AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 30053–30067.
DOI: 10.3934/math.20251321
Received: 23 September 2025
Revised: 01 December 2025
Accepted: 09 December 2025
Published: 22 December 2025

Research article

Approximating common fixed points of non-expansive type multi-valued mappings in convex metric space

Tanveer Hussain[1,*], **Abdul Rahim Khan**[1], **Kiran Riaz**[1] **and Hind Alamri**[2]

[1] Department of Mathematics and Statistics, University of Southern Punjab, Multan, Pakistan
[2] Department of Mathematics, College of Science, Taif University, Taif 21944, Saudi Arabia

* **Correspondence:** Email: uspmaths@gmail.com.

Abstract: We propose a one-step iterative scheme to approximate common fixed points of multi-valued non-expansive mappings in a convex metric space Υ, under certain mild boundary conditions. We present a numerical example to verify our convergence result obtained herein. We also establish some convergence results for multi-valued asymptotically non-expansive mappings in this context, which either improve or generalize results obtained for multi-valued non-expansive mappings on Υ. Furthermore, we demonstrate how these findings can be applied to solve a system of two equations defined by multi-valued asymptotically non-expansive mappings on a convex metric space.

Keywords: common fixed point; multi-valued mappings; non-expansive mappings; convex metric space; fixed point iteration; asymptotically non-expansive
Mathematics Subject Classification: 47H09, 47H10, 54E35

1. Introduction

It is widely recognized that fixed point theory is a decisive factor in nonlinear analysis. Banach proved the well-known Banach fixed point theorem in 1922, which has remarkable applications in mathematics and engineering.

Let E be a non-empty closed and bounded subset of a metric space (Υ, Ψ), and $\Omega(\Upsilon)$ represents the class of compact subsets of Υ. Denote by $CB(\Upsilon)$ the family of all closed and bounded subsets of Υ. Suppose that $\mathcal{D}$ is the Hausdorff metric derived from Ψ, i.e.,

$$\mathcal{D}(U, Y) = max\left\{\sup_{\mathfrak{g}\in U} \Psi(\mathfrak{g}, Y), \sup_{\varrho\in Y} \Psi(\varrho, U)\right\}$$

for $U, Y \in CB(\Upsilon)$ and $\Psi(\mathfrak{g}, Y) = \inf\{\Psi(\mathfrak{g}, \varrho) : \varrho \in Y\}$.

A multi-valued map $N : E \longrightarrow CB(E)$ is called non-expansive if

$$\mathcal{D}(N\mathfrak{g}, N\varrho) \leq \Psi(\mathfrak{g}, \varrho),$$

$\forall \mathfrak{g}, \varrho \in E$. A point $\mathfrak{g} \in \Upsilon$ is a fixed point of N if $\mathfrak{g} \in N\mathfrak{g}$. The collection of fixed points of N is expressed by $F(N)$. A point $\mathfrak{g}' \in \Upsilon$ is a common fixed point of M and N if

$$\mathfrak{g}' \in M(\mathfrak{g}') \quad \textit{and} \quad \mathfrak{g}' \in N(\mathfrak{g}').$$

The collection of common fixed points of the maps M and N is expressed by $F(M, N)$, or alternatively by F_1 (for simplicity). We shall write the term common fixed point as CFP throughout the paper.

Takahashi [1] proposed the idea of a convexity in a metric space Υ, characterized by a map $\mathcal{W} : \Upsilon \times \Upsilon \times I \longrightarrow \Upsilon$ satisfying

$$\Psi(s, \mathcal{W}(\mathfrak{g}, \varrho, \eta)) \leq \eta\Psi(s, \mathfrak{g}) + (1 - \eta)\Psi(s, \varrho), \tag{1.1}$$

for all $\mathfrak{g}, \varrho, s \in \Upsilon$ and $\eta \in I = [0, 1]$. A metric space Υ having such structure $\mathcal{W}$ is called a convex metric space (CMS). We denote a CMS by $(\Upsilon, \Psi, \mathcal{W})$ or simply by Υ. A non-empty subset E of Υ is convex if $\mathcal{W}(\mathfrak{g}, \varrho, \eta) \in E$ for all $\mathfrak{g}, \varrho \in E$ and $\eta \in I$.

A CMS, $(\Upsilon, \Psi, \mathcal{W})$, is uniformly CMS if, for every $\varepsilon > 0$, one can find a non-negative function $\mu(\varepsilon)$ such that $\forall \zeta > 0$ and $\mathfrak{g}, \varrho, \tau \in \Upsilon$ with $\Psi(\tau, \mathfrak{g}) \leq \zeta$, $\Psi(\tau, \varrho) \leq \zeta$, and $\Psi(\mathfrak{g}, \varrho) \geq \zeta$, we have

$$\Psi\left(\tau, \mathcal{W}\left(\mathfrak{g}, \varrho, \frac{1}{2}\right)\right) \leq \zeta(1 - \mu).$$

Fixed point results for multi-valued maps in a uniformly CMS have been obtained by Kaewcharoen and Panyanak [2].

Many researchers have investigated results related to multi-valued maps across various spaces by employing different iterative algorithms [3–5]. For further details on numerical approaches to iterative algorithms, interested readers can consult [6].

In 2010, Khan et al. [7] established convergence theorems of a one-step iteration scheme for multi-valued non-expansive maps. Although, this scheme is simple, it requires the condition (C): $\Psi(\mathfrak{g}, \varrho) \leq \Psi(\tau, \varrho)$ for $\varrho \in M\mathfrak{g}$ and $\tau \in N\mathfrak{g}$. Khan and Ahmed [8] studied an iterative scheme in CMS and proved that this scheme converges to a unique CFP of a finite class of asymptotically quasi-non-expansive maps.

Abbas et al. [9] presented weak and strong convergence results under certain fundamental boundary conditions within a real uniformly convex Banach space (UCBS) for multi-valued maps $M, N : E \to CB(E)$ that are non-expansive. Their iterative process goes as follows:

$$\begin{cases} \mathfrak{g}_1 \in E, \\ \mathfrak{g}_{n+1} = g_n\mathfrak{g}_n + h_n\varrho_n + j_n\tau_n, \end{cases}$$

where $\varrho_n \in N\mathfrak{g}_n$ and $\tau_n \in M\mathfrak{g}_n$ such that $\| \varrho_n - \sigma \| \leq \Psi(\sigma, M\mathfrak{g}_n)$ and $\| \tau_n - \sigma \| \leq \Psi(\sigma, N\mathfrak{g}_n)$ whenever σ is a fixed point of both maps M and N, and $\{g_n\}, \{h_n\}$ and $\{j_n\}$ are sequences in $(0, 1)$ satisfying $g_n + h_n + j_n \leq 1$.

Recently, Ahmed et al. [10] studied the family of multi-valued generalized α-non-expansive maps within Banach spaces. They proposed a new iteration scheme designed to approximate fixed points of these maps and established weak and strong convergence results under slightly weaker assumptions.

Fukhar-ud-Din [11] proposed a one-step iterative scheme for non-expansive maps $M, N : E \rightarrow CB(E)$ as follows:

$$\mathfrak{g}_{n+1} = \mathcal{W}\left(N\mathfrak{g}_n, \mathcal{W}\left(M\mathfrak{g}_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}\right), \eta_n\right), \quad (1.2)$$

where $0 < a \leq \eta_n, \theta_n \leq b < 1$, and $\eta_n + \theta_n < 1$.

