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Research article

Strong DMP inverse

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Abstract: A new-type matrix inverse based on the Hartwig-Spindelböck decomposition was investigated, which is related to the DMP inverse, is the abbreviation of the Drazin Moore-Penrose inverse, here we call this generalized inverse as the strong DMP inverse. We established the relationships between this new-type inverse and other matrix generalized inverses. We proved that the strong DMP inverse of a square complex matrix is a special type of outer inverse, where the restricted column space is $\mathcal{R}(A^k)$ and restricted null space is $\mathcal{N}(AA^D\alpha_A)$, where $\alpha_A = AA^\dagger + (A^\dagger)^*(E_n - AA^\dagger)$. The one-sided strong DMP inverses were introduced from the perspectives of the column space, null space, and index of a matrix. The general expressions of the left (right) strong DMP inverse of A were given. We answered the question of when the left strong DMP inverse is consistent with the right strong DMP inverse. Based on the relationship between the left strong DMP inverse and the right DMP inverse, the necessary and sufficient conditions for the existence of the strong DMP inverse were given.

Keywords: DMP inverse; Hartwig-Spindelböck decomposition; one-sided strong DMP inverse; column space; null space
Mathematics Subject Classification: 15A09

1. Introduction

Let $\mathbb{C}$ be the complex field. The set $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices over $\mathbb{C}$. The symbol A^* denotes the conjugate transpose of $A \in \mathbb{C}^{m\times n}$. The symbol E_n denotes the identity matrix of size n. Notations $\mathcal{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}$ and $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$ will be used in the sequel. The smallest positive integer k such that rank $(A^k) = \text{rank}(A^{k+1})$ is called the index of $A \in \mathbb{C}^{n\times n}$ and denoted by ind(A). The index can be used in definiton of the Drazin inverse [1,2]. Let $A \in \mathbb{C}^{m\times n}$. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$, then X is called the Moore-Penrose inverse of A [3,4] and denoted by $X = A^\dagger$. If $AXA = A$ holds, we say that X is a $\{1\}$-inverse of A (see [5]).

Every matrix $A \in \mathbb{C}^{n\times n}$ of rank r can be represented in the form

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*, \tag{1.1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \sigma_1 E_{r_1} \oplus \sigma_2 E_{r_2} \oplus \cdots \oplus \sigma_t E_{r_t}$ is the diagonal matrix of the nonzero singular values of A, where $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r\times r}$ and $L \in \mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^* + LL^* = E_r.$$

The decomposition in (1.1) is known as the Hartwig-Spindelböck decomposition (or called the Σ-K-L decomposition) [6].

Lemma 1.1. *[7, p. 2799 (1.4)] Let $A \in \mathbb{C}^{n\times n}$. If A has the Hartwig-Spindelböck decomposition as given in* (1.1)*, then the expression of the Moore-Penrose inverse is*

$$A^{\dagger} = U\begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix}U^*. \tag{1.2}$$

For $A \in \mathbb{C}^{n\times n}$, the DMP inverse of A is introduced by using the Drazin and the Moore-Penrose inverses of A in [8], and the formula of the DMP inverse of A is $A^{D,\dagger} = A^D AA^{\dagger}$ [8, Theorem 2.2]. The iterative method of the DMP inverse can be seen in [9].

Definition 1.2. *[8, Theorem 2.1] Let $A \in \mathbb{C}^{n\times n}$. If $X \in \mathbb{C}^{n\times n}$ satisfies*

$$XAX = X,\ XA = A^D A,\ A^k X = A^k A^{\dagger}, \tag{1.3}$$

we call X the DMP inverse of A. The DMP inverse is unique and denoted by $A^{D,\dagger}$.

Lemma 1.3. *[8, Theorem 2.5] Let $A \in \mathbb{C}^{n\times n}$. If A has the Hartwig-Spindelböck decomposition as given in* (1.1)*, then*

$$A^{D,\dagger} = U\begin{bmatrix} (\Sigma K)^D & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

