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Research article

Applications of fixed point theorems in extended cone b-metric spaces over Banach algebras to integral equations arising in signal processing

Nura Alotaibi[1] **and Jamshaid Ahmad**[2,3,*]

[1] Mathematics Department, College of Science, Jouf University, Sakaka P.O. Box 2014, Saudi Arabia; nthmalabubi@ju.edu.sa

[2] Department of Mathematics, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia

[3] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Pretoria 0204, South Africa; jkhan@uj.edu.sa

* **Correspondence:** Email: jkhan@uj.edu.sa.

Abstract: The aim of this research article is to explore the notion of extended cone b-metric spaces over Banach algebras and to establish new fixed point theorems for both single-valued mappings and multi-valued mappings. These results extend and generalize several well-known findings in metric fixed point theory. To demonstrate the applicability of our theoretical results, we provide illustrative examples that highlight the distinct features of our approach. Additionally, we showcase the practical relevance of our primary theorem by solving a Fredholm integral equation of the second kind, which arises in the context of radar and sonar signal processing.

Keywords: fixed point; extended cone b-metric spaces; single-valued mappings; Banach algebras; multivalued mappings; Fredholm integral equations
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1. Introduction

Fixed point (FP) theory is a broad and diverse branch of mathematics, generally categorized into three primary areas: metric, topological, and discrete FP theory. Of these, metric FP theory serves as a fundamental framework, emphasizing the establishment of FP's existence and uniqueness for self-mappings defined within metric spaces (MSs). This branch relies heavily on the principles of distance and convergence, which are integral characteristics of MSs.

The concept of a MS was first introduced by Maurice Fréchet [1] in 1906, characterized as a set paired with a metric, or distance function, that quantifies the distance between any two points. Over time, this idea has been expanded and refined by numerous mathematicians, leading to the formulation of more advanced structures. Significant advancements include partial metric spaces (PMSs) by Matthews [2], b-metric spaces (b-MSs) by Bakhtin [3], extended b-metric spaces (Eb-MSs) by Kamran [4] and rectangular metric spaces (RMSs) by Branciari [5].

A b-MS is an extension of the traditional concept of a MS. In standard MSs, the distance between points satisfies a specific rule called the triangle inequality. This rule states that the direct distance between two points should always be less than or equal to the sum of their distances through a third point. However, in a b-MS, this rule is slightly relaxed by introducing a constant $b \geq 1$. This constant is applied as a multiplier to the sum of distances, allowing the distance measure to be more flexible. Despite this modification, b-MSs maintain other important properties, such as symmetry and non-negativity. Eb-MSs take the generalization a step further. Instead of using a fixed constant $b \geq 1$, these spaces allow the multiplier to vary depending on the points involved. This variable multiplier is represented as a function $\varphi(\mathfrak{l}, z) \geq 1$ based on the specific configuration of two points. By replacing the fixed constant with a flexible function, extended b-MSs can model more complex scenarios where the relationship between distances is not uniform across the space.

In 2007, Huang et al. [6] initiated cone metric space (CMS) as a generalization of traditional MSs, where instead of real numbers, the elements of an ordered Banach space (BS) are used. Building on this concept, they were able to define Cauchy sequences and the convergence of sequences within these spaces in this context. Faried et al. [7] investigated and established a number of coupled coincidence theorems involving generalized admissible mappings defined on partial satisfactory CMSs. Subsequently, Liu et al. [8, 9] extended the concept of CMSs by replacing the underlying BS with a Banach algebra (BA), leading to the introduction of CMSs over BAs. They provided a generalized definition of cones over a BA and explored partial orderings in terms of the interior points of these cones. In addition, they presented an example to demonstrate that FP theorems in CMSs over BAs are generally not analogous to those in traditional MSs. While Liu and Xu worked with normal cones, Xu et al. [10] avoided the normality condition of cones and instead used solid cones (SCs) in CMSs over BAs. Abou Bakr et al. [11] proved coupled coincidence point results in the setting of vector quasi CMSs over BAs. Building on similar ideas, Huang et al. [12] gave the idea of cone b-metric spaces (Cb-MSs) over BAs by combining the concepts of CMSs over BAs and b-MSs. Since b-MSs possess several distinctive properties not found in standard MSs, it follows that Cb-MSs over BAs inherit these unique properties as well. Afterwards, Huang et al. [13] strengthened this above concept of Cb-MSs over BAs with coefficient $b \geq 1$ and proved a sharp FP result by taking the spectral radius $\rho(\varpi)$ of the generalized Lipschitz constant ϖ from $\rho(\varpi) \in \left[0, \frac{1}{b}\right)$ to $\rho(\varpi) \in [0, 1)$. Shatanawi et al. [14] established Hardy-Rogers type FP theorems for single-valued mappings in the framework of Cb-MSs over BAs. In a recent study, Roy et al. [15] defined the conception of extended Cb-MS (ECb-MS) defined over BAs, thereby extending existing frameworks.

A visionary mathematician Stefan Banach [16] laid the foundation of modern FP theory by formulating the celebrated Banach contraction principle (BCP). This fundamental theorem, introduced in 1922, asserts that any contraction mapping on a complete MS has a unique FP. The principle not only guarantees the existence and uniqueness of a FP but also provides a constructive method to approximate it through iterative processes. Berinde's work in [17] introduced a significant generalization, initially termed 'weak contraction' and subsequently renamed as 'almost contraction'. The concept of α-admissibility was given by Samet et al. [18] in their efforts to generalize FP theory by relaxing the conditions under which FPs can be established. This innovative idea broadens the scope of FP results by introducing an auxiliary function, α, to control the behavior of mappings, allowing the theory to encompass a wider range of applications. Nadler [19] significantly extended the renowned BCP by establishing a FP theorem for multivalued mappings. This groundbreaking work, published in 1969, broadened the scope of FP theory and opened up new avenues of research. Following Cho et al. [20], who established FP results for multivalued mappings in CMSs, Azam et al. [21] further extended these findings by proving FP theorems within the framework of Cb-MSs without the assumption of normality of the cone. Their work built upon and generalized the earlier contributions of Cho, offering a broader perspective on the interplay between multivalued mappings and Cb-MSs. Building on this groundwork, Kutbi et al. [22] extended the theory by proving multivalued FP theorems specifically in Cb-MSs over BAs. For additional details on this topic, readers are referred to references [23–27].

On the other hand, FP theorems form a foundational pillar in functional analysis, offering a robust and systematic framework for establishing the existence and uniqueness of solutions to a wide range of mathematical problems, including nonlinear and integral equations. Their utility becomes especially pronounced when addressing complex models, such as Fredholm integral equations of the second kind, which frequently arise in applied sciences. These theorems provide a rigorous analytical toolset to confirm the solvability of such equations and to guarantee the uniqueness of their solutions under suitable conditions. In particular, Fredholm integral equations of the second kind are central to various applications in radar and sonar signal processing. They are instrumental in capturing essential phenomena such as wave propagation, echo reflection, and target localization. The mathematical structure of these equations aligns naturally with physical models, making FP techniques an invaluable resource in analyzing and solving real-world problems in signal detection and interpretation. For further information in this direction, the readers are encouraged to consult references [28–30].

In this research article, we establish new FP theorems for both single-valued mappings and multivalued mappings within the framework of Cb-MSs over BAs. These results significantly extend classical FP theory by incorporating a more generalized setting that accounts for both the algebraic structure of BAs and the ordered nature of CMSs. The key contributions of this work are summarized as follows:

We investigate the concept of ECb-MSs over BAs, which generalizes various existing spaces, including CMSs over BA, Cb-MSs over BA, Eb-MSs, b-MSs, and classical MSs. This structure provides a broader analytical framework for studying FP theory in settings involving algebraic operations and ordered normed spaces.

In this newly introduced space, we prove some new FP theorems for both single-valued and multi-valued mappings. Our results extend and generalize classical FP theorems by incorporating contraction conditions that account for the underlying cone structure and the algebraic properties of Banach spaces.

We refine the spectral radius condition for generalized Lipschitz constants, leading to stronger existence and uniqueness results. This improvement provides new insights into contraction mappings, making the results more flexible for a wider range of applications.

We introduce a generalized Hausdorff distance function in the setting of ECb-MSs over BAs, which is not previously established in the literature. This allows us to prove multi-valued FP theorems that account for the additional complexities introduced by Banach algebraic operations and ordering in cone structures.

Finally, we apply our FP results to solve a Fredholm integral equation of the second kind, thereby demonstrating the practical relevance of our theoretical findings. This application highlights the broader utility of FP theorems, extending their significance from abstract analysis to concrete problems in applied mathematics, particularly in modeling signal processing phenomena in radar and sonar systems.

2. Preliminaries

Let us start by reviewing some fundamental definitions and concepts that will be essential for presenting the key results in the following sections.

Definition 2.1. *[31] Let $\mathbb{A}$ be a BA, which is defined as a real Banach space (BS) equipped with a multiplication operation that satisfies the following properties:*
(i) $\iota(\varsigma z) = (\iota\varsigma)z$,
(ii) $\iota(\varsigma + z) = \iota\varsigma + \iota z$ *and* $(\iota + \varsigma)z = \iota z + \varsigma z$,
(iii) $a(\iota\varsigma) = (a\iota)\varsigma = \iota(a\varsigma)$,
(iv) $\|\iota\varsigma\| \leq \|\iota\|\|\varsigma\|$,
for all $\iota, \varsigma, z \in \mathbb{A}$ *and* $a \in \mathbb{R}$

A BA $\mathbb{A}$ over a field $\mathbb{R}$ is said to have a unity element (or unit) if there exists an element $e_{\mathbb{A}} \in \mathbb{A}$ such that $e_{\mathbb{A}}\iota = \iota e_{\mathbb{A}} = \iota$, for all $\iota \in \mathbb{A}$. Here,

$e_{\mathbb{A}}$ is the multiplicative identity of the BA $\mathbb{A}$. The additive identity of a real BA $\mathbb{A}$ is an element $\theta_{\mathbb{A}} \in \mathbb{A}$ such that $\theta_{\mathbb{A}} + \iota = \iota + \theta_{\mathbb{A}} = \iota$, for all $\iota \in \mathbb{A}$. An element ι within $\mathbb{A}$ is said to be invertible if there is an element ς within $\mathbb{A}$, referred to as the inverse of ι, such that $\iota\varsigma = \varsigma\iota = e_{\mathbb{A}}$. The inverse of ι is conventionally denoted as ι^{-1}.

Proposition 2.1. *[31] Consider $\mathbb{A}$, a real BA endowed with a unity element $e_{\mathbb{A}}$. The spectral radius of an arbitrary element ι within $\mathbb{A}$, denoted by $\rho(\iota)$, is formally defined as follows:*

$$\rho(\iota) = \sup_{r \in \sigma(\iota)} |r| = \lim_{n \to \infty} \|\iota^n\|^{\frac{1}{n}},$$

where $\sigma(\iota)$ is the spectrum of $\iota \in \mathbb{A}$. If $\rho(\iota) < 1$, then $(e_{\mathbb{A}} - \iota)$ is invertible and

$$(e_{\mathbb{A}} - \iota)^{-1} = e_{\mathbb{A}} + \sum_{i=1}^{\infty} \iota^i.$$

Remark 2.1. *[31] If $\rho(\iota) < 1$, then $\|\iota^n\| \to 0$ (or equivalently $\iota^n \to \theta_{\mathbb{A}}$) as $n \to \infty$.*

Definition 2.2. *[12] Let $\mathbb{A}$ be a real BA endowed with a unity element $e_{\mathbb{A}}$, and let $\mathbb{P}$ be a subset of $\mathbb{A}$. The set $\mathbb{P}$ is said to be a cone if it satisfies the following conditions:*

(i) $\mathbb{P}$ is non empty, closed and $\{e_{\mathbb{A}}, \theta_{\mathbb{A}}\} \subset \mathbb{P}$,
(ii) $\mathbb{P}^2 = \mathbb{P}\mathbb{P} \subset \mathbb{P}$,
(iii) $\mathbb{P} \cap (-\mathbb{P}) = \{\theta_{\mathbb{A}}\}$,
(iv) $a, b \in \mathbb{R}, a, b \geq 0, \iota, \varsigma \in \mathbb{P}$ implies $a\iota + b\varsigma \in \mathbb{P}$.

Definition 2.3. *[12] Let $\mathbb{A}$ be a real BA endowed with a unity element $e_{\mathbb{A}}$ and $\mathbb{P}$ be a cone in $\mathbb{A}$, we introduce a partial order $\preceq$ on $\mathbb{P}$ as follows:*

- $\iota \preceq \varsigma \Longleftrightarrow \varsigma - \iota \in \mathbb{P}$,

- $\iota < \varsigma$ if $\iota \leq \varsigma$ and $\iota \neq \varsigma$,
- $\iota \ll \varsigma$ if $\varsigma - \iota \in int\mathbb{P}$, where $int\mathbb{P}$ signifies the interior of $\mathbb{P}$.

If $int\mathbb{P} \neq \emptyset$, then $\mathbb{P}$ is termed a SC. Given a norm $\|\cdot\|$ on $\mathbb{A}$, a cone $\mathbb{P}$ is characterized as normal if there exists a positive constant M such that for all elements ι and ς within $\mathbb{A}$, the following inequality holds:

$$\theta_{\mathbb{A}} \leq \iota \leq \varsigma \Longrightarrow \|\iota\| \leq M\|\varsigma\|.$$

Throughout this work, we assume that $\mathbb{A}$ represents a real BA with the unity element $e_{\mathbb{A}}$, $\mathbb{P}$ denotes a SC and $\leq$ signifies the partial ordering induced by $\mathbb{P}$.

