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Research article

Decoupled time filtered method for the time dependent Stokes–Biot problem

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Abstract: In this paper, we propose a novel decoupled time filtered finite element method for the coupled Stokes–Biot problem. The key innovation of our method is the combination of a decoupling strategy with a time filtered technique that can enhance computational efficiency while maintaining numerical accuracy. Specifically, applying the time filter to the backward Euler scheme elevates its temporal accuracy to second order. At each time step, we first solve the Stokes problem, then use the computed Stokes velocity to solve the Biot problem. We rigorously analyze the proposed scheme to establish its stability and derive error estimates. Furthermore, some numerical tests are presented to validate the theoretical findings and demonstrate the efficiency, accuracy, and robustness of our method.

Keywords: Stokes–Biot problem; decoupled; time filtered; stability; error estimates
Mathematics Subject Classification: 76S05, 76D07, 65M6, 65M12

1. Introduction

The interaction between a free incompressible Newtonian fluid and a fluid within a poroelastic medium is a complex multiphysics problem. This coupled phenomenon is widely employed to predict and control processes such as groundwater flow in fractured aquifers and oil or gas extraction. In this work, the free fluid flow is modeled by the Stokes equations, while the coupled deformation and fluid flow in the poroelastic medium are described by the Biot system. At the interface, the Stokes–Biot model features both the Stokes–Darcy coupling [1–6] and the fluid-structure interaction [7–13]. Numerical algorithms for these two types of coupling problems have been extensively studied in the cited literature.

The Stokes–Biot problem combines all the challenges of both the Stokes–Darcy problem and fluid-structure interaction (FSI). A wide variety of numerical methods have been studied for the Stokes–Biot problems or the Navier–Stokes–Biot problems. These include coupled finite element methods [14, 15], discontinuous finite element methods [16–18], Lagrange multiplier methods [19, 20], decoupled finite element method [21–24], a Nitsche-based cut finite element method [25], mixed finite element methods [26–28], virtual element numerical schemes [29], among others.

The backward Euler method is a fundamental approach to solving unsteady problems. By adding one line of code to the backward Euler method, Guzel and Layton [30] proposed a time filtered method. This method can increase the time accuracy from first to second order and can be easily adapted to variable time steps. For constant time steps, time-filtered methods have been successfully implemented in various coupled systems, including the Stokes–Darcy problem [31], Navier–Stokes/Darcy problem [32], Natural Convection Problems [33], MHD problem [34], the stabilized incompressible diffusive Peterlin viscoelastic fluid model [35], and Navier-Stokes problem [36]. The pursuit of greater computational efficiency has led to the investigation of variable step size techniques [37–40]. These techniques are designed to dynamically adjust the time step according to the numerical results obtained earlier.

Solving all unknown variables simultaneously requires substantial computational costs in terms of CPU time and memory. It often leads to a failed solution process due to memory limitations when the mesh is finely partitioned. Generally speaking, coupled algorithms exhibit better stability. However, decoupled algorithms have lower computational costs and can achieve a comparable level of accuracy [5]. Motivated by these trade-offs, this paper investigates a decoupled, time-filtered method based on the backward Euler scheme for solving the Stokes–Biot problem.The proposed algorithm proceeds in three sequential steps: (i) solve the Stokes problem using lagged interface terms to obtain the fluid velocity and pressure; (ii) solve the Biot system using the updated Stokes velocity as the interface condition; and (iii) apply a time filter to correct the previously computed unknowns. This algorithm effectively combines the numerical efficiency of decoupled schemes with time-filtered techniques, while preserving the stability and accuracy of the coupled system.

This paper is structured as follows. Section 2 introduces the time-dependent Stokes–Biot model and presents its weak formulation. In Section 3, we propose a novel decoupled time filtered algorithm. The stability of the scheme is analyzed in Section 4. Based on the stability analysis, Section 5 provides a detailed error estimate. Numerical experiments are presented in Section 6 to demonstrate the performance of the proposed method. Finally, some conclusions are summarized in Section 7.

2. The Stokes–Biot model and weak formulation

2.1. The Stokes–Biot model

Consider a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ partitioned into two non-overlapping subdomains: the fluid region Ω_f and the poroelastic region Ω_p. Let $\rho_f > 0$ be the density of the fluid, $\mu_f > 0$ be the constant fluid viscosity and $\boldsymbol{f}_f$ be the body force term. In the fluid region Ω_f, the velocity $\boldsymbol{u}$ and pressure p_f are governed by the time-dependent incompressible Stokes system:

$$\rho_f \partial_t \boldsymbol{u} - \nabla \cdot \boldsymbol{\sigma}_f(\boldsymbol{u}, p_f) = \boldsymbol{f}_f, \qquad \text{in } \Omega_f \times (0, T], \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \text{in } \Omega_f \times (0, T], \tag{2.1b}$$

where the deformation rate tensor $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$, and the Cauchy stress tensor $\boldsymbol{\sigma}_f(\boldsymbol{u}, p_f) = 2\mu_f \mathbb{D}(\boldsymbol{u}) - p_f \mathbb{I}$. Let $\rho_p > 0$ be the density of the saturated medium, $s_0 > 0$ be the constrained specific storage coefficient, $0 < \alpha \leq 1$ be the Biot-Willis constant, and $\mathbb{K}$ be the hydraulic conductivity. Furthermore, let $\boldsymbol{f}_p$ be the body force term and g_p be the external source or sink terms. In the poroelasticity region Ω_p, the displacement $\boldsymbol{\eta}$, the velocity of the poroelastic $\boldsymbol{\xi} = \partial_t \boldsymbol{\eta}$, and the fluid pore pressure p_p are governed by the Biot model [41,42]:

$$\rho_p \partial_t \boldsymbol{\xi} - \nabla \cdot \boldsymbol{\sigma}_p(\boldsymbol{\eta}, p_p) = \boldsymbol{f}_p, \qquad \text{in } \Omega_p \times (0, T], \tag{2.2a}$$
$$s_0 \partial_t p_p + \alpha \nabla \cdot \boldsymbol{\xi} - \nabla \cdot (\mathbb{K} \nabla p_p) = g_p, \qquad \text{in } \Omega_p \times (0, T], \tag{2.2b}$$

where $\boldsymbol{\sigma}_p(\boldsymbol{\eta}, p_p) = 2\mu_p \mathbb{D}(\boldsymbol{\eta}) + \lambda_p (\nabla \cdot \boldsymbol{\eta}) \mathbb{I} - \alpha p_p \mathbb{I}$, $\mu_p > 0$, and $\lambda_p > 0$ are the Lamé constants for the solid skeleton. Denote the largest (smallest) eigenvalue of $\mathbb{K}$ by $K_{max}(K_{min} > 0)$.

Denote the interface by $\Gamma = \partial\Omega_f \cap \partial\Omega_p$, we assume the following interface conditions on $\Gamma \times (0, T]$:

$$\boldsymbol{u} \cdot \boldsymbol{n}_f = (\boldsymbol{\xi} - \mathbb{K} \nabla p_p) \cdot \boldsymbol{n}_f, \tag{2.3a}$$
$$\boldsymbol{\tau}_f^i \cdot (\boldsymbol{\sigma}_f(\boldsymbol{u}, p_f) \boldsymbol{n}_f) = -\gamma \boldsymbol{\tau}_f^i \cdot (\boldsymbol{u} - \boldsymbol{\xi}), \text{ for } i = 1, \ldots, d-1, \tag{2.3b}$$
$$\boldsymbol{n}_f \cdot (\boldsymbol{\sigma}_f(\boldsymbol{u}, p_f) \boldsymbol{n}_f) = -p_p, \tag{2.3c}$$
$$\boldsymbol{\sigma}_f(\boldsymbol{u}, p_f) \boldsymbol{n}_f = \boldsymbol{\sigma}_p(\boldsymbol{\eta}, p_p) \boldsymbol{n}_f, \tag{2.3d}$$

where $\{\boldsymbol{\tau}_f^i\}_{i=1,\ldots,d-1}$ is a linearly independent set of vectors tangent to the interface Γ, and γ is the resistance parameter in the tangential direction.

Let $\Gamma_f = \partial\Omega_f \cap \partial\Omega$, $\Gamma_p = \partial\Omega_p \cap \partial\Omega$. We assume the following boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0}, \text{on } \Gamma_f \times (0, T], \qquad p_p = 0, \text{on } \Gamma_p^D \times (0, T], \tag{2.4}$$
$$\mathbb{K} \nabla p_p \cdot \boldsymbol{n}_p = 0, \text{on } \Gamma_p^N \times (0, T], \qquad \boldsymbol{\eta} = \boldsymbol{0}, \text{ on } \Gamma_p \times (0, T], \tag{2.5}$$

where $\Gamma_p = \Gamma_p^D \cup \Gamma_p^N$, Γ_p^D and Γ_p^N are the Dirichlet and Neumann conditions, respectively.

2.2. *Weak formulation*

Let $H^l(\Omega_\star)$ be the standard Sobolev space equipped with the norm $\|\cdot\|_{H^l(\Omega_\star)}$, where $\star$ may be f or p. Let $\|\cdot\|_X$ be the standard L^2 norm, and $(\cdot, \cdot)_X$ denote the standard $L^2(X)$ inner product, where X may be $\Omega_\star$ or Γ. Let $\boldsymbol{H}^l(\Omega_\star) = (H^l(\Omega_\star))^d$ be the vector-value Sobolev space. We introduce the following spaces:

$$\boldsymbol{H}_f = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_f) | \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_f\}, \quad Q_f = L^2(\Omega_f),$$
$$\boldsymbol{H}_p = \{\boldsymbol{\chi} \in \boldsymbol{H}^1(\Omega_p) | \boldsymbol{\chi} = \boldsymbol{0} \text{ on } \Gamma_p\}, \quad Q_p = \{\zeta \in H^1(\Omega_p) | \zeta = 0 \text{ on } \Gamma_p^D\}.$$

For the given $(\boldsymbol{f}_f, \boldsymbol{f}_p, g_p) \in \boldsymbol{L}^2(\Omega_f) \times \boldsymbol{L}^2(\Omega_p) \times L^2(\Omega_p)$, the variational formulation of the Stokes–Biot system is to find $(\boldsymbol{u}, p_f, \boldsymbol{\eta}, \boldsymbol{\xi}, p_p) \in \boldsymbol{H}_f \times Q_f \times \boldsymbol{H}_p \times \boldsymbol{H}_p \times Q_p$ with $\boldsymbol{\xi} = \partial_t \boldsymbol{\eta}$ such that for any $(\boldsymbol{v}, q, \boldsymbol{\chi}, \zeta) \in \boldsymbol{H}_f \times Q_f \times \boldsymbol{H}_p \times Q_p$ and $t \in (0, T]$, there hold

$$\rho_f (\partial_t \boldsymbol{u}, \boldsymbol{v})_{\Omega_f} + a_f(\boldsymbol{u}, \boldsymbol{v}) + b_f(\boldsymbol{v}, p_f) + (p_p, \boldsymbol{v} \cdot \boldsymbol{n}_f)_\Gamma + \gamma((\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{\tau}_f, \boldsymbol{v} \cdot \boldsymbol{\tau}_f)_\Gamma = (\boldsymbol{f}_f, \boldsymbol{v})_{\Omega_f}, \tag{2.6a}$$
$$b_f(\boldsymbol{u}, q) = 0. \tag{2.6b}$$
$$\rho_p (\partial_t \boldsymbol{\xi}, \boldsymbol{\chi})_{\Omega_p} + a_\eta(\boldsymbol{\eta}, \boldsymbol{\chi}) + b_p(\boldsymbol{\chi}, p_p) - (p_p, \boldsymbol{\chi} \cdot \boldsymbol{n}_f)_\Gamma - \gamma((\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{\tau}_f, \boldsymbol{\chi} \cdot \boldsymbol{\tau}_f)_\Gamma = (\boldsymbol{f}_p, \boldsymbol{\chi})_{\Omega_p}, \tag{2.6c}$$
$$s_0 (\partial_t p_p, \zeta)_{\Omega_p} - b_p(\boldsymbol{\xi}, \zeta) + a_p(p_p, \zeta) - ((\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n}_f, \zeta)_\Gamma = (g_p, \zeta)_{\Omega_p}, \tag{2.6d}$$

where

$$a_f(\boldsymbol{u},\boldsymbol{v}) = 2\mu_f(\mathbb{D}(\boldsymbol{u}),\mathbb{D}(\boldsymbol{v}))_{\Omega_f}, \quad b_f(\boldsymbol{v},q) = -(\nabla\cdot\boldsymbol{v},q)_{\Omega_f}, \tag{2.7}$$

$$b_p(\boldsymbol{\chi},\zeta) = -\alpha(\nabla\cdot\boldsymbol{\chi},\zeta)_{\Omega_p}, \qquad a_p(p_p,\zeta) = (\mathbb{K}\nabla p_p,\nabla\zeta)_{\Omega_p}, \tag{2.8}$$

$$a_\eta(\boldsymbol{\eta},\boldsymbol{\chi}) = 2\mu_p(\mathbb{D}(\boldsymbol{\eta}),\mathbb{D}(\boldsymbol{\chi}))_{\Omega_p} + \lambda_p(\nabla\cdot\boldsymbol{\eta},\nabla\cdot\boldsymbol{\chi})_{\Omega_p}. \tag{2.9}$$

Let $\boldsymbol{V} = (\boldsymbol{v},\boldsymbol{\chi},\zeta)$ and $\boldsymbol{F} = (\boldsymbol{f}_f,\boldsymbol{f}_p,g_p)$. Define the norms by

$$\begin{aligned}
\|\boldsymbol{V}\|^2 &= \|\boldsymbol{v}\|^2_{\Omega_f} + \|\boldsymbol{\chi}\|^2_{\Omega_p} + \|\zeta\|^2_{\Omega_p},\\
\|\boldsymbol{V}\|^2_S &= \rho_f\|\boldsymbol{v}\|^2_{\Omega_f} + \rho_p\|\boldsymbol{\chi}\|^2_{\Omega_p} + s_0\|\zeta\|^2_{\Omega_p},\\
\|\boldsymbol{F}\|^2 &= \|\boldsymbol{f}_f\|^2_{\Omega_f} + \|\boldsymbol{f}_p\|^2_{\Omega_p} + \|g_p\|^2_{\Omega_p}.
\end{aligned}$$

Then it is easy to get

$$\tilde{C}_S\|\boldsymbol{V}\|_S \le \|\boldsymbol{V}\| \le C_S\|\boldsymbol{V}\|_S. \tag{2.10}$$

Moreover, let $\boldsymbol{U} = (\boldsymbol{u},\boldsymbol{\xi},p_p)$. We introduce the following bilinear forms:

$$\langle\langle\partial_t\boldsymbol{U},\boldsymbol{V}\rangle\rangle = \rho_f(\partial_t\boldsymbol{u},\boldsymbol{v})_{\Omega_f} + \rho_p(\partial_t\boldsymbol{\xi},\boldsymbol{\chi})_{\Omega_p} + s_0(\partial_t p_p,\zeta)_{\Omega_p}, \tag{2.11}$$

$$(\boldsymbol{F},\boldsymbol{V}) = (\boldsymbol{f}_f,\boldsymbol{v})_{\Omega_f} + (\boldsymbol{f}_p,\boldsymbol{\chi})_{\Omega_p} + (g_p,\zeta)_{\Omega_p}, \tag{2.12}$$

$$a(\boldsymbol{U},\boldsymbol{V}) = a_f(\boldsymbol{u},\boldsymbol{v}) + a_p(p_p,\zeta), \tag{2.13}$$

$$a_b(\boldsymbol{U},\boldsymbol{V}) = a_{b,f}(\boldsymbol{u},\boldsymbol{v}) + a_{b,p}((\boldsymbol{\xi},p_p);(\boldsymbol{\chi},\zeta)), \tag{2.14}$$

$$a_{b,f}(\boldsymbol{u},\boldsymbol{v}) = \gamma(\boldsymbol{u}\cdot\boldsymbol{\tau}_f,\boldsymbol{v}\cdot\boldsymbol{\tau}_f)_\Gamma, \tag{2.15}$$

$$a_{b,p}((\boldsymbol{\xi},p_p);(\boldsymbol{\chi},\zeta)) = (\zeta,\boldsymbol{\xi}\cdot\boldsymbol{n}_f)_\Gamma - (p_p,\boldsymbol{\chi}\cdot\boldsymbol{n}_f)_\Gamma + \gamma(\boldsymbol{\xi}\cdot\boldsymbol{\tau}_f,\boldsymbol{\chi}\cdot\boldsymbol{\tau}_f)_\Gamma, \tag{2.16}$$

$$a_\Gamma((\boldsymbol{u},\boldsymbol{\xi},p_p);(\boldsymbol{v},\boldsymbol{\chi},\zeta)) = (p_p,\boldsymbol{v}\cdot\boldsymbol{n}_f)_\Gamma - (\zeta,\boldsymbol{u}\cdot\boldsymbol{n}_f)_\Gamma - \gamma(\boldsymbol{\xi}\cdot\boldsymbol{\tau}_f,\boldsymbol{v}\cdot\boldsymbol{\tau}_f)_\Gamma - \gamma(\boldsymbol{u}\cdot\boldsymbol{\tau}_f,\boldsymbol{\chi}\cdot\boldsymbol{\tau}_f)_\Gamma, \tag{2.17}$$

where

$$\boldsymbol{v}\cdot\boldsymbol{\tau}_f = \sum_{i=1}^{d-1}(\boldsymbol{v}\cdot\boldsymbol{\tau}_f^i)\boldsymbol{\tau}_f^i.$$

Adding (2.6a), (2.6c), and (2.6d) yields

$$\begin{aligned}
&\langle\langle\partial_t\boldsymbol{U},\boldsymbol{V}\rangle\rangle + a(\boldsymbol{U},\boldsymbol{V}) + a_\eta(\boldsymbol{\eta},\boldsymbol{\chi}) + b_f(\boldsymbol{v},p_f) + b_p(\boldsymbol{\chi},p_p) - b_p(\boldsymbol{\xi},\zeta)\\
&\quad + a_b(\boldsymbol{U},\boldsymbol{V}) + a_\Gamma((\boldsymbol{u},\boldsymbol{\xi},p_p);(\boldsymbol{v},\boldsymbol{\chi},\zeta)) = (\boldsymbol{F},\boldsymbol{V}),
\end{aligned} \tag{2.18a}$$

$$b_f(\boldsymbol{u},q) = 0. \tag{2.18b}$$

3. The numerical scheme

Let $t_n = n\Delta t$ for $n = 0,1,\ldots,N = T/\Delta t$, where $\Delta t > 0$ is a time step. In the later analysis, we define the following discrete time norms:

$$\|f\|_{l^2(0,T;H^l(X))} = \max\left\{\left(\Delta t\sum_{n=1}^{N}\|f^n\|^2_{H^l(X)}\right)^{\frac{1}{2}},\left(\Delta t\sum_{n=0}^{N-1}\|f^n\|^2_{H^l(X)}\right)^{\frac{1}{2}}\right\}.$$

Let $\mathcal{T}_h^f(\mathcal{T}_h^p)$ be a quasi-uniform partition of $\Omega_f(\Omega_p)$, and let $\boldsymbol{H}_{fh} \subset \boldsymbol{H}_f, Q_{fh} \subset Q_f$, $\boldsymbol{H}_{ph} \subset \boldsymbol{H}_p$, $Q_{ph} \subset Q_p$ be the conforming finite element spaces. Assume that there exists a constant β such that

$$\inf_{q_h \in Q_{fh}} \sup_{\boldsymbol{v}_h \in \boldsymbol{H}_{fh}} \frac{b_f(\boldsymbol{v}_h, q_h)}{\|q_h\|_{\Omega_f} \|\boldsymbol{v}_h\|_{H^1(\Omega_f)}} \geq \beta. \tag{3.1}$$

This is the classical discrete Ladyzhenskaya–Babuška–Brezzi condition, which is necessary for the stability of the discrete Stokes problem [43].

