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Research article

Solving Volterra fuzzy integro-differential equations using fuzzy Elzaki transform homotopy perturbation method

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Abstract: Fuzzy integro-differential equations are used for modeling real-life phenomena that involve uncertain (fuzzy) parameters or variables. The combination of the fuzzy Elzaki transform and homotopy perturbation method provides a powerful hybrid technique for solving fuzzy integro-differential equations. Therefore, the aim of this paper is to modify and apply a new hybrid method called fuzzy Elzaki transform homotopy perturbation method for the first time in literature to solve fuzzy integro-differential equations. In particular, the fuzzy Elzaki transform homotopy perturbation method is developed and applied for solving linear and non-linear second-kind fuzzy Volterra integro-differential equations, and non-linear second kind fuzzy mixed Fredholm- Volterra integro-differential equations. Finally, several examples are presented to show that the fuzzy Elzaki transform homotopy perturbation method is efficient for solving wide types of fuzzy integro-differential equations with high accuracy. The novelty of this work lies in its ease of use and its high efficiency, which allows mathematicians to obtain reliable results under fuzzy Hukuhara differentiability aspects in a short time.

Keywords: fuzzy Elzaki transform; homotopy perturbation method; fuzzy Volterra integro-differential equations; fuzzy mixed Fredholm-Volterra integro-differential equations
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1. Introduction

Integro-differential equations (IDEs) combine the characteristics of differential and integral equations, and there are significant tools for modeling numerous phenomena across various fields, including engineering, physics, and economics [1].

The general formula for IDEs is represented as follows:

$$l(s)\, z^{(n)}(s) = f(s) + \lambda \int_{\gamma(x)}^{\delta(x)} k(s,t) N\big(z(t)\big) dt.$$

IDEs are classified into two main types based on the limits of integration: that are Volterra integro-differential equations (V-IDEs), and Fredholm integro-differential equations (F-IDEs). In particular, the classification of IDEs depends on whether the limits of integration are variable or fixed. IDEs with variable limits known as V-IDEs where the name is assumed to refer to the scientist Volterra [2]. V-IDEs are considered as one of the most important tools for modeling optimal control systems, and they are often associated with initial value problems, meaning that the value of the solution at a given time depends on its previous values. The main characteristic of V-IDEs makes it an effective tool for modeling various systems across many fields, such as biology, physics, engineering, economics, and finance [3,4].

In contrast, the IDEs with constant limits are known as a Fredholm integro-differential equations (F-IDEs), where the name is assumed refer to the scientist Erik Fredholm [5]. F-IDEs are typically associated with boundary value problems, where the unknown function depends on values within fixed integration where they often make the F-IDEs more challenging or difficult to solve. Fredholm equations are used for modeling complex systems in several fields including physics, engineering, medicine, and biology [4].

In solving IDEs analytically, some transform methods are used, such as the Laplace transform, Sumudu transform method, or direct integration method. But, in solving some IDEs classical methods become cumbersome and inefficient or some classes of functions where the Laplace transform might diverge or be undefined. Therefore, for handling this issue, some mathematicians presented and applied a new approach known as the Elzaki transform method [6]. The Elzaki transform method was derived from the classical Fourier transform and is used to simplify and efficiently solve the IDEs since it offers some advantages over using the Laplace transform method. It handles non-zero initial conditions more easily since it provides simpler operational rules for derivatives and integrals and less strict convergence requirements. More specifically, the Elzaki transform method is effective for solving the Volterra and Fredholm integral equations by converting complex integral forms into simple algebraic expressions. After using the Elzaki transform method to solve IDEs, we often still cannot find an exact solution, especially if the equation is complicated or has nonlinear terms. Therefore, approximation methods are used to address these challenges. One significant approximation method is the homotopy perturbation method (HPM) introduced by (Ji-Huan He) in 1999. The homotopy perturbation method is an analytical approximate method used to solve differential and integral equations in both linear and nonlinear forms. However, it is particularly used for nonlinear equations because it is capable of isolating and processing their nonlinear components [7,8].

In reality, real-world phenomena are often imprecise and contain uncertainties. This vagueness can arise in scientific fields such as medicine, engineering, meteorology, manufacturing, and others [9–13]. The fuzziness arises in the collection of data, due to experimental errors, and measurement errors, inaccurate estimation, and it also may appear when calculating the boundary\initial conditions. These fuzzy aspects exist when collecting real data about essential materials like microbial populations, soil, water, etc. As described in [14–16], fuzzy sets are considered as an essential tool for handling or solving such problems in order to provide a better understanding or illustration of phenomena. The intention of early research in fuzzy set theory that was carried out by [17] was to generalize the classical concept of a set and provide a suggestion to explain the fuzziness. Fuzzy set theory is considered as a tool for modeling vague systems and processing uncertain information in mathematical models. These include using the fuzzy integral equations instead of deterministic (crisp) integral equations. Studies of the theory of fuzzy integro-differential equations (FIDEs) have greatly increased in recent times FIDEs are utilized in modeling engineering, quantum optics, robotics, gravity, medicine, and intelligence tests [11,18–20]. Therefore, both IDEs and uncertainty (fuzzy) play a fundamental role in handling mathematical problems. These lead to FIDEs.

The FIDEs are used for modeling the real-life phenomena that involving uncertain (fuzzy) parameters or variables in many fields such as medicine, physics, intelligence tests, gravity, engineering, and biology [21–23]. For example, in heat transfer or population growth models, the parameters such as the diffusion rate or interaction coefficient may be uncertain due to measurement errors, naturally leading to FIDE formulations [22]. Therefore, there is growing interest in obtaining exact or approximation solutions for FIDEs. In solving FIDEs analytically, some fuzzy transform methods are used such as, the Laplace transform, Sumudu transform method, or direct integration method. But, in solving some FIDEs, the classical methods become cumbersome and inefficient or some classes of functions where the Laplace transform might diverge or be undefined. Therefore, for handling this issue. the combination of the fuzzy Elzaki Transform and HPM provides a powerful hybrid technique for solving FIDEs. Our review of the literature indicates no attempts seem to have been made on this front. Therefore, the aim of this paper is to modify and apply a hybrid method called the fuzzy Elzaki transform homotopy perturbation method (FE-HPM) to solve FIDEs. The FE-HPM is developed and applied for solving linear and non-linear second kind fuzzy Volterra integro-differential equations (FVIDEs), and non-linear second kind fuzzy mixed Fredholm-Volterra integro-differential equations (FFVIDEs).

The remainder of this paper is organized as follows: Section 2 introduces the preliminaries; Section 3 develops the fuzzy Elzaki transform; Section 4 formulates the FIDEs; Section 5 presents the FE-HPM approach for solving second-kind FIDEs; Section 6 illustrates numerical examples; and Section 7 concludes the study.

2. Preliminaries

In this section, some theorems and definitions which are used later in this paper to establish the fuzzy Elzaki transform under Hukuhara differentiability are presented.

Definition 2.1. [17] A fuzzy set $\tilde{\mathcal{A}}$ is defined as a collection of ordered pairs:

$$\tilde{\mathcal{A}} = \{(x, \mu_{\tilde{\mathcal{A}}}(x)) : x \in \mathcal{X}, \mu_{\tilde{\mathcal{A}}}(x) \in [0,1]\},$$

where $\mu_{\tilde{\mathcal{A}}}(x)$ is the membership function of $\tilde{\mathcal{A}}$ and $\mathcal{X}$ denotes the universal set.

Definition 2.2. [17] A fuzzy number is a particular type of fuzzy set $\tilde{\mathcal{A}}$, which is normalized and convex having a piecewise continuous membership function.

Definition 2.3. [24] For $0 \le \alpha \le 1$ and $\tilde{\mathcal{A}}_\alpha = \{x \in \mathcal{X} \mid \mu_{\tilde{\mathcal{A}}}(x) \ge \alpha\}$, if $\tilde{\mathcal{A}}$ is a fuzzy number, then the fuzzy interval form using α-cut is given as:

$$\tilde{\mathcal{A}}_\alpha = [\underline{\mathcal{A}}_\alpha, \overline{\mathcal{A}}_\alpha], \text{ where } \underline{\mathcal{A}}_\alpha = min\{x \mid x \in \tilde{\mathcal{A}}_\alpha\} \text{ and, } \overline{\mathcal{A}}_\alpha = max\{x \mid x \in \tilde{\mathcal{A}}_\alpha\}.$$

Theorem 2.1. [24] Suppose that $\underline{z}, \overline{z}: [0,1] \to \mathbb{R}$, which satisfies the following conditions:

i) $\underline{z}$ is a bounded increasing function, and $\overline{z}$ is a bound decreasing function with $\underline{z}(1) \le \overline{z}(1)$.

ii) $\underline{z}$ and $\overline{z}$ are left-hand continuous functions at $\sigma = k$, for all $k \in (0, 1]$.

iii) $\underline{z}$ and $\overline{z}$ are right-hand continuous functions at $\sigma = 0$.

Then, $z: \mathbb{R} \to [0,1]$ defined by $z(x) = sup\{\alpha : \underline{z}(\alpha) \le x \le \overline{z}(\alpha)\}$ is a fuzzy number and the parametric form is $[\underline{w}(\alpha), \overline{w}(\alpha)]$. Otherwise, the functions $\underline{z}$ and $\overline{z}$ satisfy the conditions.

Definition 2.4. [25] A fuzzy number μ, is called a triangular fuzzy number if defined by three numbers $a < b < c$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[a, c]$ and vertex at $x = b$, and its membership function has the following form:

$$\mu(x; a, b, c) = \begin{cases} \frac{x-a}{b-a}, & if\ a < x \le b \\ \frac{c-x}{c-b}, & if\ b < x < c \\ 0, & other\ wise. \end{cases}$$

The α −cut of triangular fuzzy number is: $[\tilde{\mu}]_\alpha = [a + \alpha(b-a), c - \alpha(c-b)],\ \alpha \in [0, 1]$.