Lemma 2.1. *[10] If $\mathbb{E}$ is a real BS with a SC $\mathbb{P}$ and if $a \leq \lambda a$ with $a \in \mathbb{P}$ and $0 \leq \lambda < 1$, then $a = \theta_{\mathbb{E}}$.*

Lemma 2.2. *[25] Within a real BS $\mathbb{E}$ equipped with a SC $\mathbb{P}$, if $\theta_{\mathbb{E}} \leq u \ll c$ holds for every element $\theta \ll c$, then it necessarily follows that $u = \theta_{\mathbb{E}}$.*

Lemma 2.3. *[10] Consider a real BS $\mathbb{E}$ endowed with a SC $\mathbb{P}$. Given a sequence $\{\iota_n\}$ in $\mathbb{E}$ converging to the zero vector $\theta_{\mathbb{E}}$, and any element $\theta_{\mathbb{E}} \ll c \in int\mathbb{P}$, it is possible to find a positive integer $n_0 \in \mathbb{N}$ such that, $\iota_n \ll c$, for all $n < n_0$.*

Definition 2.4. *[8] Let $X \neq \emptyset$ and $\mathbb{A}$ be a BA. If a function $d : X \times X \to \mathbb{A}$ satisfies the following axioms:*

(C_1) $\theta_{\mathbb{A}} \leq d(\iota, \varsigma)$ and $d(\iota, \varsigma) = \theta$ if and only if $\iota = \varsigma$,
(C_2) $d(\iota, \varsigma) = d(\varsigma, \iota)$,
(C_3) $d(\iota, z) \leq d(\iota, \varsigma) + d(\varsigma, z)$,
for all $\iota, \varsigma, z \in X$, then $(X, \mathbb{A}, d)$ is called a CMS over a BA $\mathbb{A}$.

Later on, Huang et al. [12] extended the concept of CMS over a BA $\mathbb{A}$ to Cb-MS over a BA $\mathbb{A}$ in this way.

Definition 2.5. *[12] Let $X \neq \emptyset$, $b \geq 1$ and $\mathbb{A}$ be a real unital BA with a SC $\mathbb{P}$. If a function $d : X \times X \to \mathbb{A}$ satisfies the following conditions*

(C_{b1}) : $\theta_{\mathbb{A}} \leq d(\iota, \varsigma)$ and $d(\iota, \varsigma) = \theta_{\mathbb{A}}$ if and only if $\iota = \varsigma$,
(C_{b2}) : $d(\iota, \varsigma) = d(\varsigma, \iota)$,
(C_{b3}) : $d(\iota, z) \leq b[d(\iota, \varsigma) + d(\varsigma, z)]$,
for all $\iota, \varsigma, z \in X$, then $(X, \mathbb{A}, d)$ is identified as a Cb-MS over a BA $\mathbb{A}$.

Recently, Roy et al. [15] defined the notion of ECb-MS over a BA $\mathbb{A}$ in the following manner.

Definition 2.6. *[15] Let $X \neq \emptyset$, $\varphi : X \times X \to [1, +\infty)$ and $\mathbb{A}$ be a real unital BA with a SC $\mathbb{P}$. If a function $d : X \times X \to \mathbb{A}$ satisfies the following conditions:*

(c_{e1}) : $\theta_{\mathbb{A}} \leq d(\iota, \varsigma)$ and $d(\iota, \varsigma) = \theta_{\mathbb{A}}$ if and only if $\iota = \varsigma$,
(c_{e2}) : $d(\iota, \varsigma) = d(\varsigma, \iota)$,
(c_{e3}) : $d(\iota, z) \leq \varphi(\iota, z)(d(\iota, \varsigma) + d(\varsigma, z))$,
for all $\iota, \varsigma, z \in X$, then $(X, \mathbb{A}, d)$ is called an ECb-MS over a BA $\mathbb{A}$.

Remark 2.2. *[15] An ECb-MS over a BA $\mathbb{A}$ serves as a generalization of several well-known metric structures, including:*

1) Cone metric spaces over a Banach algebra: By setting $\varphi(\iota, z) = 1$, for all $\iota, z \in X$ in the definition of the ECb-MS, it reduces to a CMS over a BA $\mathbb{A}$.

2) Cone b-metric spaces over a Banach algebra: By setting $\varphi(\iota, z) = b \geq 1$, for all $\iota, z \in X$, the ECb-MS reduces to a Cb-MS over a BA $\mathbb{A}$.

3) Extended b-metric space: If $\mathbb{A} = \mathbb{R}$ with the cone $\mathbb{P} = [0, +\infty)$, then EC$b$-MSs simplify to E$b$-MSs.

4) b-metric space: By setting $\varphi(\iota, z) = b \geq 1$, for all $\iota, z \in X$, and letting $\mathbb{A} = \mathbb{R}$ with the cone $\mathbb{P} = [0, +\infty)$ in the definition of the ECb-MS, then it reduces to b-MS.

5) Metric space: By taking $\varphi(\iota, z) = 1$, for all $\iota, z \in X$, and setting $\mathbb{A} = \mathbb{R}$ with the cone $\mathbb{P} = [0, +\infty)$, the EC$b$-MS reduces to a standard MS.

Example 2.1. *[15] Let $X = \mathbb{R}$ and $\mathbb{A} = C[0, 1]$ be the usual unital BA with the sup norm. Let*

$$\mathbb{P} = \{f \in C[0, 1] : f(t) \geq 0 \text{ for all } t \in [0, 1]\}.$$

Define $\varphi : X \times X \to [1, +\infty)$ by $\varphi(\ell, \hbar) = 2 + |\ell| + |\hbar|$. Now define $d : X \times X \to \mathbb{A}$ by

$$d(\iota, \varsigma)(t) = \left(\sup_{t\in[0,1]} |\iota(t) - \varsigma(t)|^2\right) e^t.$$

Then $(X, \mathbb{A}, d)$ is an ECb-MS over a BA $\mathbb{A}$.

Definition 2.7. *[15] Let $(X, \mathbb{A}, d)$ be an ECb-MS over a BA $\mathbb{A}$, and let $\{\iota_n\}$ be a sequence in X. Then*

(i) A sequence $\{\iota_n\}$ is said to be convergent if there exists an element $\iota \in X$ such that, for every $c \in \mathbb{A}$ with $\theta_{\mathbb{A}} \ll c$, there exists a natural number n_0 satisfying

$$d(\iota_n, \iota) \ll c,$$

for all $n \geq n_0$.

(ii) A sequence $\{\iota_n\}$ is called Cauchy if, for every $c \in \mathbb{A}$ with $\theta_{\mathbb{A}} \ll c$, there is a natural number n_0 such that

$$d(\iota_n, \iota_m) \ll c,$$

holds true for all $n, m \geq n_0$.

(iii) The space $(X, \mathbb{A}, d)$ is said to be complete ECb-MS if every Cauchy sequence in X is convergent.

Definition 2.8. *[15] Let $(X, \mathbb{A}, d)$ be an ECb-MS over a BA $\mathbb{A}$ and $\Im : X \to X$. Then $\Im$ is termed orbitally continuous at a point $\iota \in X$ if for any sequence $\{\iota_n\}$ within the orbit of ι converging to a point ι^*, that is $\iota_n \to \iota^*$ implies the sequence $\Im\iota_n$ converges to $\Im\iota^*$ as $n \to \infty$.*

Lemma 2.4. *[15] Let $\mathbb{A}$ be a real BA and $\varpi\,(\neq \theta_{\mathbb{A}}) \in \mathbb{A}$. If $\lim_{n\to\infty} \frac{\|\varpi^{n+1}\|}{\|\varpi^n\|}$ exists then this limit is equal to $\rho(\varpi)$. We represent a subset of $\mathbb{A}$ in this way*

$$\mathbb{P}^* = \left\{\varpi\,(\neq \theta_{\mathbb{A}}) \in \mathbb{A} : \lim_{n\to\infty} \frac{\|\varpi^{n+1}\|}{\|\varpi^n\|} \text{ exists}\right\}.$$

Theorem 2.1. *[15] Let $(X, \mathbb{A}, d)$ be a complete ECb-MS over a BA $\mathbb{A}$. Let $\mathbb{P}$ be the underlying SC in $\mathbb{A}$ and $\mathfrak{J} : X \to X$ be a mapping such that*

$$d(\mathfrak{J}\mathfrak{l}, \mathfrak{J}\varsigma) \preceq \varpi d(\mathfrak{l}, \varsigma),$$

for all $\mathfrak{l}, \varsigma \in X$, where $\varpi \in \mathbb{P}^*$ with $0 < \rho(\varpi) < 1$. Suppose that $\lim_{n,m} \varphi(\mathfrak{l}_n, \mathfrak{l}_m) < \frac{1}{\rho(\varpi)}$, for the Picard iterating sequence $\mathfrak{l}_n = \mathfrak{J}^n\mathfrak{l}_0$ generated by $\mathfrak{l}_0 \in X$. Then $\mathfrak{J}$ has at least one FP in X provided that $\mathfrak{J}$ is orbitally continuous in X.

Lemma 2.5. *[15] Let (X, d) be an ECb-MS over a BA $\mathbb{A}$ with a SC $\mathbb{P}$. The following properties hold:*

(P_1) If $\ell \ll \hbar$ and $\hbar \preceq \varphi$, then $\ell \ll \varphi$.
(P_2) If $\ell \preceq \hbar$ and $\hbar \ll \varphi$, then $\ell \ll \varphi$.
(P_3) If $\ell \ll \hbar$ and $\hbar \ll \varphi$, then $\ell \ll \varphi$.
(P_4) If $\theta_{\mathbb{A}} \preceq \ell \ll \hbar$ for each $\hbar \in Int\mathbb{P}$, then $\ell = \theta_{\mathbb{A}}$.
(P_5) If $\ell \preceq \hbar + \varphi$ for each $\varphi \in Int\mathbb{P}$, then $\ell \preceq \hbar$.
(P_6) If $\ell \preceq \lambda\ell$ where $0 \leq \lambda < 1$ and $\ell \in \mathbb{P}$, then $\ell = \theta_{\mathbb{A}}$.
(P_7) If $\ell_n \in \mathbb{A}$ and $\ell_n \to \theta_{\mathbb{A}}$ as $n \to \infty$ and $\hbar \in int\mathbb{P}$, then there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$, we have $\ell_n \ll \hbar$.

In this study, we explore EC*b*-MSs over BAs and obtain novel FP theorems for both single-valued mappings and multivalued mappings. Additionally, we validate the real-world relevance of our prime result by employing it to address solutions for a Fredholm integral equation of the second kind.

3. Main results

In this section, we define the concept of Banach and Kannan-type almost contractions in the setting of EC*b*-MSs over BAs and establish some FP results for single-valued mappings. These results generalize and extend some previous FP theorems in EC*b*-MSs, C*b*-MSs, and CMSs over BAs. Our theorems provide a robust theoretical foundation for various applications, including differential and integral equations.

Definition 3.1. *Let $(X, \mathbb{A}, d)$ be an ECb-MS over a BA $\mathbb{A}$ and let $\mathbb{P}$ be an underlying SC in $\mathbb{A}$. A single-valued mapping $\mathfrak{J} : X \to X$ is said to be Banach-type almost contraction if there exist $\varpi \in \mathbb{P}^*$ with $0 < \rho(\varpi) < 1$ and $L \succeq \theta_{\mathbb{A}}$ such that*

$$d(\mathfrak{J}\mathfrak{l}, \mathfrak{J}\varsigma) \preceq \varpi d(\mathfrak{l}, \varsigma) + Ld(\varsigma, \mathfrak{J}\mathfrak{l}), \tag{3.1}$$

for all $\mathfrak{l}, \varsigma \in X$.

Theorem 3.1. *Let $(X, \mathbb{A}, d)$ be a complete ECb-MS over a BA $\mathbb{A}$ and $\mathbb{P}$ be an underlying SC in $\mathbb{A}$. Suppose that $\mathfrak{J} : X \to X$ be a Banach-type almost contraction and $\lim_{n,m\to\infty} \varphi(\mathfrak{l}_n, \mathfrak{l}_m) < \frac{1}{\rho(\varpi)}$, for the Picard iterating sequence $\mathfrak{l}_n = \mathfrak{J}^n\mathfrak{l}_0$ generated by $\mathfrak{l}_0 \in X$. Then $\mathfrak{J}$ has a FP in X provided that $\mathfrak{J}$ is orbitally continuous in X. Furthermore, if $\rho(\varpi + L) < 1$, then the FP is unique.*

Proof. Select an arbitrary point $\mathfrak{l}_0$ within the set X. Define a sequence $\{\mathfrak{l}_n\}$ by $\mathfrak{l}_n = \mathfrak{J}^n\mathfrak{l}_0 = \mathfrak{J}\mathfrak{l}_{n-1}$, for all $n \in \mathbb{N}$. By (3.1), we have

$$d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) = d(\mathfrak{J}\mathfrak{l}_{n-1}, \mathfrak{J}\mathfrak{l}_n) \preceq \varpi d(\mathfrak{l}_{n-1}, \mathfrak{l}_n) + Ld(\mathfrak{l}_n, \mathfrak{J}\mathfrak{l}_{n-1})$$

$$
\begin{aligned}
&= \varpi d(\mathfrak{l}_{n-1}, \mathfrak{l}_n) + L\theta_{\mathbb{A}} \\
&= \varpi d(\mathfrak{l}_{n-1}, \mathfrak{l}_n) + \theta_{\mathbb{A}} \\
&= \varpi d(\mathfrak{l}_{n-1}, \mathfrak{l}_n),
\end{aligned}
$$

hence,

$$d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) \preceq \varpi d(\mathfrak{l}_{n-1}, \mathfrak{l}_n). \tag{3.2}$$

Similarly,

$$
\begin{aligned}
d(\mathfrak{l}_{n-1}, \mathfrak{l}_n) &= d(\mathfrak{J}\mathfrak{l}_{n-2}, \mathfrak{J}\mathfrak{l}_{n-1}) \preceq \varpi d(\mathfrak{l}_{n-2}, \mathfrak{l}_{n-1}) + Ld(\mathfrak{l}_{n-1}, \mathfrak{J}\mathfrak{l}_{n-2}) \\
&= \varpi d(\mathfrak{l}_{n-2}, \mathfrak{l}_{n-1}) + L\theta_{\mathbb{A}} \\
&= \varpi d(\mathfrak{l}_{n-2}, \mathfrak{l}_{n-1}) + \theta_{\mathbb{A}} \\
&= \varpi d(\mathfrak{l}_{n-2}, \mathfrak{l}_{n-1}),
\end{aligned}
$$

therefore,

$$d(\mathfrak{l}_{n-1}, \mathfrak{l}_n) \preceq \varpi d(\mathfrak{l}_{n-2}, \mathfrak{l}_{n-1}). \tag{3.3}$$