Let

$$\{\boldsymbol{v}(\boldsymbol{x}, t_n)\}_{n=0}^N = \{\boldsymbol{v}^n\}_{n=0}^N, \;\; d_t \boldsymbol{v}^{n+1} = \frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^n}{\Delta t}, \;\; D_t \boldsymbol{v}^{n+2} = \frac{3\boldsymbol{v}^{n+2} - 4\boldsymbol{v}^{n+1} + \boldsymbol{v}^n}{2\Delta t}.$$

By explicitly treating a part of the coupling terms at the interface, we present three decoupled finite element methods as follows: the first two are similar to those reported in existing literature [21–23], while the third one is the focus of our study.

Remark 3.1. *All three algorithms are decoupled finite element methods, with core differences in their time-discretization schemes and whether time filtering is incorporated. Algorithm 1 is a basic implicit time-discretization scheme using the first-order backward Euler formula. It features low computational cost and straightforward implementation. Algorithm 2 is a higher-order improved version of Algorithm 1, using the second-order backward differentiation formula (BDF2). However, the solution of this method at each step depends on the results from the previous two time steps. Algorithm 3 is an optimized version of the previous two algorithms, adding a time-filtering (TF) step to the BDF1 decoupled method.*

Remark 3.2. *For Algorithm 3, Steps 1 and 2 employ the backward Euler method to compute the approximations* $(\hat{\boldsymbol{u}}_h^{n+2}, p_{fh}^{n+2}, \hat{\boldsymbol{\xi}}_h^{n+2}, \hat{\boldsymbol{\eta}}_h^{n+2}, \hat{p}_{ph}^{n+2})$*. Then, in Step 3, a time filter is applied to update the values of* $(\hat{\boldsymbol{u}}_h^{n+2}, \hat{\boldsymbol{\xi}}_h^{n+2}, \hat{\boldsymbol{\eta}}_h^{n+2}, \hat{p}_{ph}^{n+2})$*. The time filter acts as a corrector step. It post-processes the solution by combining solutions at time levels* t^{n+2}*,* t^{n+1} *and* t^n*. This combination is designed to cancel the truncation error term of the backward Euler method. This paper investigates the time-dependent Stokes–Biot problem, with a specific focus on the Biot model which couples parabolic and hyperbolic dynamics. Employing a high-order temporal discretization scheme is key to achieving high accuracy in the solutions of this coupled system.*

A simple calculation yields

$$\hat{\boldsymbol{u}}_h^{n+2} = \frac{3}{2}\boldsymbol{u}_h^{n+2} - \boldsymbol{u}_h^{n+1} + \frac{1}{2}\boldsymbol{u}_h^n, \quad \frac{\hat{\boldsymbol{u}}_h^{n+2} - \boldsymbol{u}_h^{n+1}}{\Delta t} = D_t \boldsymbol{u}_h^{n+2}. \tag{3.2}$$

According to Algorithm 3, for any $\boldsymbol{V}_h \in \boldsymbol{H}_{fh} \times \boldsymbol{H}_{ph} \times Q_{ph}$ and $q_h \in Q_{fh}$, we have

$$\begin{aligned}
&\langle\langle D_t \boldsymbol{U}_h^{n+2}, \boldsymbol{V}_h \rangle\rangle + a(\hat{\boldsymbol{U}}_h^{n+2}, \boldsymbol{V}_h) + a_b(\hat{\boldsymbol{U}}_h^{n+2}, \boldsymbol{V}_h) + b_f(\boldsymbol{v}_h, p_{fh}^{n+2}) + a_\eta(\hat{\boldsymbol{\eta}}_h^{n+2}, \boldsymbol{\chi}_h) \\
&\quad + b_p(\boldsymbol{\chi}_h, \hat{p}_{ph}^{n+2}) - b_p(\hat{\boldsymbol{\xi}}_h^{n+2}, \zeta_h) + a_\Gamma((\hat{\boldsymbol{u}}_h^{n+2}, 2\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^n, 2p_{ph}^{n+1} - p_{ph}^n); (\boldsymbol{v}_h, \boldsymbol{\chi}_h, \zeta_h)) \\
&= \langle \boldsymbol{F}^{n+2}, \boldsymbol{V}_h \rangle, \qquad (3.3a) \\
&b_f(\hat{\boldsymbol{u}}_h^{n+2}, q_h) = 0. \qquad (3.3b)
\end{aligned}$$

Algorithm 1 A backward Euler Scheme (BDF1) based decoupled finite element method.

Step 1. For any $\boldsymbol{v}_h \in \boldsymbol{H}_{fh}$, $q_h \in Q_{fh}$, find $\boldsymbol{u}_h^{n+1} \in \boldsymbol{H}_{fh}$, $p_{fh}^{n+1} \in Q_{fh}, n = 0, \ldots, N-1$ such that

$$\begin{aligned}
&\rho_f(d_t\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)_{\Omega_f} + a_f(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + b_f(\boldsymbol{v}_h, p_{fh}^{n+1}) + a_{b,f}(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) \\
&\quad + (p_{ph}^n, \boldsymbol{v}_h \cdot \boldsymbol{n}_f)_\Gamma - \gamma(\boldsymbol{\xi}_h^n \cdot \boldsymbol{\tau}_f, \boldsymbol{v}_h \cdot \boldsymbol{\tau}_f)_\Gamma = (\boldsymbol{f}_f^{n+1}, \boldsymbol{v}_h)_{\Omega_f}, \qquad (3.4a)\\
&b_f(\boldsymbol{u}_h^{n+1}, q_h) = 0. \qquad (3.4b)
\end{aligned}$$

Step 2. Given $\boldsymbol{u}_h^{n+1}$ from Step 1, for any $\boldsymbol{\chi}_h \in \boldsymbol{H}_{ph}$, $\zeta_h \in Q_{ph}$, find $\boldsymbol{\eta}_h^{n+1}, \boldsymbol{\xi}_h^{n+1} \in \boldsymbol{H}_{ph}$ and $p_{ph}^{n+1} \in Q_{ph}, n = 0, \ldots, N-1$ such that

$$\begin{aligned}
&d_t\boldsymbol{\eta}_h^{n+1} = \boldsymbol{\xi}_h^{n+1}, \qquad (3.4c)\\
&\rho_p(d_t\boldsymbol{\xi}_h^{n+1}, \boldsymbol{\chi}_h)_{\Omega_p} + s_0(d_t p_{ph}^{n+1}, \zeta_h)_{\Omega_p} + a_\eta(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{\chi}_h) + b_p(\boldsymbol{\chi}_h, p_{ph}^{n+1}) \\
&\quad + a_p(p_{ph}^{n+1}, \zeta_h) - b_p(\boldsymbol{\xi}_h^{n+1}, \zeta_h) + a_{b,p}((\boldsymbol{\xi}_h^{n+1}, p_{ph}^{n+1}); (\boldsymbol{\chi}_h, \zeta_h)) - (\boldsymbol{u}_h^{n+1} \cdot \boldsymbol{n}_f, \zeta_h)_\Gamma \\
&\quad - \gamma(\boldsymbol{u}_h^{n+1} \cdot \boldsymbol{\tau}_f, \boldsymbol{\chi}_h \cdot \boldsymbol{\tau}_f)_\Gamma \\
&= (\boldsymbol{f}_p^{n+1}, \boldsymbol{\chi}_h)_{\Omega_p} + (g_p^{n+1}, \zeta_h)_{\Omega_p}. \qquad (3.4d)
\end{aligned}$$

Algorithm 2 A second order backward differentiation formula(BDF2) based decoupled finite element method.

Step 1. For any $\boldsymbol{v}_h \in \boldsymbol{H}_{fh}$, $q_h \in Q_{fh}$, find $\boldsymbol{u}_h^{n+2} \in \boldsymbol{H}_{fh}$, $p_{fh}^{n+2} \in Q_{fh}, n = 0, \ldots, N-2$ such that

$$\begin{aligned}
&\rho_f(D_t\boldsymbol{u}_h^{n+2}, \boldsymbol{v}_h)_{\Omega_f} + a_f(\boldsymbol{u}_h^{n+2}, \boldsymbol{v}_h) + b_f(\boldsymbol{v}_h, p_{fh}^{n+2}) + a_{b,f}(\boldsymbol{u}_h^{n+2}, \boldsymbol{v}_h) \\
&\quad + (2p_{ph}^{n+1} - p_{ph}^n, \boldsymbol{v}_h \cdot \boldsymbol{n}_f)_\Gamma - \gamma((2\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^n) \cdot \boldsymbol{\tau}_f, \boldsymbol{v}_h \cdot \boldsymbol{\tau}_f)_\Gamma = (\boldsymbol{f}_f^{n+2}, \boldsymbol{v}_h)_{\Omega_f}, \qquad (3.5a)\\
&b_f(\boldsymbol{u}_h^{n+2}, q_h) = 0. \qquad (3.5b)
\end{aligned}$$

Step 2. Given $\boldsymbol{u}_h^{n+2}$ from Step 1, for any $\boldsymbol{\chi}_h \in \boldsymbol{H}_{ph}$, $\zeta_h \in Q_{ph}$, find $\boldsymbol{\eta}_h^{n+2}, \boldsymbol{\xi}_h^{n+2} \in \boldsymbol{H}_{ph}$ and $p_{ph}^{n+2} \in Q_{ph}, n = 0, \ldots, N-2$ such that

$$\begin{aligned}
&D_t\boldsymbol{\eta}_h^{n+2} = \boldsymbol{\xi}_h^{n+2}, \qquad (3.5c)\\
&\rho_p(D_t\boldsymbol{\xi}_h^{n+2}, \boldsymbol{\chi}_h)_{\Omega_p} + s_0(D_t p_{ph}^{n+2}, \zeta_h)_{\Omega_p} + a_\eta(\boldsymbol{\eta}_h^{n+2}, \boldsymbol{\chi}_h) + b_p(\boldsymbol{\chi}_h, p_{ph}^{n+2}) \\
&\quad + a_p(p_{ph}^{n+2}, \zeta_h) - b_p(\boldsymbol{\xi}_h^{n+2}, \zeta_h) + a_{b,p}((\boldsymbol{\xi}_h^{n+2}, p_{ph}^{n+2}); (\boldsymbol{\chi}_h, \zeta_h)) - (\boldsymbol{u}_h^{n+2} \cdot \boldsymbol{n}_f, \zeta_h)_\Gamma \\
&\quad - \gamma(\boldsymbol{u}_h^{n+2} \cdot \boldsymbol{\tau}_f, \boldsymbol{\chi}_h \cdot \boldsymbol{\tau}_f)_\Gamma \\
&= (\boldsymbol{f}_p^{n+2}, \boldsymbol{\chi}_h)_{\Omega_p} + (g_p^{n+2}, \zeta_h)_{\Omega_p}. \qquad (3.5d)
\end{aligned}$$

Algorithm 3 Decoupled time filtered finite element method (BDF1+TF).

Step 1. For any $\boldsymbol{v}_h \in \boldsymbol{H}_{fh}$, $q_h \in Q_{fh}$, find $\hat{\boldsymbol{u}}_h^{n+2} \in \boldsymbol{H}_{fh}$, $p_{fh}^{n+2} \in Q_{fh}, n = 0, \ldots, N-2$ such that

$$\begin{aligned}
&\rho_f(\frac{\hat{\boldsymbol{u}}_h^{n+2} - \boldsymbol{u}_h^{n+1}}{\Delta t}, \boldsymbol{v}_h)_{\Omega_f} + a_f(\hat{\boldsymbol{u}}_h^{n+2}, \boldsymbol{v}_h) + b_f(\boldsymbol{v}_h, p_{fh}^{n+2}) + a_{b,f}(\hat{\boldsymbol{u}}_h^{n+2}, \boldsymbol{v}_h) \\
&\quad + (2p_{ph}^{n+1} - p_{ph}^{n}, \boldsymbol{v}_h \cdot \boldsymbol{n}_f)_\Gamma - \gamma((2\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^{n}) \cdot \boldsymbol{\tau}_f, \boldsymbol{v}_h \cdot \boldsymbol{\tau}_f)_\Gamma = (\boldsymbol{f}_f^{n+2}, \boldsymbol{v}_h)_{\Omega_f}, \qquad (3.6a)\\
&b_f(\hat{\boldsymbol{u}}_h^{n+2}, q_h) = 0. \qquad (3.6b)
\end{aligned}$$

Step 2. Given $\hat{\boldsymbol{u}}_h^{n+2}$ from Step 1, for any $\boldsymbol{\chi}_h \in \boldsymbol{H}_{ph}$, $\zeta_h \in Q_{ph}$, find $\hat{\boldsymbol{\eta}}_h^{n+2}, \hat{\boldsymbol{\xi}}_h^{n+2} \in \boldsymbol{H}_{ph}$ and $\hat{p}_{ph}^{n+2} \in Q_{ph}, n = 0, \ldots, N-2$ such that

$$\begin{aligned}
&\frac{\hat{\boldsymbol{\eta}}_h^{n+2} - \boldsymbol{\eta}_h^{n+1}}{\Delta t} = \hat{\boldsymbol{\xi}}_h^{n+2}, \qquad (3.6c)\\
&\rho_p(\frac{\hat{\boldsymbol{\xi}}_h^{n+2} - \boldsymbol{\xi}_h^{n+1}}{\Delta t}, \boldsymbol{\chi}_h)_{\Omega_p} + s_0(\frac{\hat{p}_{ph}^{n+2} - p_{ph}^{n+1}}{\Delta t}, \zeta_h)_{\Omega_p} + a_\eta(\hat{\boldsymbol{\eta}}_h^{n+2}, \boldsymbol{\chi}_h) + b_p(\boldsymbol{\chi}_h, \hat{p}_{ph}^{n+2}) \\
&\quad + a_p(\hat{p}_{ph}^{n+2}, \zeta_h) - b_p(\hat{\boldsymbol{\xi}}_h^{n+2}, \zeta_h) + a_{b,p}((\hat{\boldsymbol{\xi}}_h^{n+2}, \hat{p}_{ph}^{n+2}); (\chi_h, \zeta_h)) - (\hat{\boldsymbol{u}}_h^{n+2} \cdot \boldsymbol{n}_f, \zeta_h)_\Gamma \\
&\quad - \gamma(\hat{\boldsymbol{u}}_h^{n+2} \cdot \boldsymbol{\tau}_f, \boldsymbol{\chi}_h \cdot \boldsymbol{\tau}_f)_\Gamma \\
&= (\boldsymbol{f}_p^{n+2}, \boldsymbol{\chi}_h)_{\Omega_p} + (g_p^{n+2}, \zeta_h)_{\Omega_p}. \qquad (3.6d)
\end{aligned}$$

Step 3. Update the $(\hat{\boldsymbol{u}}_h^{n+2}, \hat{\boldsymbol{\xi}}_h^{n+2}, \hat{\boldsymbol{\eta}}_h^{n+2}, \hat{p}_{ph}^{n+2})$ for $n = 0, 1, \ldots, N-2$

$$\boldsymbol{u}_h^{n+2} = \hat{\boldsymbol{u}}_h^{n+2} - \frac{1}{3}(\hat{\boldsymbol{u}}_h^{n+2} - 2\boldsymbol{u}_h^{n+1} + \boldsymbol{u}_h^{n}), \qquad (3.7)$$

$$\boldsymbol{\eta}_h^{n+2} = \hat{\boldsymbol{\eta}}_h^{n+2} - \frac{1}{3}(\hat{\boldsymbol{\eta}}_h^{n+2} - 2\boldsymbol{\eta}_h^{n+1} + \boldsymbol{\eta}_h^{n}), \qquad (3.8)$$

$$\boldsymbol{\xi}_h^{n+2} = \hat{\boldsymbol{\xi}}_h^{n+2} - \frac{1}{3}(\hat{\boldsymbol{\xi}}_h^{n+2} - 2\boldsymbol{\xi}_h^{n+1} + \boldsymbol{\xi}_h^{n}), \qquad (3.9)$$

$$p_{ph}^{n+2} = \hat{p}_{ph}^{n+2} - \frac{1}{3}(\hat{p}_{ph}^{n+2} - 2p_{ph}^{n+1} + p_{ph}^{n}). \qquad (3.10)$$

4. Stability analysis

The following standard inequalities are adopted in our analysis. The bounds (4.1)–(4.4) can be found in [6, 15, 44], while the inverse estimate (4.5) is provided by [45].