The expression of the DMP inverse of A is $A^{D,\dagger} = U\begin{bmatrix} (\Sigma K)^D & 0 \\ 0 & 0 \end{bmatrix}U^*$ by Lemma 1.3. Though the classical Hartwig-Spindelböck decomposition, the corresponding exact expression of the strong DMP inverse is given in Theorem 2.5 (see Section 2), which is $A_s^{D,\dagger} = U\begin{bmatrix} (\Sigma K)^D & (\Sigma K)^D\Sigma^{-1}L \\ 0 & 0 \end{bmatrix}U^*$. In this paper, we mainly propose the name of the strong DMP inverse from the perspective of the expression $A^{D,\dagger} = U\begin{bmatrix} (\Sigma K)^D & 0 \\ 0 & 0 \end{bmatrix}U^*$ and the expression $A_s^{D,\dagger} = U\begin{bmatrix} (\Sigma K)^D & (\Sigma K)^D\Sigma^{-1}L \\ 0 & 0 \end{bmatrix}U^*$. Of course, we have also proven that the expression for the strong DMP inverse is $A_s^{D,\dagger} = A^{D,\dagger}\alpha_A$. Note that we can use the DMP inverse to find the expression of the strong DMP inverse.

Let $A, B, C \in \mathbb{C}^{n\times n}$. A matrix $Y \in \mathbb{C}^{n\times n}$ is said to be the (B, C)-inverse [10, 11] of A if

$$YAB = B,\ CAY = C,\ \mathcal{N}(C) \subseteq \mathcal{N}(Y),\ \text{and}\ \mathcal{R}(Y) \subseteq \mathcal{R}(B).$$

If such Y exists, then it is unique. Note that the (B, C)-inverse is introduced by Drazin in the setting of semigroups [12]. The symbol of the (B, C)-inverse of A is $Y = A^{\|(B,C)}$.

The structure of this paper is: In Section 1, we present the basic definitions and the Hartwig-Spindelböck decomposition. This is the motivation for studying the strong DMP inverse. In Section 2, we introduce the strong DMP inverse of a square complex matrix. The criteria and expressions of the strong DMP inverse are obtained. In Section 3, the one-sided strong DMP inverse is introduced. We answer the question of when the left strong DMP inverse is consistent with the right strong DMP inverse.

2. Strong DMP inverse

In this section, we introduce the strong DMP inverse of a square complex matrix and prove this inverse is unique. The criteria for the strong DMP invertibility of a matrix and the exact expressions of the strong DMP inverse are obtained. The distinction between the strong DMP inverse and various types of generalized inverses is established though a specific example. We prove that the strong DMP inverse of a square complex matrix is a special type of outer inverse, where the restricted column space is $\mathcal{R}(A^k)$ and restricted null space is $\mathcal{N}(AA^D\alpha_A)$.

Using Lemma 1.1, it is not difficult to obtain the following lemma.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ and $\alpha_A = AA^\dagger + (A^\dagger)^*(E_n - AA^\dagger)$. If A has the Hartwig-Spindelböck decomposition as given in* (1.1)*, then*

$$\alpha_A = U \begin{bmatrix} E_r & \Sigma^{-1}L \\ 0 & 0 \end{bmatrix} U^*. \tag{2.1}$$

Definition 2.2. *Let $A, X \in \mathbb{C}^{n\times n}$ and $\alpha_A = AA^\dagger + (A^\dagger)^*(E_n - AA^\dagger)$. If*

$$XAX = X,\ XA = A^{D,\dagger}A,\ AX = AA^{D,\dagger}\alpha_A, \tag{2.2}$$

then X is called the strong DMP inverse of A and denoted by $X = A_s^{D,\dagger}$.

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$. Then the strong DMP inverse of A is unique. Moreover, the formula of the strong DMP inverse is $A_s^{D,\dagger} = A^{D,\dagger}\alpha_A$.*

Proof.

$$X = XAX = A^{D,\dagger}AX = A^{D,\dagger}AA^{D,\dagger}\alpha_A = A^{D,\dagger}\alpha_A$$

by Definition 1.2. □

Example 2.4. *The strong DMP inverse is different from the Drazin inverse A^D* [1]*, the Moore-Penrose inverse $A^\dagger$* [3, 4]*, the core-EP inverse $A^{\textcircled{\dagger}}$* [13]*, the MPWC inverse A°* [14]*, the CMP inverse $A^{c,\dagger}$* [15]*, the MPCEP inverse $A^{\dagger,\textcircled{\dagger}}$* [16]*, the MPBT inverse $A^{\dagger,\diamond}$* [17]*, and the gMP inverse $A^{\diamond}$* [18]*. Let*

$$A = \begin{bmatrix} -1 & 1 & -i & -1 \\ -1+i & 0 & i & -i \\ 1-i & i & -i & -1+i \\ -1-i & 0 & -1 & 1 \end{bmatrix} \in \mathbb{C}^{4\times 4}.$$