By (3.2) and (3.3), we have

$$d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) \preceq \varpi d(\mathfrak{l}_{n-1}, \mathfrak{l}_n) \preceq \varpi^2 d(\mathfrak{l}_{n-2}, \mathfrak{l}_{n-1}).$$

Continuing in this way, we have

$$d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) \preceq \varpi d(\mathfrak{l}_{n-1}, \mathfrak{l}_n) \preceq \varpi^2 d(\mathfrak{l}_{n-2}, \mathfrak{l}_{n-1}) \preceq ... \preceq \varpi^n d(\mathfrak{l}_0, \mathfrak{l}_1),$$

for all $n \in \mathbb{N}$. Now, for all $n \in \mathbb{N}$ and for any $m = 1, 2, ...$ we have

$$
\begin{aligned}
d(\mathfrak{l}_n, \mathfrak{l}_{n+m}) &\preceq \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \left[d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) + d(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m})\right] \\
&= \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) + \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) d(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m}) \\
&\preceq \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) \\
&\quad + \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m}) \left[d(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+2}) + d(\mathfrak{l}_{n+2}, \mathfrak{l}_{n+m})\right] \\
&= \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) \\
&\quad + \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m}) d(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+2}) \\
&\quad + \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m}) d(\mathfrak{l}_{n+2}, \mathfrak{l}_{n+m}) \\
&\preceq ... \preceq \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) d(\mathfrak{l}_n, \mathfrak{l}_{n+1}) \\
&\quad + \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m}) d(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+2}) \\
&\quad + \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m}) \cdots \varphi(\mathfrak{l}_{n+m-2}, \mathfrak{l}_{n+m}) \\
&\quad \left[d(\mathfrak{l}_{n+m-2}, \mathfrak{l}_{n+m-1}) + d(\mathfrak{l}_{n+m-1}, \mathfrak{l}_{n+m})\right] \\
&\preceq \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varpi^n d(\mathfrak{l}_0, \mathfrak{l}_1) + \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m}) \varpi^{n+1} d(\mathfrak{l}_0, \mathfrak{l}_1) \\
&\quad + \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_{n+1}, \mathfrak{l}_{n+m}) \cdots \varphi(\mathfrak{l}_{n+m-1}, \mathfrak{l}_{n+m}) \varpi^{n+m-1} d(\mathfrak{l}_0, \mathfrak{l}_1),
\end{aligned}
$$

which implies

$$d(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \preceq \varphi(\mathfrak{l}_1, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_2, \mathfrak{l}_{n+m}) \cdots \varphi(\mathfrak{l}_{n-1}, \mathfrak{l}_{n+m}) \varphi(\mathfrak{l}_n, \mathfrak{l}_{n+m}) \varpi^n d(\mathfrak{l}_0, \mathfrak{l}_1)$$

$$
\begin{aligned}
&+\varphi(\mathfrak{l}_1,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_2,\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\varpi^{n+1}d(\mathfrak{l}_0,\mathfrak{l}_1)\\
&+...+\varphi(\mathfrak{l}_1,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_2,\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\\
&\cdots\varphi(\mathfrak{l}_{n+m-1},\mathfrak{l}_{n+m})\varpi^{n+m-1}d(\mathfrak{l}_0,\mathfrak{l}_1)\\
=\;&\left[\sum_{i=n}^{n+m-1}\varpi^i\prod_{j=1}^{i}\varphi(\mathfrak{l}_j,\mathfrak{l}_{n+m})\right]d(\mathfrak{l}_0,\mathfrak{l}_1). \qquad (3.4)
\end{aligned}
$$

Now for $i\in\mathbb{N}$ and each fixed $m\in\mathbb{N}$, define

$$a_i^{(n+m)}=\prod_{j=1}^{i}\varphi(\mathfrak{l}_j,\mathfrak{l}_{n+m})\left\|\varpi^i\right\|.$$

Then

$$\frac{a_{n+1}^{(n+m)}}{a_n^{(n+m)}}=\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\frac{\left\|\varpi^{n+1}\right\|}{\left\|\varpi^n\right\|}.$$

Since $\varpi\in\mathbb{P}^*$, it follows that

$$\lim_{n\to\infty}\frac{\left\|\varpi^{n+1}\right\|}{\left\|\varpi^n\right\|}=\rho(\varpi).$$

By hypothesis $\lim_{n,m\to\infty}\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})<\frac{1}{\rho(\varpi)}$, hence for the fixed m above

$$\lim_{n\to\infty}\frac{a_{n+1}^{(n+m)}}{a_n^{(n+m)}}=\lim_{n\to\infty}\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\rho(\varpi)<1.$$

Hence, by the ratio test, the series

$$\sum_{i=n}^{n+m-1}\varpi^i\prod_{j=1}^{i}\varphi(\mathfrak{l}_j,\mathfrak{l}_{n+m}),$$

is convergent in the BA $\mathbb{A}$ for every fixed m.

Consequently,

$$\lim_{n\to\infty}\left\|\sum_{i=n}^{n+m-1}\varpi^i\prod_{j=1}^{i}\varphi(\mathfrak{l}_j,\mathfrak{l}_{n+m})\right\|\le\lim_{n\to\infty}\sum_{i=n}^{n+m-1}\prod_{j=1}^{i}\varphi(\mathfrak{l}_j,\mathfrak{l}_{n+m})\left\|\varpi^i\right\|=0, \qquad (3.5)$$

for each fixed m. It follows from the inequality (3.4) that

$$\left[\sum_{i=n}^{n+m-1}\varpi^i\prod_{j=1}^{i}\varphi(\mathfrak{l}_j,\mathfrak{l}_{n+m})\right]d(\mathfrak{l}_0,\mathfrak{l}_1)\to\theta_{\mathbb{A}},$$

as $n,m\to\infty$. Now for a given $c\gg\theta_{\mathbb{A}}$, then by (P_1) and (P_7), one can find some m_0 such that

$$\left[\sum_{i=n}^{n+m-1}\varpi^i\prod_{j=1}^{i}\varphi(\mathfrak{l}_j,\mathfrak{l}_{n+m})\right]d(\mathfrak{l}_0,\mathfrak{l}_1)\ll c,$$

for all $n, m > m_0$. Thus

$$d(\iota_n, \iota_{n+m}) \preceq \left[\sum_{i=n}^{n+m-1} \varpi^i \prod_{j=1}^{i} \varphi(\iota_j, \iota_{n+m})\right] d(\iota_0, \iota_1) \ll c,$$

for all $n, m > m_0$. It means that the sequence $\{\iota_n\}$ is Cauchy in X. By the completeness of X, this sequence $\{\iota_n\}$ converges to some point $\iota^* \in X$ such that $\iota_n \to \iota^*$ as $n \to \infty$. Now since the self mapping $\mathfrak{J}$ is orbitally continuous in X, so

$$\iota^* = \lim_{n\to\infty} \iota_{n+1} = \lim_{n\to\infty} \mathfrak{J}\iota_n = \mathfrak{J}\iota^*.$$

Hence ι^* is a FP of $\mathfrak{J}$. Assume that there is another FP $\iota^{/}$ of mapping $\mathfrak{J}$, that is, $\mathfrak{J}\iota^{/} = \iota^{/}$, then by (3.1), we have

$$\begin{aligned} d(\iota^*, \iota^{/}) &= d(\mathfrak{J}\iota^*, \mathfrak{J}\iota^{/}) \preceq \varpi d(\iota^*, \iota^{/}) + Ld(\iota^{/}, \mathfrak{J}\iota^*) \\ &= \varpi d(\iota^*, \iota^{/}) + Ld(\iota^{/}, \iota^*) \\ &= (\varpi + L)\, d(\iota^*, \iota^{/}). \end{aligned} \tag{3.6}$$

Let $\varrho = \varpi + L$, then the inequality (3.6) becomes

$$d(\iota^*, \iota^{/}) \preceq \varrho d(\iota^*, \iota^{/}).$$

Iterating above inequality gives

$$d(\iota^*, \iota^{/}) \preceq \varrho^n d(\iota^*, \iota^{/}),$$

for all $\mathbb{N}$. By assumption $\rho(\varrho) < 1$, then by the Neumann series argument in a Banach algebra,

$$\lim_{n\to\infty} \varrho^n = \theta_{\mathbb{A}},$$

and hence

$$d(\iota^*, \iota^{/}) \preceq \lim_{n\to\infty} \varrho^n d(\iota^*, \iota^{/}) = \theta_{\mathbb{A}}.$$

Therefore,

$$d(\iota^*, \iota^{/}) = \theta_{\mathbb{A}}$$

implies $\iota^* = \iota^{/}$. Thus the FP is unique. □

Example 3.1. *Let $X = [0, 1]$ and $\mathbb{A}$ be the space of continuously differentiable real-valued functions on $[0, 1]$, that is, $\mathbb{A} = C^1_{\mathbb{R}}[0, 1]$. The norm on $\mathbb{A}$ is defined as follows*

$$\|u\| = \|u\|_\infty + \left\|u^{/}\right\|_\infty .$$

Let $\mathbb{A}$ be equipped with pointwise multiplication. This endows $\mathbb{A}$ with the structure of a real BA possessing unity element $e_{\mathbb{A}}$, where $e_{\mathbb{A}}(t) = 1$, for all $t \in X$. Define the subset $\mathbb{P}$ of $\mathbb{A}$ as follows:

$$\mathbb{P} = \{u \in \mathbb{A} : \; u(t) \geq 0, \text{for all } t \in X\} .$$

It is established that $\mathbb{P}$ forms a SC that does not exhibit normality (see [23] for further elaboration). Define a function $d : X \times X \to \mathbb{A}$ by

$$d(\imath, \varsigma) = |\imath - \varsigma|^2 e^t.$$

Then $(X, \mathbb{A}, d)$ is a complete ECb-MS with the function $\varphi : X \times X \to [1, +\infty)$ defined by $\varphi(\imath, \varsigma) = 2 + |\imath| + |\varsigma|$. Define a self mapping $\mathfrak{J} : X \to X$ by $\mathfrak{J}\imath = \frac{1}{\sqrt{3}}\imath$. Now we take $\varpi = \frac{1}{3} + \frac{1}{9}t$ for $t \in [0, 1]$, then the spectrum of ϖ is $\sigma(\varpi) = [\frac{1}{3}, \frac{4}{9}]$, that is,

$$\rho(\varpi) = \frac{4}{9} < 1.$$

Consider $L = \frac{2}{\sqrt{3}}$. Then by simple calculations, we have

$$\begin{aligned} d(\mathfrak{J}\imath, \mathfrak{J}\varsigma) &= \frac{1}{3} |\imath - \varsigma|^2 e^t, \\ \varpi d(\imath, \varsigma) &= \left(\frac{1}{3} + \frac{1}{9}t\right) |\imath - \varsigma|^2 e^t, \\ d(\varsigma, \mathfrak{J}\imath) &= \left|\varsigma - \frac{1}{\sqrt{3}}\imath\right|^2 e^t \\ &= \left(|\varsigma|^2 - \frac{2}{\sqrt{3}}\imath\varsigma + \frac{1}{3}|\imath|^2\right) e^t. \end{aligned}$$

Thus all the hypotheses of the Theorem 3.1 are satisfied and 0 is the unique FP of mapping $\mathfrak{J}$.

We now proceed to derive a result that parallels one of the findings presented by Ullah et al. [26], utilizing the framework of Theorem 3.1.

Corollary 3.1. *[26] Let $(X, \mathbb{A}, d)$ be a complete ECb-MS over a BA $\mathbb{A}$. Let $\mathbb{P}$ be the underlying SC $\mathbb{P}$ in $\mathbb{A}$ and $\mathfrak{J} : X \to X$ be a mapping such that*

$$d(\mathfrak{J}\imath, \mathfrak{J}\varsigma) \preceq \varpi d(\imath, \varsigma)$$

for all $\imath, \varsigma \in X$, where $\varpi \in \mathbb{P}^*$ with $0 < \rho(\varpi) < 1$. Suppose that $\lim_{n,m} \varphi(\imath_n, \imath_m) < \frac{1}{\rho(\varpi)}$ for the Picard iterating sequence $\imath_n = \mathfrak{J}^n \imath_0$ generated by $\imath_0 \in X$. Then $\mathfrak{J}$ has a unique FP in X provided that $\mathfrak{J}$ is orbitally continuous in X.

Proof. Take $L = \theta_{\mathbb{A}}$ (the additive identity in the BA $\mathbb{A}$) in Theorem 3.1. □

Remark 3.1. *Taking $\varphi(\imath, z) = b \geq 1$, for all $\imath, z \in X$ in Definition 2.6, then the notion of ECb-MS over a BAs reduces to a Cb-MS over a BA. Building upon the preceding Corollary 3.1, we proceed to derive a result that aligns with the primary Theorem established by Huang et al. [12].*

Remark 3.2. *Taking $\varphi(\imath, z) = 1$, for all $\imath, z \in X$ and $\mathbb{A} = \mathbb{R}$ with the cone $\mathbb{P} = [0, +\infty)$ in Definition 2.6, then the notion of ECb-MS over a BAs reduces to a CMS, then Corollary 3.1 reduced to the prime result of Huang et al. [6].*

To further generalize classical FP theorems, we introduce a new class of almost contractions in the setting of EC*b*-MS over a BA $\mathbb{A}$. Specifically, we define a Kannan-type almost contraction that extends existing results in this framework.