Definition 2.5. [26] For arbitrary fuzzy numbers u and v on $\mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \to \mathbb{R}^+ \cup \{0\}$, we define the Hausdorff distance by the mapping $D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ such that:

$$D(v, u) = sup_{0 \le \alpha \le 1} max\{|\underline{v}(\alpha) - \underline{u}(\alpha)|, |\overline{v}(\alpha) - \overline{u}(\alpha)|\}.$$

Note: $\mathbb{R}_\mathcal{F}$ denotes the set of fuzzy numbers on the real number($\mathbb{R}$).

Theorem 2.2. [26] The space $(\mathbb{R}_\mathcal{F}, D)$ is a complete fuzzy metric space.

Some arithmetic operations in fuzzy mathematics:

(1) $[v]_\alpha + [u]_\alpha = [\underline{v}(\alpha) + \underline{u}(\alpha), \overline{v}(\alpha) + \overline{u}(\alpha)]$ for each $v, u \in \mathbb{R}_F$ and $0 \le \alpha \le 1$.

(2) $[v]_\alpha \ominus [u]_\alpha = [\underline{v}(\alpha) - \overline{u}(\alpha), \overline{v}(\alpha) - \underline{u}(\alpha)]$, is called the Hukuhara difference (H-difference) of v and u.

Additionally, we always use "$\ominus$" to refer to the Hukuhara difference, and it is important to note that

$$v \ominus u \neq v + (-1)u. \text{ [27]}$$

Definition 2.6. [28] If $\tilde{z}: I \to \mathbb{R}_F$ and $y_0 \in I$, where $I \in [t_0, T]$, $\tilde{z}$ is said to be Hukuhara differentiable at y_0, if there exists an element $[\widetilde{z'}]_\alpha \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\tilde{z}(y_0 + h; \alpha) \ominus \tilde{z}(y_0; \alpha)$and $\tilde{z}(y_0; \alpha) \ominus \tilde{z}(y_0 - h; \alpha)$ exists where the limits are taken in the metric space $(\mathbb{R}_F, D)$.

$$\lim_{h \to 0+} \frac{\tilde{z}(y_0 + h; \alpha) \ominus \tilde{z}(y_0; \alpha)}{h} = \lim_{h \to 0+} \frac{\tilde{z}(y_0; \alpha) \ominus \tilde{z}(y_0 - h; \alpha)}{h} = \widetilde{z'}(y_0).$$

The fuzzy set $\left[\widetilde{z'}(y_0)\right]_\alpha$ is called the Hukuhara derivative of $\left[\widetilde{z'}\right]_\alpha$ at y_0. Taking in account that this limit corresponds to the Hukuhara derivative, as defined in the standard literature, and does not represent the classical derivative, then we have the following theorem.

Theorem 2.3. [29] Let $\tilde{z}: [t_0 + \alpha, T] \to \mathbb{R}_F$ be Hukuhara differentiable and denote:

$$\left[\widetilde{z'}(t)\right]_\alpha = \left[\underline{z}'(t), \overline{z}'(t)\right]_\alpha = \left[\underline{z}'(t;\alpha), \overline{z}'(t;\alpha)\right].$$

Then, the boundary functions $\underline{z}'(t;\alpha), \overline{z}'(t;\alpha)$ are both differentiable

$$\left[\widetilde{z'}(t)\right]_\alpha = \left[\left(\underline{z}(t;\alpha)\right)', \left(\overline{z}(t;\alpha)\right)'\right],\ \forall\, \alpha \in [0,1].$$

Theorem 2.4. [28] Let $\tilde{z}: [t_0 + \alpha, T] \to \mathbb{R}_F$ be Hukuhara differentiable and denote:

$$\left[\widetilde{z'}(t)\right]_\alpha = \left[\underline{z}'(t), \overline{z}'(t)\right]_\alpha = \left[\underline{z}'(t;\alpha), \overline{z}'(t;\alpha)\right].$$

Then, both boundary functions $\underline{z}'(t;\alpha), \overline{z}'(t;\alpha)$ are differentiable, and we can write for n^{th}-order fuzzy derivative

$$\left[\tilde{z}^{(n)}(t)\right]_\alpha = \left[\left(\underline{z}^{(n)}(t;\alpha)\right)', \left(\overline{z}^{(n)}(t;\alpha)\right)'\right],\ \forall\, \alpha \in [0,1].$$

Definition 2.7. [30] Suppose that $z: [a,b] \to \mathbb{R}_\mathcal{F}$ is be a fuzzy-valued function. For each partition $p = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and $\xi_i \in [x_{i-1}, x_i], 1 \le i \le n$, assume that $R_P = \sum_{i=1}^{n} z(\xi_i)\,(x_i - x_{i-1})$ and $\Delta = \max_{1\le i\le n} |x_{i-1}, x_i|$. The definite integral of $z(x)$ over $[a,b]$ is $\int_a^b z(x)dx = \lim_{\Delta\to 0} R_P$ such that the limit exists in $(\mathbb{R}_\mathcal{F}, d)$. If the fuzzy function $z(x)$ is continuous in the metric d, its definite integral exists [24]. Moreover,

$$\underline{\left(\int_a^b z(x;\alpha)dx\right)} = \int_a^b \underline{z}(x;\alpha)dx,$$

$$\overline{\left(\int_a^b z(x;\alpha)dx\right)} = \int_a^b \overline{z}(x;\alpha)dx,$$

where the underline denotes the lower bound and the upper line denotes the upper bound of the fuzzy integral at each α-cut level set.

Theorem 2.4. [31] Let $\tilde{z}(x)$ be a fuzzy-value function on $[c,\infty)$ and it is represented by $\left(\underline{z}(x,\alpha), \overline{z}(x,\alpha)\right)$ for any fixed $\alpha \in [0,1]$, assume $\underline{z}(x,\alpha)$and $\overline{z}(x,\alpha)$ are Riemann-integrable on $[c,d]$ for every $d \ge c$, and assume there are two positive $\underline{M}(\alpha)$ and $\overline{M}(\alpha)$ such that $\int_c^d \left|\underline{z}(x,\alpha)\right| dx \le \underline{M}(\alpha)$ and $\int_c^d |\overline{z}(x,\alpha)|\, dx \le \overline{M}(\alpha)$ for every $d \ge c$. Then, $\tilde{z}(x)$ is improper fuzzy Riemann integrable on $[c,\infty)$, and the improper fuzzy Riemann integral is a fuzzy number.

Furthermore, we have:

$$\int_0^\infty \tilde{z}(x)dx = \left(\int_0^\infty \underline{z}(x,\alpha)dx\,, \int_0^\infty \overline{z}(x,\alpha)dx\right).$$

3. The fuzzy Elzaki transform

In this section, the main definitions and theorem of the fuzzy Elzaki transform are written for the first time in literature under Hukuhara differentiability. In the formulation process the classical Elzaki is developed and reformulated under fuzzy set aspects to get the main related definitions and the theorem of fuzzy Elzaki transform. The fuzzy Elzaki transform works on fuzzy-number-valued functions via α-cut representations, effectively and efficiently preserving uncertainty throughout calculations.

Definition 3.1. [6,32] Let $\mathcal{L}, \xi_1, and\ \xi_2$ be constants and consider the set

$$\mathcal{L} = \left\{z(x): \exists \mathcal{L}, \xi_1, \xi_2 > 0, |z(x)| < \mathcal{L}e^{\frac{|x|}{\xi_j}}, x \in (-1)^j \times [0,\infty], \text{where } j = 1,2\right\}.$$

Then, the Elzaki transform is defined as:

$$E[z(x)] = v\int_0^\infty z(x)e^{\frac{-x}{v}}\,dx, \qquad x > 0, \xi_1 \le v \le \xi_2. \tag{1}$$

Theorem 3.1. Linearity property for the Elzaki transforms [6,32]: Let $\psi_1(x)$ and $\psi_2\,(x)$ be two functions with the constants σ and δ. Then,

$$E\big(\,\sigma\psi_1(x)\ \pm \delta\psi_2(x)\big)\ =\ \sigma E\big(\psi_1(x)\big)\ \pm \delta E\big(\psi_2(x)\big).$$

Theorem 3.2. Convolution Theorem for the Elzaki transform[33,34]: Let $\psi_1(x)$ and $\psi_2\,(x)$ are two functions with the constants σ and δ then:

$$E\big(\psi_1(x) *\ \psi_2(x)\big) = \frac{1}{v}E\big(\psi_1(x)\big)\mathrm{E}\big(\psi_2(x)\big).$$

Theorem 3.3. [6, 32] Let $Z(v)$ be the Elzaki transform of $z(x)$ such that $E[z(x)]=\ Z(v)$. Then,

(1) $E[z'(x)] = \dfrac{Z(v)}{v} - vz(0)$,

(2) $E[z''(x)] = \dfrac{Z(v)}{v^2} - z(0) - vz'(0)$,

(3) $E\big[z^{(n)}(x)\big] = \dfrac{Z(v)}{v^n} - \displaystyle\sum_{k=0}^{n-1} v^{2-n+k}\, z^{(k)}(0)$.

Definition 3.2. Let $\tilde{z}(x)$ be a continuous fuzzy-valued function. Suppose that $\tilde{z}(x) \odot e^{\frac{-x}{v}}$ is improper fuzzy Riemann integrable on$[0,\infty)$. then, $v\int_0^\infty \tilde{z}(x) \odot e^{\frac{-x}{v}}dx$ is called the fuzzy Elzaki transform and it is denoted as:

$$\tilde{E}\big(\tilde{z}(x)\big) = \int_0^\infty v\tilde{z}(x) \odot e^{\frac{-x}{v}} dx.$$

From Theorem 2.4, the parameterized form of the fuzzy Elzaki transform is

$$\int_0^\infty v\tilde{z}(x) \odot e^{\frac{-x}{v}} dx = \left(\int_0^\infty v\underline{z}(x,\alpha)e^{\frac{-x}{v}} dx\,, \int_0^\infty v\overline{z}(x,\alpha)e^{\frac{-x}{v}} dx\right),$$

where $0 \leq \alpha \leq 1$.