Then it is easy to check that

$$A^{\dagger} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{8} - \frac{1}{8}i & 0 & -\frac{1}{8} + \frac{1}{8}i \\ \frac{3}{20} + \frac{1}{5}i & -\frac{1}{40} - \frac{13}{40}i & -\frac{1}{5} - \frac{3}{5}i & -\frac{13}{40} + \frac{1}{40}i \\ -\frac{1}{20} + \frac{2}{5}i & -\frac{7}{40} - \frac{11}{40}i & -\frac{2}{5} - \frac{1}{5}i & -\frac{11}{40} + \frac{7}{40}i \\ -\frac{1}{5} + \frac{3}{20}i & -\frac{7}{40} - \frac{1}{40}i & -\frac{2}{5} - \frac{1}{5}i & -\frac{1}{40} + \frac{7}{40}i \end{bmatrix},$$

$$AA^{\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2}i \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2}i & 0 & \frac{1}{2} \end{bmatrix},$$

$$A^3(A^3)^{\dagger} = \begin{bmatrix} \frac{1}{7} & \frac{1}{7} + \frac{1}{7}i & -\frac{1}{7} - \frac{1}{7}i & \frac{1}{7} - \frac{1}{7}i \\ \frac{1}{7} - \frac{1}{7}i & \frac{2}{7} & -\frac{2}{7} & -\frac{2}{7}i \\ -\frac{1}{7} + \frac{1}{7}i & -\frac{2}{7} & \frac{2}{7} & \frac{2}{7}i \\ \frac{1}{7} + \frac{1}{7}i & \frac{2}{7}i & -\frac{2}{7}i & \frac{2}{7} \end{bmatrix},$$

and

$$\alpha_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \frac{1}{4}i & 0 & -\frac{1}{4} - \frac{1}{2}i \\ 0 & \frac{1}{2}i & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{4} + \frac{1}{2}i & 0 & \frac{1}{2} - \frac{1}{4}i \end{bmatrix}.$$

Thus

$$A_s^{D,\dagger} = \begin{bmatrix} 1 & \frac{1}{2} + \frac{5}{4}i & 2 & -\frac{5}{4} - \frac{1}{2}i \\ 1-i & \frac{7}{4} + \frac{3}{4}i & 2-2i & -\frac{7}{4} + \frac{3}{4}i \\ -1+i & -\frac{7}{4} - \frac{3}{4}i & -2+2i & \frac{7}{4} - \frac{3}{4}i \\ 1+i & -\frac{3}{4} + \frac{7}{4}i & 2+2i & -\frac{3}{4} - \frac{7}{4}i \end{bmatrix},$$

however

$$A^D = \begin{bmatrix} 1 & -2+i & 2 & -1-3i \\ 1-i & -1+3i & 2-2i & -4-2i \\ -1+i & 1-3i & -2+2i & 4+2i \\ 1+i & -3-i & 2+2i & 2-4i \end{bmatrix},$$

$$A^{\textcircled{\dagger}} = \begin{bmatrix} \frac{1}{7}i & -\frac{1}{7} + \frac{1}{7}i & \frac{1}{7} - \frac{1}{7}i & \frac{1}{7} + \frac{1}{7}i \\ \frac{1}{7} + \frac{1}{7}i & \frac{2}{7}i & -\frac{2}{7}i & \frac{2}{7} \\ -\frac{1}{7} - \frac{1}{7}i & -\frac{2}{7}i & \frac{2}{7}i & -\frac{2}{7} \\ -\frac{1}{7} + \frac{1}{7}i & -\frac{2}{7} & \frac{2}{7} & \frac{2}{7}i \end{bmatrix},$$

$$A^{\circ} = \begin{bmatrix} \frac{3}{4}i & \frac{3}{28} - \frac{3}{14}i & \frac{3}{14} + \frac{9}{28}i & -\frac{3}{14} - \frac{3}{28}i \\ -\frac{1}{4}i & -\frac{1}{28} + \frac{1}{14}i & -\frac{1}{14} - \frac{3}{28}i & \frac{1}{14} + \frac{1}{28}i \\ \frac{1}{4}i & \frac{1}{28} - \frac{1}{14}i & \frac{1}{14} + \frac{3}{28}i & -\frac{1}{14} - \frac{1}{28}i \\ \frac{1}{4} & -\frac{1}{14} - \frac{1}{28}i & \frac{3}{28} - \frac{1}{14}i & -\frac{1}{28} + \frac{1}{14}i \end{bmatrix},$$