Definition 3.2. *Let $(X, \mathbb{A}, d)$ be an ECb-MS over a BA $\mathbb{A}$ and let $\mathbb{P}$ be an underlying SC in $\mathbb{A}$. A single-valued mapping $\mathfrak{I} : X \to X$ is said to be Kannan-type almost contraction if there exist $\varpi \in \mathbb{P}^*$ with $\mu = (e_{\mathbb{A}} - \varpi)^{-1} \varpi$ and $0 < \rho(\mu) < \frac{1}{2}$ and $L \succeq \theta_{\mathbb{A}}$ such that*

$$d(\mathfrak{I}\iota, \mathfrak{I}\varsigma) \preceq \varpi \left(d(\iota, \mathfrak{I}\iota) + d(\varsigma, \mathfrak{I}\varsigma)\right) + Ld(\varsigma, \mathfrak{I}\iota), \tag{3.7}$$

for all $\iota, \varsigma \in X$.

Theorem 3.2. *Let $(X, \mathbb{A}, d)$ be a complete ECb-MS over a BA $\mathbb{A}$ and $\mathbb{P}$ be an underlying SC in $\mathbb{A}$. Suppose that $\mathfrak{I} : X \to X$ be a Kannan type almost contraction and $\lim_{n,m\to\infty} \varphi(\iota_n, \iota_m) < \frac{1}{\rho(\mu)}$, for the Picard iterating sequence $\iota_n = \mathfrak{I}^n \iota_0$ generated by $\iota_0 \in X$. Then $\mathfrak{I}$ has a FP in X provided that $\mathfrak{I}$ is orbitally continuous in X. Moreover, if $\rho(L) < 1$, then the FP is unique.*

Proof. Consider an arbitrary point ι_0 belonging to X. Define a sequence $\{\iota_n\}$ by $\iota_n = \mathfrak{I}^n \iota_0 = \mathfrak{I}\iota_{n-1}$, for all $n \in \mathbb{N}$. By (3.7), we have

$$\begin{aligned} d(\iota_n, \iota_{n+1}) &= d(\mathfrak{I}\iota_{n-1}, \mathfrak{I}\iota_n) \preceq \varpi \left(d(\iota_{n-1}, \mathfrak{I}\iota_{n-1}) + d(\iota_n, \mathfrak{I}\iota_n)\right) + Ld(\iota_n, \mathfrak{I}\iota_{n-1}) \\ &= \varpi \left(d(\iota_{n-1}, \iota_n) + d(\iota_n, \iota_{n+1})\right) + L\theta_{\mathbb{A}} \\ &= \varpi d(\iota_{n-1}, \iota_n) + \varpi d(\iota_n, \iota_{n+1}) + \theta_{\mathbb{A}} \\ &= \varpi d(\iota_{n-1}, \iota_n) + \varpi d(\iota_n, \iota_{n+1}), \end{aligned}$$

which implies hence

$$\begin{aligned} d(\iota_n, \iota_{n+1}) &\preceq (e_{\mathbb{A}} - \varpi)^{-1} \varpi d(\iota_{n-1}, \iota_n) \\ &= \mu d(\iota_{n-1}, \iota_n). \end{aligned} \tag{3.8}$$

Similarly,

$$\begin{aligned} d(\iota_{n-1}, \iota_n) &= d(\mathfrak{I}\iota_{n-2}, \mathfrak{I}\iota_{n-1}) \preceq \varpi \left(d(\iota_{n-2}, \mathfrak{I}\iota_{n-2}) + d(\iota_{n-1}, \mathfrak{I}\iota_{n-1})\right) + Ld(\iota_{n-1}, \mathfrak{I}\iota_{n-2}) \\ &= \varpi \left(d(\iota_{n-2}, \iota_{n-1}) + d(\iota_{n-1}, \iota_n)\right) + L\theta_{\mathbb{A}} \\ &= \varpi d(\iota_{n-2}, \iota_{n-1}) + \varpi d(\iota_{n-1}, \iota_n) + \theta_{\mathbb{A}} \\ &= \varpi d(\iota_{n-2}, \iota_{n-1}) + \varpi d(\iota_{n-1}, \iota_n), \end{aligned}$$

which implies

$$\begin{aligned} d(\iota_{n-1}, \iota_n) &\preceq (e_{\mathbb{A}} - \varpi)^{-1} \varpi d(\iota_{n-2}, \iota_{n-1}) \\ &= \mu d(\iota_{n-2}, \iota_{n-1}). \end{aligned} \tag{3.9}$$

By (3.8) and (3.9), we have

$$d(\iota_n, \iota_{n+1}) \preceq \mu d(\iota_{n-1}, \iota_n) \preceq \mu^2 d(\iota_{n-2}, \iota_{n-1}).$$

Continuing in this way, we have

$$d(\iota_n, \iota_{n+1}) \preceq \mu d(\iota_{n-1}, \iota_n) \preceq \mu^2 d(\iota_{n-2}, \iota_{n-1}) \preceq \ldots \preceq \mu^n d(\iota_0, \iota_1),$$

for all $n \in \mathbb{N}$. □

Continuing the same procedure as we did in Theorem 3.1, we obtain that $\mathfrak{J}$ has a FP. Assume that there is another FP $\imath^{/}$ of mapping $\mathfrak{J}$, that is, $\mathfrak{J}\imath^{/} = \imath^{/}$. Then by (3.7), we have

$$\begin{aligned}
d(\imath^*, \imath^{/}) &= d(\mathfrak{J}\imath^*, \mathfrak{J}\imath^{/}) \preceq \varpi\left(d(\imath^*, \mathfrak{J}\imath^*) + d(\imath^{/}, \mathfrak{J}\imath^{/})\right) + Ld(\imath^{/}, \mathfrak{J}\imath^*) \\
&= \varpi(\theta_{\mathbb{A}} + \theta_{\mathbb{A}}) + Ld(\imath^{/}, \imath^*) \\
&= Ld(\imath^{/}, \imath^*) \\
&= Ld(\mathfrak{J}\imath^*, \mathfrak{J}\imath^{/}) \\
&\preceq L\left(\varpi\left(d(\imath^*, \mathfrak{J}\imath^*) + d(\imath^{/}, \mathfrak{J}\imath^{/})\right) + Ld(\imath^{/}, \mathfrak{J}\imath^*)\right) \\
&= L\left(\varpi(\theta_{\mathbb{A}} + \theta_{\mathbb{A}}) + Ld(\imath^{/}, \mathfrak{J}\imath^*)\right) \\
&= L^2 d(\imath^{/}, \imath^*) \\
&\preceq \cdots \preceq L^n d(\imath^{/}, \imath^*).
\end{aligned}$$

If $\rho(L) < 1$, then by the Neumann series argument in a Banach algebra,

$$\lim_{n\to\infty} L^n = \theta_{\mathbb{A}},$$

and hence

$$d(\imath^*, \imath^{/}) \preceq \lim_{n\to\infty} L^n d(\imath^*, \imath^{/}) = \theta_{\mathbb{A}}.$$

Therefore,

$$d(\imath^*, \imath^{/}) = \theta_{\mathbb{A}},$$

implies $\imath^* = \imath^{/}$. Thus, the FP is unique.

We now derive a result, which is one of the findings presented by Ullah et al. [26] from Theorem 3.2.

Corollary 3.2. *Let $(X, \mathbb{A}, d)$ be a complete ECb-MS over a BA $\mathbb{A}$. Let $\mathbb{P}$ be the underlying SC in $\mathbb{A}$ and $\mathfrak{J} : X \to X$ be a mapping such that*

$$d(\mathfrak{J}\imath, \mathfrak{J}\varsigma) \preceq \varpi\left(d(\imath, \mathfrak{J}\imath) + d(\varsigma, \mathfrak{J}\varsigma)\right),$$

for all $\imath, \varsigma \in X$, where $\varpi \in \mathbb{P}^*$ with $\mu = (e_{\mathbb{A}} - \varpi)^{-1}\varpi$ and $0 < \rho(\mu) < \frac{1}{2}$. Suppose that

$$\lim_{n,m} \varphi(\imath_n, \imath_m) < \frac{1}{\rho(\mu)},$$

for the Picard iterating sequence $\imath_n = \mathfrak{J}^n \imath_0$ generated by $\imath_0 \in X$. Then $\mathfrak{J}$ has a unique FP in X provided that $\mathfrak{J}$ is orbitally continuous in X.

Remark 3.3. *Taking $\varphi(\imath, z) = b \geq 1$, for all $\imath, z \in X$ in Definition 2.6, then the notion of ECb-MS over a BAs reduces to a Cb-MS over a BA, then we can deduce the main Theorem presented by Huang et al. [12] as a specific case of Corollary 3.2.*

4. Fixed point results for multivalued mappings

In this section, we extend our analysis to the existence of FPs for multivalued mappings in the framework of ECb-MS over a BAs. By introducing appropriate contractive conditions, we establish FP theorems that generalize classical results to a broader setting. These results provide a unified approach to studying nonlinear problems involving set-valued operators, which frequently arise in optimization, control theory, and differential inclusions.

In 2011, Cho [20] defined the generalized Hausdorff distance function in the context of CMSs and proved FP theorems for multivalued mappings. Subsequently, Azam et al. [21] further advanced the theory by proving FP theorems in the setting of Cb-MSs, extending the earlier results to cases where the normality of the cone is not a necessary condition. Kutbi et al. [22], drawing on this prior work, broadened the theoretical framework by demonstrating multivalued FP theorems applicable to Cb-MSs over BAs.

Consistent with the approach of Kutbi et al. [22], we define some primary concepts in the framework of ECb-MS over a BAs to obtain FP theorems for multivalued mappings.

Let $(X, \mathbb{A}, d)$ be an ECb-MS over a BA $\mathbb{A}$, $\mathbb{P}$ be the underlying SC. We represent non-empty subsets of X by 2^X, the closed subsets of X by $C(X)$ and closed and bounded subsets of X by $CB(X)$. Now we define a function $s : \mathbb{A} \to \mathbb{A}$ by

$$s(\imath) = \{\varsigma \in \mathbb{A} : \imath \preceq \varsigma\} \text{ for } \varsigma \in \mathbb{A},$$
$$s(\imath, B) = \bigcup_{\varsigma \in B} s(d(\imath, \varsigma)) = \bigcup_{\varsigma \in B} \{z \in \mathbb{A} :\ d(\imath, \varsigma) \preceq z\},$$

for $\imath \in X$ and $B \in 2^X$. For $A, B \in CB(X)$, we define the generalized Hausdorff distance as follows

$$s(A, B) = \left(\bigcap_{\imath \in A} s(\imath, B) \right) \cap \left(\bigcap_{\varsigma \in B} s(\varsigma, A) \right).$$

Since the multivalued mappings often involve the use of closed and bounded subsets, it is crucial to consider the topological framework induced by the ECb-metric d. The concepts of closedness and boundedness in ECb-MSs may differ from classical MSs due to the underlying BA structure and the associated cone. Therefore, when defining and analyzing multivalued mappings, we carefully ensure that the closed subsets $C(X)$ and closed and bounded subsets $CB(X)$ satisfy the necessary topological properties to support our FP results.

Lemma 4.1. *Let $(X, \mathbb{A}, d)$ be an ECb-MS over a BA $\mathbb{A}$, $\mathbb{P}$ be the underlying SC.*

(i) Let $\imath, \varsigma \in \mathbb{A}$. If $\imath \preceq \varsigma$, $s(\varsigma) \subset s(\imath)$.

(ii) Let $\imath \in X$ and $A \in 2^X$. If $\theta \in s(\imath, A)$, then $\imath \in A$.

(iii) Let $z \in \mathbb{P}$ and let $A, B \in CB(X)$ and $\imath \in A$. If $z \in s(A, B)$, then $z \in s(\imath, B)$ for all $\imath \in A$ or $z \in s(A, \varsigma)$ for all $\varsigma \in B$.

(iv) Let $\varsigma \in \mathbb{P}$ and let $\lambda \geq 0$, then $\lambda s(\varsigma) \subseteq s(\lambda \varsigma)$.

Remark 4.1. *Let $(X, \mathbb{A}, d)$ be an ECb-MS over a BA $\mathbb{A}$, $\mathbb{P}$ be the underlying SC. If $\mathbb{A} = \mathbb{R}$ and $\mathbb{P} = [0, +\infty)$, then $(X, \mathbb{A}, d)$ is an Eb-MS. Moreover, for $A, B \in CB(X)$,*

$$H(A, B) = \inf s(A, B)$$

is the Hausdorff distance derived from by d.

The concept of α-admissible mappings was defined by Samet et al. in 2012 within their work [18].

Definition 4.1. *[18] Let $\Im : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. Then, $\Im$ is termed an α-admissible mapping when*

$$\iota, \varsigma \in X, \quad \alpha(\iota, \varsigma) \geq 1 \quad \Longrightarrow \quad \alpha(\Im\iota, \Im\varsigma) \geq 1.$$

Abbas et al. [32] gave the concept of α-closed mappings in this way.

Definition 4.2. *[32] Let $X \neq \emptyset$, $\alpha : X \times X \to [0, +\infty)$ and $\Im : X \to 2^X$. Consequently, $\Im$ is deemed an α-closed if*

$$\iota, \varsigma \in X, \quad \alpha(\iota, \varsigma) \geq 1 \quad \Longrightarrow \quad \alpha(u, v) \geq 1 \text{ for any } u \in \Im\iota \text{ and } v \in \Im\varsigma.$$

Definition 4.3. *[32] Let $X \neq \emptyset$, $\alpha : X \times X \to [0, +\infty)$ and $\mathcal{R}, \Im : X \to 2^X$. Then the pair $(\mathcal{R}, \Im)$ is said to be α-closed if*

$$\iota, \varsigma \in X, \quad \alpha(\iota, \varsigma) \geq 1 \quad \Longrightarrow \quad \alpha(u, v) \geq 1 \text{ for any } u \in \mathcal{R}\iota \text{ and } v \in \Im\varsigma.$$

In 2018, Kutbi et al. [22] established the following result in Cb-MS over a BA in this way.