(1) For any $\boldsymbol{v} \in \boldsymbol{H}_f, \boldsymbol{\chi} \in \boldsymbol{H}_p$ and $\zeta \in Q_p$, we have the following trace inequalities:

$$\|\boldsymbol{v}\|_\Gamma \le C_{T,1}\|\boldsymbol{v}\|_{\Omega_f}^{1/2}\|\nabla\boldsymbol{v}\|_{\Omega_f}^{1/2}, \quad \|\boldsymbol{v}\|_\Gamma \le \tilde{C}_{T,1}\|\nabla\boldsymbol{v}\|_{\Omega_f}, \qquad (4.1)$$

$$\|\boldsymbol{\chi}\|_\Gamma \le C_{T,2}\|\boldsymbol{\chi}\|_{\Omega_p}^{1/2}\|\nabla\boldsymbol{\chi}\|_{\Omega_p}^{1/2}, \quad \|\boldsymbol{\chi}\|_\Gamma \le \tilde{C}_{T,2}\|\nabla\boldsymbol{\chi}\|_{\Omega_p}, \qquad (4.2)$$

$$\|\zeta\|_\Gamma \le C_{T,3}\|\zeta\|_{\Omega_p}^{1/2}\|\nabla\zeta\|_{\Omega_p}^{1/2}, \quad \|\zeta\|_\Gamma \le \tilde{C}_{T,3}\|\nabla\zeta\|_{\Omega_p}. \qquad (4.3)$$

(2) For any $\boldsymbol{v} \in \boldsymbol{H}_f, \boldsymbol{\chi} \in \boldsymbol{H}_p$ and $\zeta \in Q_p$, we have the following Poincaré inequalities:

$$\|\boldsymbol{v}\|_{\Omega_f} \leq C_{P,1}\|\nabla \boldsymbol{v}\|_{\Omega_f},\ \|\boldsymbol{\chi}\|_{\Omega_p} \leq C_{P,2}\|\nabla \boldsymbol{\chi}\|_{\Omega_p},\ \|\zeta\|_{\Omega_p} \leq C_{P,3}\|\nabla \zeta\|_{\Omega_p}. \tag{4.4}$$

(3) Inverse inequality: Let $\boldsymbol{V}_h^k(\mathcal{T}_\star)$ is a space of polynomials of order k on $\Omega_\star(\star = f, p)$,then

$$\|\nabla \boldsymbol{w}_h\|_{\Omega_\star} \leq C_{inv}h^{-1}\|\boldsymbol{w}_h\|_{\Omega_\star}, \quad \forall \boldsymbol{w}_h \in \boldsymbol{V}_h^k(\mathcal{T}_\star), \tag{4.5}$$

where C_{inv} depends on the angles in the finite element mesh and the polynomials of order k.

Using the basic inequalities, we can derive the following results.

Lemma 4.1 ([22])**.** *For the bilinear forms $a(\cdot,\cdot)$ and $a_\eta(\cdot,\cdot)$ defined in* (2.13) *and* (2.9)*, there exist three positive constants μ_1, μ_2 and μ_3 such that*

$$\begin{aligned} &a(\boldsymbol{V},\boldsymbol{V}) \geq \mu_1\|\nabla \boldsymbol{v}\|_{\Omega_f}^2 + \mu_2\|\nabla\zeta\|_{\Omega_p}^2, \quad \forall\ \boldsymbol{V} \in \boldsymbol{H}_f \times \boldsymbol{H}_p \times Q_p, &(4.6)\\ &a_\eta(\boldsymbol{\chi},\boldsymbol{\chi}) \geq \mu_3\|\nabla\boldsymbol{\chi}\|_{\Omega_p}^2, \qquad\qquad \forall\ \boldsymbol{\chi} \in \boldsymbol{H}_p, &(4.7)\end{aligned}$$

where $\mu_1 = \frac{2\mu_f}{C_{K,f}^2}, \mu_2 = K_{min}$ and $\mu_3 = \frac{2\mu_p}{C_{K,p}^2}$. Here, $C_{K,f}$ and $C_{K,p}$ are the constants in the Korn inequalities.

For later analysis, we show an upper bound on $-a_\Gamma(\cdot,\cdot)$.

Lemma 4.2. *Let $B(\boldsymbol{V}_h^{n+2}) = \boldsymbol{V}_h^{n+2} - 2\boldsymbol{V}_h^{n+1} + \boldsymbol{V}_h^n$, for any $\boldsymbol{V}_h \in \boldsymbol{H}_{fh} \times \boldsymbol{H}_{ph} \times Q_{ph}$ and $\epsilon_1 > 0$, we have*

$$\begin{aligned} &- a_\Gamma((\hat{\boldsymbol{v}}_h^{n+2}, 2\boldsymbol{\chi}_h^{n+1} - \boldsymbol{\chi}_h^n, 2\zeta_h^{n+1} - \zeta_h^n); (\hat{\boldsymbol{v}}_h^{n+2}, \hat{\boldsymbol{\chi}}_h^{n+2}, \hat{\zeta}_h^{n+2}))\\ &\leq 3\epsilon_1\|\nabla\hat{\boldsymbol{v}}_h^{n+2}\|_{\Omega_f}^2 + \gamma\|\hat{\boldsymbol{\chi}}_h^{n+2}\cdot\boldsymbol{\tau}_f\|_\Gamma^2 + \gamma\|\hat{\boldsymbol{v}}_h^{n+2}\cdot\boldsymbol{\tau}_f\|_\Gamma^2 + \frac{C_1}{h}\|B(\boldsymbol{V}_h^{n+2})\|_S^2, \end{aligned} \tag{4.8}$$

where

$$C_1 = \max\{\frac{3\tilde{C}_{T,1}^2C_{T,3}^2C_{inv}}{8s_0\epsilon_1}, \frac{3\tilde{C}_{T,1}^2C_{T,2}^2C_{inv}\gamma^2}{8\rho_p\epsilon_1}\}.$$

Proof. As defined in (2.17) for $a_\Gamma(\cdot;\cdot)$, it follows that

$$\begin{aligned} &- a_\Gamma((\hat{\boldsymbol{v}}_h^{n+2}, 2\boldsymbol{\chi}_h^{n+1} - \boldsymbol{\chi}_h^n, 2\zeta_h^{n+1} - \zeta_h^n); (\hat{\boldsymbol{v}}_h^{n+2}, \hat{\boldsymbol{\chi}}_h^{n+2}, \hat{\zeta}_h^{n+2}))\\ &\quad = (\hat{\zeta}_h^{n+2} - 2\zeta_h^{n+1} + \zeta_h^n, \hat{\boldsymbol{v}}_h^{n+2}\cdot\boldsymbol{n}_f)_\Gamma + \gamma((2\boldsymbol{\chi}_h^{n+1} - \boldsymbol{\chi}_h^n + \hat{\boldsymbol{\chi}}_h^{n+2})\cdot\boldsymbol{\tau}_f, \hat{\boldsymbol{v}}_h^{n+2}\cdot\boldsymbol{\tau}_f)_\Gamma. \end{aligned} \tag{4.9}$$

Using (3.2), we have

$$\begin{aligned} &\hat{\zeta}_h^{n+2} - \zeta_h^{n+2} = \frac{1}{2}(\zeta_h^{n+2} - 2\zeta_h^{n+1} + \zeta_h^n) = \frac{1}{2}B(\zeta_h^{n+2}), &(4.10)\\ &-2\zeta_h^{n+1} + \zeta_h^n = 2\hat{\zeta}_h^{n+2} - 3\zeta_h^{n+2}. &(4.11)\end{aligned}$$

Then the first term on the right hand side of (4.9) can be rewritten as

$$(\hat{\zeta}_h^{n+2} - 2\zeta_h^{n+1} + \zeta_h^n, \hat{\boldsymbol{v}}_h^{n+2}\cdot\boldsymbol{n}_f)_\Gamma = (3\hat{\zeta}_h^{n+2} - 3\zeta_h^{n+2}, \hat{\boldsymbol{v}}_h^{n+2}\cdot\boldsymbol{n}_f)_\Gamma = \frac{3}{2}(B(\zeta_h^{n+2}), \hat{\boldsymbol{v}}_h^{n+2}\cdot\boldsymbol{n}_f)_\Gamma. \tag{4.12}$$

It follows from the Cauchy-Schwarz inequality, trace inequalities (4.1) and (4.3), inverse inequality (4.5) and Young's inequality that

$$
\begin{aligned}
(\hat{\zeta}_h^{n+2} - 2\zeta_h^{n+1} + \zeta_h^n, \hat{\boldsymbol{v}}_h^{n+2} \cdot \boldsymbol{n}_f)_\Gamma &\leq \frac{3}{2}\tilde{C}_{T,1}C_{T,3}C_{inv}^{\frac{1}{2}}h^{-\frac{1}{2}}\|\nabla\hat{\boldsymbol{v}}_h^{n+2}\|_{\Omega_f}\|B(\zeta_h^{n+2})\|_{\Omega_p} \\
&\leq \frac{3\epsilon_1}{2}\|\nabla\hat{\boldsymbol{v}}_h^{n+2}\|_{\Omega_f}^2 + \frac{3\tilde{C}_{T,1}^2C_{T,3}^2C_{inv}}{8\epsilon_1 h}\|B(\zeta_h^{n+2})\|_{\Omega_p}^2. \qquad (4.13)
\end{aligned}
$$

Using (4.10) and (4.11), the second term on the right hand side of (4.9) can be rewritten as

$$
\begin{aligned}
\gamma((2\boldsymbol{\chi}_h^{n+1} - \boldsymbol{\chi}_h^n + \hat{\boldsymbol{\chi}}_h^{n+2}) \cdot \boldsymbol{\tau}_f, \hat{\boldsymbol{v}}_h^{n+2} \cdot \boldsymbol{\tau}_f)_\Gamma &= \gamma((3\boldsymbol{\chi}_h^{n+2} - \hat{\boldsymbol{\chi}}_h^{n+2}) \cdot \boldsymbol{\tau}_f, \hat{\boldsymbol{v}}_h^{n+2} \cdot \boldsymbol{\tau}_f)_\Gamma \\
&= \gamma((2\hat{\boldsymbol{\chi}}_h^{n+2} - \frac{3}{2}B(\boldsymbol{\chi}_h^{n+2})) \cdot \boldsymbol{\tau}_f, \hat{\boldsymbol{v}}_h^{n+2} \cdot \boldsymbol{\tau}_f)_\Gamma. \qquad (4.14)
\end{aligned}
$$

Using the trace inequality (4.1) and Young's inequality, we have

$$
\begin{aligned}
&\gamma((2\boldsymbol{\chi}_h^{n+1} - \boldsymbol{\chi}_h^n + \hat{\boldsymbol{\chi}}_h^{n+2}) \cdot \boldsymbol{\tau}_f, \hat{\boldsymbol{v}}_h^{n+2} \cdot \boldsymbol{\tau}_f)_\Gamma \\
\leq&\gamma\|\hat{\boldsymbol{\chi}}_h^{n+2} \cdot \boldsymbol{\tau}_f\|_\Gamma^2 + \gamma\|\hat{\boldsymbol{v}}_h^{n+2} \cdot \boldsymbol{\tau}_f\|_\Gamma^2 + \frac{3}{2}\gamma\tilde{C}_{T,1}C_{T,2}C_{inv}^{\frac{1}{2}}h^{-\frac{1}{2}}\|B(\boldsymbol{\chi}_h^{n+2})\|_{\Omega_p}\|\nabla\hat{\boldsymbol{v}}_h^{n+2}\|_{\Omega_f} \\
\leq&\gamma\|\hat{\boldsymbol{\chi}}_h^{n+2} \cdot \boldsymbol{\tau}_f\|_\Gamma^2 + \gamma\|\hat{\boldsymbol{v}}_h^{n+2} \cdot \boldsymbol{\tau}_f\|_\Gamma^2 + \frac{3\epsilon_1}{2}\|\nabla\hat{\boldsymbol{v}}_h^{n+2}\|_{\Omega_f}^2 + \frac{3\tilde{C}_{T,1}^2C_{T,2}^2C_{inv}\gamma^2}{8\epsilon_1 h}\|B(\boldsymbol{\chi}_h^{n+2})\|_{\Omega_p}^2. \qquad (4.15)
\end{aligned}
$$

Substituting (4.13) and (4.15) into (4.9), we get the lemma. □

Next, we will analyze the stability of Algorithm 3. For the integer $n \geq 0$, we define

$$
\begin{aligned}
\mathcal{E}^{n+1} = &\|\boldsymbol{U}_h^{n+1}\|_S^2 + \|2\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^n\|_S^2 + \|\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^n\|_S^2 + a_\eta(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{\eta}_h^{n+1}) \\
&+ a_\eta(2\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n, 2\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n) + a_\eta(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n, \boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^n). \qquad (4.16)
\end{aligned}
$$

Lemma 4.3. *If the time step restriction*

$$
\Delta t \leq \min\{\frac{12s_0\mu_1 h}{9s_0\mu_1 h + 32\tilde{C}_{T,1}^2C_{T,3}^2C_{inv}}, \frac{12\rho_p\mu_1 h}{9\rho_p\mu_1 h + 32\tilde{C}_{T,1}^2C_{T,2}^2C_{inv}\gamma^2}\} \qquad (4.17)
$$

is satisfied, for any $0 \leq m \leq N-2$*, we have*

$$
\mathcal{E}^{m+2} + \mu_1\Delta t\sum_{n=0}^{m}\|\nabla\hat{\boldsymbol{u}}_h^{n+2}\|_{\Omega_f}^2 + \mu_2\Delta t\sum_{n=0}^{m}\|\nabla\hat{p}_{ph}^{n+2}\|_{\Omega_p}^2 \leq C_3, \qquad (4.18)
$$

where μ_1 *and* μ_2 *are defined in Lemma 4.1,* C_3 *is defined in* (4.28).

