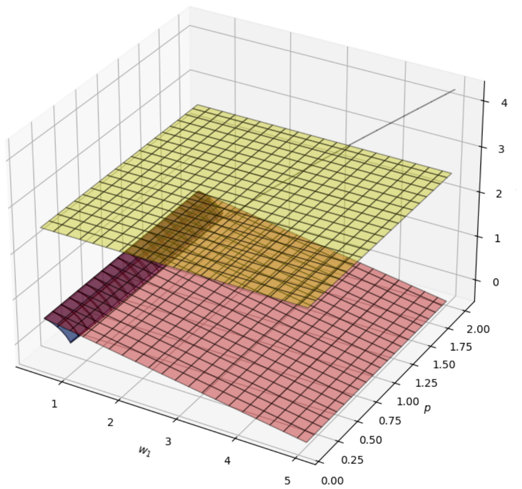
$$\begin{array}{ccccccc} \frac{1}{\left(\sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w}_k^p}\right)^{\frac{1}{p}}} & \leq & \frac{1}{\sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{x_k}}} \leq \prod_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{w}_k^{\mathrm{a_k}} \leq \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a_k w}_k, \\ H_p.M & \leq & H.M \leq G.M \leq A.M. \end{array} \tag{4.18}$$

The numerical validity of inequality (4.18) is illustrated in Table 10.

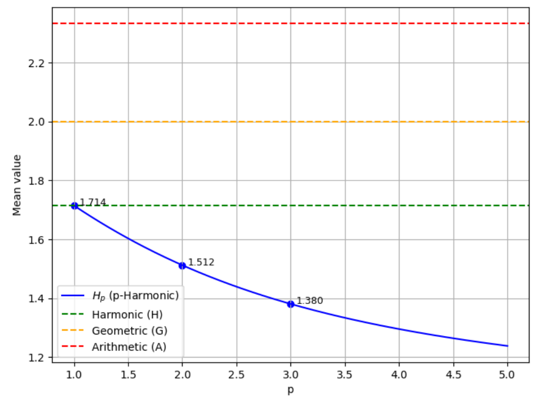
Table 10. Comparison of $H_p.M$, $H.M$, $G.M$, and $A.M$ for $\mathbf{w} = (1, 2, 4)$ with equal weights.

p	$H_p.M$	$H.M$	$G.M$	$A.M$
1	1.714	1.714	2.000	2.333
2	1.511	1.714	2.000	2.333
3	1.374	1.714	2.000	2.333

The graphical representation of inequality (4.18) is shown in Figure 8.



(a) 3D plot illustrating the inequality visualization



(b) 2D plot illustrating the inequality visualization

Figure 8. Inequality visualization for $p = 1, 2, 3$.

Consider $\Phi : [\mathrm{w}_1, \mathrm{w}_2] \subset (0, \infty)$ maps into $\mathbb{R}$, such that Φ is p-harmonic convex function defined by $\Phi(\mathrm{w}) = \ln \mathrm{w}$, where $p \geq 1$, $\mathrm{w_k} \in [\mathrm{w}_1, \mathrm{w}_2], \mathrm{a_k} \geq 0$, for $\mathrm{k} \in \{1, \ldots, \mathrm{n}\}$ and $\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a_k} = 1$. Applying inequality (3.18), we get

$$\begin{aligned} &\frac{1}{2}\left[\ln\left(\frac{1}{\left(\frac{1}{\mathrm{w}_1{}^p} + \frac{1}{\mathrm{w}_2{}^p} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w_k^{\mathit{p}}}}\right)^{\frac{1}{p}}}\right) + \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a_k} \ln(\mathrm{w_k})\right] - \ln\left(\frac{2^{\frac{1}{p}} \mathrm{w}_1 \mathrm{w}_2}{(\mathrm{w}_1{}^p + \mathrm{w}_2{}^p)^{\frac{1}{p}}}\right) \qquad (4.19) \\ \geq\ & \frac{1}{2}\left[\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a_k} \ln(\mathrm{w}_k) - \ln\left(\frac{1}{\sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w_k^{\mathit{p}}}}}\right)^{\frac{1}{p}}\right] - \frac{1}{2} \min_{1\leq \mathrm{k} \leq \mathrm{n}} \{\mathrm{a_k}\}\left[\sum_{\mathrm{k}=1}^{\mathrm{n}} \ln(\mathrm{w_k}) - \mathrm{n} \ln\left(\frac{1}{\frac{1}{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{\mathrm{w_k^{\mathit{p}}}}}\right)^{\frac{1}{p}}\right] \\ \geq\ & 0, \end{aligned}$$

which is equivalent to

$$\left(\frac{\prod_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{w_k^{a_k}}}{\left(\frac{1}{\mathrm{w}_1{}^p} + \frac{1}{\mathrm{w}_2{}^p} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w}_k^{p}}\right)^{-\frac{1}{p}}}\right)^{\frac{1}{2}} \left(\frac{\mathrm{w}_1{}^p + \mathrm{w}_2{}^p}{2\mathrm{w}_1{}^p \mathrm{w}_2{}^p}\right)^{\frac{1}{p}} \geq \left(\frac{\left(\prod_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{w}_k^{\mathrm{a_k}}\right)\left(\sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w}_k^{p}}\right)^{-\frac{1}{p}}}{\left[\left(\prod_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{w}_k\right)\left(\frac{1}{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{\mathrm{w}_k^{p}}\right)^{-\frac{\mathrm{n}}{p}}\right]^{\min_{1\leq\mathrm{k}\leq\mathrm{n}}\{\mathrm{a_k}\}}}\right)^{\frac{1}{2}} \geq 1. \qquad (4.20)$$