Theorem 4.1. *[22] Let $(X, \mathbb{A}, d)$ be a complete Cb-MS over a BA $\mathbb{A}$, $\mathbb{P}$ be the underlying SC and $\Im : X \to CB(X)$. The following conditions are assumed to be true:*

(i) there exists $\alpha : X \times X \to [0, +\infty)$ such that

$$\varpi d(\iota, \varsigma) \in \alpha(\iota, \varsigma) s(\Im\iota, \Im\varsigma),$$

for all $\iota, \varsigma \in X$, where $\varpi \in \mathbb{P}^*$ and $\rho(\varpi) < 1$,

(ii) $\Im$ is α-closed,

(iii) there exist $\iota_0 \in X$, $\iota_1 \in \Im\iota_0$ such that $\alpha(\iota_0, \iota_1) \geq 1$,

(iv) if $\{\iota_n\}$ is a sequence in X such that $\alpha(\iota_n, \iota_{n+1}) \geq 1$ for all n and $\iota_n \to \iota^*$ as $n \to \infty$, then $\alpha(\iota_n, \iota^*) \geq 1$ for all $n \in \mathbb{N}$.

Then there exists a point $\iota^* \in X$ such that $\iota^* \in \Im\iota^*$.

Kutbi et al. [24] proved the following result in CMS through this approach.

Theorem 4.2. *[24] Let (X, d) be a complete CMS, $\mathbb{P}$ be the underlying SC and $\Im : X \to CB(X)$. The following conditions are assumed to be true:*

(i) there exists $\alpha : X \times X \to [0, +\infty)$ such that

$$\varpi d(\iota, \varsigma) \in \alpha(\iota, \varsigma) s(\Im\iota, \Im\varsigma),$$

for all $\iota, \varsigma \in X$, where $0 < \varpi < 1$,

(ii) $\Im$ is α-closed,

(iii) there exist $\iota_0 \in X$, $\iota_1 \in \Im\iota_0$ such that $\alpha(\iota_0, \iota_1) \geq 1$,

(iv) if $\{\iota_n\}$ is a sequence in X such that $\alpha(\iota_n, \iota_{n+1}) \geq 1$, for all n and $\iota_n \to \iota^*$ as $n \to \infty$, then $\alpha(\iota_n, \iota^*) \geq 1$, for all $n \in \mathbb{N}$.

Then there exists a point $\iota^* \in X$ such that $\iota^* \in \Im\iota^*$.

We now present the main result of this section.

Theorem 4.3. *Let $(X, \mathbb{A}, d)$ be a complete ECb-MS over a BA $\mathbb{A}$, $\mathbb{P}$ be the underlying SC and $\mathcal{R}, \mathfrak{J} : X \to CB(X)$. The following conditions are assumed to be true:*

(i) there exists $\alpha : X \times X \to [0, +\infty)$ such that

$$\varpi d(\iota, \varsigma) \in \alpha(\iota, \varsigma) s(\mathcal{R}\iota, \mathfrak{J}\varsigma), \tag{4.1}$$

for all $\iota, \varsigma \in X$, where $\varpi \in \mathbb{P}^*$ and $\rho(\varpi) < 1$ with $\lim_{n,m\to\infty} \varphi(\iota_n, \iota_m) < \frac{1}{\rho(\varpi)}$ for a sequence $\{\iota_n\}$ in X;

(ii) $(\mathcal{R}, \mathfrak{J})$ and $(\mathfrak{J}, \mathcal{R})$ are α-closed;

(iii) there exist $\iota_0 \in X$, $\iota_1 \in \mathcal{R}\iota_0$ such that $\alpha(\iota_0, \iota_1) \geq 1$;

(iv) if $\{\iota_n\}$ is a sequence in X such that $\alpha(\iota_n, \iota_{n+1}) \geq 1$, for all n and $\iota_n \to \iota^*$ as $n \to \infty$, then $\alpha(\iota_{2n}, \iota^*) \geq 1$ and $\alpha(\iota_{2n+1}, \iota^*) \geq 1$, for all $n \in \mathbb{N}$.

Then there exists a point $\iota^* \in X$ such that $\iota^* \in \mathcal{R}\iota^* \cap \mathfrak{J}\iota^*$.

Proof. Let ι_0 be an arbitrary point in X , then $\mathcal{R}\iota_0 \neq \emptyset$ and $\mathcal{R}\iota_0 \in C(X)$. So there exists $\iota_1 \in X$ such that $\iota_1 \in \mathcal{R}\iota_0$. By (4.1), we have

$$\varpi d(\iota_0, \iota_1) \in \alpha(\iota_0, \iota_1) s(\mathcal{R}\iota_0, \mathfrak{J}\iota_1).$$

By the definition of "s", we have

$$\varpi d(\iota_0, \iota_1) \in \alpha(\iota_0, \iota_1) \left(\bigcap_{\iota \in \mathcal{R}\iota_0} s(\iota, \mathfrak{J}\iota_1) \right) \cap \left(\bigcap_{\varsigma \in \mathfrak{J}\iota_1} s(\varsigma, \mathcal{R}\iota_0) \right),$$

which implies

$$\varpi d(\iota_0, \iota_1) \in \alpha(\iota_0, \iota_1) \ (s(\iota, \mathfrak{J}\iota_1)),$$

for all $\iota \in \mathcal{R}\iota_0$. Since $\iota_1 \in \mathcal{R}\iota_0$, we have

$$\varpi d(\iota_0, \iota_1) \in \alpha(\iota_0, \iota_1) \ (s(\iota_1, \mathfrak{J}\iota_1)).$$

This implies that

$$\varpi d(\iota_0, \iota_1) \in \alpha(\iota_0, \iota_1) \ (s(\iota_1, \mathfrak{J}\iota_1)) = \alpha(\iota_0, \iota_1) s\left(\bigcup_{\iota \in \mathfrak{J}\iota_1} d(\iota_1, \iota) \right).$$

Since $\mathfrak{J}\iota_1 \neq \emptyset$, so there exists $\iota_2 \in X$ such that $\iota_2 \in \mathfrak{J}\iota_1$, we get

$$\varpi d(\iota_0, \iota_1) \in \alpha(\iota_0, \iota_1) s(d(\iota_1, \iota_2)).$$

By Lemma 4.1 (iv), we have

$$\varpi d(\iota_0, \iota_1) \in \alpha(\iota_0, \iota_1) s(d(\iota_1, \iota_2)) = s(\alpha(\iota_0, \iota_1) d(\iota_1, \iota_2)).$$

Therefore,

$$\alpha(\iota_0, \iota_1) d(\iota_1, \iota_2) \preceq \varpi d(\iota_0, \iota_1).$$

Since $\alpha(\iota_0, \iota_1) \geq 1$, we obtain

$$0 \prec d(\iota_1, \iota_2) \preceq \alpha(\iota_0, \iota_1) d(\iota_1, \iota_2) \preceq \varpi d(\iota_0, \iota_1). \tag{4.2}$$

If $\mathfrak{l}_1 = \mathfrak{l}_2$,then $\mathfrak{l}_1$ is the required FP. And we have nothing to prove. So we suppose that $\mathfrak{l}_1 \neq \mathfrak{l}_2$. As $\alpha(\mathfrak{l}_0, \mathfrak{l}_1) \geq 1$ and the pairs $(\mathcal{R}, \mathfrak{J})$ and $(\mathfrak{J}, \mathcal{R})$ are α-closed, so $\alpha(\mathfrak{l}_1, \mathfrak{l}_2) \geq 1$. Now from (4.1), we have

$$\varpi d(\mathfrak{l}_1, \mathfrak{l}_2) \in \alpha(\mathfrak{l}_1, \mathfrak{l}_2) s(\mathfrak{J}\mathfrak{l}_1, \mathcal{R}\mathfrak{l}_2).$$

This implies that

$$\varpi d(\mathfrak{l}_1, \mathfrak{l}_2) \in \alpha(\mathfrak{l}_1, \mathfrak{l}_2) \left(\left(\bigcap_{\mathfrak{l} \in \mathfrak{J}\mathfrak{l}_1} s(\mathfrak{l}, \mathcal{R}\mathfrak{l}_2) \right) \cap \left(\bigcap_{\varsigma \in S\mathfrak{l}_2} s(\varsigma, \mathfrak{J}\mathfrak{l}_1) \right) \right).$$

By definition of "s", we have

$$\varpi d(\mathfrak{l}_1, \mathfrak{l}_2) \in \alpha(\mathfrak{l}_1, \mathfrak{l}_2) \left(s(\mathfrak{l}, \mathcal{R}\mathfrak{l}_2) \right),$$

for all $\mathfrak{l} \in \mathfrak{J}\mathfrak{l}_1$. Since $\mathfrak{l}_2 \in \mathfrak{J}\mathfrak{l}_1$, we have

$$\varpi d(\mathfrak{l}_1, \mathfrak{l}_2) \in \alpha(\mathfrak{l}_1, \mathfrak{l}_2) \ \left(s(\mathfrak{l}_2, \mathcal{R}\mathfrak{l}_2) \right).$$

This implies that

$$\varpi d(\mathfrak{l}_1, \mathfrak{l}_2) \in \alpha(\mathfrak{l}_1, \mathfrak{l}_2) \ \left(s(\mathfrak{l}_2, \mathcal{R}\mathfrak{l}_2) \right) = \alpha(\mathfrak{l}_1, \mathfrak{l}_2) s \left(\bigcup_{\mathfrak{l} \in \mathcal{R}\mathfrak{l}_2} d(\mathfrak{l}_2, \mathfrak{l}) \right).$$

Since $\mathcal{R}\mathfrak{l}_2 \neq \emptyset$, so there exists $\mathfrak{l}_3 \in X$ such that $\mathfrak{l}_3 \in \mathcal{R}\mathfrak{l}_2$, we have

$$\varpi d(\mathfrak{l}_1, \mathfrak{l}_2) \in \alpha(\mathfrak{l}_1, \mathfrak{l}_2) \ s\left(d(\mathfrak{l}_2, \mathfrak{l}_3) \right).$$

By Lemma 4.1 (iv), we have

$$\varpi d(\mathfrak{l}_1, \mathfrak{l}_2) \in \alpha(\mathfrak{l}_1, \mathfrak{l}_2) \ s\left(d(\mathfrak{l}_2, \mathfrak{l}_3) \right) = s\left(\alpha(\mathfrak{l}_1, \mathfrak{l}_2) d(\mathfrak{l}_2, \mathfrak{l}_3) \right).$$

Therefore,

$$\alpha(\mathfrak{l}_1, \mathfrak{l}_2) d(\mathfrak{l}_2, \mathfrak{l}_3) \preceq \varpi d(\mathfrak{l}_1, \mathfrak{l}_2).$$

Since $\alpha(\mathfrak{l}_1, \mathfrak{l}_2) \geq 1$, we have

$$0 \prec d(\mathfrak{l}_2, \mathfrak{l}_3) \preceq \alpha(\mathfrak{l}_1, \mathfrak{l}_2) d(\mathfrak{l}_2, \mathfrak{l}_3) \preceq \varpi d(\mathfrak{l}_1, \mathfrak{l}_2). \tag{4.3}$$

If $\mathfrak{l}_2 = \mathfrak{l}_3$,then $\mathfrak{l}_2$ is the required FP. And we have nothing to prove. So we suppose that $\mathfrak{l}_2 \neq \mathfrak{l}_3$. As $\alpha(\mathfrak{l}_1, \mathfrak{l}_2) \geq 1$ and the pairs $(\mathcal{R}, \mathfrak{J})$ and $(\mathfrak{J}, \mathcal{R})$ are α-closed, so $\alpha(\mathfrak{l}_2, \mathfrak{l}_3) \geq 1$. Now from (4.1), we have

$$\varpi d(\mathfrak{l}_2, \mathfrak{l}_3) \in \alpha(\mathfrak{l}_2, \mathfrak{l}_3) s(\mathcal{R}\mathfrak{l}_2, \mathfrak{J}\mathfrak{l}_3).$$

This implies that

$$\varpi d(\mathfrak{l}_2, \mathfrak{l}_3) \in \alpha(\mathfrak{l}_2, \mathfrak{l}_3) \left(\left(\bigcap_{\mathfrak{l} \in \mathcal{R}\mathfrak{l}_2} s(\mathfrak{l}, \mathfrak{J}\mathfrak{l}_3) \right) \cap \left(\bigcap_{\varsigma \in \mathfrak{J}\mathfrak{l}_3} s(\varsigma, \mathcal{R}\mathfrak{l}_2) \right) \right).$$

By definition of "s", we have

$$\varpi d(\mathfrak{l}_2, \mathfrak{l}_3) \in \alpha(\mathfrak{l}_2, \mathfrak{l}_3) \ \left(s(\mathfrak{l}, \mathfrak{J}\mathfrak{l}_3) \right)$$

for all $\mathfrak{l} \in \mathcal{R}\mathfrak{l}_2$. Since $\mathfrak{l}_3 \in \mathcal{R}\mathfrak{l}_2$, we have

$$\varpi d(\mathfrak{l}_2, \mathfrak{l}_3) \in \alpha(\mathfrak{l}_2, \mathfrak{l}_3) \ \left(s(\mathfrak{l}_3, \mathfrak{J}\mathfrak{l}_3) \right).$$

This implies that

$$\varpi d(\mathfrak{l}_2,\mathfrak{l}_3)\in\alpha(\mathfrak{l}_2,\mathfrak{l}_3)\ \left(s\left(\mathfrak{l}_3,\mathfrak{J}\mathfrak{l}_3\right)\right)=\alpha(\mathfrak{l}_2,\mathfrak{l}_3)s\left(\bigcup_{\mathfrak{l}\in\mathfrak{J}\mathfrak{l}_3}d(\mathfrak{l}_3,\mathfrak{l})\right).$$

Since $\mathfrak{J}\mathfrak{l}_3\neq\emptyset$, so there exists $\mathfrak{l}_4\in X$ such that $\mathfrak{l}_4\in\mathfrak{J}\mathfrak{l}_3$, we have

$$\varpi d(\mathfrak{l}_2,\mathfrak{l}_3)\in\alpha(\mathfrak{l}_2,\mathfrak{l}_3)s\left(d\left(\mathfrak{l}_3,\mathfrak{l}_4\right)\right).$$