Proof. For $\Omega_\star = \Omega_f, \Omega_p$, we can easily get the following identity, see, e.g., [31,46],

$$
\begin{aligned}
&4(\frac{3a^{n+2} - 4a^{n+1} + a^n}{2}, \frac{3}{2}a^{n+2} - a^{n+1} + \frac{1}{2}a^n)_{\Omega_\star} \\
&= \|a^{n+2}\|_{\Omega_\star}^2 + \|2a^{n+2} - a^{n+1}\|_{\Omega_\star}^2 + \|a^{n+2} - a^{n+1}\|_{\Omega_\star}^2 - \|a^{n+1}\|_{\Omega_\star}^2 - \|2a^{n+1} - a^n\|_{\Omega_\star}^2 \\
&\quad - \|a^{n+1} - a^n\|_{\Omega_\star}^2 + 3\|a^{n+2} - 2a^{n+1} + a^n\|_{\Omega_\star}^2. \qquad (4.19)
\end{aligned}
$$

Set $\boldsymbol{V}_h = 4\Delta t\hat{\boldsymbol{U}}_h^{n+2} = 4\Delta t(\hat{\boldsymbol{u}}_h^{n+2}, \hat{\boldsymbol{\xi}}_h^{n+2}, \hat{p}_{ph}^{n+2})$ in (3.3a), from $\frac{\hat{\boldsymbol{\eta}}_h^{n+2}-\boldsymbol{\eta}_h^{n+1}}{\Delta t} = \hat{\boldsymbol{\xi}}_h^{n+2}$ and (4.19), we have

$$\begin{aligned}
&\|\boldsymbol{U}_h^{n+2}\|_S^2 + \|2\boldsymbol{U}_h^{n+2} - \boldsymbol{U}_h^{n+1}\|_S^2 + \|\boldsymbol{U}_h^{n+2} - \boldsymbol{U}_h^{n+1}\|_S^2 - \|\boldsymbol{U}_h^{n+1}\|_S^2 - \|2\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^{n}\|_S^2 \\
&\quad - \|\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^{n}\|_S^2 + 3\|B(\boldsymbol{U}_h^{n+2})\|_S^2 + a_\eta(\boldsymbol{\eta}_h^{n+2}, \boldsymbol{\eta}_h^{n+2}) + a_\eta(2\boldsymbol{\eta}_h^{n+2} - \boldsymbol{\eta}_h^{n+1}, 2\boldsymbol{\eta}_h^{n+2} - \boldsymbol{\eta}_h^{n+1}) \\
&\quad + a_\eta(\boldsymbol{\eta}_h^{n+2} - \boldsymbol{\eta}_h^{n+1}, \boldsymbol{\eta}_h^{n+2} - \boldsymbol{\eta}_h^{n+1}) - a_\eta(\boldsymbol{\eta}_h^{n+1}, \boldsymbol{\eta}_h^{n+1}) - a_\eta(2\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^{n}, 2\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^{n}) \\
&\quad - a_\eta(\boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^{n}, \boldsymbol{\eta}_h^{n+1} - \boldsymbol{\eta}_h^{n}) + 3a_\eta(B(\boldsymbol{\eta}_h^{n+2}), B(\boldsymbol{\eta}_h^{n+2})) \\
&\quad + 4\Delta t\left(a(\hat{\boldsymbol{U}}_h^{n+2}, \hat{\boldsymbol{U}}_h^{n+2}) + a_b(\hat{\boldsymbol{U}}_h^{n+2}, \hat{\boldsymbol{U}}_h^{n+2})\right) \\
&= 4\Delta t(\boldsymbol{F}^{n+2}, \hat{\boldsymbol{U}}_h^{n+2}) - 4\Delta t a_\Gamma((\hat{\boldsymbol{u}}_h^{n+2}, 2\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^{n}, 2p_{ph}^{n+1} - p_{ph}^{n}); (\hat{\boldsymbol{u}}_h^{n+2}, \hat{\boldsymbol{\xi}}_h^{n+2}, \hat{p}_{ph}^{n+2})). \qquad (4.20)
\end{aligned}$$

By recalling the definition in (4.16), we can rewrite (4.20) as

$$\mathcal{E}^{n+2} + 3I_1 + 4\Delta t I_2 = \mathcal{E}^{n+1} + 4\Delta t I_3 + 4\Delta t I_4, \qquad (4.21)$$

where

$$\begin{aligned}
I_1 &= \|B(\boldsymbol{U}_h^{n+2})\|_S^2 + a_\eta(B(\boldsymbol{\eta}_h^{n+2}), B(\boldsymbol{\eta}_h^{n+2})), \\
I_2 &= a(\hat{\boldsymbol{U}}_h^{n+2}, \hat{\boldsymbol{U}}_h^{n+2}) + a_b(\hat{\boldsymbol{U}}_h^{n+2}, \hat{\boldsymbol{U}}_h^{n+2}), \\
I_3 &= (\boldsymbol{F}^{n+2}, \hat{\boldsymbol{U}}_h^{n+2}), \\
I_4 &= -a_\Gamma((\hat{\boldsymbol{u}}_h^{n+2}, 2\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^{n}, 2p_{ph}^{n+1} - p_{ph}^{n}); (\hat{\boldsymbol{u}}_h^{n+2}, \hat{\boldsymbol{\xi}}_h^{n+2}, \hat{p}_{ph}^{n+2})).
\end{aligned}$$

Using Lemma 4.1 and the definition of $a_b(\cdot,\cdot)$, we have

$$I_2 \geq \mu_1\|\nabla\hat{\boldsymbol{u}}_h^{n+2}\|_{\Omega_f}^2 + \mu_2\|\nabla\hat{p}_{ph}^{n+2}\|_{\Omega_p}^2 + \gamma\|\hat{\boldsymbol{u}}_h^{n+2}\cdot\boldsymbol{\tau}_f\|_\Gamma^2 + \gamma\|\hat{\boldsymbol{\xi}}_h^{n+2}\cdot\boldsymbol{\tau}_f\|_\Gamma^2. \qquad (4.22)$$

Using (4.11), a simple calculation yields

$$\|\hat{\boldsymbol{\xi}}_h^{n+2}\|_{\Omega_p}^2 = \|\frac{3}{2}B(\boldsymbol{\xi}_h^{n+2}) + 2\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^{n}\|_{\Omega_p}^2 \leq \frac{9}{2}\|B(\boldsymbol{\xi}_h^{n+2})\|_{\Omega_p}^2 + 2\|2\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^{n}\|_{\Omega_p}^2. \qquad (4.23)$$

An application of the Cauchy-Schwarz inequality, Poincaré inequality, and Young inequalities, along with (4.23), we get

$$\begin{aligned}
|I_3| &\leq C_{P,1}\|\boldsymbol{f}_f^{n+2}\|_{\Omega_f}\|\nabla\hat{\boldsymbol{u}}_h^{n+2}\|_{\Omega_f} + \|\boldsymbol{f}_p^{n+2}\|_{\Omega_p}\|\hat{\boldsymbol{\xi}}_h^{n+2}\|_{\Omega_p} + C_{P,3}\|g_p^{n+2}\|_{\Omega_p}\|\nabla\hat{p}_{ph}^{n+2}\|_{\Omega_p} \\
&\leq \epsilon_1\|\nabla\hat{\boldsymbol{u}}_h^{n+2}\|_{\Omega_f}^2 + \frac{C_{P,1}^2}{4\epsilon_1}\|\boldsymbol{f}_f^{n+2}\|_{\Omega_f}^2 + \epsilon_2\|\nabla\hat{p}_{ph}^{n+2}\|_{\Omega_p}^2 + \frac{C_{P,3}^2}{4\epsilon_2}\|g_p^{n+2}\|_{\Omega_p}^2 \\
&\quad + \frac{9\rho_p}{16}\|B(\boldsymbol{\xi}_h^{n+2})\|_{\Omega_p}^2 + \frac{\rho_p}{4}\|2\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^{n}\|_{\Omega_p}^2 + \frac{2}{\rho_p}\|\boldsymbol{f}_p^{n+2}\|_{\Omega_p}^2. \qquad (4.24)
\end{aligned}$$

The term I_4 can be estimated by setting $\boldsymbol{V}_h = \boldsymbol{U}_h$ in Lemma 4.2, which, together with (4.22) and (4.24), yields

$$\begin{aligned}
&\mathcal{E}^{n+2} + 4(\mu_1 - 4\epsilon_1)\Delta t\|\nabla\hat{\boldsymbol{u}}_h^{n+2}\|_{\Omega_f}^2 + 4(\mu_2 - \epsilon_2)\Delta t\|\nabla\hat{p}_{ph}^{n+2}\|_{\Omega_p}^2 + (3 - C_4\Delta t)\,\|B(\boldsymbol{U}_h^{n+2})\|_S^2 \\
&\leq \mathcal{E}^{n+1} + \Delta t\|2\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^{n}\|_S^2 + C_5\Delta t\|\boldsymbol{F}^{n+2}\|^2, \qquad (4.25)
\end{aligned}$$

where

$$C_4 = \frac{9}{4} + \frac{4C_1}{h} \text{ and } C_5 = \max\{\frac{C_{P,1}^2}{\epsilon_1}, \frac{C_{P,3}^2}{\epsilon_2}, \frac{8}{\rho_p}\}.$$

Let $\epsilon_1 = \frac{3\mu_1}{16}$, $\epsilon_2 = \frac{3\mu_2}{4}$. Since the time-step restriction (4.17) is satisfied, we have $3 - C_4\Delta t \geq 0$. Thus

$$\mathcal{E}^{n+2} + \mu_1\Delta t\|\nabla\hat{\boldsymbol{u}}_h^{n+2}\|_{\Omega_f}^2 + \mu_2\Delta t\|\nabla\hat{p}_{ph}^{n+2}\|_{\Omega_p}^2 \leq \mathcal{E}^{n+1} + \Delta t\|2\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^n\|_S^2 + C_5\Delta t\|\boldsymbol{F}^{n+2}\|^2. \quad (4.26)$$

Summing (4.26) from $n = 0$ to $n = m$, we get

$$\mathcal{E}^{m+2} + \mu_1\Delta t\sum_{n=0}^{m}\|\nabla\hat{\boldsymbol{u}}_h^{n+2}\|_{\Omega_f}^2 + \mu_2\Delta t\sum_{n=0}^{m}\|\nabla\hat{p}_{ph}^{n+2}\|_{\Omega_p}^2 \leq \Delta t\sum_{n=0}^{m}\mathcal{E}^{n+1} + C_6, \quad (4.27)$$

where $C_6 = \mathcal{E}^1 + C_5\Delta t\sum_{n=0}^{m}\|\boldsymbol{F}^{n+2}\|^2$. Using the Gronwall inequality (see, e.g., [47]), we have

$$\mathcal{E}^{m+2} + \mu_1\Delta t\sum_{n=0}^{m}\|\nabla\hat{\boldsymbol{u}}_h^{n+2}\|_{\Omega_f}^2 + \mu_2\Delta t\sum_{n=0}^{m}\|\nabla\hat{p}_{ph}^{n+2}\|_{\Omega_p}^2 \leq C(T)C_6 = C_3. \quad (4.28)$$

Then we get the lemma. □

Remark 4.1. *The use of the inverse inequality to bound the interface terms introduces a theoretically restrictive condition on the time step. However, numerical experiments demonstrate that the scheme maintains both stability and accuracy even with slightly larger time steps (see Table 2 in Subsection 6.1).*

5. Error estimates

Let $(S_h, P_h) : \boldsymbol{H}_f \times Q_f \to \boldsymbol{H}_{fh} \times Q_{fh}$ be the Stokes projection operator satisfying

$$a_f(S_h\boldsymbol{u}, \boldsymbol{v}_h) + b_f(\boldsymbol{v}_h, P_h p_f) + a_{b,f}(S_h\boldsymbol{u}, \boldsymbol{v}_h) = a_f(\boldsymbol{u}, \boldsymbol{v}_h) + b_f(\boldsymbol{v}_h, p_f) + a_{b,f}(\boldsymbol{u}, \boldsymbol{v}_h), \quad (5.1)$$

$$b_f(S_h\boldsymbol{u}, \boldsymbol{v}_h) = 0. \quad (5.2)$$

Then if $\boldsymbol{u} \in \boldsymbol{H}^{k_f+1}(\Omega_f)$ and $p_f \in \boldsymbol{H}^{k_f}(\Omega_f)$, we have

$$\|\boldsymbol{u} - S_h\boldsymbol{u}\|_{\Omega_f} + h\|\nabla(\boldsymbol{u} - S_h\boldsymbol{u})\|_{\Omega_f} + \|p_f - P_h p_f\|_{\Omega_f} \leq Ch^{k_f+1}(\|\boldsymbol{u}\|_{H^{k_f+1}(\Omega_f)} + \|p_f\|_{H^{k_f}(\Omega_f)}). \quad (5.3)$$

Using (5.3), if $h < 1$, it is easy to obtain

$$\|P_h p_f\|_{\Omega_f} \leq C(\|\boldsymbol{u}\|_{H^2(\Omega_f)} + \|p_f\|_{H^1(\Omega_f)}). \quad (5.4)$$

Let I_h be the Lagrangian interpolation operator onto $\boldsymbol{H}_{ph}$, then for any $\boldsymbol{\xi} \in \boldsymbol{H}^{k_s+1}(\Omega_p)$, we have

$$\|\boldsymbol{\xi} - I_h\boldsymbol{\xi}\|_{\Omega_p} + h\|\nabla(\boldsymbol{\xi} - I_h\boldsymbol{\xi})\|_{\Omega_p} \leq Ch^{k_s+1}\|\boldsymbol{\xi}\|_{H^{k_s+1}(\Omega_p)}. \quad (5.5)$$

Let R_h be the Ritz projection operator onto $\boldsymbol{H}_{ph}$ such that for all $\boldsymbol{\eta} \in \boldsymbol{H}^{k_s+1}(\Omega_p)$, there holds

$$a_\eta(R_h\boldsymbol{\eta}, \boldsymbol{\chi}_h) = a_\eta(\boldsymbol{\eta}, \boldsymbol{\chi}_h), \quad \forall\boldsymbol{\chi}_h \in \boldsymbol{H}_{ph}. \quad (5.6)$$

Then

$$\|\nabla(\boldsymbol{\eta} - R_h\boldsymbol{\eta})\|_{\Omega_p} \leq Ch^{k_s}\|\boldsymbol{\eta}\|_{H^{k_s+1}(\Omega_p)}. \tag{5.7}$$

Let Π_h be the Ritz projection operator onto Q_{ph} such that for all $p_p \in H^{k_p+1}(\Omega_p)$, there holds

$$a_p(p_p, \zeta_h) = a_p(\Pi_h p_p, \zeta_h), \quad \forall \zeta_h \in Q_{ph}. \tag{5.8}$$

Then

$$\|p_p - \Pi_h p_p\|_{\Omega_p} + h\|\nabla(p_p - \Pi_h p_p)\|_{\Omega_p} \leq Ch^{k_p+1}\|p_p\|_{H^{k_p+1}(\Omega_p)}. \tag{5.9}$$

Denote the errors at the time $t = t_n$ by

$$\begin{aligned}
\boldsymbol{e}_u^n &= \boldsymbol{u}_h^n - \boldsymbol{u}^n = (\boldsymbol{u}_h^n - S_h\boldsymbol{u}^n) + (S_h\boldsymbol{u}^n - \boldsymbol{u}^n) = \boldsymbol{\theta}_u^n + \boldsymbol{\sigma}_u^n,\\
e_{fp}^n &= p_{fh}^n - p_f^n = (p_{fh}^n - P_h p_f^n) + (P_h p_f^n - p_f^n) = \theta_{fp}^n + \sigma_{fp}^n,\\
\boldsymbol{e}_\xi^n &= \boldsymbol{\xi}_h^n - \boldsymbol{\xi}^n = (\boldsymbol{\xi}_h^n - I_h\boldsymbol{\xi}^n) + (I_h\boldsymbol{\xi}^n - \boldsymbol{\xi}^n) = \boldsymbol{\theta}_\xi^n + \boldsymbol{\sigma}_\xi^n,\\
\boldsymbol{e}_\eta^n &= \boldsymbol{\eta}_h^n - \boldsymbol{\eta}^n = (\boldsymbol{\eta}_h^n - R_h\boldsymbol{\eta}^n) + (R_h\boldsymbol{\eta}^n - \boldsymbol{\eta}^n) = \boldsymbol{\theta}_\eta^n + \boldsymbol{\sigma}_\eta^n,\\
e_{pp}^n &= p_{ph}^n - p_p^n = (p_{ph}^n - \Pi_h p_p^n) + (\Pi_h p_p^n - p_p^n) = \theta_{pp}^n + \sigma_{pp}^n.
\end{aligned}$$

Similar to (4.16), we define

$$\begin{aligned}
\mathcal{E}_\theta^{n+1} = {}& \|\boldsymbol{\Theta}^{n+1}\|_S^2 + \|2\boldsymbol{\Theta}^{n+1} - \boldsymbol{\Theta}^n\|_S^2 + \|\boldsymbol{\Theta}^{n+1} - \boldsymbol{\Theta}^n\|_S^2 + a_\eta(\boldsymbol{\theta}_\eta^{n+1}, \boldsymbol{\theta}_\eta^{n+1})\\
&+ a_\eta(2\boldsymbol{\theta}_\eta^{n+1} - \boldsymbol{\theta}_\eta^n, 2\boldsymbol{\theta}_\eta^{n+1} - \boldsymbol{\theta}_\eta^n) + a_\eta(\boldsymbol{\theta}_\eta^{n+1} - \boldsymbol{\theta}_\eta^n, \boldsymbol{\theta}_\eta^{n+1} - \boldsymbol{\theta}_\eta^n).
\end{aligned} \tag{5.10}$$

Lemma 5.1. *For $n \geq 0$, the following estimates hold*

$$\|D_t\boldsymbol{v}^{n+2} - \partial_t\boldsymbol{v}^{n+2}\|_{\Omega_\star}^2 \leq C\Delta t^3 \int_{t_n}^{t_{n+2}} \|\partial_{ttt}\boldsymbol{v}\|_{\Omega_\star}^2 dt, \tag{5.11}$$

$$\|B(\boldsymbol{v}^{n+2})\|_{\Omega_\star}^2 \leq C\Delta t^3 \int_{t_n}^{t_{n+2}} \|\partial_{tt}\boldsymbol{v}\|_{\Omega_\star}^2 dt, \tag{5.12}$$

$$\|D_t\boldsymbol{v}^{n+2}\|_{\Omega_\star}^2 \leq \frac{C}{\Delta t} \int_{t_n}^{t_{n+2}} \|\partial_t\boldsymbol{v}\|_{\Omega_\star}^2 dt, \tag{5.13}$$

$$\|\hat{\boldsymbol{v}}^{n+2} - \boldsymbol{v}^{n+2}\|_{\Omega_\star}^2 \leq C\Delta t^3 \int_{t_n}^{t_{n+2}} \|\partial_{tt}\boldsymbol{v}\|_{\Omega_\star}^2 dt, \tag{5.14}$$

$$\|D_t\boldsymbol{v}^{n+2} - \partial_t\hat{\boldsymbol{v}}^{n+2}\|_{\Omega_\star}^2 \leq C\Delta t^3 \int_{t_n}^{t_{n+2}} \|\partial_{ttt}\boldsymbol{v}\|_{\Omega_\star}^2 dt, \tag{5.15}$$

where $\Omega_\star$ may be Ω_f or Ω_p.