The algorithm (1.2) for multi-valued non-expansive maps in a CMS is described as follows:

Let N and M be a pair of multi-valued non-expansive maps from E into $CB(E)$, where E is a convex subset of a convex metric space. Consider and $\{\theta_n\}$ as sequences satisfying $0 < a \leq \eta_n, \theta_n \leq b < 1$, and $\eta_n + \theta_n < 1$. Then for $\mathfrak{g}_1 \in E$, construct $\{\mathfrak{g}_n\}$ as

$$\mathfrak{g}_{n+1} = \mathcal{W}(\varrho_n, \mathcal{W}(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}), \eta_n), \quad (1.3)$$

where $\varrho_n \in N\mathfrak{g}_n$ and $\tau_n \in M\mathfrak{g}_n$ such that $\Psi(\varrho_n, \sigma) \leq \Psi(\sigma, N\mathfrak{g}_n)$ and $\Psi(\tau_n, \sigma) \leq \Psi(\sigma, M\mathfrak{g}_n)$ whenever σ is a fixed point of the maps M and N.

In the Banach space setting, (1.3) becomes a one-step iterative scheme of Yao and Chen [12]:

$$\mathfrak{g}_{n+1} = \eta_n N\mathfrak{g}_n + \theta_n \mathfrak{g}_n + (1 - \eta_n - \theta_n)\mathfrak{g}_n.$$

When $M = I$ in (1.2), it reduces to the well-known Mann iterative scheme:

$$\mathfrak{g}_{n+1} = \mathcal{W}(N\mathfrak{g}_n, \mathfrak{g}_n, \eta_n).$$

It is remarked that results of a one-step iteration process for non-expansive maps on a $CAT(0)$ space have been obtained by Uddin et al. [13]. In 2022, Tassaddiq et al. [14] obtained results concerning fixed points of both single-valued and multivalued maps within the framework of a strong b-metric space.

Hussain et al. [15] proposed a multi-valued F-iteration method aimed at approximating fixed points of a certain family of generalized non-expansive multi-valued maps within Banach spaces. For the latest results on this topic, the reader is referred to [16–18].

In this paper, we focus on approximating CFPs of multi-valued non-expansive type maps in a CMS by employing sequences defined in Eqs (1.3) and (3.2).

We require the following technical result.

Lemma 1.1. [11] Suppose Υ is a uniformly CMS with a continuous structure $\mathcal{W}$. Let $\mathfrak{g} \in \Upsilon$ and $\{\eta_n\}$ be a sequence in $[k, t]$ for some $k, t \in (0, 1)$. If $\{\kappa_n\}$ and $\{\omega_n\}$ are sequences in Υ such that $\lim_{n\longrightarrow\infty} \sup \Psi(\kappa_n, \mathfrak{g}) \leq \nu$, $\lim_{n\longrightarrow\infty} \sup \Psi(\omega_n, \mathfrak{g}) \leq \nu$, and $\lim_{n\longrightarrow\infty} \Psi\left(\mathcal{W}\left(\kappa_n, \omega_n, \eta_n\right), \mathfrak{g}\right) = \nu$ for some $\nu \geq 0$, then

$$\lim_{n\longrightarrow\infty} \Psi(\kappa_n, \omega_n) = 0.$$

2. Main results

The existence of a CFP of multi-valued non-expansive maps is provided in the example to follow.

Example 2.1. **(i)** Consider $C = [0, 1]$ with the usual metric $\Psi(\mathfrak{g}, \varrho) = |\mathfrak{g} - \varrho|$. Define two multi-valued maps $N, M : C \to CB(\mathbb{R}^2)$ by

$$N(\mathfrak{g}) = \left\{ \begin{bmatrix} \frac{\mathfrak{g}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\mathfrak{g}}{2} \\ \frac{\mathfrak{g}}{4} \end{bmatrix} \right\}, \quad M(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 \\ \frac{\mathfrak{g}}{3} \end{bmatrix}, \begin{bmatrix} \frac{\mathfrak{g}}{6} \\ \frac{\mathfrak{g}}{3} \end{bmatrix} \right\},$$

for each $\mathfrak{g} \in C$.

To show that N and M are non-expansive, we compute the Hausdorff distance between their images for any $\mathfrak{g}, \varrho \in C$.

For N, the points in $N(\mathfrak{g})$ and $N(\varrho)$ are

$$p_1 = \begin{bmatrix} \frac{\mathfrak{g}}{2} \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} \frac{\mathfrak{g}}{2} \\ \frac{\mathfrak{g}}{4} \end{bmatrix}, \quad q_1 = \begin{bmatrix} \frac{\varrho}{2} \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} \frac{\varrho}{2} \\ \frac{\varrho}{4} \end{bmatrix}.$$

The minimum distance from p_1 to $N(\varrho)$ is

$$\min \|p_1 - q_i\| = \left| \frac{\mathfrak{g}}{2} - \frac{\varrho}{2} \right| = \frac{|\mathfrak{g} - \varrho|}{2}.$$

Similarly, the minimum distance from p_2 to $N(\varrho)$ is

$$\min \|p_2 - q_i\| = \left| \frac{\mathfrak{g}}{4} - \frac{\varrho}{4} \right| = \frac{|\mathfrak{g} - \varrho|}{4}.$$

Taking the supremum over points in $N(\mathfrak{g})$, we get

$$\sup_{p \in N(\mathfrak{g})} \inf_{q \in N(\varrho)} \|p - q\| = \max\left(\frac{|\mathfrak{g} - \varrho|}{2}, \frac{|\mathfrak{g} - \varrho|}{4} \right) = \frac{|\mathfrak{g} - \varrho|}{2}.$$

By symmetry, the same holds for $\sup_{q \in N(\varrho)} \inf_{p \in N(\mathfrak{g})} \|q - p\|$. Therefore,

$$D(N(\mathfrak{g}), N(\varrho)) = \frac{|\mathfrak{g} - \varrho|}{2} \leq |\mathfrak{g} - \varrho|.$$

For M, the points in $M(\mathfrak{g})$ and $M(\varrho)$ are

$$r_1 = \begin{bmatrix} 0 \\ \frac{\mathfrak{g}}{3} \end{bmatrix}, \quad r_2 = \begin{bmatrix} \frac{\mathfrak{g}}{6} \\ \frac{\mathfrak{g}}{3} \end{bmatrix}, \quad s_1 = \begin{bmatrix} 0 \\ \frac{\varrho}{3} \end{bmatrix}, \quad s_2 = \begin{bmatrix} \frac{\varrho}{6} \\ \frac{\varrho}{3} \end{bmatrix}.$$

The minimal distances from points in $M(\mathfrak{g})$ to $M(\varrho)$ are

$$\min \|r_1 - s_i\| = \frac{|\mathfrak{g} - \varrho|}{3}, \quad \min \|r_2 - s_i\| = \frac{|\mathfrak{g} - \varrho|}{6}.$$

Taking the supremum over points in $M(\mathfrak{g})$, we have

$$\sup_{r \in M(\mathfrak{g})} \inf_{s \in M(\varrho)} \|r - s\| = \max\left(\frac{|\mathfrak{g} - \varrho|}{3}, \frac{|\mathfrak{g} - \varrho|}{6} \right) = \frac{|\mathfrak{g} - \varrho|}{3}.$$

By symmetry, the same holds for $\sup_{s \in M(\varrho)} \inf_{r \in M(\mathfrak{g})} \|s - r\|$. Hence,

$$D(M(\mathfrak{g}), M(\varrho)) = \frac{|\mathfrak{g} - \varrho|}{3} \leq |\mathfrak{g} - \varrho|.$$

Since for all $\mathfrak{g}, \varrho \in C$,

$$D(N(\mathfrak{g}), N(\varrho)) \leq |\mathfrak{g} - \varrho|, \quad D(M(\mathfrak{g}), M(\varrho)) \leq |\mathfrak{g} - \varrho|.$$

Therefore, maps N and M are multi-valued non-expansive maps from C into $CB(\mathbb{R}^2)$ and $F_1 = \{0\}$.