$$A^{c,\dagger} = \begin{bmatrix} -\frac{3}{4}i & -\frac{3}{8}i & -\frac{3}{2}i & -\frac{3}{8} \\ \frac{1}{4}i & \frac{1}{8}i & \frac{1}{2}i & \frac{1}{8} \\ -\frac{1}{4}i & -\frac{1}{8}i & -\frac{1}{2}i & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{2} & -\frac{1}{8}i \end{bmatrix},$$

$$A^{\dagger,\textcircled{\dagger}} = \begin{bmatrix} -\frac{3}{28} & -\frac{3}{28} - \frac{3}{28}i & \frac{3}{28} + \frac{3}{28}i & -\frac{3}{28} + \frac{3}{28}i \\ \frac{1}{28} & \frac{1}{28} + \frac{1}{28}i & -\frac{1}{28} - \frac{1}{28}i & \frac{1}{28} - \frac{1}{28}i \\ -\frac{1}{28} & -\frac{1}{28} - \frac{1}{28}i & \frac{1}{28} + \frac{1}{28}i & -\frac{1}{28} + \frac{1}{28}i \\ \frac{1}{28}i & -\frac{1}{28} + \frac{1}{28}i & \frac{1}{28} - \frac{1}{28}i & \frac{1}{28} + \frac{1}{28}i \end{bmatrix},$$

$$A^{\dagger,\diamond} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{12} - \frac{1}{12}i & \frac{1}{12} + \frac{1}{12}i & -\frac{1}{12} + \frac{1}{12}i \\ \frac{3}{20} + \frac{1}{5}i & \frac{1}{20} - \frac{1}{60}i & -\frac{1}{20} + \frac{1}{60}i & -\frac{1}{60} - \frac{1}{20}i \\ \frac{1}{20} + \frac{2}{5}i & \frac{1}{60} - \frac{7}{60}i & -\frac{1}{60} + \frac{7}{60}i & -\frac{7}{60} - \frac{1}{60}i \\ -\frac{1}{5} + \frac{3}{20}i & \frac{1}{60} + \frac{1}{20}i & -\frac{1}{60} - \frac{1}{20}i & \frac{1}{20} - \frac{1}{60}i \end{bmatrix},$$

$$\text{and } A^{\diamond} = \begin{bmatrix} -\frac{7}{80} & -\frac{7}{80} - \frac{7}{80}i & \frac{7}{80} + \frac{7}{80}i & -\frac{7}{80} + \frac{7}{80}i \\ \frac{1}{40} + \frac{1}{80}i & \frac{1}{80} + \frac{3}{80}i & -\frac{1}{80} - \frac{3}{80}i & \frac{3}{80} - \frac{1}{80}i \\ -\frac{3}{80} - \frac{1}{40}i & -\frac{1}{80} - \frac{1}{16}i & \frac{1}{80} + \frac{1}{16}i & -\frac{1}{16} + \frac{1}{80}i \\ \frac{3}{80} + \frac{1}{40}i & \frac{1}{80} + \frac{1}{16}i & -\frac{1}{80} - \frac{1}{16}i & \frac{1}{16} - \frac{1}{80}i \end{bmatrix}.$$

Though the classical Hartwig-Spindelböck decomposition, the corresponding exact expression of the strong DMP inverse is given in the following theorem.

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$. If A has the Hartwig-Spindelböck decomposition as given in* (1.1)*, then the expression of the strong DMP inverse of A is*

$$A_s^{D,\dagger} = U \begin{bmatrix} (\Sigma K)^D & (\Sigma K)^D \Sigma^{-1} L \\ 0 & 0 \end{bmatrix} U^*. \tag{2.3}$$

Proof. Direct calculation based on $A_s^{D,\dagger} = A^{D,\dagger}\alpha_A$ yields the results. □

In the following theorem, we will discuss when the DMP inverse and the strong DMP inverse are consistent from the perspective of the Hartwig-Spindelböck decomposition.