So, by using the definition of the classical Elzaki transform we have:

$$E\left(\underline{z}(x,\alpha)\right) = \int_0^\infty v\underline{z}(x,\alpha)e^{\frac{-x}{v}} dx \quad \text{and} \quad E\big(\overline{z}(x,\alpha)\big) = \int_0^\infty v\overline{z}(x,\alpha)e^{\frac{-x}{v}} dx.$$

Then, we have

$$\tilde{E}[\tilde{z}(x)] = \big(E\big[\underline{z}(x,\alpha)\big], E[\overline{z}(x,\alpha)]\big),$$

where $x > 0$, and it exists if ψ and γ are piecewise continuous functions.

Theorem 3.4. Let $\widetilde{w}_1(x)$ and $\widetilde{w}_2\,(x)$ are continuous fuzzy-valued functions with the constants σ and δ. Then,

$$\tilde{E}[\sigma\widetilde{w}_1(x) \odot \oplus\, \delta \odot \widetilde{w}_2(x)\,] \;=\; \sigma \odot \tilde{E}[\widetilde{w}_1(x)] \oplus\; \delta \odot \tilde{E}[\widetilde{w}_2(x)].$$

Proof. Let $\widetilde{w}_1(x) = \big[\underline{w_1}(x;\alpha), \overline{w}_1(x;\alpha)\big]$, $\widetilde{w}_2(x) = \big[\underline{w_2}(x;\alpha), \overline{w}_2(x;\alpha)\big]$.

From Definition 3.2, the following was obtained:

$$E\left[\left(\sigma\underline{w_1}(x;\alpha)\right) + \left(\delta\underline{w_2}(x;\alpha)\right)\right]$$
$$= v\int_0^\infty \sigma\underline{w_1}(x;\alpha)e^{\frac{-x}{v}} dx \; + v\int_0^\infty \delta\underline{w_2}(x;\alpha)e^{\frac{-x}{v}} dx$$
$$= \sigma\int_0^\infty v\underline{w_1}(x;\alpha)e^{\frac{-x}{v}} dx \; + \delta\int_0^\infty v\underline{w_2}(x;\alpha)e^{\frac{-x}{v}} dx$$
$$= \sigma E\big[\underline{w_1}(x;\alpha)\big] + \delta E\big[\underline{w_2}(x;\alpha)\big],$$

and

$$E\big[\big(\sigma\overline{w}_1(x;\alpha)\big) + \big(\delta\overline{w}_2(x;\alpha)\big)\big]$$
$$= v\int_0^\infty \sigma\overline{w}_1(x;\alpha)e^{\frac{-x}{v}} dx \; + v\int_0^\infty \delta\overline{w}_2(x;\alpha)e^{\frac{-x}{v}} dx$$
$$= \sigma\int_0^\infty v\overline{w}_1(x;\alpha)e^{\frac{-x}{v}} dx \; + \delta\int_0^\infty v\overline{w}_2(x;\alpha)e^{\frac{-x}{v}} dx$$
$$= \sigma E[\overline{w}_1(x;\alpha)] + \delta E[\overline{w}_2(x;\alpha)].$$

So,

$= \sigma \odot \tilde{E}[\tilde{w}_1(x)] \oplus \delta \odot \tilde{E}[\tilde{w}_2(x)]$.

Theorem 3.5. (Convolution Theorem): Let $\tilde{w}_1(x)$ and $\tilde{w}_2(x)$ be continuous fuzzy-valued functions with the constants σ and δ. Then,

$$\tilde{E}[\tilde{w}_1(x) \odot \tilde{w}_2(x)] = \frac{1}{v} \odot \tilde{E}[\tilde{w}_1(x)] \odot \tilde{E}[\tilde{w}_2(x)].$$

Proof. The fuzzy functions can be represented by their α-cut:

$$\tilde{w}_1(x) = [\underline{w}_1(x;\alpha), \overline{w}_1(x;\alpha)], \quad \tilde{w}_2(x) = [\underline{w}_2(x;\alpha), \overline{w}_2(x;\alpha)] \text{ for each } \alpha \in [0,1].$$

Using the definition of the Elzaki transform of the lower bounds,

$$E[\underline{w}_1(x;\alpha)] = v\int_0^\infty \underline{w}_1(x;\alpha)e^{\frac{-x}{v}}dx \text{ and } E[\underline{w}_2(x;\alpha)] = v\int_0^\infty w_2(r;\alpha)e^{\frac{-r}{v}}dr.$$

Therefore,

$$\frac{1}{v}E[\underline{w}_1(x;\alpha)]E[\underline{w}_2(x;\alpha)] = \frac{1}{v}\left(v\int_0^\infty \underline{w}_1(x;\alpha)e^{\frac{-x}{v}}dx\right)\left(v\int_0^\infty \underline{w}_2(r;\alpha)e^{\frac{-r}{v}}dr\right)$$

$$= \frac{1}{v}\left(v^2\int_0^\infty\int_0^\infty \left(\underline{w}_1(x;\alpha)\underline{w}_2(r;\alpha)\right)e^{\frac{-(x+r)}{v}}drdx\right)$$

$$= v\int_0^\infty\int_0^\infty \left(\underline{w}_1(x;\alpha)\underline{w}_2(r;\alpha)\right)e^{\frac{-(r+x)}{v}}drdx.$$

Taking $s = r + x$ and $ds = dx$, we have

$$= v\int_0^\infty\int_0^\infty \left(\underline{w}_1(s-r;\alpha)\underline{w}_2(r;\alpha)\right)e^{\frac{-s}{v}}drdx = E[\underline{w}_1(x;\alpha) * \underline{w}_2(x;\alpha)],$$

and

$$E[\overline{w}_1(x;\alpha)] = v\int_0^\infty \overline{w}_1(x;\alpha)e^{\frac{-x}{v}}dx \text{ and } E[\overline{w}_2(x;\alpha)] = v\int_0^\infty \overline{w}_2(r;\alpha)e^{\frac{-r}{v}}dr$$

$$= \frac{1}{v}\left(v\int_0^\infty \overline{w}_1(x;\alpha)e^{\frac{-x}{v}}dx\right)\left(v\int_0^\infty \overline{w}_2(r;\alpha)e^{\frac{-r}{v}}dr\right)$$

$$= \frac{1}{v}\left(v^2\int_0^\infty\int_0^\infty (\overline{w}_1(x;\alpha)\overline{w}_2(r;\alpha))e^{\frac{-(x+r)}{v}}drdx\right)$$

$$= v\int_0^\infty\int_0^\infty (\overline{w}_1(x;\alpha)\overline{w}_2(r;\alpha))e^{\frac{-(r+x)}{v}}drdx.$$

Take $s = r + x$ and $ds = dx$ then we have

$$= v\int_0^\infty\int_0^\infty (\overline{w}_1(s-r;\alpha)\overline{w}_2(r;\alpha))e^{\frac{-s}{v}}drdx = E[\overline{w}_1(x;\alpha) * \overline{w}_2(x;\alpha)].$$

So,

$= \frac{1}{v} \odot \tilde{E}[\widetilde{w}_1(x)] \odot \tilde{E}[\widetilde{w}_2(x)].$

Lemma 3.1. Let $\widetilde{w}(x)$ be a continuous fuzzy-valued function on the interval $[0,1)$ and $\sigma \in \mathbb{R}$. Then, $\tilde{E}[\sigma \odot \widetilde{w}(x)] = \sigma \odot \tilde{E}[\widetilde{w}(x)]$.

Theorem 3.6. Let $\tilde{z}^{(n)}(x;\alpha)$ be an integrable fuzzy-valued function, and $\tilde{z}^{(n-1)}(x;\alpha)$ is a prime of $\tilde{z}^{(n)}(x;\alpha)$ on $[0,\infty)$. Then,

(1) $\tilde{E}[\tilde{z}'(x)] = \frac{\tilde{z}(v)}{v} \ominus v\tilde{z}(0)$.

Proof. Assume $\tilde{z}$ is Hukuhara differentiable. Then, by Theorem 2.3 can we obtain that

$$\tilde{z}'^{(x;\alpha)} = \left[\underline{z}'(x;\alpha), \overline{z}'(x;\alpha)\right].$$

Therefore, it follows that

$$\tilde{E}[\tilde{z}'(x;\alpha)] = \tilde{E}\left[\underline{z}'(x;\alpha), \overline{z}'(x;\alpha)\right].$$

By using Definition 3.2, we can get

$$\tilde{E}\left[\underline{z}'(x;\alpha), \overline{z}'(x;\alpha)\right] = \left(E\left[\underline{z}'(x;\alpha)\right], E[\overline{z}'(x;\alpha)]\right).$$

By using the classical Elzaki transform the first derivative, we to get the following

$$\left(E\left[\underline{z}'(x;\alpha)\right], E[\overline{z}'(x;\alpha)]\right) = \left(\frac{\underline{Z}(v;\alpha)}{v} - v\underline{z}(0;\alpha), \frac{\overline{Z}(v;\alpha)}{v} - v\overline{z}(0;\alpha)\right).$$

Thus,

$$\tilde{E}[\tilde{z}'(x;\alpha)] = \left(\frac{\underline{Z}(v;\alpha)}{v} - v\underline{z}(0;\alpha), \frac{\overline{Z}(v;\alpha)}{v} - v\overline{z}(0;\alpha)\right).$$

(2) $\tilde{E}[\tilde{z}''(x)] = \frac{\tilde{Z}(v)}{v} \ominus \tilde{z}(0) \ominus v\tilde{z}'(0)$.