Proof. The first three inequalities can be proved by the integral form of Taylor's Theorem, see (5.9)–(5.11) in [6]. The estimate (5.14) can be found in (3.33) of [34]. Therefore, we only need to prove (5.15), using the Taylor formula of the integral remainder yields

$$\begin{aligned}
&\|D_t\boldsymbol{v}^{n+2} - \partial_t\hat{\boldsymbol{v}}^{n+2}\|_{\Omega_\star}^2\\
&= \left\| -\frac{1}{\Delta t}\int_{t_{n+2}}^{t_{n+1}} (t^{n+1} - t)^2\partial_{ttt}\boldsymbol{v}dt + \frac{1}{4\Delta t}\int_{t_{n+2}}^{t_n} (t^n - t)^2\partial_{ttt}\boldsymbol{v}dt \right.
\end{aligned}$$

$$+\int_{t_{n+2}}^{t_{n+1}}(t^{n+1}-t)\partial_{ttt}\boldsymbol{v}dt-\frac{1}{2}\int_{t_{n+2}}^{t_n}(t^n-t)\partial_{ttt}\boldsymbol{v}dt\Big\|_{\Omega_\star}^2. \tag{5.16}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|-\frac{1}{\Delta t}\int_{t_{n+1}}^{t_{n+2}}(t^{n+1}-t)^2\partial_{ttt}\boldsymbol{v}dt\|_{\Omega_\star}^2 &\le \frac{C}{\Delta t^2}\int_{\Omega_\star}(\int_{t_{n+1}}^{t_{n+2}}(t^{n+1}-t)^4dt\int_{t_{n+1}}^{t_{n+2}}(\partial_{ttt}\boldsymbol{v})^2dt)dx\\ &\le C\Delta t^3\int_{t_{n+1}}^{t_{n+2}}\|\partial_{ttt}\boldsymbol{v}\|_{\Omega_\star}^2dt.\end{aligned} \tag{5.17}$$

Similarly

$$\|\frac{1}{4\Delta t}\int_{t_{n+2}}^{t_n}(t^{n+1}-t)^2\partial_{ttt}\boldsymbol{v}dt\|_{\Omega_\star}^2 \le C\Delta t^3\int_{t_n}^{t_{n+2}}\|\partial_{ttt}\boldsymbol{v}\|_{\Omega_\star}^2dt, \tag{5.18}$$

$$\|\int_{t_{n+2}}^{t_{n+1}}(t^{n+1}-t)\partial_{ttt}\boldsymbol{v}dt\|_{\Omega_\star}^2 \le C\Delta t^3\int_{t_{n+1}}^{t_{n+2}}\|\partial_{ttt}\boldsymbol{v}\|_{\Omega_\star}^2dt, \tag{5.19}$$

$$\|-\frac{1}{2}\int_{t_{n+2}}^{t_n}(t^n-t)\partial_{ttt}\boldsymbol{v}dt\|_{\Omega_\star}^2 \le C\Delta t^3\int_{t_n}^{t_{n+2}}\|\partial_{ttt}\boldsymbol{v}\|_{\Omega_\star}^2dt. \tag{5.20}$$

Substituting (5.17)–(5.20) into (5.16), using the triangle inequality, we get (5.15). □

Firstly, we present some estimates for $a_{b,p}((\cdot,\cdot);(\cdot,\cdot))$ and $a_\Gamma(\cdot,\cdot)$.

Lemma 5.2. *Let $a_{b,p}((\cdot,\cdot);(\cdot,\cdot))$ and $a_\Gamma(\cdot,\cdot)$ be defined as in* (2.16) *and* (2.17)*, respectively. For any $\delta_1,\delta_2,\delta_3,\delta_4>0$, the following estimates hold*

$$\begin{aligned}&-a_{b,p}((\hat{\boldsymbol{\sigma}}_\xi^{n+2},\hat{\sigma}_{pp}^{n+2});(\hat{\boldsymbol{\theta}}_\xi^{n+2},\hat{\theta}_{pp}^{n+2}))\\ &\le \delta_2\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}^2+\frac{9\delta_4\rho_p}{4}\|B(\boldsymbol{\theta}_\xi^{n+2})\|_{\Omega_p}^2+\delta_4\rho_p\|2\boldsymbol{\theta}_\xi^{n+1}-\boldsymbol{\theta}_\xi^n\|_{\Omega_p}^2\\ &\quad+Ch^{2k_p}\|\hat{p}_p^{n+2}\|_{H^{k_p+1}(\Omega_p)}^2+Ch^{2k_s}\|\hat{\boldsymbol{\xi}}^{n+2}\|_{H^{k_s+1}(\Omega_p)}^2.\end{aligned} \tag{5.21}$$

$$\begin{aligned}&-a_\Gamma((\hat{\boldsymbol{\sigma}}_u^{n+2},2\boldsymbol{\sigma}_\xi^{n+1}-\boldsymbol{\sigma}_\xi^n,2\sigma_{pp}^{n+1}-\sigma_{pp}^n);(\hat{\boldsymbol{\theta}}_u^{n+2},\hat{\boldsymbol{\theta}}_\xi^{n+2},\hat{\theta}_{pp}^{n+2}))\\ &\le 2\delta_1\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}^2+\delta_2\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}^2+\frac{9\delta_4\rho_p}{4}\|B(\boldsymbol{\theta}_\xi^{n+2})\|_{\Omega_p}^2+\delta_4\rho_p\|2\boldsymbol{\theta}_\xi^{n+1}-\boldsymbol{\theta}_\xi^n\|_{\Omega_p}^2\\ &\quad+Ch^{2k_f}\left(\|\hat{\boldsymbol{u}}^{n+2}\|_{H^{k_f+1}(\Omega_f)}^2+\|\hat{p}_f^{n+2}\|_{H^{k_f}(\Omega_f)}\right)\\ &\quad+Ch^{2k_s+1}\left(\|\boldsymbol{\xi}^{n+1}\|_{H^{k_s+1}(\Omega_p)}^2+\|\boldsymbol{\xi}^n\|_{H^{k_s+1}(\Omega_p)}^2\right)\\ &\quad+Ch^{2k_p+1}\left(\|p_p^{n+1}\|_{H^{k_p+1}(\Omega_p)}^2+\|p_p^n\|_{H^{k_p+1}(\Omega_p)}^2\right).\end{aligned} \tag{5.22}$$

Proof. Using the trace inequalities, (5.3), (5.5) and (5.9), we get

$$\|\hat{\boldsymbol{\sigma}}_u^n\|_\Gamma \le CC_{T,1}h^{k_f+\frac{1}{2}}(\|\hat{\boldsymbol{u}}^n\|_{H^{k_f+1}(\Omega_f)}+\|\hat{p}_f^n\|_{H^{k_f}(\Omega_f)}), \tag{5.23}$$

$$\|\hat{\boldsymbol{\sigma}}_\xi^n\|_\Gamma \le CC_{T,2}h^{k_s+\frac{1}{2}}\|\hat{\boldsymbol{\xi}}^n\|_{H^{k_s+1}(\Omega_p)}, \tag{5.24}$$

$$\|\hat{\sigma}_{pp}^n\|_\Gamma \le CC_{T,3}h^{k_p+\frac{1}{2}}\|\hat{p}_p^n\|_{H^{k_p+1}(\Omega_p)}. \tag{5.25}$$

It follows from the Cauchy-Schwarz inequality, trace inequalities (4.2) and (4.3), inverse inequality (4.5) and (5.25) that

$$
\begin{aligned}
&- a_{b,p}((\hat{\boldsymbol{\sigma}}_\xi^{n+2}, \hat{\sigma}_{pp}^{n+2}); (\hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})) \\
&\leq \|\hat{\theta}_{pp}^{n+2}\|_\Gamma \|\hat{\boldsymbol{\sigma}}_\xi^{n+2} \cdot \boldsymbol{n}_f\|_\Gamma + \|\hat{\sigma}_{pp}^{n+2}\|_\Gamma \|\hat{\boldsymbol{\theta}}_\xi^{n+2} \cdot \boldsymbol{n}_f\|_\Gamma + \gamma \|\hat{\boldsymbol{\sigma}}_\xi^{n+2} \cdot \boldsymbol{\tau}_f\|_\Gamma \|\hat{\boldsymbol{\theta}}_\xi^{n+2} \cdot \boldsymbol{\tau}_f\|_\Gamma \\
&\leq C C_{T,2} h^{k_s+\frac{1}{2}} \left(\tilde{C}_{T,3} \|\nabla \hat{\theta}_{pp}^{n+2}\|_{\Omega_p} + \gamma C_{T,2} C_{inv}^{\frac{1}{2}} h^{-\frac{1}{2}} \|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p} \right) \|\hat{\boldsymbol{\xi}}^{n+2}\|_{H^{k_s+1}(\Omega_p)} \\
&\quad + C C_{T,2} C_{T,3} C_{inv}^{\frac{1}{2}} h^{k_p} \|\hat{p}_p^{n+2}\|_{H^{k_p+1}(\Omega_p)} \|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p} \\
&\leq \delta_2 \|\nabla \hat{\theta}_{pp}^{n+2}\|_{\Omega_p}^2 + \frac{\delta_4 \rho_p}{2} \|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p}^2 + C h^{2k_p} \|\hat{p}_p^{n+2}\|_{H^{k_p+1}(\Omega_p)}^2 \\
&\quad + C h^{2k_s} \|\hat{\boldsymbol{\xi}}^{n+2}\|_{H^{k_s+1}(\Omega_p)}^2.
\end{aligned}
\tag{5.26}
$$

Then (5.21) can be obtained from (5.26) and (4.23).

Using the Cauchy-Schwarz inequality and the trace inequalities yields

$$
\begin{aligned}
&- a_\Gamma((\hat{\boldsymbol{\sigma}}_u^{n+2}, 2\boldsymbol{\sigma}_\xi^{n+1} - \boldsymbol{\sigma}_\xi^n, 2\sigma_{pp}^{n+1} - \sigma_{pp}^n); (\hat{\boldsymbol{\theta}}_u^{n+2}, \hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})) \\
&\leq \tilde{C}_{T,1} \|2\sigma_{pp}^{n+1} - \sigma_{pp}^n\|_\Gamma \|\nabla \hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f} + \tilde{C}_{T,3} \|\hat{\boldsymbol{\sigma}}_u^{n+2}\|_\Gamma \|\nabla \hat{\theta}_{pp}^{n+2}\|_{\Omega_p} \\
&\quad + \gamma \tilde{C}_{T,1} \|2\boldsymbol{\sigma}_\xi^{n+1} - \boldsymbol{\sigma}_\xi^n\|_\Gamma \|\nabla \hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f} + \gamma C_{T,2} \|\hat{\boldsymbol{\sigma}}_u^{n+2}\|_\Gamma \|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p}^{\frac{1}{2}} \|\nabla \hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p}^{\frac{1}{2}}.
\end{aligned}
\tag{5.27}
$$

The last term on the right hand of (5.27) can be estimated by the inverse inequality, (5.23) and (4.23)

$$
\begin{aligned}
\gamma C_{T,2} \|\hat{\boldsymbol{\sigma}}_u^{n+2}\|_\Gamma \|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p}^{\frac{1}{2}} \|\nabla \hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p}^{\frac{1}{2}} &\leq C_{T,2} C_{inv}^{\frac{1}{2}} h^{-\frac{1}{2}} \gamma \|\hat{\boldsymbol{\sigma}}_u^{n+2}\|_\Gamma \|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p} \\
&\leq C C_{T,2} C_{inv}^{\frac{1}{2}} h^{k_f} \gamma (\|\hat{\boldsymbol{u}}^{n+2}\|_{\Omega_f} + \|\hat{p}_f^{n+2}\|) \|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p} \\
&\leq \frac{9\delta_4 \rho_p}{4} \|B(\boldsymbol{\theta}_\xi^{n+2})\|_{\Omega_p}^2 + \delta_4 \rho_p \|2\boldsymbol{\theta}_\xi^{n+1} - \boldsymbol{\theta}_\xi^n\|_{\Omega_p}^2 \\
&\quad + C h^{2k_f} \left(\|\hat{\boldsymbol{u}}^{n+2}\|_{H^{k_f+1}(\Omega_f)}^2 + \|\hat{p}_f^{n+2}\|_{H^{k_f}(\Omega_f)} \right).
\end{aligned}
\tag{5.28}
$$

then we get (5.22) by (5.23)–(5.25), (5.28) and Young's inequality. □

Now, we study the error estimates for Algorithm 3.

Theorem 5.1. *Let the body terms $\partial_{tt}\boldsymbol{f}_f \in L^2(0,T;\boldsymbol{L}^2(\Omega_f)), \partial_{tt}\boldsymbol{f}_p \in L^2(0,T;\boldsymbol{L}^2(\Omega_p))$ and $\partial_{tt}g_p \in L^2(0,T;L^2(\Omega_p))$. If the solution $(\boldsymbol{u}, p_f, \boldsymbol{\xi}, \boldsymbol{\eta}, p_p)$ of problems (2.18a) and (2.18b) satisfies the following hypotheses:*

$$\boldsymbol{u} \in H^3(0,T;\boldsymbol{L}^2(\Omega_f)) \cap H^1(0,T;\boldsymbol{H}^{k_s+1}(\Omega_f)), \tag{5.29}$$
$$p_f \in H^1(0,T;H^{k_f}(\Omega_f)), \tag{5.30}$$
$$\boldsymbol{\xi} \in H^3(0,T;\boldsymbol{L}^2(\Omega_p)) \cap H^2(0,T;\boldsymbol{H}^1(\Omega_p)) \cap H^1(0,T;\boldsymbol{H}^{k_s+1}(\Omega_p)), \tag{5.31}$$
$$p_p \in H^3(0,T;L^2(\Omega_p)) \cap H^2(0,T;H^1(\Omega_p)) \cap H^1(0,T;H^{k_p+1}(\Omega_p)), \tag{5.32}$$

where $k_f, k_s, k_p \geq 1$.If the time step restriction

$$\Delta t \leq \min\{\frac{s_0\mu_1 h}{s_0\mu_1 h + 4\tilde{C}_{T,1}^2 C_{T,3}^2 C_{inv}}, \frac{\rho_p\mu_1 h}{\rho_p\mu_1 h + 4\tilde{C}_{T,1}^2 C_{T,2}^2 C_{inv}\gamma^2}\} \tag{5.33}$$

is satisfied,

$$\|\boldsymbol{E}^{m+2}\|^2 \leq C(\Delta t^4 + h^{2k_f} + h^{2k_s} + h^{2k_p}), \tag{5.34}$$

$$a_\eta(\boldsymbol{e}_\eta^{m+2}, \boldsymbol{e}_\eta^{m+2}) \leq C(\Delta t^4 + h^{2k_f} + h^{2k_s} + h^{2k_p}). \tag{5.35}$$

Proof. For any $\boldsymbol{v}_h \in \boldsymbol{H}_{fh} \subset \boldsymbol{H}_f$, using (3.6a), (3.6b) and (5.1), we have

$$\begin{aligned}
&\rho_f(D_t\boldsymbol{u}^{n+2}, \boldsymbol{v}_h)_{\Omega_f} + a_f(S_h\hat{\boldsymbol{u}}^{n+2}, \boldsymbol{v}_h) + b_f(\boldsymbol{v}_h, P_h p_f^{n+2}) + a_{b,f}(S_h\hat{\boldsymbol{u}}^{n+2}, \boldsymbol{v}_h)\\
&\quad + \left(2p_p^{n+1} - p_p^n, \boldsymbol{v}_h \cdot \boldsymbol{n}_f\right)_\Gamma - \gamma\left((2\boldsymbol{\xi}^{n+1} - \boldsymbol{\xi}^n) \cdot \boldsymbol{\tau}_f, \boldsymbol{v}_h \cdot \boldsymbol{\tau}_f\right)_\Gamma = (\boldsymbol{f}_f^{n+2}, \boldsymbol{v}_h)_{\Omega_f} + E_1(\boldsymbol{v}_h), && (5.36)\\
&b_f(S_h\hat{\boldsymbol{u}}^{n+2}, q_h) = 0, && (5.37)
\end{aligned}$$

where

$$\begin{aligned}
E_1(\boldsymbol{v}_h) &= (\hat{\boldsymbol{f}}_f^{n+2} - \boldsymbol{f}_f^{n+2}, \boldsymbol{v}_h)_{\Omega_f} + \rho_f(D_t\boldsymbol{u}^{n+2} - \partial_t\hat{\boldsymbol{u}}^{n+2}, \boldsymbol{v}_h)_{\Omega_f} + b_f(\boldsymbol{v}_h, P_h p_f - P_h\hat{p}_f^{n+2})\\
&\quad - \left(\hat{p}_p^{n+2} - 2p_p^{n+1} + p_p^n, \boldsymbol{v}_h \cdot \boldsymbol{n}_f\right)_\Gamma + \gamma\left((\hat{\boldsymbol{\xi}}^{n+2} - 2\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n) \cdot \boldsymbol{\tau}_f, \boldsymbol{v}_h \cdot \boldsymbol{\tau}_f\right)_\Gamma\\
&= (\hat{\boldsymbol{f}}_f^{n+2} - \boldsymbol{f}_f^{n+2}, \boldsymbol{v}_h)_{\Omega_f} + \rho_f(D_t\boldsymbol{u}^{n+2} - \partial_t\hat{\boldsymbol{u}}^{n+2}, \boldsymbol{v}_h)_{\Omega_f} + b_f(\boldsymbol{v}_h, P_h p_f - P_h\hat{p}_f^{n+2})\\
&\quad - \left(B(p_p^{n+2}) + \hat{p}_p^{n+2} - p_p^{n+2}, \boldsymbol{v}_h \cdot \boldsymbol{n}_f\right)_\Gamma + \gamma\left((B(\boldsymbol{\xi}^{n+2}) + \hat{\boldsymbol{\xi}}^{n+2} - \boldsymbol{\xi}^{n+2}) \cdot \boldsymbol{\tau}_f, \boldsymbol{v}_h \cdot \boldsymbol{\tau}_f\right)_\Gamma.
\end{aligned}$$