By Lemma 4.1 (iv), we have

$$\varpi d(\mathfrak{l}_2,\mathfrak{l}_3)\in\alpha(\mathfrak{l}_2,\mathfrak{l}_3)s\left(d\left(\mathfrak{l}_3,\mathfrak{l}_4\right)\right)=s\left(\alpha(\mathfrak{l}_2,\mathfrak{l}_3)d\left(\mathfrak{l}_3,\mathfrak{l}_4\right)\right).$$

Therefore,

$$\alpha(\mathfrak{l}_2,\mathfrak{l}_3)d\left(\mathfrak{l}_3,\mathfrak{l}_4\right)\preceq\varpi d(\mathfrak{l}_2,\mathfrak{l}_3).$$

Since $\alpha(\mathfrak{l}_2,\mathfrak{l}_3)\geq 1$, we have

$$0\prec d\left(\mathfrak{l}_3,\mathfrak{l}_4\right)\preceq\alpha(\mathfrak{l}_2,\mathfrak{l}_3)d\left(\mathfrak{l}_3,\mathfrak{l}_4\right)\preceq\varpi d(\mathfrak{l}_2,\mathfrak{l}_3).\tag{4.4}$$

Iterating this procedure yields a sequence $\{\mathfrak{l}_n\}$ in X such that

$$\mathfrak{l}_{2n+1}\in\mathcal{R}\mathfrak{l}_{2n}\text{ and }\mathfrak{l}_{2n+2}\in\mathfrak{J}\mathfrak{l}_{2n+1},$$

and $\alpha(\mathfrak{l}_n,\mathfrak{l}_{n+1})\geq 1$ and

$$d(\mathfrak{l}_n,\mathfrak{l}_{n+1})\preceq\varpi d(\mathfrak{l}_{n-1},\mathfrak{l}_n)\preceq...\preceq\varpi^n d(\mathfrak{l}_0,\mathfrak{l}_1),\tag{4.5}$$

for all n. Now, for all $n\in\mathbb{N}$ and for any $m=1,2,...,$

$$\begin{array}{rcl}
d(\mathfrak{l}_n,\mathfrak{l}_{n+m}) & \preceq & \varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\left[d(\mathfrak{l}_n,\mathfrak{l}_{n+1})+d(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\right]\\
& = & \varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})d(\mathfrak{l}_n,\mathfrak{l}_{n+1})+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})d(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\\
& \preceq & \varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})d(\mathfrak{l}_n,\mathfrak{l}_{n+1})+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\left[d(\mathfrak{l}_{n+1},\mathfrak{l}_{n+2})+d(\mathfrak{l}_{n+2},\mathfrak{l}_{n+m})\right]\\
& = & \varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})d(\mathfrak{l}_n,\mathfrak{l}_{n+1})+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})d(\mathfrak{l}_{n+1},\mathfrak{l}_{n+2})\\
& & +\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})d(\mathfrak{l}_{n+2},\mathfrak{l}_{n+m})\\
& \preceq & \varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})d(\mathfrak{l}_n,\mathfrak{l}_{n+1})+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})d(\mathfrak{l}_{n+1},\mathfrak{l}_{n+2})+...\\
& & +\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_{n+m-2},\mathfrak{l}_{n+m})d(\mathfrak{l}_{n+m-2},\mathfrak{l}_{n+m-1})\\
& & +\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_{n+m-1},\mathfrak{l}_{n+m})d(\mathfrak{l}_{n+m-1},\mathfrak{l}_{n+m})\\
& \preceq & \varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varpi^n d(\mathfrak{l}_0,\mathfrak{l}_1)+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\varpi^{n+1}d(\mathfrak{l}_0,\mathfrak{l}_1)+...\\
& & +\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_{n+m-2},\mathfrak{l}_{n+m})\varpi^{n+m-2}d(\mathfrak{l}_0,\mathfrak{l}_1)\\
& & +\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_{n+m-1},\mathfrak{l}_{n+m})\varpi^{n+m-1}d(\mathfrak{l}_0,\mathfrak{l}_1)\\
& = & \left[\begin{array}{c}\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varpi^n+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\varpi^{n+1}+...\\+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_{n+m-2},\mathfrak{l}_{n+m})\varpi^{n+m-2}\\+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_{n+m-1},\mathfrak{l}_{n+m})\varpi^{n+m-1}\end{array}\right]d(\mathfrak{l}_0,\mathfrak{l}_1)\\
& \preceq & \left[\begin{array}{c}\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varpi^n+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\varpi^{n+1}+...\\+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_{n+m-2},\mathfrak{l}_{n+m})\varpi^{n+m-2}\\+\varphi(\mathfrak{l}_n,\mathfrak{l}_{n+m})\varphi(\mathfrak{l}_{n+1},\mathfrak{l}_{n+m})\cdots\varphi(\mathfrak{l}_{n+m-1},\mathfrak{l}_{n+m})\varpi^{n+m-1}\end{array}\right]d(\mathfrak{l}_0,\mathfrak{l}_1)
\end{array}$$

$$= \left[\sum_{i=n}^{n+m-1} \varpi^i \prod_{j=1}^{i} \varphi(\imath_j, \imath_{n+m})\right] d(\imath_0, \imath_1). \tag{4.6}$$

We note that for any fixed $m \in \mathbb{N}$, the following holds

$$\frac{a_{n+1}^{(n+m)}}{a_n^{(n+m)}} = \varphi(\imath_{n+1}, \imath_{n+m}) \frac{\left\|\varpi^{n+1}\right\|}{\|\varpi^n\|}.$$

Since $\varpi \in \mathbb{P}^*$, it follows that

$$\lim_{n\to\infty} \frac{\left\|\varpi^{n+1}\right\|}{\|\varpi^n\|} = \rho(\varpi).$$

Consequently for every fixed m, we obtain

$$\lim_{n\to\infty} \frac{a_{n+1}^{(n+m)}}{a_n^{(n+m)}} = \lim_{n\to\infty} \varphi(\imath_{n+1}, \imath_{n+m})\rho(\varpi) < 1,$$

because, by assumption, $\lim_{n,m\to\infty} \varphi(\imath_{n+1}, \imath_{n+m}) < \frac{1}{\rho(\varpi)}$. Therefore, by applying the ratio test, the corresponding series

$$\sum_{i=n}^{n+m-1} \varpi^i \prod_{j=1}^{i} \varphi(\imath_j, \imath_{n+m})$$

is convergent in the BA $\mathbb{A}$, for fixed m above. Consequently,

$$\lim_{n\to\infty} \left\| \sum_{i=n}^{n+m-1} \varpi^i \prod_{j=1}^{i} \varphi(\imath_j, \imath_{n+m}) \right\| \le \lim_{n\to\infty} \sum_{i=n}^{n+m-1} \prod_{j=1}^{i} \varphi(\imath_j, \imath_{n+m}) \left\|\varpi^i\right\| = 0, \tag{4.7}$$

for every fixed m. It follows from the inequality (4.6) that

$$\left[\sum_{i=n}^{n+m-1} \varpi^i \prod_{j=1}^{i} \varphi(\imath_j, \imath_{n+m})\right] d(\imath_0, \imath_1) \to \theta_{\mathbb{A}},$$

as $n, m \to \infty$. Now for a given $c \gg \theta_{\mathbb{A}}$, then by (P_1) and (P_7), one can find some m_0 such that

$$\left[\sum_{i=n}^{n+m-1} \varpi^i \prod_{j=1}^{i} \varphi(\imath_j, \imath_{n+m})\right] d(\imath_0, \imath_1) \ll c,$$

for all $m, n > m_0$. Thus

$$d(\imath_n, \imath_{n+m}) \le \left[\sum_{i=n}^{n+m-1} \varpi^i \prod_{j=1}^{i} \varphi(\imath_j, \imath_{n+m})\right] d(\imath_0, \imath_1) \ll c,$$

for all $m, n > m_0$. It means that the sequence $\{\imath_n\}$ is Cauchy in X. By the completeness of X, this sequence $\{\imath_n\}$ converges to some point $\imath^* \in X$ such that $\imath_n \to \imath^*$ as $n \to \infty$. Since $\alpha(\imath_n, \imath_{n+1}) \ge 1$, for all n and $\imath_n \to \imath^*$ as $n \to +\infty$, so by the assumption (iv), we have $\alpha(\imath_{2n}, \imath^*) \ge 1$, for all n. By (4.1), we have

$$\varpi d(\imath_{2n}, \imath^*) \in \alpha(\imath_{2n}, \imath^*) s(\mathcal{R}\imath_{2n}, \mathfrak{J}\imath^*),$$

for all $n \in \mathbb{N}$. By definition of "s", we have

$$\varpi d(\imath_{2n}, \imath^*) \in \alpha(\imath_{2n}, \imath^*) \left(\left(\bigcap_{\imath \in \mathcal{R}\imath_{2n}} s\,(\imath, \mathfrak{J}\imath^*) \right) \cap \left(\bigcap_{\varsigma \in \mathfrak{J}\imath^*} s\,(\varsigma, \mathcal{R}\imath_{2n}) \right) \right),$$

which implies that

$$\varpi d(\imath_{2n}, \imath^*) \in \alpha(\imath_{2n}, \imath^*) \;\left(s\,(\imath, \mathfrak{J}\imath^*)\right),$$

for all $\imath \in \mathcal{R}\imath_{2n}$. Since $\imath_{2n+1} \in \mathcal{R}\imath_{2n}$, we have

$$\varpi d(\imath_{2n}, \imath^*) \in \alpha(\imath_{2n}, \imath^*) \;\left(s\,(\imath_{2n+1}, \mathfrak{J}\imath^*)\right).$$

This implies that

$$\varpi d(\imath_{2n}, \imath^*) \in \alpha(\imath_{2n}, \imath^*) \;\left(s\,(\imath_{2n+1}, \mathfrak{J}\imath^*)\right) = \alpha(\imath_{2n}, \imath^*) s \left(\bigcup_{\imath \in \mathfrak{J}\imath^*} d(\imath_{2n+1}, \imath) \right).$$

Since $\mathfrak{J}\imath^* \neq \emptyset$, so there exists $v_n \in X$ such that $v_n \in \mathfrak{J}\imath^*$, we have

$$\varpi d(\imath_{2n}, \imath^*) \in \alpha(\imath_{2n}, \imath^*) \; s\,(d(\imath_{2n+1}, v_n)).$$

By Lemma 4.1 (iv), we have

$$\varpi d(\imath_{2n}, \imath^*) \in \alpha(\imath_{2n}, \imath^*) \; s\,(d(\imath_{2n+1}, v_n)) = s\,(\alpha(\imath_{2n}, \imath^*) d(\imath_{2n+1}, v_n)).$$

Therefore,

$$\alpha(\imath_{2n}, \imath^*) d(\imath_{2n+1}, v_n) \preceq \varpi d(\imath_{2n}, \imath^*).$$

Since $\alpha(\imath_{2n}, \imath^*) \geq 1$, we have

$$0 \prec d(\imath_{2n+1}, v_n) \preceq \alpha(\imath_{2n}, \imath^*) d(\imath_{2n+1}, u_n) \preceq \varpi d(\imath_{2n}, \imath^*). \tag{4.8}$$

As $\imath_n \to \imath^*$ as $n \to +\infty$, so for a given $c \in Int\mathbb{P}$, there exists $k \in \mathbb{N}$ such that $d(\imath_{2n}, \imath^*) \ll \frac{c}{2\varpi\varphi(\imath^*, v_n)}$ and $d(\imath_{2n+1}, \imath^*) \ll \frac{c}{2\varphi(\imath^*, v_n)}$, for $n \geq k = k(c)$. Now from triangle inequality, we have

$$\begin{aligned} d(\imath^*, v_n) &\preceq \varphi(\imath^*, v_n)\,(d(\imath^*, \imath_{2n+1}) + d(\imath_{2n+1}, v_n)) \\ &= \varphi(\imath^*, v_n) d(\imath^*, \imath_{2n+1}) + \varphi(\imath^*, v_n) d(\imath_{2n+1}, v_n) \\ &\preceq \varphi(\imath^*, v_n) d(\imath^*, \imath_{2n+1}) + \varphi(\imath^*, v_n) \varpi d(\imath_{2n}, \imath^*) \\ &\preceq \varphi(\imath^*, v_n) \frac{c}{2\varphi(\imath^*, v_n)} + \varphi(\imath^*, v_n) \varpi \frac{c}{2\varpi\varphi(\imath^*, v_n)} \\ &\ll \frac{c}{2} + \frac{c}{2} = c, \text{ for } n \geq k = k(c). \end{aligned}$$

Hence, $\lim_{n\to\infty} v_n = \imath^*$. Since $\mathfrak{J}\imath^*$ is closed, so $\imath^* \in \mathfrak{J}\imath^*$. This implies that $\imath^*$ is a FP of $\mathfrak{J}$. Since the sequence $\{\imath_n\}$ in X satisfies $\alpha(\imath_n, \imath_{n+1}) \geq 1$ for all n and $\imath_n \to \imath^*$ as $n \to +\infty$, by assumption, we have $\alpha(\imath_{2n+1}, \imath^*) \geq 1$ for all n. By (4.1), we have

$$\varpi d(\imath_{2n+1}, \imath^*) \in \alpha(\imath_{2n+1}, \imath^*) s(\mathfrak{J}\imath_{2n+1}, \mathcal{R}\imath^*),$$

for all $n \in \mathbb{N}$. By definition of "s", we have

$$\varpi d(\imath_{2n+1}, \imath^*) \in \alpha(\imath_{2n+1}, \imath^*) \left(\bigcap_{\imath \in \mathfrak{J}\imath_{2n+1}} s(\imath, \mathcal{R}\imath^*) \right) \cap \left(\bigcap_{\varsigma \in \mathfrak{J}\imath^*} s(\varsigma, \mathfrak{J}\imath_{2n+1}) \right),$$

which implies that

$$\varpi d(\imath_{2n+1}, \imath^*) \in \alpha(\imath_{2n+1}, \imath^*) \ (s(\imath, \mathcal{R}\imath^*)),$$

for all $\imath \in \mathfrak{J}\imath_{2n+1}$. Since $\imath_{2n+2} \in \mathfrak{J}\imath_{2n+1}$, we have

$$\varpi d(\imath_{2n+1}, \imath^*) \in \alpha(\imath_{2n+1}, \imath^*) (s(\imath_{2n+2}, \mathcal{R}\imath^*)).$$