Proof. Assume $\tilde{z}$ and $\tilde{z}'$are Hukuhara differentiable. Then, by Theorem 2.4 can we obtain that

$$\tilde{z}''^{(x;\alpha)} = \left[\underline{z}''^{(x;\alpha)}, \overline{z}''^{(x;\alpha)}\right].$$

Therefore, it follows that

$$\tilde{E}[\tilde{z}''(x;\alpha)] = \tilde{E}\left[\underline{z}''(x;\alpha), \overline{z}''(x;\alpha)\right].$$

By using Definition 3.2, we can get

$$\tilde{E}\left[\underline{z}''(x;\alpha), \overline{z}''(x;\alpha)\right] = \left(E\left[\underline{z}''(x;\alpha)\right], E[\overline{z}''(x;\alpha)]\right).$$

By using the classical Elzaki transform the second derivative, we get

$$\left(E\left[\underline{z}''(x;\alpha)\right], E[\overline{z}''(x;\alpha)]\right) = \left(\frac{\underline{Z}(v;\alpha)}{v} - \underline{z}(0;\alpha) - v\underline{z}'^{(0;\alpha)}, \frac{\overline{Z}(v;\alpha)}{v} - \overline{z}(0;\alpha) - v\overline{z}'(0;\alpha)\right).$$

Thus,

$$\tilde{E}[\tilde{z}''(x;\alpha)] = \left(\frac{\underline{Z}(v;\alpha)}{v} - \underline{z}(0;\alpha) - v\underline{z}'^{(0;\alpha)}, \frac{\overline{Z}(v;\alpha)}{v} - \overline{z}(0;\alpha) - v\overline{z}'(0;\alpha)\right).$$

4. Integro-differential equations in fuzzy environment

Consider the one-dimensional FIDEs of the second kind under Hukuhara derivative with the initial and boundary conditions:

$$\tilde{z}^{(n)}\,(s,\alpha) = \tilde{f}(s,\alpha\,) + \tilde{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\widetilde{N}\big(\tilde{z}(t,\alpha)\big)dt. \tag{2}$$

Using the fuzzy set theory based on the Zadeh extension principle and the α -cut level set approach, as well as based on the approach in [30], Eq (7) can be rewritten as follows:

$$\begin{cases} \underline{z}^{(n)}(s,\alpha) = \underline{f}(s,\alpha) + \underline{\lambda}\displaystyle\int_{\gamma(x)}^{\delta(x)} \underline{k(s,t)\;N(z(t,\alpha))}dt, \\ \overline{z}^{(n)}(s,\alpha) = \overline{f}(s,\alpha) + \overline{\lambda}\displaystyle\int_{\gamma(x)}^{\delta(x)} \overline{k(s,t)N(z(t,\alpha))}dt\,, \end{cases} \tag{3}$$

where [30],

$$\begin{cases} \underline{k(s,t)\;z(t,\alpha)} = k(s,t)\underline{z}(t,\alpha) \qquad k(s,t) \geq 0, \\ \overline{k(s,t)z(t,\alpha)} = \;k(s,t)\overline{z}(t,\alpha) \qquad k(s,t) \geq 0, \end{cases} \tag{4}$$

such that $(x) \leq s \;\leq\; \delta(x)$, $\tilde{\lambda} > 0$, $k(s,t)$ is a known function called the kernel function, and α refers to the fuzzy parameter between [0,1]. Equation (3) represents the general formula of the FIDEs of second kind in single parametric form of a fuzzy number under the Hukuhara derivative. The H-derivative is chosen because it is the most widely used fuzzy derivative in analytical fuzzy differential equations, and allows the FE-HPM method to be applied directly to α-cut representations. Other fuzzy derivatives exist (e.g., generalized Hukuhara and strongly generalized derivatives), but their applicability to integral-type fuzzy equations is more restricted.

5. Solving the second kind FIDEs using the FE-HPM

To solve Eq (3) by FE-HPM, at the first the HPM is implemented as follows:

$$\begin{cases} H(\underline{Z},p,\alpha) = (1-p)[\underline{Z}^{(n)}(s,\alpha) - \underline{z}_0(s,\alpha)] + p\left[\underline{Z}^{(n)}(s,\alpha) - \underline{f}(s,\alpha) - \underline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\underline{N}\left(\underline{Z}(t,\alpha)\right)dt\right] = 0, \\ H(\overline{Z},p,\alpha) = (1-p)\left[\overline{Z}^{(n)}(s,\alpha) - \overline{z}_0(s,\alpha)\right] + p\left[\overline{Z}^{(n)}(s,\alpha) - \overline{f}(s,\alpha) - \overline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\overline{N}\left(\overline{Z}(t,\alpha)\right)dt\right] = 0. \end{cases} \tag{5}$$

Thus, the initial approximation is taken as:

$$\begin{cases} \underline{z}_0(s,\alpha) = \underline{f}(s,\alpha), \\ \overline{z}_0(s,\alpha) = \overline{f}(s,\alpha). \end{cases} \tag{6}$$

By substituting Eq (6) in Eq (5), we get

$$\begin{cases} \underline{Z}^n(s,\alpha) = \underline{f}(s,\alpha) + p\,\underline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\underline{N}\left(\underline{Z}(t,\alpha)\right)dt, \\ \overline{Z}^n(s,\alpha) = \overline{f}(s,\alpha) + p\,\overline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\overline{N}\left(\overline{Z}(t,\alpha)\right)dt. \end{cases} \tag{7}$$

Now, by taking the Elzaki transform and using the differential property of the Elzaki transform on both sides of Eq (7), we get the following:

$$\begin{cases} E\{\underline{Z}(s,\alpha)\} = (v^n)\left\{\sum_{i=1}^{n-1} v^{n-(i+2)}\underline{z}^{(i)}(0,\alpha) + E\left\{\underline{f}(s,\alpha) + p\,\underline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\underline{N}\left(\underline{Z}(t,\alpha)\right)dt\right\}\right\}, \\ E\{\underline{Z}(s,\alpha)\} = (v^n)\left\{\sum_{i=1}^{n-1} v^{n-(i+2)}\overline{z}^{(i)}(0,\alpha) + E\left\{\overline{f}(s,\alpha) + p\,\overline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\overline{N}\left(\overline{Z}(t,\alpha)\right)dt\right\}\right\}. \end{cases} \tag{8}$$

Now, by taking the inverse Elzaki transform on both sides of Eq (8), we get the following:

$$\begin{cases} \underline{Z}(s,\alpha) = E^{-1}\left\{(v^n)\left\{\sum_{i=1}^{n-1} v^{n-(i+2)}\underline{z}^{(i)}(0,\alpha) + E\left\{\underline{f}(s,\alpha) + p\,\underline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\underline{N}\left(\underline{Z}(t,\alpha)\right)dt\right\}\right\}\right\} \\ \underline{Z}(s,\alpha) = E^{-1}\left\{(v^n)\left\{\sum_{i=1}^{n-1} v^{n-(i+2)}\overline{z}^{(i)}(0,\alpha) + E\left\{\overline{f}(s,\alpha) + p\,\overline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\overline{N}\left(\overline{Z}(t,\alpha)\right)dt\right\}\right\}\right\} \end{cases} \tag{9}$$

To clarify the implementation of the homotopy perturbation method (HPM), an embedding parameter is introduced to gradually transform the initial approximation into the final solution. When this parameter is zero, the solution represents the initial guess, and as it approaches one, it becomes the actual solution of the fuzzy integro-differential equation [35]. The method then expresses the unknown function as a series in terms of this parameter, and the terms of the series are obtained step-by-step by substituting into the main equation and matching coefficients of similar powers. Each new term is calculated from the previous ones, and the complete approximate solution is obtained by taking the parameter equal to one [36–39].

Now there are two cases dependent on the linearity of FIDEs:

Case 1: If the unknown function appears within the integral, in a linear form. Now the solution of Eq (9) can be written as a power series p:

$$\begin{cases} \underline{Z}(s,\alpha) = \sum_{i=0}^{\infty} p^i \underline{Z}_i(s,\alpha), \\ \overline{Z}(s,\alpha) = \sum_{i=0}^{\infty} p^i \overline{Z}_i(s,\alpha). \end{cases} \tag{10}$$

Now, by substituting Eq (10) into Eq (9) and comparing coefficients such as the power of p. Therefore, using the above iterative results, the series-form solution is provided as:

$$p^o: \begin{cases} \underline{Z}_0(s,\alpha) = E^{-1}\left\{(v^n)\left\{\sum_{i=1}^{n-1} v^{n-(i+2)} \underline{z}^{(i)}(0,\alpha) + E\left\{\underline{f}(s,\alpha)\right\}\right\}\right\}, \\ \overline{Z}_o(s,\alpha) = E^{-1}\left\{(v^n)\left\{\sum_{i=1}^{n-1} v^{n-(i+2)} \overline{z}^{(i)}(0,\alpha) + E\left\{\overline{f}(s,\alpha)\right\}\right\}\right\}. \end{cases} \tag{11}$$

$$p^1: \begin{cases} \underline{Z}_1(s,\alpha) = E^{-1}\left\{(v^n)\left\{E\left\{\underline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\underline{Z}_0(t,\alpha)dt\right\}\right\}\right\}, \\ \overline{Z}_1(s,\alpha) = E^{-1}\left\{(v^n)\left\{E\left\{\overline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\overline{Z}_0(t,\alpha)dt\right\}\right\}\right\}. \end{cases} \tag{12}$$

$$p^k: \begin{cases} \underline{Z}_k(s,\alpha) = E^{-1}\left\{(v^n)\left\{E\left\{\underline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\underline{Z}_{k-1}(t,\alpha)dt\right\}\right\}\right\}, \\ \overline{Z}_k(s,\alpha) = E^{-1}\left\{(v^n)\left\{E\left\{\overline{\lambda}\int_{\gamma(x)}^{\delta(x)} k(s,t)\overline{Z}_{k-1}(t,\alpha)dt\right\}\right\}\right\}. \end{cases} \tag{13}$$

And so on….

Case 2: If the unknown function appears within the integral, in a non-linear term.