Theorem 2.6. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If A has the Hartwig-Spindelböck decomposition as given in* (1.1)*, then the DMP inverse coincides with the strong DMP inverse of A if and only if the condition $(\Sigma K)^{k-1}\Sigma^{-1}L = 0$ holds.*

Proof. By Lemma 1.3 the DMP inverse of A coincides with the strong DMP inverse of A if and only if

$$(\Sigma K)^D \Sigma^{-1} L = 0,$$

by equality (2.3). In the following, we will prove that $(\Sigma K)^D\Sigma^{-1}L = 0$ holds if and only if $(\Sigma K)^{k-1}\Sigma^{-1}L = 0$ holds. Note that the condition $\operatorname{ind}(A) = k$ gives $\operatorname{ind}(\Sigma K) = k - 1$. If $(\Sigma K)^D\Sigma^{-1}L = 0$ holds, then

$$(\Sigma K)^{k-1}\Sigma^{-1}L = (\Sigma K)^k(\Sigma K)^D\Sigma^{-1}L = 0.$$

If $(\Sigma K)^{k-1}\Sigma^{-1}L = 0$ holds, then

$$\begin{aligned} (\Sigma K)^D\Sigma^{-1}L &= (\Sigma K)^D(\Sigma K)(\Sigma K)^D\Sigma^{-1}L \\ &= (\Sigma K)^D(\Sigma K)^{k-1}\left[(\Sigma K)^D\right]^{k-1}\Sigma^{-1}L \\ &= (\Sigma K)^D\left[(\Sigma K)^D\right]^{k-1}(\Sigma K)^{k-1}\Sigma^{-1}L \\ &= 0. \end{aligned}$$

□

Theorem 2.7. *Let $A, X \in \mathbb{C}^{n\times n}$ and $\alpha_A = AA^\dagger + (A^\dagger)^*(E_n - AA^\dagger)$. Then X is the strong DMP inverse if and only if*

$$XAX = X,\ XA = A^DA,\ AX = AA^D\alpha_A. \tag{2.4}$$

Moreover, the formula of the strong DMP inverse is $A_s^{D,\dagger} = A^D\alpha_A$.

Proof. It is clear by $A^{D,\dagger}A = A^DA$ and $AA^{D,\dagger}\alpha_A = A^DA\alpha_A$. □

Theorem 2.8. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and $\alpha_A = AA^\dagger + (A^\dagger)^*(E_n - AA^\dagger)$. Then the strong DMP inverse is the $(A^k, AA^D\alpha_A)$-inverse of A.*

Proof. Let Y be the strong DMP inverse of A. By Theorem 2.7, the formula of the strong DMP inverse is $A_s^{D,\dagger} = A^D\alpha_A$, that is, $Y = A^D\alpha_A$.

$$\begin{aligned}\alpha_A A = &\ U\begin{bmatrix} E_r & \Sigma^{-1}L \\ 0 & 0\end{bmatrix}\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0\end{bmatrix}U^* = U\begin{bmatrix} E_r\Sigma K & E_r\Sigma L \\ 0 & 0\end{bmatrix}U^* \\ = &\ U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0\end{bmatrix}U^* = A\end{aligned}$$

gives $\alpha_A A = A$. Then

$$\begin{aligned} YAA^k = &\ A^D\alpha_A AA^k = A^D\alpha_A A^{k+1} = A^k, \\ AA^D\alpha_A AY = &\ AA^D\alpha_A AA^D\alpha_A = AA^DAA^D\alpha_A = AA^D\alpha_A. \end{aligned} \tag{2.5}$$

For arbitrary $x \in \mathcal{N}(AA^D\alpha_A)$, we have $AA^D\alpha_A x = 0$. Then

$$Yx = A^D\alpha_A x = A^DAA^D\alpha_A x = A^D(AA^D\alpha_A x) = 0,$$

which gives $\mathcal{N}(AA^D\alpha_A) \subseteq \mathcal{N}(Y)$. The condition

$$Y = A^D\alpha_A = A^DAA^D\alpha_A = A^k(A^D)^{k+1}\alpha_A$$

means that $\mathcal{R}(Y) \subseteq \mathcal{R}(A^k)$. Thus, the strong DMP inverse is the $(A^k, AA^D\alpha_A)$-inverse of A by [10, Theorem 7.1]. □

3. One-sided strong DMP inverse

Motivated by the ideal of the one-sided (B, C)-inverse [10], the one-sided strong DMP inverse is introduced. The one-sided strong DMP inverse is introduced from the perspectives of the column space, null space, and index of a matrix. We answer the question of when the left strong DMP inverse is consistent with the right strong DMP inverse. Based on the relationship between the left strong DMP inverse and the right DMP inverse, the necessary and sufficient conditions for the existence of the strong DMP inverse are given. The following definition can be found by [10, Definition 2.7], however, for the sake of our readers, we provide specific definitions.