For any $\boldsymbol{\chi}_h \in \boldsymbol{H}_{ph}$, $\zeta_h \in Q_{ph}$, using (3.6c), (3.6d), (5.6) and (5.8), we have

$$\begin{aligned}
&\rho_p(D_t\boldsymbol{\xi}^{n+2}, \boldsymbol{\chi}_h)_{\Omega_p} + s_0(D_t p_p^{n+2}, \zeta_h)_{\Omega_p} + a_\eta(R_h\hat{\boldsymbol{\eta}}^{n+2}, \boldsymbol{\chi}_h) + b_p(\boldsymbol{\chi}_h, \hat{p}_p^{n+2})\\
&\quad + a_p(\Pi_h\hat{p}_p^{n+2}, \zeta_h) - b_p(\hat{\boldsymbol{\xi}}^{n+2}, \zeta_h) + a_{b,p}((\hat{\boldsymbol{\xi}}^{n+2}, \hat{p}_p^{n+2}); (\boldsymbol{\chi}_h, \zeta_h))\\
&\quad - (\hat{\boldsymbol{u}}^{n+2} \cdot \boldsymbol{n}_f, \zeta_h)_\Gamma - \gamma(\hat{\boldsymbol{u}}^{n+2} \cdot \boldsymbol{\tau}_f, \boldsymbol{\chi}_h \cdot \boldsymbol{\tau}_f)_\Gamma\\
&= (\boldsymbol{f}_p^{n+2}, \boldsymbol{\chi}_h)_{\Omega_p} + (g_p^{n+2}, \zeta_h)_{\Omega_p} + E_2(\boldsymbol{\chi}_h, \zeta_h), && (5.38)
\end{aligned}$$

where

$$\begin{aligned}
E_2(\boldsymbol{\chi}_h, \zeta_h) &= (\hat{\boldsymbol{f}}_p^{n+2} - \boldsymbol{f}_p^{n+2}, \boldsymbol{\chi}_h)_{\Omega_p} + (\hat{g}_p^{n+2} - g_p^{n+2}, \zeta_h)_{\Omega_p}\\
&\quad + \rho_p(D_t\boldsymbol{\xi}^{n+2} - \partial_t\hat{\boldsymbol{\xi}}^{n+2}, \boldsymbol{\chi}_h)_{\Omega_p} + s_0(D_t p_p^{n+2} - \partial_t\hat{p}_p^{n+2}, \zeta_h)_{\Omega_p}.
\end{aligned}$$

Subtracting the sum of (5.36) and (5.38) from (3.3a), and subtracting (5.37) from (3.3b), we obtain

$$\begin{aligned}
&\langle\langle D_t\boldsymbol{E}^{n+2}, \boldsymbol{V}_h\rangle\rangle + a_f(\hat{\boldsymbol{\theta}}_u^{n+2}, \boldsymbol{v}_h) + b_f(\boldsymbol{v}_h, \theta_{fp}^{n+2}) + a_p(\hat{\theta}_{pp}^{n+2}, \zeta_h) + a_\eta(\hat{\boldsymbol{\theta}}_\eta^{n+2}, \boldsymbol{\chi}_h)\\
&\quad + b_p(\boldsymbol{\chi}_h, \hat{e}_{pp}^{n+2}) - b_p(\hat{\boldsymbol{e}}_\xi^{n+2}, \zeta_h) + a_{b,f}(\hat{\boldsymbol{\theta}}_u^{n+2}, \boldsymbol{v}_h) + a_{b,p}((\hat{\boldsymbol{e}}_\xi^{n+2}, \hat{e}_{pp}^{n+2}); (\boldsymbol{\chi}_h, \zeta_h))\\
&\quad + a_\Gamma((\hat{\boldsymbol{e}}_u^{n+2}, 2\boldsymbol{e}_\xi^{n+1} - \boldsymbol{e}_\xi^n, 2e_{pp}^{n+1} - e_{pp}^n); (\boldsymbol{v}_h, \boldsymbol{\chi}_h, \zeta_h))\\
&= E_1(\boldsymbol{v}_h) + E_2(\boldsymbol{\chi}_h, \zeta_h), && (5.39)\\
&b_f(\hat{\boldsymbol{\theta}}_u^{n+2}, q_h) = 0. && (5.40)
\end{aligned}$$

Let $\boldsymbol{V}_h = \hat{\boldsymbol{\Theta}}^{n+2} = (\hat{\boldsymbol{\theta}}_u^{n+2}, \hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})$ in (5.39), using the divergence free property (5.40), we have

$$\begin{aligned}
&\langle\langle D_t\boldsymbol{\Theta}^{n+2}, \hat{\boldsymbol{\Theta}}^{n+2}\rangle\rangle + a(\hat{\boldsymbol{\Theta}}^{n+2}, \hat{\boldsymbol{\Theta}}^{n+2}) + a_b(\hat{\boldsymbol{\Theta}}^{n+2}, \hat{\boldsymbol{\Theta}}^{n+2}) + a_\eta(\hat{\boldsymbol{\theta}}_\eta^{n+2}, \hat{\boldsymbol{\theta}}_\xi^{n+2})\\
&= -\langle\langle D_t\boldsymbol{\Sigma}^{n+2}, \hat{\boldsymbol{\Theta}}^{n+2}\rangle\rangle - b_p(\hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\sigma}_{pp}^{n+2}) + b_p(\hat{\boldsymbol{\sigma}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})
\end{aligned}$$

$$
\begin{aligned}
&- a_{b,p}((\hat{\boldsymbol{\sigma}}_\xi^{n+2}, \hat{\sigma}_{pp}^{n+2}); (\hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})) \\
&- a_\Gamma((\hat{\boldsymbol{e}}_u^{n+2}, 2\boldsymbol{e}_\xi^{n+1} - \boldsymbol{e}_\xi^n, 2e_{pp}^{n+1} - e_{pp}^n); (\hat{\boldsymbol{\theta}}_u^{n+2}, \hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})) + E_1(\boldsymbol{v}_h) + E_2(\boldsymbol{\chi}_h, \zeta_h). \qquad (5.41)
\end{aligned}
$$

Since

$$
\hat{\boldsymbol{\xi}}_h^{n+2} = \frac{\hat{\boldsymbol{\eta}}_h^{n+2} - \boldsymbol{\eta}_h^{n+1}}{\Delta t} = D_t\boldsymbol{\eta}_h^{n+2} \text{ and } \hat{\boldsymbol{\xi}}^{n+2} = \partial_t\hat{\boldsymbol{\eta}}^{n+2},
$$

we have

$$
\hat{\boldsymbol{\theta}}_\xi^{n+2} = D_t\boldsymbol{e}_\eta^{n+2} - \boldsymbol{r}^{n+2} - \hat{\boldsymbol{\sigma}}_\xi^{n+2}, \qquad (5.42)
$$

where $\boldsymbol{r}^{n+2} = \partial_t\hat{\boldsymbol{\eta}}^{n+2} - D_t\boldsymbol{\eta}^{n+2}$.

Combining (5.41), (5.42) and (4.19), recalling the definition of (5.10), similar to (4.21), we have

$$
\mathcal{E}_\theta^{n+2} + 3J_1 + 4\Delta tJ_2 = \mathcal{E}_\theta^{n+1} + 4\Delta tJ_3 + 4\Delta tJ_4 + 4\Delta tJ_5 + 4\Delta tJ_6, \qquad (5.43)
$$

where

$$
\begin{aligned}
J_1 &= \|B(\boldsymbol{\Theta}^{n+2})\|_S^2 + a_\eta(B(\boldsymbol{\theta}_\eta^{n+2}), B(\boldsymbol{\theta}_\eta^{n+2})), \\
J_2 &= a(\hat{\boldsymbol{\Theta}}^{n+2}, \hat{\boldsymbol{\Theta}}^{n+2}) + a_b(\hat{\boldsymbol{\Theta}}^{n+2}, \hat{\boldsymbol{\Theta}}^{n+2}), \\
J_3 &= -\langle\langle D_t\boldsymbol{\Sigma}^{n+2}, \hat{\boldsymbol{\Theta}}^{n+2}\rangle\rangle - a_\eta(\hat{\boldsymbol{\theta}}_\eta^{n+2}, D_t\boldsymbol{\sigma}_\eta^{n+2}), \\
J_4 &= a_\eta(\hat{\boldsymbol{\theta}}_\eta^{n+2}, \hat{\boldsymbol{\sigma}}_\xi^{n+2}) - b_p(\hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\sigma}_{pp}^{n+2}) + b_p(\hat{\boldsymbol{\sigma}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2}) \\
&= J_{41} + J_{42} + J_{43}, \\
J_5 &= -a_\Gamma((\hat{\boldsymbol{\theta}}_u^{n+2}, 2\boldsymbol{\theta}_\xi^{n+1} - \boldsymbol{\theta}_\xi^n, 2\theta_{pp}^{n+1} - \theta_{pp}^n); (\hat{\boldsymbol{\theta}}_u^{n+2}, \hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})) \\
&\quad - a_\Gamma((\hat{\boldsymbol{\sigma}}_u^{n+2}, 2\boldsymbol{\sigma}_\xi^{n+1} - \boldsymbol{\sigma}_\xi^n, 2\sigma_{pp}^{n+1} - \sigma_{pp}^n); (\hat{\boldsymbol{\theta}}_u^{n+2}, \hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})) \\
&\quad - a_{b,p}((\hat{\boldsymbol{\sigma}}_\xi^{n+2}, \hat{\sigma}_{pp}^{n+2}); (\hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2})) \\
&= J_{51} + J_{52} + J_{53}, \\
J_6 &= E_1(\hat{\boldsymbol{\theta}}_u^{n+2}) + E_2(\hat{\boldsymbol{\theta}}_\xi^{n+2}, \hat{\theta}_{pp}^{n+2}) + a_\eta(\hat{\boldsymbol{\theta}}_\eta^{n+2}, \boldsymbol{r}^{n+2}) \\
&= J_{61} + J_{62} + J_{63}.
\end{aligned}
$$

Similar to (4.22), we obtain

$$
J_2 \ge \mu_1\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}^2 + \mu_2\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}^2 + \gamma\|\hat{\boldsymbol{\theta}}_\xi^{n+2} \cdot \boldsymbol{\tau}_f\|_\Gamma^2 + \gamma\|\hat{\boldsymbol{\theta}}_u^{n+2} \cdot \boldsymbol{\tau}_f\|_\Gamma^2. \qquad (5.44)
$$

Using (5.6) yields

$$
J_3 = -\langle\langle D_t\boldsymbol{\Sigma}^{n+2}, \hat{\boldsymbol{\Theta}}^{n+2}\rangle\rangle. \qquad (5.45)
$$

For any $\delta_1, \delta_2, \delta_4 > 0$, from Young's inequality, (5.3), (5.5) and (5.9), we have

$$
\begin{aligned}
J_3 &\le \rho_f C_{P,1}\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}\|D_t\boldsymbol{\sigma}_u^{n+2}\|_{\Omega_f} + \rho_p\|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p}\|D_t\boldsymbol{\sigma}_\xi^{n+2}\|_{\Omega_p} \\
&\quad + s_0C_{P,3}\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}\|D_t\sigma_{pp}^{n+2}\|_{\Omega_p} \\
&\le \delta_1\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}^2 + \delta_2\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}^2 + \frac{9\delta_4\rho_p}{4}\|B(\boldsymbol{\theta}_\xi^{n+2})\|_{\Omega_p}^2 + \delta_4\rho_p\|2\boldsymbol{\theta}_\xi^{n+1} - \boldsymbol{\theta}_\xi^n\|_{\Omega_p}^2 \\
&\quad + \frac{Ch^{2k_f+2}}{\Delta t}\int_{t_n}^{t_{n+2}}(\|\partial_t\boldsymbol{u}\|_{H^{k_f+1}(\Omega_f)}^2 + \|\partial_t p_f\|_{H^{k_f}(\Omega_f)}^2)dt
\end{aligned}
$$

$$+\frac{Ch^{2k_s+2}}{\Delta t}\int_{t_n}^{t_{n+2}}\|\partial_t\boldsymbol{\xi}\|^2_{H^{k_s+1}(\Omega_p)}dt+\frac{Ch^{2k_p+2}}{\Delta t}\int_{t_n}^{t_{n+2}}\|\partial_t p_p\|^2_{H^{k_p+1}(\Omega_p)}dt, \tag{5.46}$$

where in the last step we use the triangle inequality and the equality

$$\hat{\boldsymbol{\theta}}_\xi^{n+2}=\frac{3}{2}B(\boldsymbol{\theta}_\xi^{n+2})+2\boldsymbol{\theta}_\xi^{n+1}-\boldsymbol{\theta}_\xi^{n}. \tag{5.47}$$

For any $\delta_5>0$, since $\hat{\boldsymbol{\theta}}_\eta^{n+2}=\frac{3}{2}B(\boldsymbol{\theta}_\eta^{n+2})+2\boldsymbol{\theta}_\eta^{n+1}-\boldsymbol{\theta}_\eta^{n}$, using Young's inequality and (5.5), we have

$$\begin{aligned}
J_{41}&\le a_\eta(\hat{\boldsymbol{\theta}}_\eta^{n+2},\hat{\boldsymbol{\theta}}_\eta^{n+2})^{\frac{1}{2}}a_\eta(\hat{\boldsymbol{\sigma}}_\xi^{n+2},\hat{\boldsymbol{\sigma}}_\xi^{n+2})^{\frac{1}{2}}\\
&\le\frac{\delta_5}{2}a_\eta(\hat{\boldsymbol{\theta}}_\eta^{n+2},\hat{\boldsymbol{\theta}}_\eta^{n+2})+Ch^{2k_s}\|\hat{\boldsymbol{\xi}}^{n+2}\|^2_{H^{k_s+1}(\Omega_p)}\\
&\le\frac{9\delta_5}{4}a_\eta(B(\boldsymbol{\theta}_\eta^{n+2}),B(\boldsymbol{\theta}_\eta^{n+2}))+\delta_5 a_\eta(2\boldsymbol{\theta}_\eta^{n+1}-\boldsymbol{\theta}_\eta^{n},2\boldsymbol{\theta}_\eta^{n+1}-\boldsymbol{\theta}_\eta^{n})+Ch^{2k_s}\|\hat{\boldsymbol{\xi}}^{n+2}\|^2_{H^{k_s+1}(\Omega_p)}.
\end{aligned} \tag{5.48}$$

Using the inverse inequality (4.5), (4.4) and (5.47), for any $\delta_2,\delta_4>0$, we have from the Young's inequality that

$$\begin{aligned}
J_{42}+J_{43}&\le CC_{inv}h^{-1}\alpha\|\hat{\sigma}_{pp}^{n+2}\|_{\Omega_p}\|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p}+\alpha C_{P,3}\|\nabla\cdot\hat{\boldsymbol{\sigma}}_\xi^{n+2}\|_{\Omega_p}\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}\\
&\le\frac{9\delta_4\rho_p}{4}\|B(\boldsymbol{\theta}_\xi^{n+2})\|^2_{\Omega_p}+\delta_4\rho_p\|2\boldsymbol{\theta}_\xi^{n+1}-\boldsymbol{\theta}_\xi^{n}\|^2_{\Omega_p}+\delta_2\|\nabla\hat{\theta}_{pp}^{n+2}\|^2_{\Omega_p}\\
&\quad+Ch^{2k_s}\|\hat{\boldsymbol{\xi}}^{n+2}\|^2_{H^{k_s+1}(\Omega_p)}+Ch^{2k_p}\|\hat{p}_p^{n+2}\|^2_{H^{k_p+1}(\Omega_p)}.
\end{aligned} \tag{5.49}$$