Now the solution of Eq (9) can be written as a power series p:

$$\begin{cases} \underline{Z}(s,\alpha) = \sum_{i=0}^{\infty} p^i \underline{Z}_i(s,\alpha), \\ \overline{Z}(s,\alpha) = \sum_{i=0}^{\infty} p^i \overline{Z}_i(s,\alpha). \end{cases} \tag{14}$$

And, the non-linear term can be represented based on [34] as follows:

$$\begin{cases} N\left(\underline{Z}(t,\alpha)\right) = \sum_{i=0}^{\infty} p^i \underline{A}_i(t,\alpha), \\ N\left(\overline{Z}(t,\alpha)\right) = \sum_{i=0}^{\infty} p^i \overline{A}_i(t,\alpha), \end{cases} \tag{15}$$

where $\underline{A}_i$ and $\overline{A}_i$ are the Adomian polynomials, and can be calculated as follows [34]:

$$\underline{A}_i(t,\alpha) = \frac{1}{i!}\frac{d^i}{d\lambda^i}\underline{\mathcal{F}}\left(\sum_{k=0}^{\infty} \lambda^k \, \underline{Z}_k(t,\alpha)\right)_{\lambda=0},$$

and

$$\overline{A}_i(t,\alpha) = \frac{1}{i!}\frac{d^i}{d\lambda^i}\overline{\mathcal{F}}\left(\sum_{k=0}^{\infty} \lambda^k \, \overline{Z}_k(t,\alpha)\right)_{\lambda=0}.$$

Now, by substituting Eqs (14) and (15) into Eq (9) and comparing coefficients such as the power of p. Therefore, using the above iterative results, the series form solution is provided as:

$$p^o: \begin{cases} \underline{Z}_0(s,\alpha) = E^{-1}\left\{(v^n)\left\{\sum_{i=1}^{n-1} v^{n-(i+2)} \underline{z}^{(i)}(0,\alpha) + E\left\{\underline{f}(s,\alpha)\right\}\right\}\right\}, \\ \overline{Z}_o(s,\alpha) = E^{-1}\left\{(v^n)\left\{\sum_{i=1}^{n-1} v^{n-(i+2)} \overline{z}^{(i)}(0,\alpha) + E\{\overline{f}(s,\alpha)\}\right\}\right\}. \end{cases} \tag{16}$$

$$p^1:\begin{cases}\underline{Z}_1(s,\alpha)=E^{-1}\left\{(v^n)\left\{E\left\{\underline{\lambda}\int_{\gamma(x)}^{\delta(x)}k(s,t)\underline{A}_0(t,\alpha)dt\right\}\right\}\right\},\\ \overline{Z}_1(s,\alpha)=E^{-1}\left\{(v^n)\left\{E\left\{\overline{\lambda}\int_{\gamma(x)}^{\delta(x)}k(s,t)\overline{A}_0(t,\alpha)dt\right\}\right\}\right\}.\end{cases}\tag{17}$$

$$p^k:\begin{cases}\underline{Z}_k(s,\alpha)=E^{-1}\left\{(v^n)\left\{E\left\{\underline{\lambda}\int_{\gamma(x)}^{\delta(x)}k(s,t)\underline{A}_{k-1}(t,\alpha)dt\right\}\right\}\right\},\\ \overline{Z}_k(s,\alpha)=E^{-1}\left\{(v^n)\left\{E\left\{\overline{\lambda}\int_{\gamma(x)}^{\delta(x)}k(s,t)\overline{A}_{k-1}(t,\alpha)dt\right\}\right\}\right\}.\end{cases}\tag{18}$$

And so on….

Thus, the general solution of FIDEs of the 2^{nd} kind solution for both cases is given as follows:

$$\begin{cases}\underline{z}(s,\alpha)=\lim_{p\to1}\underline{Z}(s,\alpha)=\sum_{i=0}^{\infty}\underline{Z_i}(s,\alpha),\\ \overline{z}(s,\alpha)=\lim_{p\to1}\overline{Z}(s,\alpha)=\sum_{i=0}^{\infty}\overline{Z}_i(s,\alpha).\end{cases}\tag{19}$$

In summary, to clarify how the theoretical method is applied, we briefly outline the procedure used in the examples. For each problem, we first express the fuzzy integro-differential equation in terms of its α-cut representation. Then, we construct the homotopy and apply the fuzzy Elzaki transform to obtain the recursive relations for the HPM terms. After that, we compute a finite number of series terms and reconstruct the lower and upper fuzzy solutions. Finally, we compare the approximate solution with the exact fuzzy solution at selected values of s and α.

6. Numerical examples

In this section, a numerical example of FIDEs is presented.

Example 1. Consider the linear FVIDE of the second kind

$$\tilde{z}'(s,\alpha)=\tilde{f}(s,\alpha)+\int_0^s \tilde{z}(t,\alpha)dt,\tag{20}$$

with $\tilde{z}(0,\alpha)=[0,0]$, where $\tilde{\lambda}=[1,1]$, $0\le t\le s$, $0\le\alpha\le1$, $k(s,t)=1$, $\tilde{f}(s,\alpha)=[\alpha-1,1-\alpha]$.

To solve Eq (20) by FE-HPM, first, the HPM is implemented as follows:

$$\begin{cases} H(\underline{Z}, p, \alpha) = \underline{Z}'(s, \alpha) - (\alpha - 1) - p\int_0^s \underline{Z}(t, \alpha)dt = 0, \\ H(\overline{Z}, p, \alpha) = \overline{Z}'(s, \alpha) - (1 - \alpha) - p\int_0^s \overline{Z}(t, \alpha)dt = 0. \end{cases} \tag{21}$$

Now, by taking the Elzaki transform and using the differential property of the Elzaki transform on both sides of Eq (21), we obtain

$$\begin{cases} E\{\underline{Z}(s, \alpha)\} = v^3(\alpha - 1) + vE\left\{p\int_0^s \underline{Z}(t, \alpha)dt\right\}, \\ E\{\overline{Z}(s, \alpha)\} = v^3(1 - \alpha) + vE\left\{p\int_0^s \overline{Z}(t, \alpha)dt\right\}. \end{cases} \tag{22}$$

Now, by taking the inverse Elzaki transform on both sides of Eq (22), we get the following:

$$\begin{cases} \underline{Z}(s, \alpha) = E^{-1}\left\{v^3(\alpha - 1) + vE\left\{p\int_0^s \underline{Z}(t, \alpha)dt\right\}\right\}, \\ \overline{Z}(s, \alpha) = E^{-1}\left\{v^3(1 - \alpha) + vE\left\{p\int_0^s \overline{Z}(t, \alpha)dt\right\}\right\}. \end{cases} \tag{23}$$

Now, the solution of Eq (23) can be written as a power series in p as follows:

$$\begin{cases} \underline{Z}(s, \alpha) = \sum_{i=0}^{\infty} p^i \underline{Z}_i(s, \alpha), \\ \overline{Z}(s, \alpha) = \sum_{i=0}^{\infty} p^i \overline{Z}_i(s, \alpha). \end{cases} \tag{24}$$

Now, by substituting Eq (24) into Eq (23) and comparing coefficients such as the power of p. Therefore, using the above iterative results, the series form solution is provided as

$$p^0: \begin{cases} \underline{Z}_0(s, \alpha) = s(\alpha - 1), \\ \overline{Z}_0(s, \alpha) = s(1 - \alpha). \end{cases} \tag{25}$$

$$p^1: \begin{cases} \underline{Z}_1(s, \alpha) = \frac{s^3}{6}(\alpha - 1), \\ \overline{Z}_1(s, \alpha) = \frac{s^3}{6}(1 - \alpha). \end{cases} \tag{26}$$

$$p^2:\begin{cases}\underline{Z}_2(s,\alpha)=\dfrac{s^5}{120}\,(\alpha-1),\\ \overline{Z}_2(s,\alpha)=\dfrac{s^5}{120}\,(1-\alpha).\end{cases} \tag{27}$$

$$p^3:\begin{cases}\underline{Z}_3(s,\alpha)=\dfrac{s^7}{5040}\,(\alpha-1),\\ \overline{Z}_3(s,\alpha)=\dfrac{s^7}{5040}\,(1-\alpha).\end{cases} \tag{28}$$

And so on….

Currently, the answer is provided as

$$\begin{cases}\underline{z}(s,\alpha)=\sum_{i=0}^{\infty}\underline{Z}_i(s,\alpha),\\ \overline{z}(s,\alpha)=\sum_{i=0}^{\infty}\overline{Z}_i(s,\alpha).\end{cases} \tag{29}$$

Therefore, using the above iterative results, the series form solution is provided as:

$$\begin{cases}\underline{z}(s,\alpha)=s(\alpha-1)+\dfrac{s^3}{6}\,(\alpha-1)+\dfrac{s^5}{120}\,(\alpha-1)+\dfrac{s^7}{5040}\,(\alpha-1)+\cdots\\ \overline{z}(s,\alpha)=s(1-\alpha)+\dfrac{s^3}{6}\,(1-\alpha)+\dfrac{s^5}{120}\,(1-\alpha)+\dfrac{s^7}{5040}\,(1-\alpha)+\cdots\end{cases} \tag{30}$$

Additionally, the exact solution is provided as:

$$\begin{cases}\underline{z}(s,\alpha)=\sinh s(\alpha-1),\\ \overline{z}(s,\alpha)=\sinh s(1-\alpha).\end{cases} \tag{31}$$

Table 1 and Figure 1 show that the approximation of Eq (20) by FE-HPM and, the exact solution at $s=0.2$ for $\alpha\in[0,1]$ attains the triangular fuzzy number shape, and thus satisfies the fuzzy number properties. Also, the results obtained using FE-HPM show that the method is accurate, and the results confirm our theoretical analysis. Furthermore, Table 2 shows that the FE-HPM method gives good results very quickly. With only a few terms, the approximate solution becomes very close to the exact solution. When we use 4 or 5 terms, the difference between the exact and approximate values becomes extremely small. This means the method converges fast, and the tiny errors in Table 1 are normal and expected.