Definition 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. We say that $X \in \mathbb{C}^{n\times n}$ is a left strong DMP inverse of A if we have*

$$\mathcal{N}(AA^D\alpha_A) \subseteq \mathcal{N}(X) \text{ and } XA^{k+1} = A^k.$$

We say that $Y \in \mathbb{C}^{n\times n}$ is a right strong DMP inverse of A if we have

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^k) \text{ and } A^DAY = A^D\alpha_A.$$

In the following theorem, an expression of the left strong DMP inverse of A is given.

Theorem 3.2. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$. *Then*

$$A^D\alpha_A + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^D\alpha_A$$

is the expression of the left strong DMP inverse of A, for any $V \in \mathbb{C}^{n\times n}$ *and any* $(A^{k+1})^- \in (A^{k+1})\{1\}$.

Proof. Let $X = A^D\alpha_A + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^D\alpha_A$. For arbitrary $t \in \mathcal{N}(AA^D\alpha_A)$, we have $AA^D\alpha_A t = 0$, and then

$$\begin{aligned} Xt &= \left\{A^D\alpha_A + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^D\alpha_A\right\}t \\ &= \left\{A^DAA^D\alpha_A + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^D\alpha_A\right\}t \\ &= \left\{A^D + V\left[E_n - A^{k+1}(A^{k+1})^-\right]\right\}AA^D\alpha_A t \\ &= 0. \end{aligned}$$

Based on the arbitrariness of t, we have $\mathcal{N}(AA^D\alpha_A) \subseteq \mathcal{N}(X)$. By the proof of Theorem 2.8, we have $\alpha_A A = A$. Then

$$\begin{aligned} XA^{k+1} &= \left\{A^D\alpha_A + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^D\alpha_A\right\}A^{k+1} \\ &= A^D\alpha_A A^{k+1} + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^D\alpha_A A^{k+1} \\ &= A^D\alpha_A AA^k + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^D\alpha_A AA^k \\ &= A^DAA^k + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^DAA^k \\ &= A^DA^{k+1} + V\left[E_n - A^{k+1}(A^{k+1})^-\right]A^{k+1}A^DA \\ &= A^k + V\left[A^{k+1} - A^{k+1}(A^{k+1})^-A^{k+1}\right]A^DA \\ &= A^k, \end{aligned}$$

which gives $XA^{k+1} = A^k$, and thus $X = A^D\alpha_A + V\left[E_n - A^{k+1}(A^{k+1})^-\right]AA^D\alpha_A$ is a general expression of the left strong DMP inverse of A by $\mathcal{N}(AA^D\alpha_A) \subseteq \mathcal{N}(X)$ and Definition 3.1. □

In the following theorem, an expression of the right strong DMP inverse of A is given.

Theorem 3.3. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$. *Then*

$$A^D\alpha_A + A^k\left[E_n - (A^{k+1})^-A^{k+1}\right]U$$

is the expression of the right strong DMP inverse of A, for any $U \in \mathbb{C}^{n\times n}$ *and any* $(A^{k+1})^- \in (A^{k+1})\{1\}$.

Proof. Let $Y = A^D\alpha_A + A^k\left[E_n - (A^{k+1})^-A^{k+1}\right]U$. By $A^D = A^k(A^D)^{k+1}$, we have

$$\begin{aligned} Y &= A^D\alpha_A + A^k\left[E_n - (A^{k+1})^-A^{k+1}\right]U \\ &= A^k(A^D)^{k+1}\alpha_A + A^k\left[E_n - (A^{k+1})^-A^{k+1}\right]U \\ &= A^k\left((A^D)^{k+1}\alpha_A + \left[E_n - (A^{k+1})^-A^{k+1}\right]U\right), \end{aligned}$$

which gives $\mathcal{R}(Y) \subseteq \mathcal{R}(A^k)$. By A^D is an outer inverse of A, we have

$$\begin{aligned} A^D AY &= A^D AA^D \alpha_A + A^D AA^k \left[E_n - (A^{k+1})^- A^{k+1}\right] U \\ &= A^D \alpha_A + A^D A^{k+1} \left[E_n - (A^{k+1})^- A^{k+1}\right] U \\ &= A^D \alpha_A + A^D \left[A^{k+1} - A^{k+1}(A^{k+1})^- A^{k+1}\right] U \\ &= A^D \alpha_A + A^D \left[A^{k+1} - A^{k+1}\right] U \\ &= A^D \alpha_A, \end{aligned}$$

which gives $A^D AY = A^D \alpha_A$, and thus $A^D \alpha_A + A^k \left[E_n - (A^{k+1})^- A^{k+1}\right] U$ is a general expression of the right strong DMP inverse of A by $\mathcal{R}(Y) \subseteq \mathcal{R}(A^k)$ and Definition 3.1. □

In the following Theorems 3.4 and 3.5, the necessary and sufficient conditions are given for when a matrix $X \in \mathbb{C}^{n\times n}$ is the left strong DMP inverse and the right strong DMP inverse of a given matrix $A \in \mathbb{C}^{n\times n}$, respectively.