(ii) We provide an example for non-interval convex metric space and multi-valued mappings.

Let $X = \{(\mathfrak{g}, \varrho) \in \mathbb{R}^2 : \mathfrak{g} \geq 0, \varrho \geq 0, \mathfrak{g} + \varrho \leq 1\}$ be the convex metric space with the Euclidean metric

$$\Psi((\mathfrak{g}, \varrho), (u, v)) = \sqrt{(\mathfrak{g} - u)^2 + (\varrho - v)^2}.$$

Define multi-valued maps $N, M : X \to CB(X)$ by

$$N(p) = \left\{ \left(\frac{\mathfrak{g}}{2} + \theta, \frac{\varrho}{3} \right), \left(\frac{\mathfrak{g}}{2}, \frac{\varrho}{3} + \theta \right) \right\}$$

and

$$M(p) = \left\{ \left(\frac{\mathfrak{g}}{3} + \eta, \frac{\varrho}{4} \right), \left(\frac{\mathfrak{g}}{3}, \frac{\varrho}{4} + \eta \right) \right\},$$

for each $p = (\mathfrak{g}, \varrho) \in X$, where $\theta, \eta \geq 0$ are fixed parameters with $\theta, \eta \leq \frac{1}{10}$.

The maps N and M are multi-valued non-expansive maps on X, i.e., for all $p, q \in X$,

$$D(N(p), N(q)) \leq \Psi(p, q), \quad D(M(p), M(q)) \leq \Psi(p, q),$$

where D denotes the Hausdorff metric induced by the Euclidean metric Ψ.

Let $p = (\mathfrak{g}, \varrho)$ and $q = (u, v)$ be arbitrary points in X. First, we compute distances between points $n_1 = \left(\frac{\mathfrak{g}}{2} + \theta, \frac{\varrho}{3}\right)$, $n_2 = \left(\frac{\mathfrak{g}}{2}, \frac{\varrho}{3} + \theta\right)$ in $N(p)$ and $m_1 = \left(\frac{u}{2} + \theta, \frac{v}{3}\right)$, $m_2 = \left(\frac{u}{2}, \frac{v}{3} + \theta\right)$ in $N(q)$.

Now, the Euclidean distance is given by

$$\|n_1 - m_1\| = \sqrt{\left(\frac{\mathfrak{g}}{2} - \frac{u}{2}\right)^2 + \left(\frac{\varrho}{3} - \frac{v}{3}\right)^2} = \sqrt{\frac{(\mathfrak{g} - u)^2}{4} + \frac{(\varrho - v)^2}{9}}.$$

Similarly,

$$\|n_2 - m_2\| = \sqrt{\frac{(\mathfrak{g} - u)^2}{4} + \frac{(\varrho - v)^2}{9}}.$$

Using the inequality for the Euclidean norm, we get

$$\|n_i - m_i\| \leq \sqrt{\max\left(\frac{1}{4}, \frac{1}{9}\right)} \cdot \Psi(p, q) = \frac{1}{2}\Psi(p, q).$$

For any point in $N(p)$, the closest point in $N(q)$ is at most $\frac{1}{2}\Psi(p, q)$ away, and vice versa. Therefore,

$$\begin{aligned} D(N(p), N(q)) &= \max\left\{ \sup_{n \in N(p)} \inf_{m \in N(q)} \|n - m\|, \sup_{m \in N(q)} \inf_{n \in N(p)} \|m - n\| \right\} \\ &\leq \frac{1}{2}\Psi(p, q) \leq \Psi(p, q). \end{aligned}$$

Now, points in $M(p)$ are $r_1 = \left(\frac{\mathfrak{g}}{3}+\eta, \frac{\varrho}{4}\right)$, $r_2 = \left(\frac{\mathfrak{g}}{3}, \frac{\varrho}{4}+\eta\right)$, and in $M(q)$ are $s_1 = \left(\frac{u}{3}+\eta, \frac{v}{4}\right)$, $s_2 = \left(\frac{u}{3}, \frac{v}{4}+\eta\right)$.

$$\|r_1 - s_1\| = \sqrt{\left(\frac{\mathfrak{g}}{3}-\frac{u}{3}\right)^2 + \left(\frac{\varrho}{4}-\frac{v}{4}\right)^2} = \sqrt{\frac{(\mathfrak{g}-u)^2}{9} + \frac{(\varrho-v)^2}{16}}.$$

Same concerns as $\|r_2 - s_2\|$.

$$\|r_i - s_i\| \leq \frac{1}{3}\Psi(p,q).$$

Thus,

$$D(M(p), M(q)) \leq \frac{1}{3}\Psi(p,q) \leq \Psi(p,q).$$

Since for all $p, q \in X$,

$$D(N(p), N(q)) \leq \Psi(p,q), \quad D(M(p), M(q)) \leq \Psi(p,q).$$

Therefore, the maps N and M are multi-valued non-expansive maps on the convex metric space X.

Lemma 2.2. Let E be a closed and convex subset of a CMS and M and N be multi-valued non-expansive maps on E with $F_1 \neq \emptyset$. Then, for the sequence $\{\mathfrak{g}_n\}$ generated by (1.3), $\lim_{n\to\infty} \Psi(\mathfrak{g}_n, \sigma)$ exists for every $\sigma \in F_1$.

Proof. Let $\sigma \in F_1$. Applying (1.3), we have

$$\begin{aligned}
\Psi(\mathfrak{g}_{n+1}, \sigma) &= \Psi\left(\mathcal{W}\left(\varrho_n, \mathcal{W}\left(\tau_n, \mathfrak{g}_n, \tfrac{\theta_n}{1-\eta_n}\right), \eta_n\right), \sigma\right) \\
&\leq \eta_n \Psi(\varrho_n, \sigma) + (1-\eta_n)\Psi(\mathcal{W}(\tau_n, \mathfrak{g}_n, \tfrac{\theta_n}{1-\eta_n}), \sigma) \\
&\leq \eta_n \Psi(\varrho_n, \sigma) + (1-\eta_n)\left[\tfrac{\theta_n}{1-\eta_n}\Psi(\tau_n, \sigma) + \left(1 - \tfrac{\theta_n}{1-\eta_n}\right)\Psi(\mathfrak{g}_n, \sigma)\right] \\
&= \eta_n \Psi(\varrho_n, \sigma) + \theta_n \Psi(\tau_n, \sigma) + (1-\eta_n-\theta_n)\Psi(\mathfrak{g}_n, \sigma) \\
&= \eta_n \mathcal{D}(N\mathfrak{g}_n, N\sigma) + \theta_n \mathcal{D}(M\mathfrak{g}_n, M\sigma) + (1-\eta_n-\theta_n)\Psi(\mathfrak{g}_n, \sigma) \\
&\leq \eta_n \Psi(\mathfrak{g}_n, \sigma) + \theta_n \Psi(\mathfrak{g}_n, \sigma) + (1-\eta_n-\theta_n)\Psi(\mathfrak{g}_n, \sigma), \\
\Psi(\mathfrak{g}_{n+1}, \sigma) &\leq \Psi(\mathfrak{g}_n, \sigma),
\end{aligned}$$

for all $\sigma \in F_1$.