Table 1. The lower and upper bounds fuzzy exact and approximation solutions of Eq (20) by FE-HPM with 4 HPM terms at $s = 0.2$ for all $\alpha \in [0,1]$.

	Lower fuzzy solution			Upper fuzzy solution		
(α)	Exact Solution	Approximation Solution	Absolute Error	Exact Solution	Approximation Solution	Absolute Error
0	-0.201336	-0.201336	1.41×10^{-12}	0.201336	0.201336	1.41×10^{-12}
0.2	-0.161068	-0.161068	1.12×10^{-12}	0.161068	0.161068	1.12×10^{-12}
0.4	-0.120801	-0.120801	8.46×10^{-13}	0.120801	0.120801	8.46×10^{-13}
0.6	-0.080534	-0.080534	5.64×10^{-13}	0.080534	0.080534	5.64×10^{-13}
0.8	-0.040267	-0.040267	2.82×10^{-13}	0.040267	0.040267	2.82×10^{-13}
1	0	0	0	0	0	0

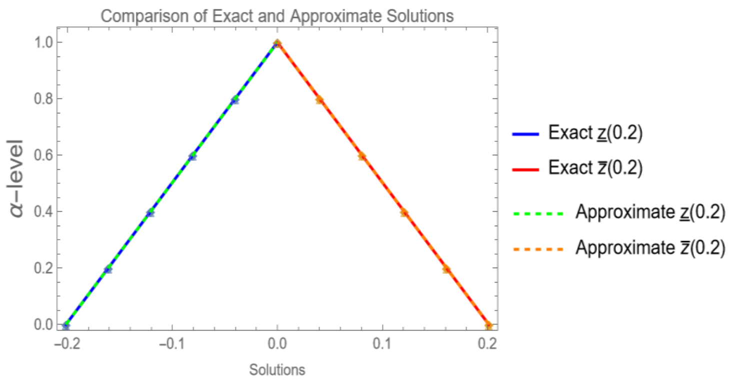

Figure 1: 2D plot comparing the fuzzy exact and approximate solutions of Eq (20) by FE-HPM with 4 HPM terms at $s = 0.2$ for all $\alpha \in [0,1]$.

Table 2. Convergence of the lower and upper bounds fuzzy FE-HPM solution for Example 1 at $s = 0.2$ and $\alpha = 0$.

	Lower fuzzy solution		Upper fuzzy solution	
Number of terms (N)	Approximation Solution	Absolute Error	Exact Solution	Absolute Error
1	-0.200000000	1.34×10^{-3}	0.200000000	1.34×10^{-3}
2	-0.201333333	2.67×10^{-6}	0.201333333	2.67×10^{-6}
3	-0.201336000	2.54×10^{-9}	0.201336000	2.54×10^{-9}
4	-0.201336003	1.41×10^{-13}	0.201336003	1.41×10^{-13}

Example 2. Consider the non-linear FVIDE of the second kind

$$\tilde{z}'(s,\alpha) = \tilde{f}(s,\alpha) + \int_0^s \tilde{z}^2(t,\alpha)dt \tag{32}$$

with $\tilde{z}(0,\alpha) = [0,0]$, where $\tilde{\lambda} = [1,1], 0 \le t \le s$, $0 \le \alpha \le 1$, $k(s,t) = 1$, and $\tilde{f}(s,\alpha) = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$.

To solve Eq (32) by FE-HPM, first, the HPM is implemented as follows:

$$\begin{cases} H(\underline{Z}, p, \alpha) = \underline{Z}'(s,\alpha) - (0.75 + 0.25\alpha) - p\int_0^s \underline{Z}^2(t,\alpha)dt = 0, \\ H(\overline{Z}, p, \alpha) = \overline{Z}'(s,\alpha) - (1.25 - 0.25\alpha) - p\int_0^s \overline{Z}^2(t,\alpha)dt = 0. \end{cases} \tag{33}$$

Now, by taking the Elzaki transform and using the differential property of the Elzaki transform on both sides of Eq (33), we obtain

$$\begin{cases} E\{\underline{Z}(s,\alpha)\} = \left\{v^3(0.75 + 0.25\alpha) + vE\left\{p\int_0^s \underline{Z}^2(t,\alpha)dt\right\}\right\}, \\ E\{\overline{Z}(s,\alpha)\} = \left\{v^3(1.25 - 0.25\alpha) + vE\left\{p\int_0^s \overline{Z}^2(t,\alpha)dt\right\}\right\}. \end{cases} \tag{34}$$

Now, by taking the inverse Elzaki transform on both sides of Eq (34), we get the following:

$$\begin{cases} \underline{Z}(s,\alpha) = E^{-1}\left\{v^3(0.75 + 0.25\alpha) + vE\left\{p\int_0^s \underline{Z}^2(t,\alpha)dt\right\}\right\}, \\ \overline{Z}(s,\alpha) = E^{-1}\left\{v^3(1.25 - 0.25\alpha) + vE\left\{p\int_0^s \overline{Z}^2(t,\alpha)dt\right\}\right\}. \end{cases} \tag{35}$$

Now, the solution of Eq (35) can be written as power series in p as follows:

$$\begin{cases} \underline{Z}(s,\alpha) = \sum_{i=0}^{\infty} p^i \underline{Z}_i(s,\alpha), \\ \overline{Z}(s,\alpha) = \sum_{i=0}^{\infty} p^i \overline{Z}_i(s,\alpha), \end{cases} \tag{36}$$

and the non-linear term can be represented as

$$\begin{cases} \underline{Z}^2(t,\alpha) = \sum_{i=0}^{\infty} p^i \underline{A}_i(s,\alpha), \\ \overline{Z}^2(t,\alpha) = \sum_{i=0}^{\infty} p^i \overline{A}_i(s,\alpha). \end{cases} \tag{37}$$

where $\underline{A}_i(t,\alpha)$ and $\overline{A}_i(t,\alpha)$ are the Adomian polynomials for the non-linear terms $\underline{Z}^2(t,\alpha)$ and $\overline{Z}^2(t,\alpha)$, respectively, and they are given by

$$\underline{A}_0(s,\alpha) = \underline{Z}_0{}^2(s,\alpha),\ \overline{A}_0(s,\alpha) = \overline{Z}_0{}^2(s,\alpha).$$

$$\underline{A}_1(s,\alpha) = 2\,\underline{Z}_0(s,\alpha)\underline{Z}_1(s,\alpha),\ \overline{A}_1(s,\alpha) = 2\,\overline{Z}_0(s,\alpha)\overline{Z}_1(s,\alpha).$$

Now, by substituting Eqs (36) and (37) into Eq (35) and comparing coefficients such as the power of p. Therefore, using the above iterative results, the series-form solution is provided as

$$p^0:\begin{cases}\underline{Z}_0(s,\alpha) = s(0.75+0.25\alpha),\\ \overline{Z}_0(s,\alpha) = s(1.25-0.25\alpha).\end{cases} \tag{38}$$

$$p^1:\begin{cases}\underline{Z}_1(s,\alpha) = \dfrac{s^4}{12}(0.75+0.25\alpha)^2,\\ \overline{Z}_1(s,\alpha) = \dfrac{s^4}{12}(1.25-0.25\alpha)^2.\end{cases} \tag{39}$$

$$p^2:\begin{cases}\underline{Z}_2(s,\alpha) = \dfrac{s^7}{252}(0.75+0.25\alpha)^3,\\ \overline{Z}_2(s,\alpha) = \dfrac{s^7}{252}(1.25-0.25\alpha)^3.\end{cases} \tag{40}$$

And so on….

Currently, the answer is provided as

$$\begin{cases}\underline{z}(s,\alpha) = \sum_{i=0}^{\infty}\underline{Z}_i(s,\alpha),\\ \overline{z}(s,\alpha) = \sum_{i=0}^{\infty}\overline{Z}_i(s,\alpha).\end{cases} \tag{41}$$

Therefore, using the above iterative results, the series form solution is provided as

$$\begin{cases}\underline{z}(s,\alpha) = s(0.75+0.25\alpha) + \dfrac{s^4}{12}(0.75+0.25\alpha)^2 + \dfrac{s^7}{252}(0.75+0.25\alpha)^3 + \cdots\\ \overline{z}(s,\alpha) = s(1.25-0.25\alpha) + \dfrac{s^4}{12}(1.25-0.25\alpha)^2 + \dfrac{s^7}{252}(1.25-0.25\alpha)^3 \ldots\end{cases} \tag{42}$$

Table 3 and Figure 2 show that the approximation of Eq (32) by FE-HPM at $s = 0.2$ for $\alpha \in [0,1]$ attains the triangular fuzzy number shape, and thus satisfies the fuzzy number properties. Also, the results obtained using FE-HPM show that the method is accurate, and the results confirm our theoretical analysis.

Table 3. The lower and upper bounds fuzzy aproximation solutions of Eq (32) by FE-HPM with 3 HPM terms at $s = 0.2$ and FL-ADM with 4 ADM terms for all $\alpha \in [0,1]$.