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. $X \in \mathbb{C}^{n\times n}$ is a left strong DMP inverse of A and only if both $\mathcal{N}(A^k \alpha_A) \subseteq \mathcal{N}(X)$ and $XA^{k+1} = A^k$.*

Proof. By Definition 3.1, it suffices to prove the following equality:

$$\mathcal{N}(A^k \alpha_A) = \mathcal{N}(AA^D \alpha_A).$$

For arbitrary $s \in \mathcal{N}(A^k \alpha_A)$, we have $A^k \alpha_A s = 0$, and then

$$AA^D \alpha_A s = (A^D)^k A^k \alpha_A s = 0,$$

which gives $s \in \mathcal{N}(AA^D \alpha_A)$, so $\mathcal{N}(A^k \alpha_A) \subseteq \mathcal{N}(AA^D \alpha_A)$ by the arbitrariness of s.

For arbitrary $t \in \mathcal{N}(AA^D \alpha_A)$, we have $AA^D \alpha_A t = 0$, and then

$$A^k \alpha_A t = A^D A^{k+1} \alpha_A t = A^k AA^D \alpha_A t = 0,$$

which gives $t \in \mathcal{N}(A^k \alpha_A)$, so $\mathcal{N}(AA^D \alpha_A) \subseteq \mathcal{N}(A^k \alpha_A)$ by the arbitrariness of t. Thus $\mathcal{N}(A^k \alpha_A) = \mathcal{N}(AA^D \alpha_A)$ holds by $\mathcal{N}(A^k \alpha_A) \subseteq \mathcal{N}(AA^D \alpha_A)$ and $\mathcal{N}(AA^D \alpha_A) \subseteq \mathcal{N}(A^k \alpha_A)$. □

Theorem 3.5. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then Y is a right strong DMP inverse of A if and only if both $Y = A^k (A^k)^\dagger Y$ and $A^{k+1} Y = A^k \alpha_A$ hold.*

Proof. "⇒" Let Y be a right strong DMP inverse of A. Then by Definition 3.1, we have

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^k) \text{ and } A^D AY = A^D \alpha_A.$$

The condition $\mathcal{R}(Y) \subseteq \mathcal{R}(A^k)$ gives $Y = A^k U$ for some $U \in \mathbb{C}^{n\times n}$, then

$$Y = A^k U = A^k (A^k)^\dagger A^k U = A^k (A^k)^\dagger Y.$$

Pre-multiplying A^{k+1} on the condition $A^D AY = A^D \alpha_A$ gives

$$A^{k+1} A^D AY = A^{k+1} A^D \alpha_A,$$

which gives $A^{k+1}Y = A^k\alpha_A$ by $A^{k+1}A^D = A^k$.

"⇐" The condition $Y = A^k(A^k)^\dagger Y$ means $\mathcal{R}(Y) \subseteq \mathcal{R}(A^k)$ is trivial. Pre-multiplying $(A^D)^{k+1}$ on the condition $A^{k+1}Y = A^k\alpha_A$ gives $(A^D)^{k+1}A^{k+1}Y = (A^D)^{k+1}A^k\alpha_A$, which is $A^DAY = A^D\alpha_A$, and thus, Y is a right strong DMP inverse of A by Definition 3.1. □

Note that Theorem 3.5 also can be found by [10, Definition 2.7]. For the convenience of the readers, we have retained the proof of Theorem 3.5.

Corollary 3.6. *Let $A \in \mathbb{C}^{n\times n}$ with* ind $(A) = k$. *Then Y is a right strong DMP inverse of A if and only if both $Y = AA^{\textcircled{\dagger}}Y$ and $A^{k+1}Y = A^k\alpha_A$ hold.*

Proof. By [19, Corollary 3.3], we have $AA^{\textcircled{\dagger}} = A^k(A^k)^\dagger$, and thus the proof is finished by Theorem 3.5. □

A natural question is: If a complex matrix is both left strong DMP invertible and right strong DMP invertible, is it strong DMP invertible? In Theorem 3.7, we answer this question.