Applying Lemma 4.2 with $\boldsymbol{V}_h^{n+i}=\hat{\boldsymbol{\Theta}}^{n+i}$ for $i=0,1,2$ and $\epsilon_1=\delta_1$ yields an estimate for the term J_{51}. Combining the estimate for J_{51} with those for J_{52} and J_{53} from Lemma 5.2, we obtain

$$\begin{aligned}
J_5&\le 5\delta_1\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|^2_{\Omega_f}+2\delta_2\|\nabla\hat{\theta}_{pp}^{n+2}\|^2_{\Omega_p}+\gamma\|\hat{\boldsymbol{\theta}}_u^{n+2}\cdot\boldsymbol{\tau}_f\|^2_\Gamma+\gamma\|\hat{\boldsymbol{\theta}}_\xi^{n+2}\cdot\boldsymbol{\tau}_f\|^2_\Gamma\\
&\quad+\frac{9\delta_4\rho_p}{2}\|B(\boldsymbol{\theta}_\xi^{n+2})\|^2_{\Omega_p}+2\delta_4\rho_p\|2\boldsymbol{\theta}_\xi^{n+1}-\boldsymbol{\theta}_\xi^{n}\|^2_{\Omega_p}+\frac{C_7}{h}\|B(\boldsymbol{\Theta}^{n+2})\|^2_S\\
&\quad+Ch^{2k_f}\left(\|\hat{\boldsymbol{u}}^{n+2}\|^2_{H^{k_f+1}(\Omega_f)}+\|\hat{p}_f^{n+2}\|_{H^{k_f}(\Omega_f)}\right)\\
&\quad+Ch^{2k_s}\left(\|\hat{\boldsymbol{\xi}}^{n+2}\|^2_{H^{k_s+1}(\Omega_p)}+h\|\boldsymbol{\xi}^{n+1}\|^2_{H^{k_s+1}(\Omega_p)}+h\|\boldsymbol{\xi}^{n}\|^2_{H^{k_s+1}(\Omega_p)}\right)\\
&\quad+Ch^{2k_p}\left(\|\hat{p}_p^{n+2}\|^2_{H^{k_p+1}(\Omega_p)}+h\|p_p^{n+1}\|^2_{H^{k_p+1}(\Omega_p)}+h\|p_p^{n}\|^2_{H^{k_p+1}(\Omega_p)}\right),
\end{aligned} \tag{5.50}$$

where

$$C_7=\max\{\frac{3\tilde{C}_{T,1}^2C_{T,3}^2C_{inv}}{8s_0\delta_1},\frac{3\tilde{C}_{T,1}^2C_{T,2}^2C_{inv}\gamma^2}{8\rho_p\delta_1}\}.$$

Using the Poincaré inequalities (4.4), trace inequalities (4.1)–(4.3), the Cauchy-Schwarz inequality, Lemma 5.1, and (5.4), we have

$$\begin{aligned}
J_{61}&\le(C_{P,1}\|\hat{\boldsymbol{f}}_f^{n+2}-\boldsymbol{f}_f^{n+2}\|_{\Omega_f}+\rho_f C_{P,1}\|D_t\boldsymbol{u}^{n+2}-\partial_t\hat{\boldsymbol{u}}^{n+2}\|_{\Omega_f}+C\|P_hp_f-P_h\hat{p}_f\|_{\Omega_f})\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}\\
&\quad+C\tilde{C}_{T,1}\tilde{C}_{T,3}(\|\nabla B(p_p^{n+2})\|_{\Omega_p}+\|\nabla(\hat{p}_p^{n+2}-p_p^{n+2})\|_{\Omega_p})\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}\\
&\quad+C\tilde{C}_{T,1}\tilde{C}_{T,2}(\|\nabla B(\boldsymbol{\xi}^{n+2})\|_{\Omega_p}+\|\nabla(\hat{\boldsymbol{\xi}}^{n+2}-\boldsymbol{\xi}^{n+2})\|_{\Omega_p})\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}
\end{aligned}$$

$$\le \delta_1\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}^2 + C\Delta t^3\int_{t_n}^{t_{n+2}}(\|\partial_{tt}\boldsymbol{f}_f\|_{\Omega_f}^2 + \|\partial_{ttt}\boldsymbol{u}\|_{\Omega_f}^2 + \|\partial_{tt}\boldsymbol{u}\|_{H^2(\Omega_f)}^2 + \|\partial_{tt}p_f\|_{H^1(\Omega_f)}^2)dt$$
$$+ C\Delta t^3\int_{t_n}^{t_{n+2}}(\|\partial_{tt}p_p\|_{H^1(\Omega_p)}^2 + \|\partial_{tt}\boldsymbol{\xi}\|_{H^1(\Omega_p)}^2)dt. \tag{5.51}$$

Similarly, an application of (4.23) yields

$$\begin{aligned}
J_{62} &\le (\|\hat{\boldsymbol{f}}_p^{n+2} - \boldsymbol{f}_p^{n+2}\|_{\Omega_p} + \rho_p\|D_t\boldsymbol{\xi}^{n+2} - \partial_t\hat{\boldsymbol{\xi}}^{n+2}\|_{\Omega_p})\|\hat{\boldsymbol{\theta}}_\xi^{n+2}\|_{\Omega_p}\\
&\quad + C_{p,3}(\|\hat{g}_p^{n+2} - g_p^{n+2}\|_{\Omega_p} + s_0\|D_tp_p^{n+2} - \partial_t\hat{p}_p^{n+2}\|_{\Omega_p})\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}\\
&\le \frac{9\delta_4\rho_p}{4}\|B(\boldsymbol{\theta}_\xi^{n+2})\|_{\Omega_p}^2 + \delta_4\rho_p\|2\boldsymbol{\theta}_\xi^{n+1} - \boldsymbol{\theta}_\xi^n\|_{\Omega_p}^2 + \delta_2\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}\\
&\quad + C\Delta t^3\int_{t_n}^{t_{n+2}}(\|\partial_{tt}\boldsymbol{f}_p\|_{\Omega_p}^2 + \|\partial_{tt}g_p\|_{\Omega_p}^2 + \|\partial_{ttt}\boldsymbol{\xi}\|_{\Omega_p}^2 + \|\partial_{ttt}p_p\|_{\Omega_p}^2)dt.
\end{aligned} \tag{5.52}$$

Following a derivation analogous to that of (5.48) and applying (5.15), we obtain

$$\begin{aligned}
J_{63} &\le a_\eta(\hat{\boldsymbol{\theta}}_\eta^{n+2}, \hat{\boldsymbol{\theta}}_\eta^{n+2})^{\frac{1}{2}}a_\eta(\boldsymbol{r}^{n+2}, \boldsymbol{r}^{n+2})^{\frac{1}{2}}\\
&\le \frac{9\delta_5}{4}a_\eta(B(\boldsymbol{\theta}_\eta^{n+2}), B(\boldsymbol{\theta}_\eta^{n+2})) + \delta_5a_\eta(2\boldsymbol{\theta}_\eta^{n+1} - \boldsymbol{\theta}_\eta^n, 2\boldsymbol{\theta}_\eta^{n+1} - \boldsymbol{\theta}_\eta^n)\\
&\quad + C\Delta t^3\int_{t_n}^{t_{n+2}}\|\partial_{ttt}\boldsymbol{\eta}\|_{H^1(\Omega_p)}^2dt.
\end{aligned} \tag{5.53}$$

Combining all of the above estimates for $J_i, i = 1, \ldots, 6$, we obtain

$$\begin{aligned}
&\mathcal{E}_\theta^{n+2} + 4(\mu_1 - 7\delta_1)\Delta t\|\nabla\hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}^2 + 4(\mu_2 - 4\delta_2)\Delta t\|\nabla\hat{\theta}_{pp}^{n+2}\|_{\Omega_p}^2\\
&\quad + (3 - 4\tilde{C}_7\Delta t)\|B(\boldsymbol{\Theta}^{n+2})\|_S^2 + (3 - 18\delta_5\Delta t)a_\eta(B(\boldsymbol{\theta}_\eta^{n+2}), B(\boldsymbol{\theta}_\eta^{n+2}))\\
&\le \mathcal{E}_\theta^{n+1} + 4C_8\Delta t\mathcal{E}_\theta^{n+1} + C\Delta t^4A_1 + C\Delta t(h^{2k_f}A_2 + h^{2k_s}A_3 + h^{2k_p}A_4),
\end{aligned} \tag{5.54}$$

where $C_8 = \max\{5\delta_4, 2\delta_5\}$, $\tilde{C}_7 = \frac{C_7}{h} + \frac{45\delta_4}{4}$ and

$$\begin{aligned}
A_1 &= \int_{t_n}^{t_{n+2}}(\|\partial_{tt}\boldsymbol{f}_f\|_{\Omega_f}^2 + \|\partial_{tt}\boldsymbol{f}_p\|_{\Omega_p}^2 + \|\partial_{tt}g_p\|_{\Omega_p}^2 + \|\partial_{ttt}\boldsymbol{u}\|_{\Omega_f}^2 + \|\partial_{ttt}\boldsymbol{\xi}\|_{\Omega_p}^2 + \|\partial_{ttt}p_p\|_{\Omega_p}^2)dt\\
&\quad + \int_{t_n}^{t_{n+2}}(\|\partial_{tt}\boldsymbol{u}\|_{H^2(\Omega_f)}^2 + \|\partial_{tt}p_f\|_{H^1(\Omega_f)}^2 + \|\partial_{tt}p_p\|_{H^1(\Omega_p)}^2 + \|\partial_{tt}\boldsymbol{\xi}\|_{H^1(\Omega_p)}^2)dt\\
A_2 &= \|\hat{\boldsymbol{u}}^{n+2}\|_{H^{k_f+1}(\Omega_f)}^2 + \|\hat{p}_f^{n+2}\|_{H^{k_f}(\Omega_f)}\\
&\quad + \frac{h^2}{\Delta t}\int_{t_n}^{t_{n+2}}(\|\partial_t\boldsymbol{u}\|_{H^{k_f+1}(\Omega_f)}^2 + \|\partial_tp_f\|_{H^{k_f}(\Omega_f)}^2)dt,\\
A_3 &= \|\hat{\boldsymbol{\xi}}^{n+2}\|_{H^{k_s+1}(\Omega_p)}^2 + h(\|\boldsymbol{\xi}^{n+1}\|_{H^{k_s+1}(\Omega_p)}^2 + \|\boldsymbol{\xi}^n\|_{H^{k_s+1}(\Omega_p)}^2)\\
&\quad + \frac{h^2}{\Delta t}\int_{t_n}^{t_{n+2}}\|\partial_t\boldsymbol{\xi}\|_{H^{k_s+1}(\Omega_p)}^2dt,\\
A_4 &= \|\hat{p}_p^{n+2}\|_{H^{k_p+1}(\Omega_p)}^2 + h(\|p_p^{n+1}\|_{H^{k_p+1}(\Omega_p)}^2 + \|p_p^n\|_{H^{k_p+1}(\Omega_p)}^2)\\
&\quad + \frac{h^2}{\Delta t}\int_{t_n}^{t_{n+2}}\|\partial_tp_p\|_{H^{k_p+1}(\Omega_p)}^2dt.
\end{aligned}$$

We set the parameters to $\delta_1 = \frac{\mu_1}{8}$, $\delta_2 = \frac{\mu_2}{5}$, $\delta_4 = \frac{1}{15}$ and $\delta_5 = \frac{1}{6}$. Assume the time step restriction (5.33) is satisfied. Then, summing Eq (5.54) from $n = 0$ to $n = m$, we obtain

$$\begin{aligned} &\mathcal{E}_\theta^{m+2} + \frac{\mu_1 \Delta t}{2} \sum_{n=0}^{m} \|\nabla \hat{\boldsymbol{\theta}}_u^{n+2}\|_{\Omega_f}^2 + \frac{4\mu_2 \Delta t}{5} \sum_{n=0}^{m} \|\nabla \hat{\theta}_{pp}^{n+2}\|_{\Omega_p}^2 \\ &\leq 4C_8 \Delta t \sum_{n=0}^{m} \mathcal{E}_\theta^{n+1} + \mathcal{E}_\theta^1 + C\Delta t^4 \mathcal{A}_1 + C(h^{2k_f}\mathcal{A}_2 + h^{2k_s}\mathcal{A}_3 + h^{2k_p}\mathcal{A}_4), \end{aligned} \tag{5.55}$$

where

$$\begin{aligned} \mathcal{A}_1 &= \|\partial_{tt}\boldsymbol{f}_f\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\partial_{tt}\boldsymbol{f}_p\|_{L^2(0,T;L^2(\Omega_p))}^2 + \|\partial_{tt}g_p\|_{L^2(0,T;L^2(\Omega_p))}^2 \\ &\quad + \|\partial_{ttt}\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\partial_{ttt}p_p\|_{L^2(0,T;L^2(\Omega_p))}^2 + \|\partial_{ttt}\boldsymbol{\xi}\|_{L^2(0,T;L^2(\Omega_p))}^2 \\ &\quad + \|\partial_{tt}p_p\|_{L^2(0,T;H^1(\Omega_p))}^2 + \|\partial_{tt}\boldsymbol{\xi}\|_{L^2(0,T;H^1(\Omega_p))}^2, \\ \mathcal{A}_2 &= \|\boldsymbol{u}\|_{l^2(0,T;H^{k_f+1}(\Omega_f))}^2 + \|p_f\|_{l^2(0,T;H^{k_f}(\Omega_f))}^2 \\ &\quad + h^2(\|\partial_t\boldsymbol{u}\|_{L^2(0,T;H^{k_f+1}(\Omega_f))}^2 + \|\partial_t p_f\|_{L^2(0,T;H^{k_f}(\Omega_f))}^2), \\ \mathcal{A}_3 &= \|\boldsymbol{\xi}\|_{l^2(0,T;H^{k_s+1}(\Omega_p))}^2 + h^2\|\partial_t\boldsymbol{\xi}\|_{L^2(0,T;H^{k_s+1}(\Omega_p))}^2, \\ \mathcal{A}_4 &= \|p_p\|_{l^2(0,T;H^{k_p+1}(\Omega_p))} + h^2\|\partial_t p_p\|_{L^2(0,T;H^{k_p+1}(\Omega_p))}^2. \end{aligned}$$

Assume that $\boldsymbol{\theta}_u^i = 0$, $\boldsymbol{\theta}_\xi^i = 0$, $\boldsymbol{\theta}_\eta^i = 0$, $\theta_{pp}^i = 0$ for $i = 0, 1$, using the Gronwall inequality, we obtain

$$\|\boldsymbol{\Theta}^{m+2}\|^2 \leq C(\Delta t^4 + h^{2k_f} + h^{2k_s} + h^{2k_p}), \tag{5.56}$$

$$a_\eta(\boldsymbol{\theta}_\eta^{m+2}, \boldsymbol{\theta}_\eta^{m+2}) \leq C(\Delta t^4 + h^{2k_f} + h^{2k_s} + h^{2k_p}). \tag{5.57}$$

Then (5.34) and (5.35) can be obtained by the triangle inequality. □

6. Numerical example

In this section, the numerical example is implemented by using the software package FreeFem++ [48]. All three decoupled algorithms decompose the original problem into two sub-problems. For the solution of these two sub-problems, we employ the built-in keyword "problem" in FreeFem++. All configurations are set to default. A commonly used numerical example is chosen, see, e.g., [15, 19, 21].

Example 6.1. *Let* $\Omega_f = (0,1) \times (0,1)$, $\Omega_p = (-1,0) \times (0,1)$ *and the interface* $\Gamma = (0,1) \times \{0\}$*, assume that the exact solution in the fluid region is*

$$\boldsymbol{u}(\boldsymbol{x}, t) = \pi \cos(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix},$$

$$p_f(\boldsymbol{x}, t) = \exp(t) \sin(\pi x) \cos(\frac{\pi y}{2}) + 2\pi\mu_f \cos(\pi t),$$

and the exact solution in the poroelastic region is

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \pi \cos(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix},$$

$$\boldsymbol{\eta}(\boldsymbol{x},t) = \sin(\pi t)\begin{pmatrix} -3x+\cos(y) \\ y+1 \end{pmatrix},$$

$$p_p(\boldsymbol{x},t) = \exp(t)\sin(\pi x)\cos(\frac{\pi y}{2}).$$

According to the exact solution, we can choose the right hand functions, initial conditions, and boundary conditions.

6.1. Numerical verification the convergence rates for Algorithm 3

We set $\rho_f = \mu_f = \rho_p = \mu_p = \lambda_p = s_0 = \alpha = \gamma = 1$, $\mathbb{K} = \mathbb{I}$ in the Stokes–Biot problem. We test the temporal and spatial convergence rates for the proposed decoupled time filtered finite element method (Algorithm 3). We discretize the Stokes problem in space by using the Taylor-Hood(P2-P1) elements for the velocity and pressure, and discretize the Biot problem by using piecewise quadratic elements(P2) for the displacement, velocity and pressure. Combining the numerical results in Table 1 and 2, it can be concluded that whether the time step is set to $\Delta t = 0.3h$ or $\Delta t = 0.6h$, the expected temporal convergence rates $O(\Delta t^2)$ are also obtained. This verifies the stability and reliability of the temporal convergence of the proposed method under different time steps. In Table 3, let $\Delta t = O(h^{\frac{3}{2}})$. Here, we see that all unknowns in H^1 norm and L^2 norm achieve the optimal spatial error convergence rates. Although according to our Theorem 5.1, the spatial convergence rate for these unknowns is only predicted to be second-order in the L^2 norm.