	Lower fuzzy solution			Upper fuzzy solution		
(α)	FL-ADM	FE-HPM	Absolute Error	FL-ADM	FE-HPM	Absolute Error
0	0.150075	0.150075	3.66×10^{-10}	0.250208	0.250208	1.69×10^{-9}
0.2	0.160085	0.160085	4.44×10^{-10}	0.240192	0.240192	1.50×10^{-9}
0.4	0.170096	0.170096	5.33×10^{-10}	0.230176	0.23017	1.32×10^{-9}
0.6	0.180108	0.180108	6.33×10^{-10}	0.220161	0.220161	1.15×10^{-9}
0.8	0.190120	0.190120	7.45×10^{-10}	0.21014	0.210147	1.00×10^{-9}
1	0	0	0	0	0	0

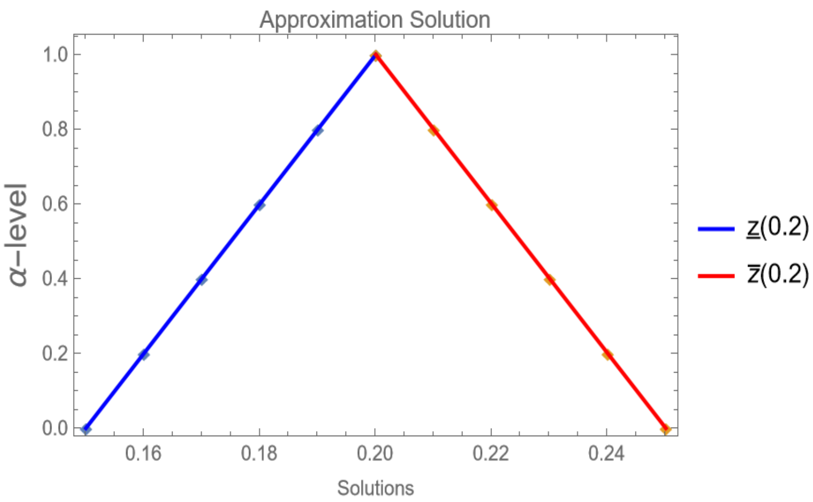

Figure 2. The lower and upper bounds fuzzy approximation solutions of Eq (43) by FE-HPM with 4 HPM terms at $s = 0.2$ for all $\alpha \in [0,1]$.

Example 3: Consider the non-linear FFVIDEs second kind:

$$z'(s,\alpha) = f(s,\alpha) + \int_0^s \int_0^1 z^2(t,\alpha)dtdr, \tag{43}$$

with $z(0,\alpha) = (0,0)$, where $0 \leq t \leq 1$, $0 \leq \alpha \leq 1$, $k(s,t) = 1$, and $f(s,\alpha) = \left(\underline{f}(s,\alpha), \overline{f}(s,\alpha)\right)$, i.e.,

$$f(s,\alpha) = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha].$$

To solve Eq (43) by FE-HPM, we first create the following homotopy:

$$\begin{cases} H(\underline{Z},p,\alpha) = \underline{Z}'(s,\alpha) - (0.75+0.25\alpha) - p\int_0^s\int_0^1 \underline{Z}^2(t,\alpha)dtdr = 0, \\ H(\overline{Z},p,\alpha) = \overline{Z}'(s,\alpha) - (1.25-0.25\alpha) - p\int_0^s\int_0^1 \overline{Z}^2(t,\alpha)dtdr = 0. \end{cases} \tag{44}$$

Now, by taking the Elzaki transform and using the differential property of the Elzaki transform on both sides of Eq (44), we get the following:

$$\begin{cases} E\{\underline{Z}(s,\alpha)\} = \left\{v^3(0.75+0.25\alpha) + vE\left\{p\int_0^s\int_0^1 \underline{Z}^2(t,\alpha)dtdr\right\}\right\}, \\ E\{\overline{Z}(s,\alpha)\} = \left\{v^3(1.25-0.25\alpha) + vE\left\{p\int_0^s\int_0^1 \overline{Z}^2(t,\alpha)dtdr\right\}\right\}. \end{cases} \tag{45}$$

Now, by taking the inverse Elzaki transform on both sides of Eq (45), we get the following:

$$\begin{cases} \underline{Z}(s,\alpha) = E^{-1}\left\{v^3(0.75+0.25\alpha) + vE\left\{p\int_0^s\int_0^1 \underline{Z}^2(t,\alpha)dtdr\right\}\right\}, \\ \overline{Z}(s,\alpha) = E^{-1}\left\{v^3(1.25-0.25\alpha) + vE\left\{p\int_0^s\int_0^1 \overline{Z}^2(t,\alpha)dtdr\right\}\right\}. \end{cases} \tag{46}$$

Now, the solution of Eq (46) can be written as power series in p as follows:

$$\begin{cases} \underline{Z}(s,\alpha) = \sum_{i=0}^{\infty} p^i \underline{Z}_i(s,\alpha), \\ \overline{Z}(s,\alpha) = \sum_{i=0}^{\infty} p^i \overline{Z}_i(s,\alpha). \end{cases} \tag{47}$$

And, let the non-linear term can be represented as

$$\begin{cases} \underline{Z}^2(t,\alpha) = \sum_{i=0}^{\infty} p^i \underline{A}_i(s,\alpha), \\ \overline{Z}^2(t,\alpha) = \sum_{i=0}^{\infty} p^i \overline{A}_i(s,\alpha), \end{cases} \tag{48}$$

where $\underline{A}_i(t,\alpha)$ and $\overline{A}_i(t,\alpha)$ are the Adomian polynomials for the non-linear terms $\underline{Z}^2(t,\alpha)$ and $\overline{Z}^2(t,\alpha)$, respectively, and they are given by

$$\underline{A}_0(s,\alpha) = \underline{Z}_0{}^2(s,\alpha), \quad \overline{A}_0(s,\alpha) = \overline{Z}_0{}^2(s,\alpha).$$

$$\underline{A}_1(s,\alpha) = 2\,\underline{Z}_0(s,\alpha)\underline{Z}_1(s,\alpha),\ \overline{A}_1(s,\alpha) = 2\,\overline{Z}_0(s,\alpha)\overline{Z}_1(s,\alpha).$$

Now, by substituting Eqs (47) and (48) into Eq (46) and comparing coefficients such as the power of p. Therefore, using the above iterative results, the series form solution is given as

$$p^0: \begin{cases} \underline{Z}_0(s,\alpha) = s\,(0.75 + 0.25\alpha), \\ \overline{Z}_0(s,\alpha) = s(1.25 - 0.25\alpha). \end{cases} \tag{49}$$

$$p^1: \begin{cases} \underline{Z}_1(s,\alpha) = \frac{s^2}{6}\,(0.75 + 0.25\alpha)^2, \\ \overline{Z}_1(s,\alpha) = \frac{s^2}{6}(1.25 - 0.25\alpha)^2. \end{cases} \tag{50}$$

$$p^2: \begin{cases} \underline{Z}_2(s,\alpha) = \frac{s^2}{24}\,(0.75 + 0.25\alpha)^3, \\ \overline{Z}_2(s,\alpha) = \frac{s^2}{24}\,(1.25 - 0.25\alpha)^3. \end{cases} \tag{51}$$

And so on….

Currently, the answer is given as

$$\begin{cases} \underline{z}(s,\alpha) = \sum_{i=0}^{\infty} \underline{Z}_i(s,\alpha), \\ \overline{z}(s,\alpha) = \sum_{i=0}^{\infty} \overline{Z}_i(s,\alpha). \end{cases} \tag{52}$$

Therefore, using the above iterative results, the series-form solution is given as

$$\begin{cases} \underline{z}(s,\alpha) = s\,(0.75 + 0.25\alpha) + \frac{s^2}{6}\,(0.75 + 0.25\alpha)^2 + \frac{s^2}{24}\,(0.75 + 0.25\alpha)^3 + \cdots \\ \overline{z}(s,\alpha) = s(1.25 - 0.25\alpha) + \frac{s^2}{6}(1.25 - 0.25\alpha)^2 + \frac{s^2}{24}\,(1.25 - 0.25\alpha)^3 + \cdots \end{cases} \tag{53}$$

Table 4 and Figure 3 show that the approximation of Eq (43) by FE-HPM at $s = 0.2$ for $\alpha \in [0,1]$ attains the triangular fuzzy number shape, and thus satisfies the fuzzy number properties. Also, the results obtained using FE-HPM show that the method is accurate, and the results confirm our theoretical analysis.

Table 4. The lower and upper bounds fuzzy aproximation solutions of Eq (43) by FE-HPM with 3 HPM terms at $s = 0.2$ and FL-ADM with 4 ADM terms for all $\alpha \in [0,1]$.

	Lower fuzzy solution			Upper fuzzy solution		
(α)	FL-ADM	FE-HPM	Absolute Error	FL-ADM	FE-HPM	Absolute Error
0	0.154453	0.154453	1.66×10^{-10}	0.258463	0.258463	1.28×10^{-12}
0.2	0.165120	0.165120	2.16×10^{-10}	0.247679	0.247679	1.09×10^{-12}
0.4	0.175840	0.175840	2.75×10^{-10}	0.236943	0.236943	9.23×10^{-12}
0.6	0.186615	0.186615	3.46×10^{-10}	0.226251	0.226251	7.72×10^{-12}
0.8	0.197445	0.197445	3.46×10^{-10}	0.216245	0.216245	6.41×10^{-12}
1	0.208333	0.208333	5.27×10^{-10}	0.205000	0.205000	5.27×10^{-12}

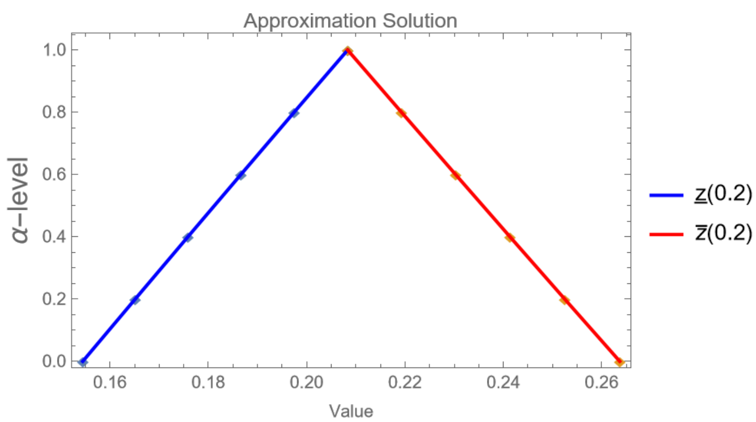

Figure 3. 2D plot of the fuzzy approximate solutions of Eq (43) by FE-HPM with 3 HPM terms at $s = 0.2, n = 4$ for all $\alpha \in [0,1]$.