Theorem 3.7. *Let $A \in \mathbb{C}^{n\times n}$. If A is both left and right strong DMP invertible, then the left strong DMP inverse of A and the right strong DMP inverse of A are unique. Moreover, the left strong DMP inverse of A coincides with the right strong DMP inverse of A.*

Proof. Let X be a left strong DMP inverse of A and Y be a right strong DMP inverse of A. Then by Definition 3.1,

$$\mathcal{N}(AA^D\alpha_A) \subseteq \mathcal{N}(X) \text{ and } XA^{k+1} = A^k \tag{3.1}$$

and

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^k) \text{ and } A^DAY = A^D\alpha_A \tag{3.2}$$

hold. Thus $X = UAA^D\alpha_A$ and $Y = A^kV$ for some $U, V \in \mathbb{C}^{n\times n}$ by (3.1) and (3.2). Note that by the proof of Theorem 2.8, we have $\alpha_AA = A$. Therefore,

$$\begin{aligned} &X = UAA^D\alpha_A = UAA^DAY = UAA^D\alpha_AAY = XAY; \\ &Y = A^kV = XA^{k+1}V = XAA^kV = XAY \end{aligned} \tag{3.3}$$

by (3.1) and (3.2). Hence $X = Y$ by (3.3). If Z is a another right strong DMP inverse of A, one can prove $X = Z$ in a similar way. Then $Y = Z$ by $X = Y$ and $X = Z$, which says the right strong DMP inverse of A is unique. One can prove the left strong DMP inverse of A is unique by a similar proof of the uniqueness of the right strong DMP inverse of A. By the above proof, we can see that the left strong DMP inverse of A coincides with the right strong DMP inverse of A. □

Theorem 3.8. *Let $A, X \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*

(1) *X is the strong DMP inverse of A;*

(2) *$\mathcal{N}(AA^D\alpha_A) \subseteq \mathcal{N}(X)$, $XA^{k+1} = A^k$, $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, and $A^DAX = A^D\alpha_A$;*

(3) *$\mathcal{N}(A^k\alpha_A) \subseteq \mathcal{N}(X)$, $XA^{k+1} = A^k$, $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, and $A^DAX = A^D\alpha_A$;*

(4) *$\mathcal{N}(AA^D\alpha_A) \subseteq \mathcal{N}(X)$, $XA^{k+1} = A^k$, $X = A^k(A^k)^\dagger X$, and $A^{k+1}X = A^k\alpha_A$;*

(5) *$\mathcal{N}(A^k\alpha_A) \subseteq \mathcal{N}(X)$, $XA^{k+1} = A^k$, $X = A^k(A^k)^\dagger X$, and $A^{k+1}X = A^k\alpha_A$;*

(6) $\mathcal{N}(AA^D\alpha_A) \subseteq \mathcal{N}(X)$*,* $XA^{k+1} = A^k$*,* $X = AA^{\textcircled{\dagger}}X$*, and* $A^{k+1}X = A^k\alpha_A$*;*

(7) $\mathcal{N}(A^k\alpha_A) \subseteq \mathcal{N}(X)$*,* $XA^{k+1} = A^k$*,* $X = AA^{\textcircled{\dagger}}X$*, and* $A^{k+1}X = A^k\alpha_A$*.*

Proof. "(1) ⇔ (2)" is trivial by Definition 3.1 and Theorem 3.7.
"(2) ⇔ (3)" by Theorem 3.4.
"(2) ⇔ (4)" and "(3) ⇔ (5)" by Theorem 3.5.
"(2) ⇔ (6)" and "(3) ⇔ (7)" by Corollary 3.6. □

Note that, in Theorem 3.8, the first two equations or the last two equations among the items (2)–(7) are sufficient to characterize $X = A^{D,\dagger}$ equivalently by Theorem 3.4 and [10, Definition 2.7].

4. Conclusions

We introduce the strong DMP inverse of a square complex matrix and prove that the strong DMP inverse of a square complex matrix is unique. The criteria for the strong DMP invertibility of a matrix and the exact expressions of the strong DMP inverse are obtained. The one-sided strong DMP inverse is introduced. We answer the question of when the left strong DMP inverse is consistent with the right strong DMP inverse, and we describe perspectives for further research:
(1) Considering the matrix partial orders based on the generalized inverses related to the strong DMP inverse.
(2) Extending the strong DMP inverse of a complex matrix to an element in rings.
(3) The column space and the null space of a complex matrix can be used to study the generalized inverses for a complex matrix.

Author contributions

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Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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