Table 1. Errors, temporal convergence rates for $T = 0.3$ and $\Delta t = 0.3\ h$.

h	1/8	1/16	1/32	1/64	1/128
$\|\|\boldsymbol{e}_u^N\|\|_{\Omega_f}$	3.0728E-02	6.7146E-03	1.6348E-03	4.0407E-04	1.0044E-04
Rate	–	2.19	2.04	2.02	2.01
$\|\|\nabla\boldsymbol{e}_u^N\|\|_{\Omega_f}$	1.1920E-01	2.6252E-02	6.4246E-03	1.5921E-03	3.9625E-04
Rate	–	2.18	2.03	2.01	2.01
$\|\|e_{fp}^N\|\|_{\Omega_f}$	2.1446E-01	6.9202E-02	1.6338E-02	3.9902E-03	9.8615E-04
Rate	–	1.63	2.08	2.03	2.02
$\|\|\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	3.7838E-03	1.1789E-03	3.1735E-04	8.1588E-05	2.0641e-05
Rate	–	1.68	1.89	1.96	1.98
$\|\|\nabla\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	2.6012E-02	7.1426E-03	1.9160E-03	5.1946E-04	1.4259E-04
Rate	–	1.86	1.90	1.88	1.87
$\|\|\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	6.5515E-03	1.9778E-03	5.3579E-04	1.3902E-04	3.5384E-05
Rate	–	1.73	1.88	1.95	1.97
$\|\|\nabla\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	1.7528E-02	5.3078E-03	1.4415E-03	3.7446E-04	9.5360E-05
Rate	–	1.72	1.88	1.94	1.97
$\|\|e_{pp}^N\|\|_{\Omega_p}$	7.6725E-03	1.9177E-03	4.7199E-04	1.1734E-04	2.92786e-05
Rate	–	2.00	2.02	2.01	2.00
$\|\|\nabla e_{pp}^N\|\|_{\Omega_p}$	5.5733E-02	1.4032E-02	3.5112E-03	8.7787E-04	2.1947E-04
Rate	–	1.99	2.00	2.00	2.00

Table 2. Errors, temporal convergence rates for $T = 0.3$ and $\Delta t = 0.6\ h$.

h	1/8	1/16	1/32	1/64	1/128
$\|\|\boldsymbol{e}_u^N\|\|_{\Omega_f}$	1.0744E-01	1.8492E-02	4.4203E-03	1.0636E-03	2.6072E-04
Rate	–	2.54	2.06	2.06	2.03
$\|\|\nabla\boldsymbol{e}_u^N\|\|_{\Omega_f}$	4.1973E-01	7.2485E-02	1.7378E-02	4.1866E-03	1.0270E-03
Rate	–	2.53	2.06	2.05	2.03
$\|\|e_{fp}^N\|\|_{\Omega_f}$	7.3452E-01	1.7930E-01	3.6883E-02	8.6123E-03	2.0793E-03
Rate	–	2.03	2.28	2.10	2.05
$\|\|\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	1.6475E-02	5.3260E-03	1.4297E-03	3.6427E-04	9.1687E-05
Rate	–	1.63	1.90	1.97	1.99
$\|\|\nabla\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	7.9036E-02	2.6132E-02	7.1557E-03	1.9193E-03	5.2011E-04
Rate	–	1.60	1.87	1.90	1.88
$\|\|\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	2.8206E-02	9.0567E-03	2.5021E-03	6.5411E-04	1.6704E-04
Rate	–	1.64	1.86	1.94	1.97
$\|\|\nabla\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	7.6832E-02	2.4794E-02	6.8796E-03	1.8015E-03	4.6031E-04
Rate	–	1.63	1.85	1.93	1.97
$\|\|e_{pp}^N\|\|_{\Omega_p}$	2.4664E-02	6.2230E-03	1.4608E-03	3.5571E-04	8.7913E-05
Rate	–	1.99	2.09	2.04	2.02
$\|\|\nabla e_{pp}^N\|\|_{\Omega_p}$	1.0302E-01	2.4110E-02	5.6678E-03	1.3782E-03	3.4021E-04
Rate	–	2.10	2.09	2.04	2.02

Table 3. Errors, spatial convergence rates with $T = 0.2$ and $\Delta t = 0.2\ h^{\frac{3}{2}}$.

h	1/9	1/16	1/25	1/36	1/49
$\|\|\boldsymbol{e}_u^N\|\|_{\Omega_f}$	1.5062E-03	2.6092E-04	6.7718E-05	2.2577E-05	8.9327E-06
Rate	–	3.05	3.02	3.01	3.01
$\|\|\nabla\boldsymbol{e}_u^N\|\|_{\Omega_f}$	5.9142E-03	1.0374E-03	2.7392E-04	9.3767E-05	3.8571e-05
Rate	–	3.03	2.98	2.94	2.88
$\|\|e_{fp}^N\|\|_{\Omega_f}$	1.3312E-02	2.4876E-03	7.6142E-04	3.1422E-04	1.5616E-04
Rate	–	2.92	2.65	2.43	2.27
$\|\|\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	5.1267E-04	9.1836E-05	2.4124E-05	8.0858E-06	3.20809e-06
Rate	–	2.99	3.00	3.00	3.00
$\|\|\nabla\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	2.6545E-03	5.2838E-04	1.5760E-04	5.98685e-05	2.71751e-05
Rate	–	2.81	2.71	2.65	2.56
$\|\|\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	9.1065E-04	1.6695E-04	4.4219E-05	1.4876E-05	5.91296e-06
Rate	–	2.95	2.98	2.99	2.99
$\|\|\nabla\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	2.5188E-03	4.6467E-04	1.2456E-04	4.2794E-05	1.7578E-05
Rate	–	2.94	2.95	2.93	2.89
$\|\|e_{pp}^N\|\|_{\Omega_p}$	5.9012E-04	1.0050E-04	2.5834E-05	8.5627E-06	3.3750E-06
Rate	–	3.08	3.04	3.03	3.02
$\|\|\nabla e_{pp}^N\|\|_{\Omega_p}$	1.6306E-02	5.1636E-03	2.1170E-03	1.0215E-03	5.5161E-04
Rate	–	2.00	2.00	2.00	2.00

6.2. *The influence in different physical parameters*

Let $\rho_f = \rho_p = \mu_p = \lambda_p = s_0 = \alpha = \gamma = 1$ and $\mathbb{K} = \mathbb{I}$ in the Stokes–Biot problem. To test our theoretical results in Theorem 5.1, we choose different values for the fluid viscosity μ_f. By setting the time step $\Delta t = O(h)$, Tables 4 and 5 present the numerical errors and corresponding convergence rates for fluid viscosities $\mu_f = 0.1$ and $\mu_f = 0.001$, respectively. We can see that the errors in both the L^2 and H^1 norms decrease monotonically with a second-order convergence temporal rate, confirming the method's accuracy and robustness.

Table 4. Errors, temporal convergence rates for $T = 0.1$, $\Delta t = 0.05\ h$ and $\mu_f = 0.1$.

h	1/8	1/16	1/32	1/64	1/128
$\|\|\boldsymbol{e}_u^N\|\|_{\Omega_f}$	7.3984E-04	1.8283E-04	4.5641E-05	1.1406E-05	2.8509E-06
Rate	–	2.02	2.00	2.00	2.00
$\|\|\nabla\boldsymbol{e}_u^N\|\|_{\Omega_f}$	7.4243E-03	1.3917E-03	3.0067E-04	7.0795E-05	1.7243E-05
Rate	–	2.42	2.21	2.09	2.04
$\|\|e_{fp}^N\|\|_{\Omega_f}$	4.4449E-03	1.0919E-03	2.7155E-04	6.7774E-05	1.6934E-05
Rate	–	2.03	2.01	2.00	2.00
$\|\|\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	1.3010E-04	3.0472E-05	7.6515E-06	1.9268E-06	4.8421E-07
Rate	–	2.09	1.99	1.99	1.99
$\|\|\nabla\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	3.3396E-03	5.7219E-04	1.5064E-04	4.1074E-05	1.1018E-05
Rate	–	2.55	1.93	1.87	1.90
$\|\|\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	1.0899E-04	2.8743E-05	7.3654E-06	1.8633E-06	4.6853E-07
Rate	–	1.92	1.96	1.98	1.99
$\|\|\nabla\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	3.2949E-04	8.5112E-05	2.1646E-05	5.4674E-06	1.3740E-06
Rate	–	1.95	1.98	1.99	1.99
$\|\|e_{pp}^N\|\|_{\Omega_p}$	3.2949E-04	8.5112E-05	2.1646E-05	5.4674E-06	1.3740E-06
Rate	–	1.95	1.98	1.99	1.99
$\|\|\nabla e_{pp}^N\|\|_{\Omega_p}$	1.6756E-02	4.2218E-03	1.0586E-03	2.6496E-04	6.6277E-05
Rate	–	1.99	2.00	2.00	2.00

Table 5. Errors, temporal convergence rates for $T = 0.1$, $\Delta t = 0.05\ h$ and $\mu_f = 0.001$.

h	1/8	1/16	1/32	1/64	1/128
$\|\|\boldsymbol{e}_u^N\|\|_{\Omega_f}$	4.0390E-03	5.9748E-04	6.4682E-05	1.1642E-05	2.7692E-06
Rate	–	2.76	3.21	2.47	2.07
$\|\|\nabla\boldsymbol{e}_u^N\|\|_{\Omega_f}$	2.4089E-01	7.0660E-02	1.2010E-02	1.9732E-03	3.4411E-04
Rate	–	1.77	2.56	2.61	2.52
$\|\|e_{fp}^N\|\|_{\Omega_f}$	4.4022E-03	1.0801E-03	2.6894E-04	6.7168E-05	1.6789E-05
Rate	–	2.03	2.01	2.00	2.00
$\|\|\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	3.4236E-04	9.3360E-05	2.2928E-05	5.6359E-06	1.4065E-06
Rate	–	1.87	2.03	2.02	2.00
$\|\|\nabla\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	6.6565E-03	1.9727E-03	6.3248E-04	1.8909E-04	5.3465E-05
Rate	–	1.75	1.64	1.74	1.82
$\|\|\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	1.0829E-04	2.8493E-05	7.2990E-06	1.8463E-06	4.6425E-07
Rate	–	1.93	1.96	1.98	1.99
$\|\|\nabla\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	5.2734E-04	1.3646E-04	3.3942E-05	8.4534E-06	2.1118E-06
Rate	–	1.95	2.01	2.01	2.00
$\|\|e_{pp}^N\|\|_{\Omega_p}$	3.1025E-04	4.8224E-05	9.4070E-06	2.1578E-06	5.2694E-07
Rate	–	2.69	2.36	2.12	2.03
$\|\|\nabla e_{pp}^N\|\|_{\Omega_p}$	1.6765E-02	4.2227E-03	1.0585E-03	2.6494E-04	6.6269E-05
Rate	–	1.99	2.00	2.00	2.00

6.3. Comparison of Algorithm 1–3

The efficiency of Algorithm 3 is evaluated against Algorithms 1 and 2. Let $\rho_f = \mu_f = \rho_p = \mu_p = \lambda_p = s_0 = \alpha = \gamma = 1$, $\mathbb{K} = \mathbb{I}$ in the Stokes–Biot problem. We discretize the Stokes problem in space by using the Taylor-Hood(P2-P1) elements for the velocity and pressure, and discretize the Biot problem by using piecewise quadratic elements(P2) for the displacement, velocity and the piecewise linear elements(P1) for pressure. Comparing Table 6 with Table 8, we see that by simply adding a time filter, the time accuracy is increased from first order to second order. The first order $O(\Delta t)$ for $||\nabla e_{pp}||_{\Omega_p}$ is due to the use of the piecewise linear elements(P1) for pressure. It can also be shown from Table 6–8 that Algorithm 1 takes the shortest CPU time, followed by Algorithm 3, and Algorithm 2 takes the longest CPU time.

Table 6. Errors and convergence rates of Algorithm 1 for $T = 1.0$ and $\Delta t = h$.

h	1/8	1/16	1/32	1/64	1/128
$\|\|\boldsymbol{e}_u^N\|\|_{\Omega_f}$	1.3624e-02	6.5651e-03	3.1877e-03	1.5573e-03	7.6425e-04
Rate	–	1.05	1.04	1.03	1.03
$\|\|\nabla\boldsymbol{e}_u^N\|\|_{\Omega_f}$	9.9144e-02	4.6833e-02	2.3344e-02	1.1653e-02	5.7855e-03
Rate	–	1.08	1.00	1.00	1.01
$\|\|e_{fp}^N\|\|_{\Omega_f}$	1.3764e+00	7.0486e-01	3.5621e-01	1.7918e-01	8.9905e-02
Rate	–	0.97	0.98	0.99	0.99
$\|\|\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	2.7874e-01	1.7200e-01	9.8078e-02	5.2944e-02	2.7639e-02
Rate	–	0.70	0.81	0.89	0.94
$\|\|\nabla\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	1.0484e+00	6.4957e-01	3.7646e-01	2.0775e-01	1.1107e-01
Rate	–	0.69	0.79	0.86	0.90
$\|\|\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	4.4854e-01	2.2207e-01	1.1036e-01	5.4978e-02	2.7429e-02
Rate	–	1.01	1.01	1.01	1.00
$\|\|\nabla\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	1.2657e+00	6.3629e-01	3.1989e-01	1.6067e-01	8.0587e-02
Rate	–	0.99	0.99	0.99	1.00
$\|\|e_{pp}^N\|\|_{\Omega_p}$	3.4551e-02	1.6520e-02	1.0731e-02	6.2982e-03	3.4372e-03
Rate	–	1.06	0.62	0.77	0.87
$\|\|\nabla e_{pp}^N\|\|_{\Omega_p}$	7.1306e-01	3.6582e-01	1.8624e-01	9.4170e-02	4.7394e-02
Rate	–	0.96	0.97	0.98	0.99
CPU(s)	0.479	3.773	31.016	285.248	2622.240

Table 7. Errors and convergence rates of Algorithm 2 for $T = 1.0$ and $\Delta t = h$.

h	1/4	1/8	1/16	1/32	1/64
$\|\|\boldsymbol{e}_u^N\|\|_{\Omega_f}$	7.9249e-03	2.1790e-03	5.5934e-04	1.4107e-04	3.5378e-05
Rate	–	1.86	1.96	1.99	2.00
$\|\|\nabla\boldsymbol{e}_u^N\|\|_{\Omega_f}$	9.8613e-02	2.6714e-02	6.8376e-03	1.7243e-03	4.3248e-04
Rate	–	1.88	1.97	1.99	2.00
$\|\|e_{fp}^N\|\|_{\Omega_f}$	1.5111e-01	4.3143e-02	1.1960e-02	3.1614e-03	8.1223e-04
Rate	–	1.81	1.85	1.92	1.96
$\|\|\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	3.2289e-02	1.2008e-02	3.5818e-03	9.6081e-04	2.4742e-04
Rate	–	1.43	1.75	1.90	1.96
$\|\|\nabla\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	1.6028e-01	5.1297e-02	1.5383e-02	4.1757e-03	1.0884e-03
Rate	–	1.64	1.74	1.88	1.94
$\|\|\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	2.8835e-02	7.2942e-03	1.8031e-03	4.4532e-04	1.1048e-04
Rate	–	1.98	2.02	2.02	2.01
$\|\|\nabla\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	1.0515e-01	2.7671e-02	6.9063e-03	1.7072e-03	4.2355e-04
Rate	–	1.93	2.00	2.02	2.01
$\|\|e_{pp}^N\|\|_{\Omega_p}$	4.9539e-02	1.2608e-02	3.2099e-03	8.1313e-04	2.0444e-04
Rate	–	1.97	1.97	1.98	1.99
$\|\|\nabla e_{pp}^N\|\|_{\Omega_p}$	6.9427e-01	3.4748e-01	1.7378e-01	8.6894e-02	4.3447e-02
Rate	–	1.00	1.00	1.00	1.00
CPU(s)	0.46	3.77	37.32	329.30	3106.90

Table 8. Errors and convergence rates of Algorithm 3 for $T = 1.0$ and $\Delta t = h$.

h	1/8	1/16	1/32	1/64	1/128
$\|\|\boldsymbol{e}_u^N\|\|_{\Omega_f}$	3.0439e-01	7.6653e-02	1.9227e-02	4.8100e-03	1.2027e-03
Rate	–	1.99	2.00	2.00	2.00
$\|\|\nabla \boldsymbol{e}_u^N\|\|_{\Omega_f}$	1.1901e+00	3.0035e-01	7.5469e-02	1.8908e-02	4.7311e-03
Rate	–	1.99	1.99	2.00	2.00
$\|\|e_{fp}^N\|\|_{\Omega_f}$	3.7200e+00	9.3142e-01	2.2920e-01	5.6620e-02	1.4060e-02
Rate	–	2.00	2.02	2.02	2.01
$\|\|\boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	3.8392e-02	1.8466e-02	6.0599e-03	1.6770e-03	4.3561e-04
Rate	–	1.06	1.61	1.85	1.94
$\|\|\nabla \boldsymbol{e}_\xi^N\|\|_{\Omega_p}$	2.3393e-01	7.1184e-02	2.3358e-02	6.9149e-03	1.8817e-03
Rate	–	1.72	1.61	1.76	1.88
$\|\|\boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	2.2713e-02	6.3155e-03	1.5038e-03	3.6229e-04	8.8900e-05
Rate	–	1.85	2.07	2.05	2.03
$\|\|\nabla \boldsymbol{e}_\eta^N\|\|_{\Omega_p}$	1.0514e-01	2.8245e-02	6.7994e-03	1.6332e-03	3.9870e-04
Rate	–	1.90	2.05	2.06	2.03
$\|\|e_{pp}^N\|\|_{\Omega_p}$	2.5788e-02	6.9021e-03	1.7972e-03	4.4192e-04	1.0949e-04
Rate	–	1.90	1.94	2.02	2.01
$\|\|\nabla e_{pp}^N\|\|_{\Omega_p}$	7.2229e-01	3.5127e-01	1.7426e-01	8.6952e-02	4.3454e-02
Rate	–	1.04	1.01	1.00	1.00
CPU(s)	0.43	3.74	31.62	292.48	2702.78

7. Conclusions

In this paper, we propose a decoupled time filtered method for the Stokes–Biot model, and derive the corresponding stability results and a priori error estimates. Through comparisons with Algorithm 1 and 2, our algorithm (Algorithm 3) retains the second-order temporal accuracy of Algorithm 2 while exhibiting the lower computational costs of Algorithm 1.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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