This shows that $\{\mathfrak{g}_n\}$ is non-increasing and bounded below; it means the sequence cannot decrease indefinitely without limit. This guarantees the sequence converges to some limit. Hence, $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, \sigma)$ exists for each $\sigma \in F_1$.

Theorem 2.3. Let E be a non-empty closed and convex subset of a CMS Υ equipped with a continuous structure $\mathcal{W}$. Suppose that M and N are multi-valued non-expansive maps on E with $F_1 \neq \emptyset$. Then, for the sequence $\{\mathfrak{g}_n\}$ generated by (1.3), the following holds:

$$\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, M\mathfrak{g}_n) = 0 = \lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, N\mathfrak{g}_n).$$

Proof. By Lemma 2.2, $\lim_{n\to\infty} \Psi(\mathfrak{g}_n, \sigma)$ exists for each $\sigma \in F_1$. Assume that $\lim_{n\to\infty} \Psi(\mathfrak{g}_n, \sigma) = c$.

If $c = 0$, then the result is straightforward.

For $c > 0$, $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_{n+1}, \sigma) = c$ gives that

$$\lim_{n\to\infty} \Psi\left(\mathcal{W}\left(\varrho_n, \mathcal{W}\left(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}\right), \eta_n\right), \sigma\right) = c. \tag{2.1}$$

As N is non-expansive,

$$\mathcal{D}\left(N\mathfrak{g}_n, N\sigma\right) \leq \Psi\left(\mathfrak{g}_n, \sigma\right),$$

and so

$$\lim_{n\longrightarrow\infty} \sup \mathcal{D}\left(N\mathfrak{g}_n, N\sigma\right) \leq \lim_{n\longrightarrow\infty} \sup \Psi\left(\mathfrak{g}_n, \sigma\right) = c. \tag{2.2}$$

Since

$$\begin{aligned} \Psi\left(\mathcal{W}\left(\tau_n, \mathfrak{g}_n, \tfrac{\theta_n}{1-\eta_n}\right), \sigma\right) &\leq \tfrac{\theta_n}{1-\eta_n}\Psi\left(\tau_n, \sigma\right) + \left(1 - \tfrac{\theta_n}{1-\eta_n}\right)\Psi\left(\mathfrak{g}_n, \sigma\right) \\ &= \tfrac{\theta_n}{1-\eta_n}\mathcal{D}\left(M\mathfrak{g}_n, M\sigma\right) + \left(1 - \tfrac{\theta_n}{1-\eta_n}\right)\Psi\left(\mathfrak{g}_n, \sigma\right) \\ &\leq \tfrac{\theta_n}{1-\eta_n}\Psi\left(\mathfrak{g}_n, \sigma\right) + \left(1 - \tfrac{\theta_n}{1-\eta_n}\right)\Psi\left(\mathfrak{g}_n, \sigma\right), \\ \Psi\left(\mathcal{W}\left(\tau_n, \mathfrak{g}_n, \tfrac{\theta_n}{1-\eta_n}\right), \sigma\right) &\leq \Psi\left(\mathfrak{g}_n, \sigma\right). \end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \sup \Psi\left(\mathcal{W}\left(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}\right), \sigma\right) \leq c. \tag{2.3}$$

By Lemma 1.1, with $\mathfrak{g} = \sigma$, $\nu = c$, $g_n = \eta_n$, $\kappa_n = \varrho_n$, $\omega_n = \mathcal{W}\left(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}\right)$ and using (2.1)–(2.3), we have

$$\lim_{n\longrightarrow\infty} \Psi\left(\varrho_n, \mathcal{W}\left(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}\right)\right) = 0. \tag{2.4}$$

Now,

$$\begin{aligned} \Psi(\mathfrak{g}_{n+1}, \varrho_n) &= \Psi\left(\mathcal{W}\left(\varrho_n, \mathcal{W}\left(\tau_n, \mathfrak{g}_n, \tfrac{\theta_n}{1-\eta_n}\right), \eta_n\right), \varrho_n\right) \\ &\leq \eta_n\mathcal{D}\left(N\mathfrak{g}_n, N\mathfrak{g}_n\right) + \left(1-\eta_n\right)\Psi\left(\mathcal{W}\left(\tau_n, \mathfrak{g}_n, \tfrac{\theta_n}{1-\eta_n}\right), \varrho_n\right) \\ &\leq \eta_n\Psi\left(\mathfrak{g}_n, \mathfrak{g}_n\right) + \left(1-\eta_n\right)\Psi\left(\mathcal{W}\left(\tau_n, \mathfrak{g}_n, \tfrac{\theta_n}{1-\eta_n}\right), \varrho_n\right) \\ &\leq \left(1-\eta_n\right)\Psi\left(\mathcal{W}\left(\tau_n, \mathfrak{g}_n, \tfrac{\theta_n}{1-\eta_n}\right), \varrho_n\right) \end{aligned}$$

given by (2.4),

$$\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_{n+1}, \varrho_n) = 0. \tag{2.5}$$

Since M is non-expansive, therefore, it follows that

$$\lim_{n\longrightarrow\infty} \sup \Psi\left(\tau_n, \sigma\right) \leq c.$$

Now by the triangular inequality, we obtain $\Psi(\mathfrak{g}_{n+1}, \sigma) \leq \Psi(\mathfrak{g}_{n+1}, \varrho_n) + \Psi(\varrho_n, \mathcal{W}(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n})) + \Psi(\mathcal{W}(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}), \sigma)$.