This implies that

$$\varpi d(\imath_{2n+1}, \imath^*) \in \alpha(\imath_{2n+1}, \imath^*) \ (s(\imath_{2n+2}, \mathcal{R}\imath^*)) = \alpha(\imath_{2n+1}, \imath^*) s \left(\bigcup_{\imath \in \mathcal{R}\imath^*} d(\imath_{2n+2}, \imath) \right).$$

Since $\mathcal{R}\imath^* \neq \emptyset$, so there exists $u_n \in X$ such that $u_n \in \mathcal{R}\imath^*$, we have

$$\varpi d(\imath_{2n+1}, \imath^*) \in \alpha(\imath_{2n+1}, \imath^*) \ s(d(\imath_{2n+2}, u_n)).$$

By Lemma 4.1 (iv), we have

$$\varpi d(\imath_{2n+1}, \imath^*) \in \alpha(\imath_{2n+1}, \imath^*) \ s(d(\imath_{2n+2}, u_n)) = s(\alpha(\imath_{2n+1}, \imath^*) d(\imath_{2n+2}, u_n)).$$

Therefore,

$$\alpha(\imath_{2n+1}, \imath^*) d(\imath_{2n+2}, u_n) \preceq \varpi d(\imath_{2n+1}, \imath^*).$$

Since $\alpha(\imath_{2n+1}, \imath^*) \geq 1$, we have

$$0 \prec d(\imath_{2n+2}, u_n) \preceq \alpha(\imath_{2n+1}, \imath^*) d(\imath_{2n+2}, u_n) \preceq \varpi d(\imath_{2n+1}, \imath^*). \tag{4.9}$$

As $\imath_n \to \imath^*$ as $n \to +\infty$, so for a given $c \in Int\mathbb{P}$, there exists $k \in \mathbb{N}$ such that $d(\imath_{2n+1}, \imath^*) \ll \frac{c}{2\varpi\varphi(\imath^*, u_n)}$ and $d(\imath_{2n+2}, \imath^*) \ll \frac{c}{2\varphi(\imath^*, u_n)}$ for $n \geq k = k(c)$. Now from triangle inequality of ECb-MS, we have

$$\begin{aligned} d(\imath^*, u_n) &\preceq \varphi(\imath^*, u_n) (d(\imath^*, \imath_{2n+2}) + d(\imath_{2n+2}, u_n)) \\ &= \varphi(\imath^*, u_n) d(\imath^*, \imath_{2n+2}) + \varphi(\imath^*, u_n) d(\imath_{2n+2}, u_n) \\ &\preceq \varphi(\imath^*, u_n) d(\imath^*, \imath_{2n+2}) + \varphi(\imath^*, u_n) \varpi d(\imath_{2n+1}, \imath^*) \\ &\preceq \varphi(\imath^*, u_n) \frac{c}{2\varphi(\imath^*, u_n)} + \varphi(\imath^*, u_n) \varpi \frac{c}{2\varpi\varphi(\imath^*, u_n)} \\ &\ll \frac{c}{2} + \frac{c}{2} = c, \ \text{for } n \geq k = k(c). \end{aligned}$$

Hence, $\lim_{n\to\infty} u_n = \imath^*$. Since $\mathcal{R}\imath^*$ is closed, so $\imath^* \in \mathcal{R}\imath^*$. This implies that $\imath^*$ is a FP of $\mathcal{R}$. This completes the proof. □

Corollary 4.1. *Let $(X, \mathbb{A}, d)$ be a complete ECb-MS over a BA $\mathbb{A}$, $\mathbb{P}$ be the underlying SC and $\mathfrak{J} : X \to CB(X)$. The following conditions are assumed to be true:*

(i) there exists $\alpha : X \times X \to [0, +\infty)$ such that

$$\varpi d(\imath, \varsigma) \in \alpha(\imath, \varsigma) s(\mathfrak{J}\imath, \mathfrak{J}\varsigma),$$

for all $\iota, \varsigma \in X$, *where* $\varpi \in \mathbb{P}^*$ *and* $\rho(\varpi) < 1$ *with* $\lim_{n,m\to\infty} \varphi(\iota_n, \iota_m) < \frac{1}{\rho(\varpi)}$ *for a sequence* $\{\iota_n\}$ *in* X,

(ii) $\mathfrak{J}$ *is* α*-closed,*

(iii) there exist $\iota_0 \in X$, $\iota_1 \in \mathfrak{J}\iota_0$ *such that* $\alpha(\iota_0, \iota_1) \geq 1$,

(iv) if $\{\iota_n\}$ *is a sequence in* X *such that* $\alpha(\iota_n, \iota_{n+1}) \geq 1$ *for all* n *and* $\iota_n \to \iota^*$ *as* $n \to \infty$, *then* $\alpha(\iota_n, \iota^*) \geq 1$ *for all* $n \in \mathbb{N}$.

Then there exists a point $\iota^* \in X$ such that $\iota^* \in \mathfrak{J}\iota^*$.

Proof. Take $\mathcal{R} = \mathfrak{J}$ in Theorem 4.3. □

Remark 4.2. *Taking* $\varphi(\iota, z) = b \geq 1$, *for all* $\iota, z \in X$ *in Definition 2.6, then the notion of ECb-MS over a BAs reduces to a Cb-MS over a BA, then we derive the leading Theorem of Kutbi et al. [22] from Corollary 4.1.*

Remark 4.3. *Taking* $\varphi(\iota, z) = 1$, *for all* $\iota, z \in X$ *and* $\mathbb{A} = \mathbb{R}$ *with the cone* $\mathbb{P} = [0, +\infty)$ *in Definition 2.6, then the notion of ECb-MS over a BAs reduces to a CMS, then a result of Kutbi et al. [24] is directly derived from Corollary 4.1.*

Corollary 4.2. *Let* $(X, \mathbb{A}, d)$ *be a complete ECb-MS over a BA* $\mathbb{A}$, $\mathbb{P}$ *be the underlying SC and* $\mathcal{R}, \mathfrak{J} : X \to CB(X)$. *Assume that there is* $\varpi \in \mathbb{P}^*$ *such that* $\rho(\varpi) < 1$ *and*

$$\varpi d(\iota, \varsigma) \in s(\mathcal{R}\iota, \mathfrak{J}\varsigma),$$

for all $\iota, \varsigma \in X$ *and* $\lim_{n,m\to\infty} \varphi(\iota_n, \iota_m) < \frac{1}{\rho(\varpi)}$ *for a sequence* $\{\iota_n\}$ *in* X. *Then there exists a point* $\iota^* \in X$ *such that* $\iota^* \in \mathcal{R}\iota^* \cap \mathfrak{J}\iota^*$.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by $\alpha(\iota, \varsigma) = 1$, for all $\iota, \varsigma \in X$ in Theorem 4.3. □

Corollary 4.3. *Let* $(X, \mathbb{A}, d)$ *be a complete ECb-MS over a BA* $\mathbb{A}$, $\mathbb{P}$ *be the underlying SC and* $\mathfrak{J} : X \to CB(X)$. *Suppose that there is* $\varpi \in \mathbb{P}^*$ *such that* $\rho(\varpi) < 1$ *and*

$$\varpi d(\iota, \varsigma) \in s(\mathfrak{J}\iota, \mathfrak{J}\varsigma),$$

for all $\iota, \varsigma \in X$ *and* $\lim_{n,m\to\infty} \varphi(\iota_n, \iota_m) < \frac{1}{\rho(\varpi)}$ *for a sequence* $\{\iota_n\}$ *in* X. *Then there exists a point* $\iota^* \in X$ *such that* $\iota^* \in \mathfrak{J}\iota^*$.

Proof. Take $\mathcal{R} = \mathfrak{J}$ in above Corollary. □

Remark 4.4. *Taking* $\varphi(\iota, z) = 1$, *for all* $\iota, z \in X$ *and* $\mathbb{A} = \mathbb{R}$ *with the cone* $\mathbb{P} = [0, +\infty)$ *in Definition 2.6, then the notion of ECb-MS over a BAs reduces to a CMS, then Corollary 4.3 reduced to the prime result of Cho et al. [20].*

By Remark 4.1, we have the following corollaries:

Corollary 4.4. *Let* (X, d) *be a complete Eb-MS and* $\mathcal{R}, \mathfrak{J} : X \to CB(X)$. *The following conditions are assumed to be true:*

(i) there exist a function $\alpha : X \times X \to [0, +\infty)$ *and a constant* $\varpi \in [0, 1)$ *such that*

$$\alpha(\iota, \varsigma) H(\mathcal{R}\iota, \mathfrak{J}\varsigma) \leq \varpi d(\iota, \varsigma),$$

for all $\iota, \varsigma \in X$, *where* $\lim_{n,m\to\infty} \varphi(\iota_n, \iota_m) < \frac{1}{\varpi}$ *for a sequence* $\{\iota_n\}$ *in* X,

(ii) $(\mathcal{R}, \Im)$ *and* $(\Im, \mathcal{R})$ *are* α*-closed,*

(iii) there exist $\iota_0 \in X$, $\iota_1 \in \mathcal{R}\iota_0$ *such that* $\alpha(\iota_0, \iota_1) \geq 1$,

(iv) if $\{\iota_n\}$ *is a sequence in* X *such that* $\alpha(\iota_n, \iota_{n+1}) \geq 1$, *for all* n *and* $\iota_n \to \iota^*$ *as* $n \to \infty$, *then* $\alpha(\iota_{2n}, \iota^*) \geq 1$ *and* $\alpha(\iota_{2n+1}, \iota^*) \geq 1$, *for all* $n \in \mathbb{N}$.

Then there exists a point $\iota^* \in X$ such that $\iota^* \in \mathcal{R}\iota^* \cap \Im\iota^*$.

Corollary 4.5. *Let* (X, d) *be a complete Eb-MS and* $\Im : X \to CB(X)$. *The following conditions are assumed to be true:*

(i) there exist a function $\alpha : X \times X \to [0, +\infty)$ *and a constant* $\varpi \in [0, 1)$ *such that*

$$\alpha(\iota, \varsigma) H(\Im\iota, \Im\varsigma) \leq \varpi d(\iota, \varsigma),$$

for all $\iota, \varsigma \in X$, *where* $\lim_{n,m\to\infty} \varphi(\iota_n, \iota_m) < \frac{1}{\varpi}$ *for a sequence* $\{\iota_n\}$ *in* X,

(ii) $\Im$ *is* α*-closed,*

(iii) there exist $\iota_0 \in X$, $\iota_1 \in \Im\iota_0$ *such that* $\alpha(\iota_0, \iota_1) \geq 1$,

(iv) if $\{\iota_n\}$ *is a sequence in* X *such that* $\alpha(\iota_n, \iota_{n+1}) \geq 1$, *for all* n *and* $\iota_n \to \iota^*$ *as* $n \to \infty$, *then* $\alpha(\iota_n, \iota^*) \geq 1$ *and* $\alpha(\iota_{n+1}, \iota^*) \geq 1$, *for all* $n \in \mathbb{N}$.

Then there exists a point $\iota^* \in X$ such that $\iota^* \in \Im\iota^*$.

Proof. Take $\mathcal{R} = \Im$ in the previous Corollary 4.4. □

Corollary 4.6. *Let* (X, d) *be a complete Eb-MS and* $\mathcal{R}, \Im : X \to CB(X)$. *Assume that there exists a constant* $\varpi \in [0, 1)$ *such that*

$$H(\mathcal{R}\iota, \Im\varsigma) \leq \varpi d(\iota, \varsigma),$$

for all $\iota, \varsigma \in X$, *where* $\lim_{n,m\to\infty} \varphi(\iota_n, \iota_m) < \frac{1}{\varpi}$ *for a sequence* $\{\iota_n\}$ *in* X. *Then there exists a point* $\iota^* \in X$ *such that* $\iota^* \in \mathcal{R}\iota^* \cap \Im\iota^*$.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by $\alpha(\iota, \varsigma) = 1$, for all $\iota, \varsigma \in X$ in Corollary 4.4. □

Example 4.1. *Let* $X = [0, 1]$ *and* $\mathbb{A}$ *be the space of continuously differentiable real-valued functions on* $[0, 1]$, *that is,* $\mathbb{A} = C^1_{\mathbb{R}}[0, 1]$. *The norm on* $\mathbb{A}$ *is defined as*

$$\|u\| = \|u\|_\infty + \left\|u'\right\|_\infty.$$

Let $\mathbb{A}$ *be equipped with pointwise multiplication. This endows* $\mathbb{A}$ *with the structure of a real BA possessing unity* $e_{\mathbb{A}}$, *where* $e_{\mathbb{A}}(t) = 1$, *for all* $t \in X$. *Define the subset* $\mathbb{P}$ *of* $\mathbb{A}$ *as follows:*

$$\mathbb{P} = \{u \in \mathbb{A} : \ u(t) \geq 0, \ t \in X\},$$

then $\mathbb{P}$ *is a SC (see [23]). Define a function* $d : X \times X \to \mathbb{A}$ *by*

$$d(\iota, \varsigma) = |\iota - \varsigma|^p e^t \text{ for } p > 1.$$

Then $(X, \mathbb{A}, d)$ *is a complete ECb-MS with the function* $\varphi : X \times X \to [1, +\infty)$ *defined by* $\varphi(\iota, \varsigma) = 2 + |\iota| + |\varsigma|$. *Define* $\Im : X \to CB(X)$ *by* $\Im\iota = [0, \frac{\iota}{10}]$. *Define* $\alpha : X \times X \to [0, +\infty)$ *by*

$$\alpha(\iota, \varsigma) = \begin{cases} \frac{10^p}{2}\left(1 - \frac{|\iota - \varsigma|}{10}\right) & \text{if } \iota \neq \varsigma, \\ 1, & \iota = \varsigma. \end{cases}$$

Then the mapping $\mathfrak{J}$ *is* α*-closed. Take* $\text{ı}_0 = \frac{1}{2} \in X$*, then* $\mathfrak{J}\text{ı}_0 = [0, \frac{1}{20}]$*. Choosing* $\text{ı}_1 = \frac{1}{20}$*, then*

$$\alpha(\text{ı}_0, \text{ı}_1) \geq 1.$$

Now we take $\varpi = \frac{1}{2}$*, then the spectrum of* ϖ *is*

$$\rho(\varpi) = \frac{1}{2} < 1.$$

Then we have

$$s(\mathcal{R}\text{ı}, \mathfrak{J}\varsigma) = s\left(\left|\frac{\text{ı}}{20} - \frac{\varsigma}{10}\right|^p e^t\right),$$
$$\varpi d(\text{ı}, \varsigma) = \left(\frac{1}{2}\right)|\text{ı} - \varsigma|^p\, e^t.$$

Since

$$\frac{10^p}{2}\left(1 - \frac{|\text{ı} - \varsigma|}{10}\right)\left|\frac{\text{ı}}{10} - \frac{\varsigma}{10}\right|^p e^t \preceq \left(\frac{1}{2}\right)|\text{ı} - \varsigma|^p\, e^t,$$

so

$$\varpi d(\text{ı}, \varsigma) = \left(\frac{1}{2}\right)|\text{ı} - \varsigma|^p\, e^t \in \frac{10^p}{2}\left(1 - \frac{|\text{ı} - \varsigma|}{10}\right)\left|\frac{\text{ı}}{10} - \frac{\varsigma}{10}\right|^p e^t = \alpha(\text{ı}, \varsigma)s(\mathfrak{J}\text{ı}, \mathfrak{J}\varsigma).$$

Thus all the conditions of Corollary 4.1 are satisfied and $\mathfrak{J}$ *has a FP, that is,* 0.