With regard to the CPU time to run the algorithm, all computations for the FE-HPM algorithm were performed using Wolfram Mathematica 13 on a Windows 10 computer equipped with an Intel Core i7 processor and 16 GB of RAM. The execution time required to obtain the approximate fuzzy solutions was minimal: each example ran in well under 0.05 seconds when using four HPM terms. These results indicate that the proposed FE-HPM method is computationally efficient and requires only minimal computational resources. Furthermore, the present study is limited to fuzzy integro-differential equations that satisfy Hukuhara differentiability, and the analysis focuses on particular types of Volterra and mixed Fredholm–Volterra structures. In addition, the convergence and stability of the FE-HPM approach have been demonstrated numerically, but a rigorous theoretical proof for the fuzzy HPM series remains an open issue. These aspects will be examined in future investigations.

7. Conclusions

In this paper, the FE-HPM is provided as an efficient and effective approach for solving several types of FIDEs under Hukuhara differentiability. The proposed method using a hybrid method between fuzzy Elzaki transform with homotopy perturbation method is provided as a new and reliable method to solve both linear and non-linear FVIDEs and FFIDEs, in addition to nonlinear FFVIDEs, which are used in fields involving unclear and uncertainty. The results obtained demonstrate that FE-HPM provides a quick convergence to exact solutions with significantly lower iterations compared to classical numerical methods. This makes it a fundamental tool for solving these problems. This method of handling fuzzy systems makes it intrinsically valuable for obtaining reliable results in a short time. Future research may extend the FE-HPM method to broader classes of fuzzy integro-differential equations, including rigorous convergence and stability analysis, performing deeper comparisons with other fuzzy numerical techniques, and applying the approach to real-world fuzzy models.

Author contributions

Hamzeh Zureigat: Conceptualization, methodology, software, validation, formal analysis, investigation, writing—original draft preparation, writing-review and editing; Murad Algazo: Formal analysis, resources, writing—original draft preparation, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence tools in the creation of this article.

Conflicts of interest

The authors declare no conflicts of interest.

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References

1. Y. N. Grigoriev, N. H. Ibragimov, V. F. Kovalev, S. V. Meleshko, *Symmetries of integro-differential equations*, Dordrecht: Springer, 2010. https://doi.org/10.1007/978-90-481-3797-8
2. V. Volterra, *Theory of functionals and of integral and integro-differential equations*, New York: Dover, 1959.
3. D. S. Naidu, *Optimal control systems*, Boca Raton: CRC Press, 2003.
4. V. Lakshmikantham, M. R. M. Rao, *Theory of integro-differential equations*, New York: CRC Press, 1995.

5. I. Fredholm, Sur une classe d'équations fonctionnelles, *Acta Math.*, **27** (1903), 365–390. https://doi.org/10.1007/BF02421317
6. T. M. Elzaki, the new integral transform Elzaki transform, *Glob. J. Pure Appl. Math.*, **7** (2011), 57–64.
7. J. H. He, Homotopy perturbation technique, *Comput. Methods. Appl. Mech. Eng.*, **178** (1999), 257–262. https://doi.org/10.1016/S0045-7825(99)00018-3
8. J. H. He, Comparison of homotopy perturbation method and homotopy analysis method, *Appl. Math. Comput.*, **156** (2004), 527–539. https://doi.org/10.1016/j.amc.2003.08.008
9. S. Kambalimath, P. C. Deka, A basic review of fuzzy logic applications in hydrology and water resources, *Appl. Water Sci.*, **10** (2020), 191. https://doi.org/10.1007/s13201-020-01276-2
10. D. T. Muhamediyeva, Approaches to the numerical solving of fuzzy differential equations, *Int. J. Res. Eng. Technol.*, **3** (2014), 335–342. https://doi.org/10.15623/ijret.2014.0307057
11. F. Baig, M. S. Khan, Y. Noor, M. Imran, Design model of fuzzy logic medical diagnosis control system, *Int. J. Comput. Sci. Eng.*, **3** (2011), 2093–2108.
12. K. Nemati, M. Matinfar, An implicit method for fuzzy parabolic partial differential equations, *The J. Nonlinear Sci. Appl.*, **1** (2008), 61–71. https://doi.org/10.22436/jnsa.001.02.02
13. H. T. Nguyen, C. Walker, E. A. Walker, *A first course in fuzzy logic*, New York: Chapman and Hall/CRC, 2018. https://doi.org/10.1201/9780429505546
14. S. P. Mondal, T. K. Roy, System of differential equation with initial value as triangular intuitionistic fuzzy number and its application, *Int. J. Appl. Comput. Math.*, **1** (2015), 449–474. https://doi.org/10.1007/s40819-015-0026-x
15. Q. Zhou, H. Li, L. Wang, R. Lu, Prescribed performance observer-based adaptive fuzzy control for nonstrict-feedback stochastic nonlinear systems, *IEEE Trans. Syst. Man Cybern. Syst.*, **48** (2017), 1747–1758. https://doi.org/10.1109/TSMC.2017.2738155
16. S. S. Behzadi, B. Vahdani, T. Allahviranloo, S. Abbasbandy, Application of fuzzy Picard method for solving fuzzy quadratic Riccati and fuzzy Painlevé I equations, *Appl. Math. Model.*, **40** (2016), 8125–8137. https://doi.org/10.1016/j.apm.2016.05.003
17. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
18. H. V. Long, N. T. K. Son, N. V. Hoa, Fuzzy fractional partial differential equations in partially ordered metric spaces, *Iran. J. Fuzzy Syst.*, **14** (2017), 107–126. https://doi.org/10.22111/ijfs.2017.3136
19. O. Akın, O. Oruc, A prey and predator model with fuzzy initial values, *Hacet. J. Math. Stat.*, **41** (2012), 387–395.
20. A. Torres, J. J. Nieto, Fuzzy logic in medicine and bioinformatics, *Biomed Res. Int.*, **2006** (2006), 091908. https://doi.org/10.1155/JBB/2006/91908
21. P. Guttorp, Fuzzy mathematical models in engineering and management science, *Technometrics*, **32** (1990), 238. https://doi.org/10.1080/00401706.1990.10484661
22. S. Chakraverty, S. Tapaswini, D. Behera, *Fuzzy differential equations and applications for engineers and scientists*, Boca Raton: CRC Press, 2016. https://doi.org/10.1201/9781315372853
23. O. Abu Arqub, R. Mezghiche, B. Maayah, Fuzzy M-fractional integrodifferential models: theoretical existence and uniqueness results, and approximate solutions utilizing the Hilbert reproducing kernel algorithm, *Front. Phys.*, **11** (2023), 1252919. https://doi.org/10.3389/fphy.2023.1252919

24. R. Goetschel, W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets Syst.*, **18** (1986), 31–43. https://doi.org/10.1016/0165-0114(86)90026-6
25. D. Dubois, H. Prade, Systems of linear fuzzy constraints, *Fuzzy Sets Syst.*, **3** (1980), 37–48. https://doi.org/10.1016/0165-0114(80)90004-4
26. M. L. Puri, D. A. Ralescu, L. Zadeh, Fuzzy random variables, In: *Readings in fuzzy sets for intelligent systems*, 1993, 265–271. https://doi.org/10.1016/b978-1-4832-1450-4.50029-8
27. M. Hukuhara, Integration des applications mesurables dont la valeur est un compact convexe, *Funkcial. Ekvac.*, **10** (1967), 205–223.
28. S. S. Mansouri, N. Ahmady, A numerical method for solving nth-order fuzzy differential equation by using characterization theorem, *Commun. Nonlinear Sci.*, **2012** (2012), cna-00054.
29. L. Stefanini, L. Sorini, M. L. Guerra, Parametric representation of fuzzy numbers and application to fuzzy calculus, *Fuzzy Sets Syst.*, **157** (2006), 2423–2455. https://doi.org/10.1016/j.fss.2006.02.002
30. M. Friedman, M. Ma, A. Kandel, Numerical solutions of fuzzy differential and integral equations, *Fuzzy Sets Syst.*, **106** (1999), 35–48. https://doi.org/10.1016/s0165-0114(98)00355-8
31. H. C. Wu, The improper fuzzy Riemann integral and its numerical integration, *Inf. Sci.*, **111** (1998), 109–137. https://doi.org/10.1016/s0020-0255(98)00016-4
32. T. M. Elzaki, S. M. Elzaki, On the Tarig Transform and ordinary differential equation with variable coefficients, *Elixir Appl. Math.*, **38** (2011), 4250–4252.
33. H. Kim, The time shifting theorem and the convolution for Elzaki transform, *Int. J. Pure Appl. Math.*, **87** (2013), 261–271. https://doi.org/10.12732/ijpam.v87i2.6
34. A. Ghorbani, Beyond Adomian polynomials: He polynomials, *Chaos Soliton Fract.*, **39** (2009), 1486–1492. https://doi.org/10.1016/j.chaos.2007.06.034
35. H. M. Ahmed, An advanced approach for numerical solution of a class of Fredholm-Volterra integro-differential equations with mixed boundary conditions, *Int. J. Mod. Phys. C*, 2025. https://doi.org/10.1142/S0129183125501542
36. S. M. Hosseini, S. Shahmorad, Numerical piecewise approximate solution of Fredholm integro-differential equations by the Tau method, *Appl. Math. Model.*, **29** (2005), 1005–1021. https://doi.org/10.1016/j.apm.2005.02.003
37. S. Liao, On the homotopy analysis method for nonlinear problems, *Appl. Math. Comput.*, **147** (2004), 499–513. https://doi.org/10.1016/S0096-3003(02)00790-7
38. O. Nave, V. Gol'dshtein, A combination of two semi-analytical methods called "singular perturbed homotopy analysis method (SPHAM)" applied to combustion of spray fuel droplets, *Cogent Math.*, **3** (2016), 1256467. https://doi.org/10.1080/23311835.2016.1256467
39. T. Liu, Porosity reconstruction based on Biot elastic model of porous media by homotopy perturbation method, *Chaos Soliton Fract.*, **158** (2022), 112007. https://doi.org/10.1016/j.chaos.2022.112007