Taking $\lim\inf_{n\to\infty}$ and using (2.4) and (2.5), we get

$$c \leq \lim_{n\longrightarrow\infty} \inf \Psi\left(\mathcal{W}(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}), \sigma\right),$$

which is given by (2.3),

$$\lim_{n\longrightarrow\infty} \Psi\left(\mathcal{W}(\tau_n, \mathfrak{g}_n, \frac{\theta_n}{1-\eta_n}), \sigma\right) = c. \tag{2.6}$$

By Lemma 1.1, it follows that

$$\lim_{n\longrightarrow\infty} \Psi\left(\mathfrak{g}_n, \tau_n\right) = 0. \tag{2.7}$$

Further note that

$$\Psi(\mathfrak{g}_{n+1},\mathfrak{g}_n)\leq\Psi(\mathfrak{g}_{n+1},\varrho_n)+\Psi(\varrho_n,\mathcal{W}(\tau_n,\mathfrak{g}_n,\tfrac{\theta_n}{1-\eta_n}))+\Psi(\mathcal{W}(\tau_n,\mathfrak{g}_n,\tfrac{\theta_n}{1-\eta_n},\mathfrak{g}_n)).$$

Taking the limit on both sides

$$\lim_{n\to\infty}\Psi(\mathfrak{g}_{n+1},\mathfrak{g}_n)\leq\lim_{n\to\infty}\Psi(\mathfrak{g}_{n+1},\varrho_n)+\lim_{n\to\infty}\Psi(\varrho_n,\mathcal{W}(\tau_n,\mathfrak{g}_n,\tfrac{\theta_n}{1-\eta_n}))+\lim_{n\to\infty}\Psi(\mathcal{W}(\tau_n,\mathfrak{g}_n,\tfrac{\theta_n}{1-\eta_n},\mathfrak{g}_n)).$$

By (2.4) and (2.5), we get

$$\begin{aligned}\lim_{n\to\infty}\Psi(\mathfrak{g}_{n+1},\mathfrak{g}_n)&\leq\lim_{n\to\infty}\Psi\left(\mathcal{W}(\tau_n,\mathfrak{g}_n,\tfrac{\theta_n}{1-\eta_n}),\mathfrak{g}_n\right)\\&\leq\lim_{n\to\infty}\tfrac{\theta_n}{1-\eta_n}\Psi(\tau_n,\mathfrak{g}_n)+\lim_{n\to\infty}(1-\tfrac{\theta_n}{1-\eta_n})\Psi(\mathfrak{g}_n,\mathfrak{g}_n).\end{aligned}$$

Using (2.7), it follows that

$$lim_{n\longrightarrow\infty}\Psi(\mathfrak{g}_{n+1},\mathfrak{g}_n)=0. \tag{2.8}$$

As

$$\Psi(\mathfrak{g}_n,\varrho_n)\leq\Psi(\mathfrak{g}_n,\mathfrak{g}_{n+1})+\Psi(\mathfrak{g}_{n+1},\varrho_n).$$

So, (2.5) and (2.8) imply

$$\lim_{n\longrightarrow\infty}\Psi(\mathfrak{g}_n,\varrho_n)=0. \tag{2.9}$$

Now,

$$\Psi(\mathfrak{g}_n,N\mathfrak{g}_n)\leq\Psi(\mathfrak{g}_n,\varrho_n),$$

and

$$\Psi(\mathfrak{g}_n,M\mathfrak{g}_n)\leq\Psi(\mathfrak{g}_n,\tau_n).$$

Hence, by (2.7) and (2.9),

$$\lim_{n\to\infty}\Psi(\mathfrak{g}_n,M\mathfrak{g}_n)=0=\lim_{n\to\infty}\Psi(\mathfrak{g}_n,N\mathfrak{g}_n).$$

Theorem 2.4. Let E be a non-empty closed and bounded subset of a complete CMS and let M and N be multi-valued non-expansive maps on E with $F_1\neq\emptyset$. Then, $\{\mathfrak{g}_n\}$ in (1.3) converges strongly to a CFP of M and N iff

$$\lim_{n\longrightarrow\infty}\inf\Psi(\mathfrak{g}_n,F_1)=0.$$

Proof. The necessity is straightforward. Conversely, suppose that

$$\lim_{n\longrightarrow\infty}\inf\Psi(\mathfrak{g}_n,F_1)=0.$$

As in the proof of Lemma 2.2, we have

$$\Psi(\mathfrak{g}_{n+1},\sigma)\leq\Psi(\mathfrak{g}_n,\sigma),$$

for all $\sigma\in F_1$. This implies that $\Psi(\mathfrak{g}_{n+1},F_1)\leq\Psi(\mathfrak{g}_n,F_1)$ so that $\lim_{n\to\infty}\Psi(\mathfrak{g}_n,F_1)$ exists.

We show that $\{\mathfrak{g}_n\}$ is Cauchy. Let $\varepsilon>0$ be an arbitrary real number.

By $\lim_{n\to\infty}\Psi(\mathfrak{g}_n,F_1)=0$, $\exists$ a positive integer n_0 such that $\Psi(\mathfrak{g}_n,F_1)<\frac{\varepsilon}{4}$, $\forall n\geq n_0$. In particular,

$$\inf\{\Psi(\mathfrak{g}_{n_0},\sigma)\,;\,\sigma\in F_1\}<\frac{\varepsilon}{4}.$$

Thus, $\exists$ a $\sigma* \in F_1$ such that $\Psi(\mathfrak{g}_{n_0}, \sigma*) < \frac{\varepsilon}{2}$. Now, $\forall m, n \geq n_0$, we have

$$\begin{aligned}\Psi(\mathfrak{g}_{m+n}, \mathfrak{g}_n) &\leq \Psi(\mathfrak{g}_{m+n}, \sigma*) + \Psi(\sigma*, \mathfrak{g}_n) \leq 2\Psi(\mathfrak{g}_{n_0}, \sigma*) \leq 2(\tfrac{\varepsilon}{2}),\\ \Psi(\mathfrak{g}_{m+n}, \mathfrak{g}_n) &\leq \varepsilon.\end{aligned}$$

Hence, $\{\mathfrak{g}_n\}$ is Cauchy. As Υ is complete so, $\{\mathfrak{g}_n\}$ converges to $\jmath$.

$$\begin{aligned}\Psi(\jmath, M\jmath) &\leq \Psi(\jmath, \mathfrak{g}_n) + \Psi(\mathfrak{g}_n, M\mathfrak{g}_n) + \mathcal{D}(M\mathfrak{g}_n, M\jmath)\\ &\leq \Psi(\jmath, \mathfrak{g}_n) + \Psi(\mathfrak{g}_n, M\mathfrak{g}_n) + \Psi(\mathfrak{g}_n, \jmath).\end{aligned}$$

In this inequality, taking limit as $n \to \infty$ and using Theorem 2.3, we have

$$\Psi(\jmath, M\jmath) = 0,$$

which is given by the closeness of $M\jmath$ that $\jmath \in F(M)$.

In the same way, we can prove that $\Psi(\jmath, N\jmath) = 0$ and $\jmath \in F(N)$.

Theorem 2.5. Suppose Υ is a complete CMS and E is a non-empty compact and convex subset of Υ. If $M, N : E \to CB(E)$ are non-expansive multi-valued maps and $F_1 \neq \phi$. Then, the iterative process (1.3) is convergent to an element of F_1.

Proof. By Lemma 2.2, $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, \sigma)$ exists, and $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, N\mathfrak{g}_n) = 0$ by Theorem 2.3. Since E is compact; therefore, a subsequence $\{\mathfrak{g}_{n_k}\}$ of $\{\mathfrak{g}_n\}$ exists such that $\lim_{n\longrightarrow\infty} \{\mathfrak{g}_{n_k}\} = \sigma \in E$.

Now,

$$\begin{aligned}\Psi(\sigma, N\sigma) &\leq \Psi(\sigma, \mathfrak{g}_{n_k}) + \Psi(\mathfrak{g}_{n_k}, N\sigma)\\ &= \Psi(\sigma, \mathfrak{g}_{n_k}) + \mathcal{D}(N\mathfrak{g}_{n_k}, N\sigma)\\ &\leq \Psi(\sigma, \mathfrak{g}_{n_k}) + \Psi(\sigma, \mathfrak{g}_{n_k})\\ &= 2\Psi(\sigma, \mathfrak{g}_{n_k})\end{aligned}$$

gives that $\Psi(\sigma, N\sigma) \leq 0$, which in turn gives that $\sigma \in F(N)$. Similarly, $\sigma \in F(M)$. As $\{\mathfrak{g}_{n_k}\}$ converges strongly to σ and $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, \sigma)$ exists, so the sequence $\{\mathfrak{g}_n\}$ converges strongly to $\sigma \in F_1$.