Remark 4.5. *The study of FP theorems in the framework of ECb-MSs over a BAs presents several unique challenges that are not encountered in classical MSs. These difficulties arise due to the interplay between the cone structure, the algebraic operations in BAs, and the need for generalized contractive conditions. Below, we present the primary challenges in this setting:*

- *Unlike classical MSs, ECb-MSs incorporate a cone-valued distance function, which affects contraction conditions and the structure of FP proofs. The nonlinear structure of BAs imposes additional algebraic restrictions that must be carefully handled. Standard techniques in FP theory are often insufficient, requiring new analytical approaches to establish contractive conditions.*
- *Ensuring the existence and uniqueness of FPs in this setting requires new contractive conditions that are compatible with the algebraic structure of Banach algebras. The cone structure introduces an ordering, making it necessary to redefine standard contraction principles in this space. Unlike in classical MSs, proving convergence demands additional constraints on the spectral radius of generalized Lipschitz constants.*
- *The definition of a multi-valued contraction must be adapted to preserve the ordering induced by the cone. A generalized Hausdorff distance function needs to be introduced in the setting of ECb-MSs over a BA, which has not been previously developed in the literature. Establishing compactness and continuity conditions for multi-valued mappings in this framework is more intricate than in classical spaces.*

5. Applications

FP theorems constitute a cornerstone of functional analysis, offering a robust framework for establishing the existence and uniqueness of solutions to a diverse array of mathematical equations, including integral equations. These theorems prove particularly invaluable when tackling intricate problems such as Fredholm integral equations, providing a systematic approach to ascertain solvability and guarantee uniqueness under specific conditions.

In recent years, several authors have extended classical FP principles to generalized structures. For example, Xinjie et al. [33] established FP theorems in extended cone b-metric-like spaces over Banach algebras and discussed the existence of solutions for Urysohn I-type integral equations. Similarly, Shereen et al. [34] explored new extended cone b-metric-like spaces over a real Banach algebra and solved nonlinear functional integral equations by applying suitable FP theorems. For additional applications of FP results to integral equations, readers are referred to [35, 36].

However, these studies did not address the Fredholm integral equation within the ECb-MS over Banach algebras framework. The present work bridges this gap by showing how ECb-MS provides a more general and algebraically rich setting to study Fredholm-type equations, particularly when the underlying operator or kernel can take values in a Banach algebra. This framework not only generalizes many existing metric-type results but also ensures a stronger and unified analytical foundation for proving convergence and existence results.

The subsequent section delves into the investigation of the following Fredholm integral equation of the second kind:

$$\gimel(x) = f(x) + \lambda \int_a^b K(x,t)\, \gimel(t) dt, \tag{5.1}$$

where $\gimel(x)$ represents the unknown function that we are trying to find, $f(x)$ is a known function, often referred to as the "forcing function" or "inhomogeneous term", λ is a parameter (often real or complex) and $K(t, \gimel(r))$ is the kernel of the integral equation, which is a given function defined on $[a, b] \times [a, b]$.

The Fredholm integral equation of the second kind plays a crucial role in modeling and solving problems in radar and sonar signal processing, particularly in contexts involving wave propagation, echo formation, and target detection. In radar and sonar systems, a wave (signal) is transmitted, and a portion of it reflects off objects and returns to the receiver-these reflected components are known as echoes.

However, the received echo is not merely a straightforward replica of the transmitted signal. It is modified by the medium through which it travels, influenced by factors such as multipath reflections, signal delay, and attenuation. As a result, the received signal is often a superposition of multiple delayed and altered versions of the original signal. This complex interaction can be effectively modeled by the Fredholm integral equation of the second kind:

$$\gimel(x) = f(x) + \lambda \int_a^b K(x,t)\, \gimel(t) dt,$$

where,

- $\gimel(x)$ denotes the measured echo signal at time or position x;

- $f(x)$ represents the initial or known component of the signal;
- λ is a scaling constant accounting for attenuation;
- $K(x,t)$ is the echo kernel, which characterizes how a signal originating from point t contributes to the observation at point x, incorporating effects like delay, spatial spreading, and reflection intensity;
- The integral $\int_a^b$ accumulates contributions from all relevant points in the domain (e.g., time or spatial interval $[a,b]$).

Let $C[a,b]$ be the set of all continuous functions on $[a,b]$, where $a,b \in \mathbb{R}$. Let $\mathbb{A} = \mathbb{R}^2$ and $\mathbb{P} = \{(\imath, \varsigma) \in \mathbb{A} : \imath, \varsigma \geq 0\}$ with the norm given as

$$\|\imath\| = \|(\imath_1, \imath_2)\| = |\imath_1| + |\imath_2|,$$

for $\imath = (\imath_1, \imath_2) \in \mathbb{A}$ and the multiplication given as

$$\imath\varsigma = (\imath_1, \imath_2)(\varsigma_1, \varsigma_2) = (\imath_1\varsigma_1, \imath_1\varsigma_2 + \imath_2\varsigma_1),$$

for $\imath = (\imath_1, \imath_2), \varsigma = (\varsigma_1, \varsigma_2) \in \mathbb{A}$, and the partial order $\preceq$ on $\mathbb{A}$ as

$$\imath \preceq \varsigma \text{ if and only if } \imath_1 \leq \varsigma_1 \text{ and } \imath_2 \leq \varsigma_2.$$

Define $d : C[a,b] \times C[a,b] \to \mathbb{A}$ by

$$d(\imath, \varsigma) = \left(\sup_{x\in[a,b]} |\imath(x) - \varsigma(x)|^2, \sup_{x\in[a,b]} |\imath(x) - \varsigma(x)|^2 \right) e^t,$$

$\forall\ \imath, \varsigma \in C[a,b]$ and $t \in [a,b]$ and $\varphi : C[a,b] \times C[a,b] \to [1, +\infty)$ by

$$\varphi(\imath, \varsigma) = 2 + |\imath(x)| + |\varsigma(x)|.$$

Then $(C[a,b], \mathbb{A}, d)$ is a complete ECb-MS over a BA $\mathbb{A}$.

Theorem 5.1. *Let $(C[a,b], \mathbb{A}, d)$ be a complete ECb-MS over a BA $\mathbb{A}$. Consider a integral equation (5.1), where $\imath, f \in C[a,b]$ and $K : [a,b] \times [a,b] \to \mathbb{R}$. Assume that the following conditions are satisfied:*

(i) The kernel function $K(x,t)$ is continuous on $[a,b] \times [a,b]$, and there exists a constant

$$M = \sup_{(x,t)\in[a,b]^2} |K(x,t)| < \infty.$$

(ii) The parameter λ satisfies the inequality

$$|\lambda| < \frac{1}{\sqrt{2}M(b-a)}.$$

Under these assumptions, the integral equation (5.1) admits a solution.

Proof. Define the mapping $\Im : C[a,b] \to C[a,b]$ by

$$\Im(\imath)(x) = f(x) + \lambda \int_a^b K(x,t)\,\imath(t)dt.$$

For any $\imath, \varsigma \in C[a,b]$, we have

$$\begin{aligned} |\Im(\imath)(x) - \Im(\varsigma)(x)| &= \left| \lambda \int_a^b K(x,t)(\imath(t) - \varsigma(t))\,dt \right| \\ &\leq |\lambda| \int_a^b |K(x,t)|\,|\imath(t) - \varsigma(t)|\,dt \\ &\leq |\lambda|\,M \int_a^b |\imath(t) - \varsigma(t)|\,dt. \end{aligned}$$

Now use:

$$|\imath(t) - \varsigma(t)| \leq \sup_{s\in[a,b]} |\imath(s) - \varsigma(s)| = \|\imath - \varsigma\|_\infty .$$

So

$$|\Im(\imath)(x) - \Im(\varsigma)(x)| \leq |\lambda|\,M(b-a)\,\|\imath - \varsigma\|_\infty .$$

Take supremum:

$$\sup_{x\in[a,b]} |\Im(\imath)(x) - \Im(\varsigma)(x)| \leq |\lambda|\,M(b-a)\,\|\imath - \varsigma\|_\infty .$$

Now square both sides:

$$\sup_{x\in[a,b]} |\Im(\imath)(x) - \Im(\varsigma)(x)|^2 \leq [|\lambda|\,M(b-a)]^2\,\|\imath - \varsigma\|_\infty^2 .$$

Now recalling the ECb-MS over a BA $\mathbb{A}$, we have

$$d(\imath,\varsigma) = (\Theta,\Theta)\,e^t, \quad \Theta = \sup_{x\in[a,b]} |\imath(x) - \varsigma(x)| .$$

So we get

$$d(\Im(\imath), \Im(\varsigma)) = \left(\Theta^/, \Theta^/\right) e^t,$$

where

$$\Theta^/ \leq [|\lambda|\,M(b-a)]^2 \sup_{x\in[a,b]} |\imath(x) - \varsigma(x)|^2 .$$

So finally:

$$d(\Im(\imath), \Im(\varsigma)) \leq \varpi d(\imath,\varsigma),$$

where

$$\varpi = \left([|\lambda|\,M(b-a)]^2, [|\lambda|\,M(b-a)]^2\right).$$

Recall the norm on $\mathbb{A}$ is:

$$\|\varpi\| = |\varpi_1| + |\varpi_2| = 2\left[|\lambda|\, M\,(b-a)\right]^2.$$

Now since $|\lambda| < \frac{1}{\sqrt{2}M(b-a)}$, so $\|\varpi\| < 1$ and $\varpi \in \mathbb{P}$. Since the mapping $\mathfrak{J}$ satisfies all the axioms of Theorem 3.1 for any $L \geq \theta_{\mathbb{A}}$, so $\mathfrak{J}$ has a unique FP $\mathrm{I} \in C[a,b]$ that satisfies the integral equation (5.1). Thus, the integral equation (5.1) has a unique solution. □

Example 5.1. *The nonlinear integral equation*

$$\mathrm{I}(x) = \sin x + 0.1\int_0^1 \frac{1}{2} xt\mathrm{I}(t)dt$$

satisfies all the conditions of Theorem 5.1, and has a unique solution.

6. Conclusions and future work

In this study, we investigated the notion of ECb-MSs defined over BAs and presented novel FP theorems for both single-valued and multi-valued mappings operating within this framework. Our findings serve to generalize and expand upon a range of existing results within this domain, thereby enriching our comprehension of such spaces. A key contribution of this work is the development of generalized contractive conditions tailored to the algebraic and ordered structure of ECb-MSs over BAs. Unlike classical MSs, the interplay between the cone structure and the algebraic operations introduces additional challenges, requiring refined analytical techniques. For multi-valued mappings, we introduced a generalized Hausdorff distance in the framework of ECb-MSs over BAs, a concept that was previously unexplored in the literature. To demonstrate the significance and applicability of our main results, we provided an illustrative example illustrating the novelty of our FP theorems. Additionally, we applied our findings to solve a Fredholm integral equation of the second kind, which arises in the context of radar and sonar signal processing. This highlights the potential for our results to contribute to broader applications in mathematical analysis, functional equations, and applied sciences.

As a future direction, our results can be extended to multi-valued, fuzzy, and L-fuzzy mappings for generalized contractive conditions in ECb-MSs over BAs. This could lead to further developments in the study of differential and integral equations in Banach algebras. Another promising direction is the application of our FP results to fractional differential equations, particularly in the context of Volterra-Fredholm integro-differential equations, which arise in mathematical physics and engineering models.

Moreover, our work has potential implications for controllability problems in fractional delay integro-differential systems of order $1 < r < 2$, particularly those involving Sobolev-type conditions. This aligns with recent studies on approximate and exact controllability in fractional systems (see [37, 38]). Future research could explore the extension of our results to impulsive fractional mixed Volterra-Fredholm integro-differential equations, further connecting FP theory with modern applications in dynamical systems and control theory.

Author contributions

Nura Alotaibi: Conceptualization, formal analysis, investigation, visualization, writing–original draft, writing–review and editing; Jamshaid Ahmad: Conceptualization, formal analysis, methodology, validation, writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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