Next, let us take $\Phi : [\mathrm{w}_1, \mathrm{w}_2] \subset (0, \infty)$, mapping into $\mathbb{R}$ such that Φ is p-harmonic convex function defined by $\Phi(\mathrm{w}) = \mathrm{w}$, $p > 0$, $\mathrm{w_k} \in [\mathrm{w}_1, \mathrm{w}_2], \mathrm{a_k} \geq 0$, for $\mathrm{k} \in \{1, \ldots, \mathrm{n}\}$. By applying inequalities (3.18) and (2.8), it follows that

$$\frac{1}{2}\left[\left(\frac{1}{\left(\frac{1}{\mathrm{w}_1{}^p} + \frac{1}{\mathrm{w}_2{}^p} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{a_k}}{\mathrm{w_k^{\mathit{p}}}}\right)^{\frac{1}{p}}}\right) + \frac{1}{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{w_k}\right] \geq \left(\frac{2\mathrm{w}_1{}^p \mathrm{w}_2{}^p}{\mathrm{w}_1{}^p + \mathrm{w}_2{}^p}\right)^{\frac{1}{p}}, \qquad (4.21)$$

and

$$\frac{1}{2}\left[\left(\frac{1}{\left(\frac{1}{w_1{}^p}+\frac{1}{w_2{}^p}-\sum_{k=1}^{n}\frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right)+\frac{1}{n}\sum_{k=1}^{n}w_k\right] \leq \left(\frac{w_1{}^p+w_2{}^p}{2}\right)^{\frac{1}{p}}. \tag{4.22}$$

From (4.21) and (4.22), we get

$$\left(\frac{2w_1{}^p w_2{}^p}{w_1{}^p+w_2{}^p}\right)^{\frac{1}{p}} \leq \frac{1}{2}\left[\left(\frac{1}{\left(\frac{1}{w_1{}^p}+\frac{1}{w_2{}^p}-\sum_{k=1}^{n}\frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right)+\frac{1}{n}\sum_{k=1}^{n}w_k\right] \leq \left(\frac{w_1{}^p+w_2{}^p}{2}\right)^{\frac{1}{p}}, \tag{4.23}$$

and hence, using inequality (4.23), we get another improvement of inequality (2.2) presented in [18] as follows:

$$\begin{aligned}\left(\frac{2w_1{}^p w_2{}^p}{w_1{}^p+w_2{}^p}\right)^{\frac{2}{p}} &\leq \frac{1}{4}\left[\left(\frac{1}{\left(\frac{1}{w_1{}^p}+\frac{1}{w_2{}^p}-\sum_{k=1}^{n}\frac{a_k}{w_k^p}\right)^{\frac{1}{p}}}\right)+\frac{1}{n}\sum_{k=1}^{n}w_k\right]^2 \\ &\leq \left(\frac{w_1{}^p+w_2{}^p}{2}\right)^{\frac{2}{p}} \leq \frac{1}{3}(w_1{}^{2p}+w_1{}^p w_2{}^p+w_2{}^{2p})^{\frac{1}{p}} \\ &\leq \left(\frac{w_1{}^{2p}+w_2{}^{2p}}{2}\right)^{\frac{1}{p}}.\end{aligned} \tag{4.24}$$

5. Conclusions

In this study, we developed refined Hermite-Hadamard and Fejér-type inequalities for p-harmonic convex functions and extended these results to a more general class of functions satisfying p-harmonic convexity. Sharp bounds are established for this generalized framework, offering deeper insights into the structural characteristics of p-harmonic convex functions. To illustrate the effectiveness and applicability of the proposed results, several examples are provided, supported by numerical tables and graphical representations. These applications not only validate the accuracy of the derived bounds, but also emphasize their potential in advancing analytical techniques. Overall, the results and methodologies presented in this work contribute meaningfully to the growing theory of p-harmonic convexity and are expected to stimulate further research in areas such as real analysis, fractional calculus, integral inequalities, and optimization theory.

Author contributions

F. Azhar, M. I. Asjad and I. A. Baloch: Conceptualization, Methodology; F. Azhar, I. A. Baloch and M. De la Sen: Investigation; F. Azhar: Software, Writing–original draft; M. De la Sen: Funding; M. I. Asjad and I. A. Baloch: Supervision; F. Azhar, M. I. Asjad, I. A. Baloch and M. De la Sen: Writing–review & editing. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors are grateful to the Basque Government for funding this research work through Grant IT1555-22.

Conflict of interest

The authors declare no conflicts of interest.

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