Abbas et al. [9] introduced a multi-valued variant of condition (A'), which is less restrictive than the compactness, as follows:

Let $M, N : E \to CB(E)$ be two multi-valued non-expansive maps. These maps fulfill condition (A') if $\exists$ a non-decreasing function $\phi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0$, $\phi(r) > 0$ for all $r \in (0, \infty)$ such that either $\Psi(\mathfrak{g}, N\mathfrak{g}) \geq \phi(\Psi(\mathfrak{g}, F_1))$ or $\Psi(\mathfrak{g}, M\mathfrak{g}) \geq \phi(\Psi(\mathfrak{g}, F_1))$, $\forall \mathfrak{g} \in E$.

Theorem 2.6. Let Υ be a complete CMS, E and $\{\mathfrak{g}_n\}$ be as in the statement of Lemma 2.2. Let M and N be non-expansive maps on E which satisfy (A'). Then $\{\mathfrak{g}_n\}$ converges strongly to a CFP of M and N.

Proof. By Lemma 2.2, $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, \sigma)$ exists $\forall\ \sigma \in F_1$. Now, $\Psi(\mathfrak{g}_{n+1}, \sigma) \leq \Psi(\mathfrak{g}_n, \sigma)$ gives that

$$\inf_{\sigma\in F_1} \Psi(\mathfrak{g}_{n+1}, \sigma) \leq \inf_{\sigma\in F_1} \Psi(\mathfrak{g}_n, \sigma),$$

which means that $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, F_1)$ exists. By condition (A'), either

$$\lim_{n\longrightarrow\infty} \phi(\Psi(\mathfrak{g}_n, F_1)) \leq \lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, N\mathfrak{g}_n) = 0$$

or

$$\lim_{n\longrightarrow\infty} \phi(\Psi(\mathfrak{g}_n, F_1)) \leq \lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, M\mathfrak{g}_n) = 0.$$

In both cases, we have

$$\lim_{n\longrightarrow\infty}\phi\left(\Psi\left(\mathfrak{g}_n,F_1\right)\right)=0.$$

Since ϕ is monotonically increasing and $\phi(0)=0$, it follows that

$$\lim_{n\longrightarrow\infty}\phi\left(\Psi\left(\mathfrak{g}_n,F_1\right)\right)=0.$$

The remaining part of the proof is that of Theorem 2.4 and is thus omitted.

We give an example in support of Theorem 2.5.

Example 2.7. Consider $E=[0,1]$ with the usual metric $\Psi(\mathfrak{g},\varrho)=|\mathfrak{g}-\varrho|^{\sigma}$, where $\sigma\geq 1$. Define multi-valued maps $N,M:E\to CB(E)$ as

$$N\mathfrak{g}=\left[\frac{\mathfrak{g}}{2},\frac{\mathfrak{g}+1}{2}\right],\quad M\mathfrak{g}=\left[\frac{\mathfrak{g}}{3},\frac{\mathfrak{g}+1}{3}\right].$$

It can be easily verified that common fixed point $\sigma\in[0,\frac{1}{2}]$. For any $\mathfrak{g},\varrho\in E$,

$$\begin{aligned}\mathcal{D}(N\mathfrak{g},N\varrho)&=\max\left(\left|\tfrac{\mathfrak{g}}{2}-\tfrac{\varrho}{2}\right|^{\sigma},\left|\tfrac{\mathfrak{g}+1}{2}-\tfrac{\varrho+1}{2}\right|^{\sigma}\right)=\left(\tfrac{|\mathfrak{g}-\varrho|}{2}\right)^{\sigma}=\tfrac{|\mathfrak{g}-\varrho|^{\sigma}}{2^{\sigma}}\\&=\tfrac{\Psi(\mathfrak{g},\varrho)}{2^{\sigma}}\leq\Psi(\mathfrak{g}-\varrho)\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}(M\mathfrak{g},M\varrho)&=\max\left(\left|\tfrac{\mathfrak{g}}{3}-\tfrac{\varrho}{3}\right|^{\sigma},\left|\tfrac{\mathfrak{g}+1}{3}-\tfrac{\varrho+1}{3}\right|^{\sigma}\right)=\left(\tfrac{|\mathfrak{g}-\varrho|}{3}\right)^{\sigma}=\tfrac{|\mathfrak{g}-\varrho|^{\sigma}}{3^{\sigma}}\\&=\tfrac{\Psi(\mathfrak{g},\varrho)}{3^{\sigma}}\leq\Psi(\mathfrak{g}-\varrho)\end{aligned}$$

are multi-valued non-expansive maps.

Set $\mathcal{W}(\mathfrak{g},\varrho,t)=(1-t)\mathfrak{g}+t\varrho$. Choose $\eta_n=\theta_n=\frac{1}{3}$, where $\eta_n+\theta_n<1$. Then using (1.3), we get

$$\mathfrak{g}_{n+1}=\frac{2}{3}\varrho_n+\frac{1}{3}\left(\frac{1}{2}z_n+\frac{1}{2}\mathfrak{g}_n\right)=\frac{2}{3}\varrho_n+\frac{1}{6}z_n+\frac{1}{6}\mathfrak{g}_n. \tag{2.10}$$

Let $\varrho_n\in N\mathfrak{g}_n=\frac{2\mathfrak{g}_n+1}{4}$ and $z_n\in M\mathfrak{g}_n=\frac{2\mathfrak{g}_n+1}{6}$.

Substitute the value of ϱ_n and z_n in (2.10) to get

$$\mathfrak{g}_{n+1}=\frac{2}{3}\cdot\frac{2\mathfrak{g}_n+1}{4}+\frac{1}{6}\cdot\frac{2\mathfrak{g}_n+1}{6}+\frac{1}{6}\mathfrak{g}_n=\frac{2\mathfrak{g}_n+1}{6}+\frac{2\mathfrak{g}_n+1}{36}+\frac{1}{6}\mathfrak{g}_n=\frac{5}{9}\mathfrak{g}_n+\frac{7}{36}.$$

The coefficient of g_n is $\frac{5}{9}<1$, so the iteration is a contraction on the interval $[0,1]$. Since $[0,1]$ with the usual metric is complete, the sequence $\{g_n\}$ converges to a unique fixed point g^*. Now, the fixed point $\mathfrak{g}^*$ is found as follows:

$$\mathfrak{g}^*=\frac{7}{16}$$

and

$$N(\mathfrak{g}^*)=\left[\frac{7}{32},\frac{23}{32}\right].$$

As $\dfrac{7}{16}\in\left[\frac{7}{32},\frac{23}{32}\right]$, so $\mathfrak{g}^*\in N(\mathfrak{g}^*)$. In the same way,

$$M(\mathfrak{g}^*)=\left[\frac{7}{48},\frac{23}{48}\right].$$

As $\dfrac{7}{16}\in\left[\frac{7}{48},\frac{23}{48}\right]$, so $\mathfrak{g}^*\in M(\mathfrak{g}^*)$, which shows that (1.3) converges to a common fixed point of M and N.

3. Multi-valued asymptotically non-expansive maps

The family of asymptotically non-expansive (ANE) maps has garnered consideration in the realm of fixed point theory since the seminal work of Goebel and Kirk [19]. Kirk and Xu [20] investigated these maps within the setting of uniformly convex Banach spaces. Their findings were further generalized by Hussain and Khamsi [21] to encompass metric spaces. Subsequently, Khamsi and Kozlowski [22] extended these results to the broader context of modular function spaces.

Consider E to be a closed and convex subset of a CMS. A mapping $N : E \longrightarrow CB(E)$ is said to be multi-valued ANE if there is a sequence $\mathcal{K}_n \subset [1, \infty)$ such that $\lim_{n\longrightarrow\infty}\mathcal{K}_n = 1$ and

$$\mathcal{D}(N^n \mathfrak{g}, N^n \varrho) \leq \mathcal{K}_n \Psi(\mathfrak{g}, \varrho). \tag{3.1}$$

In this section, we introduce a modification of iteration scheme (1.2) for multivalued asymptotically non-expansive mappings as follows:

$$\mathfrak{g}_{n+1} = \mathcal{W}\left(N^n \mathfrak{g}_n, \mathcal{W}\left(M^n \mathfrak{g}_n, \mathfrak{g}_n, \frac{\eta_n}{1-\theta_n}\right), \theta_n\right), \tag{3.2}$$

where $0 < a \leq \theta_n,\ \eta_n \leq b < 1$, and $\theta_n + \eta_n < 1$.

We generalize, Theorem 2.3 for two asymptotically non-expansive mappings on a complete convex metric space.

Theorem 3.1. Let E be a nonempty closed, bounded, and convex subset of a complete convex metric space Υ. Let $M, N : E \longrightarrow CB(E)$ be multivalued asymptotically non-expansive maps with $F_1 \neq \phi$. Then

$$\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, M^n \mathfrak{g}_n) = 0 = \lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, N^n \mathfrak{g}_n) \tag{3.3}$$

for the sequence $\{\mathfrak{g}_n\}$ in (3.2).

Proof. For $\sigma \in F_1$,

$$\begin{aligned}
\Psi(\mathfrak{g}_{n+1}, \sigma) &\leq \Psi\left(\mathcal{W}\left(N^n \mathfrak{g}_n, \mathcal{W}\left(M^n \mathfrak{g}_n, \mathfrak{g}_n, \tfrac{\eta_n}{1-\theta_n}\right), \theta_n\right), \sigma\right) \\
&\leq \theta_n \Psi(N^n \mathfrak{g}_n, \sigma) + (1-\theta_n)\Psi\left(\mathcal{W}(M^n \mathfrak{g}_n, \mathfrak{g}_n, \tfrac{\eta_n}{1-\theta_n}), \sigma\right) \\
&\leq \theta_n \mathcal{D}(N^n \mathfrak{g}_n, N\sigma) + (1-\theta_n)\{\tfrac{\eta_n}{1-\theta_n}\Psi(M^n \mathfrak{g}_n, \sigma) + (1 - \tfrac{\eta_n}{1-\theta_n})\Psi(\mathfrak{g}_n, \sigma)\} \\
&\leq \theta_n \mathcal{K}_n \Psi(\mathfrak{g}_n, \sigma) + (1-\theta_n)\{\tfrac{\eta_n}{1-\theta_n}\mathcal{D}(M^n \mathfrak{g}_n, M\sigma) + (1 - \tfrac{\eta_n}{1-\theta_n})\Psi(\mathfrak{g}_n, \sigma)\} \\
&= \theta_n \mathcal{K}_n \Psi(\mathfrak{g}_n, \sigma) + \mathcal{K}_n \eta_n \Psi(\mathfrak{g}_n, \sigma) + (1 - \theta_n - \eta_n)\Psi(\mathfrak{g}_n, \sigma) \\
&\leq [\theta_n \mathcal{K}_n + \mathcal{K}_n \eta_n + (1 - \theta_n - \eta_n)]\Psi(\mathfrak{g}_n, \sigma).
\end{aligned}$$

Taking the limit on both sides and $\mathcal{K}_n \to 1$ as $n \to \infty$ gives

$$\Psi(\mathfrak{g}_{n+1}, \sigma) \leq \Psi(\mathfrak{g}_n, \sigma).$$

This gives that $\{\mathfrak{g}_n\}$ is a non-increasing and bounded sequence of numbers and hence is convergent. So $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, \sigma)$ exists for each $\sigma \in F_1$.

Now, to prove $\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, M^n \mathfrak{g}_n) = 0 = \lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, N^n \mathfrak{g}_n)$, we follow the procedure used in proof of Theorem 2.3.

By applying Theorem 3.1, we derive the following convergence result:

Theorem 3.2. Let E be a non-empty, complete, and compact convex subset of complete CMS Υ. Let M, N, and $\{\mathfrak{g}_n\}$ be as in the statement of Theorem 3.1. If $F_1 \neq \phi$, then there is a subsequence of $\{\mathfrak{g}_n\}$ that converges to a CFP of M and N.
Proof. By Theorem 3.1, we have

$$\lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, M^n\mathfrak{g}_n) = 0 = \lim_{n\longrightarrow\infty} \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n).$$

As E is complete and compact, we have a subsequence $\{\mathfrak{g}_{n_k}\}$ of $\{\mathfrak{g}_n\}$ with $\mathfrak{g}_{n_k} \longrightarrow q$ in E. Continuity of M and N implies $N\mathfrak{g}_{n_k} \longrightarrow Nq$ and $M\mathfrak{g}_{n_k} \longrightarrow Mq$ as $n_k \longrightarrow \infty$.

Thus,

$$\Psi(Mq, q) = 0 = \Psi(Nq, q).$$

Therefore,

$$Nq = Mq = q.$$

In 1994, Rhoades [23, Theorem 2.2] studied strong convergence of a sequence for a single-valued completely continuous ANE self-map on a UCBS. A mapping $N : E \rightarrow CB(E)$ is completely continuous if it is continuous and, for any bounded subset of E, its image under N is relatively compact within E.

We now obtain a multi-valued version of the above theorem by using our iteration scheme (3.2).

Theorem 3.3. Let Υ be a complete CMS, and let E be a non-empty closed, bounded, and convex subset of Υ. Suppose that N is a completely continuous multi-valued ANE mapping of E with $\{\mathcal{K}_n\}$ satisfying $\mathcal{K}_n \geq 1$ and $\sum_{n=1}^{\infty}\left(\mathcal{K}_n^2 - 1\right) < \infty$. Let $\{\eta_n\}, \{\theta_n\} \subset [0, 1]$ satisfy

(i) $0 < \liminf_{n\to\infty} \eta_n \leq \limsup_{n\to\infty} \eta_n < 1$,
(ii) $\limsup_{n\to\infty} \theta_n < 1$.

Then, $\{\mathfrak{g}_n\}$ in (3.2) is strongly convergent to a fixed point of N.
Proof. By Theorem 3.1,

$$\lim_{n\to\infty} \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) = 0,$$

$$\begin{aligned}
\Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) &\leq \Psi(\mathfrak{g}_{n+1}, \mathfrak{g}_n) + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + \Psi(N^n\mathfrak{g}_n, N^n\mathfrak{g}_{n+1}) \\
&= \Psi(\mathfrak{g}_{n+1}, \mathfrak{g}_n) + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + \mathcal{D}(N^n\mathfrak{g}_n, N^n\mathfrak{g}_{n+1}) \\
&\leq \Psi(\mathfrak{g}_{n+1}, \mathfrak{g}_n) + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + \mathcal{K}_n\Psi(\mathfrak{g}_n, \mathfrak{g}_{n+1}) \\
&= (1 + \mathcal{K}_n)\Psi(\mathfrak{g}_n, \mathfrak{g}_{n+1}) + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n),
\end{aligned}$$

$$\begin{aligned}
\Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) &\leq (1 + \mathcal{K}_n)\Psi(\mathfrak{g}_n, \mathfrak{g}_{n+1}) + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) \\
&\leq (1 + \mathcal{K}_n)\Psi\left(\mathfrak{g}_n, \mathcal{W}(N^n\mathfrak{g}_n, \mathcal{W}(M^n\mathfrak{g}_n, \mathfrak{g}_n, \tfrac{\eta_n}{1-\theta_n}), \theta_n\right) + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) \\
&\leq (1 + \mathcal{K}_n)\{\theta_n\Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + (1 - \theta_n)\Psi(\mathfrak{g}_n, \mathcal{W}(M^n\mathfrak{g}_n, \mathfrak{g}_n, \tfrac{\eta_n}{1-\theta_n})\} + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) \\
&= (1 + \mathcal{K}_n)\theta_n\Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + (1 + \mathcal{K}_n)(1 - \theta_n)\{\tfrac{\eta_n}{1-\theta_n}\Psi(\mathfrak{g}_n, M^n\mathfrak{g}_n) \\
&\quad + (1 - \tfrac{\eta_n}{1-\theta_n})\Psi(\mathfrak{g}_n, \mathfrak{g}_n)\} + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) \\
&= (1 + \mathcal{K}_n)\left[\eta_n\Psi(\mathfrak{g}_n, M^n\mathfrak{g}_n) + (1 - \theta_n - \eta_n)\Psi(\mathfrak{g}_n, \mathfrak{g}_n)\right] \\
&\quad + \Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + (1 + \mathcal{K}_n)\theta_n\Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n).
\end{aligned}$$

Taking the limit and using (3.3), we get

$$\lim_{n\to\infty} \Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) = 0. \tag{3.4}$$

Thus,

$$\begin{aligned}
\Psi(\mathfrak{g}_n, N\mathfrak{g}_n) &\leq \Psi(\mathfrak{g}_n, \mathfrak{g}_{n+1}) + \Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) + \Psi(N^n\mathfrak{g}_{n+1}, N\mathfrak{g}_n) \\
&\leq \Psi\left(\mathfrak{g}_n, \mathcal{W}(N^n\mathfrak{g}_n, \mathcal{W}(M^n\mathfrak{g}_n, \mathfrak{g}_n, \tfrac{\eta_n}{1-\theta_n}), \theta_n\right) + \Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) + \Psi(N^n\mathfrak{g}_{n+1}, N\mathfrak{g}_n) \\
&\leq \theta_n\Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + (1-\theta_n)\Psi(\mathfrak{g}_n, \mathcal{W}(M^n\mathfrak{g}_n, \mathfrak{g}_n, \tfrac{\eta_n}{1-\theta_n}) + \Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) + \Psi(N^n\mathfrak{g}_{n+1}, \mathfrak{g}_{n+1}) \\
&\leq \theta_n\Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + (1-\theta_n)\{\tfrac{\eta_n}{1-\theta_n}\Psi(\mathfrak{g}_n, M^n\mathfrak{g}_n) + (1 - \tfrac{\eta_n}{1-\theta_n})\Psi(\mathfrak{g}_n, \mathfrak{g}_n)\} \\
&\quad + \Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) + \Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) \\
&= \theta_n\Psi(\mathfrak{g}_n, N^n\mathfrak{g}_n) + \eta_n\Psi(\mathfrak{g}_n, M^n\mathfrak{g}_n) + (1-\theta_n-\eta_n)\Psi(\mathfrak{g}_n, \mathfrak{g}_n) + \Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}) \\
&\quad + \Psi(\mathfrak{g}_{n+1}, N^n\mathfrak{g}_{n+1}).
\end{aligned}$$

Taking the limit in the above inequality and using (3.3) and (3.4), we get

$$\lim_{n\to\infty} \Psi(\mathfrak{g}_n, N\mathfrak{g}_n) = 0. \tag{3.5}$$

Since N is completely continuous and $\{\mathfrak{g}_n\}$ is bounded, there exists a subsequence $\{\mathfrak{g}_{n_k}\}$ of $\{\mathfrak{g}_n\}$ such that $\{N\mathfrak{g}_{n_k}\}$ converges. Therefore, from (3.4), $\{\mathfrak{g}_{n_k}\}$ converges. Let $\lim_{k\to\infty} \mathfrak{g}_{n_k} = \sigma$; it follows from the continuity of N and (3.3) that $\sigma \in N\sigma$ (i.e., σ is a fixed point of N). We know that $\lim_{n\to\infty} \Psi(\mathfrak{g}_n, \sigma)$ exists and $\{N\mathfrak{g}_{n_k}\}$ converges to σ, so $\lim_{n\to\infty} \Psi(\mathfrak{g}_n, \sigma) = 0$; that is, $\lim_{n\to\infty} \mathfrak{g}_n = \sigma$.

Remark 3.4. Based on Kuhfittig's work [24, p. 137], our iteration scheme (3.2) for ANE maps can be reduced to his scheme (1), which he employed to solve a system of equations of the form

$$\mathfrak{g} - M_i\mathfrak{g} = f_i, \quad i = 1, 2, 3, \ldots, n,$$

where each M_i is a non-expansive self-mapping on Υ, and each f_i is a fixed element of Υ. As in Kuhfittig's work [24], our iteration scheme (3.2) can be applied to find solutions of similar systems of the type

$$\mathfrak{g} - M_i^n\mathfrak{g} = f_i, \quad i = 1, 2,$$

where the maps M_i form a pair of ANE maps on a CMS.

4. Conclusions

In this paper, we have established common fixed point results for multi-valued non-expansive and asymptotically non-expansive maps using a one-step iteration scheme. We have extended the results of Fukhar-ud-Din [11] from single-valued to multi-valued mappings. Two examples, one for R^2 and the other for non-interval convex metric spaces, are presented. Example 2.7 is presented, which validates our one-step iteration scheme for multi-valued non-expansive mappings. In Remark 3.4, we have provided an avenue for the application of our work on asymptotically non-expansive maps on convex metric spaces.

Author contributions

Tanveer Hussain and Abdul Rahim Khan: Conceptualization, Methodology, Writing–review and editing; Tanveer Hussain and Kiran Riaz: Formal analysis, Writing–original draft preparation; Abdul Rahim Khan and Hind Alamri: Validation, Supervision and funding. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to acknowledge the Deanship of Graduate Studies and Scientific Research, Taif University for the financial assistance during this research work.

Conflict of interest

The authors declare no conflicts of interest.

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