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Research article

Exponential Hardy spaces and applications

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Abstract: We introduce some Hardy spaces built on exponential Orlicz functions. We use these Hardy-type spaces to study the mapping properties of the Cesáro operators and the Cauchy transform.

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1. Introduction

In this paper, we introduce the Hardy spaces built on some exponential Orlicz functions. We call these Hardy-type spaces the exponential Hardy spaces. They are generalizations of the classical Hardy spaces H^p.

The classical Hardy spaces were introduced by Riesz in [1]. Since the introduction of Hardy space, it has became one of the most important function spaces in complex analysis, real analysis, and harmonic analysis. The reader is referred to [2–4] for the history, development, and results on the classical Hardy spaces. In particular, one of the major topics of the Hardy spaces is the study of the operators on Hardy spaces, such as the Hankel operator [5, 6], the Toeplitz operator, the Cauchy transform [7, 8], and the Cesáro operator [9, 10].

Notice that the mapping properties of the above operators are for H^p with $p < \infty$, such as the Cauchy transform and the Cesáro operator [1,8,11]. In this paper, we use the exponential Hardy spaces to extend the mapping properties of the Cesáro operators and the Cauchy transform when $p \to \infty$. In particular, we establish some mapping properties of the Cesáro operators and the Cauchy transform for H^∞. We obtain these extensions by using a characterization of the exponential Orlicz spaces. These characterizations already yield the mapping properties for the Bergman projections and the Berezin transforms on the exponential Bergman spaces in [12].

This paper is organized as follows. Section 2 contains the definitions and the boundedness of the Cesáro operator and the Cauchy transform on the classical Hardy spaces. The exponential Hardy spaces are defined in Section 3. It also contains the mapping properties of the Cesáro operator and the Cauchy transform on the exponential Hardy spaces.

2. Preliminaries and definitions

Let $\mathbb{D}$ and $\partial\mathbb{D} = \mathbb{T}$ be the unit disc and the unit circle on $\mathbb{C}$, respectively. Let $H(\mathbb{D})$ be the set of holomorphic functions on $\mathbb{D}$. Let $d\sigma$ be the normalized Lebesgue measure on $\partial\mathbb{D}$. That is, $\int_{\partial\mathbb{D}} d\sigma = 1$. When $\partial\mathbb{D}$ is parameterized by $\theta \to e^{i\theta}$, the measure $d\sigma$ is given by $\frac{d\theta}{2\pi}$.

We recall the definition of the classical Hardy space H^p.

Definition 2.1. Let $p \in (0, \infty)$. The Hardy space H^p consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H^p} = \sup_{r\in(0,1)} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty.$$

The Hardy space H^∞ consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H^\infty} = \sup_{r\in(0,1),\theta\in(0,2\pi]} |f(re^{i\theta})| < \infty.$$

For the studies of the Hardy spaces H^p, the reader is referred to [1, 2, 13].

For any $f \in H(\mathbb{D})$, the Cesáro operator Cf is defined as

$$Cf(z) = \frac{1}{z} \int_0^z \frac{f(\xi)}{1-\xi} d\xi, \quad z \in \mathbb{D},$$

see [9, 10] and [14, Section 14].

The reader is referred to [10, 14] for the applications and related topics of the Cesáro operator. The following are the boundedness of the Cesáro operator on the Hardy spaces. It is proved in [15] by using the semigroups of weighted composition operators.

Theorem 2.1. *Let $p \in [1, \infty)$. The Cesáro operator $\mathsf{C} : H^p \to H^p$ is bounded.*

(1) If $p \in [2, \infty)$, then $\|\mathsf{C}\|_{H^p \to H^p} = p$.
(2) If $p \in [1, 2)$, then $p \leq \|\mathsf{C}\|_{H^p \to H^p} \leq 2$.

The Cauchy transform is defined as

$$Cg(z) = \int_{\partial\mathbb{D}} \frac{g(\eta)}{1 - \bar{\eta} z} d\sigma(\eta), \quad g \in L^1(\partial\mathbb{D}).$$

It is well known that the Cauchy transform C is bounded from $L^p(\partial\mathbb{D})$ to H^p.

The following result gives the mapping properties and the norm estimate for the Cauchy transform.

Theorem 2.2. *Let $1 < p < \infty$. For any $f \in L^p(\partial\mathbb{D})$*

$$\|Cf\|_{H^p} \leq \csc \frac{\pi}{p} \|f\|_{L^p(\partial\mathbb{D})}.$$

The proof of Theorem 2.2 follows the norm estimate for the Riesz projection obtained in [11]. In [16], it is shown that the norm estimate for the Cauchy transform is equivalent to the Riesz projection, and the norm estimate for the Riesz projection was obtained in [11].

3. Exponential Hardy spaces

In this section, we introduce the exponential Orlicz spaces and the exponential Hardy spaces. We use the exponential Hardy spaces to establish the mapping properties of the Cauchy transform and the Cesáro operator on the exponential Hardy spaces. Especially, we obtain the mapping properties of the Cauchy transform and the Cesáro operator on H^∞.

We begin with the definitions of the exponential Orlicz spaces on $\partial\mathbb{D}$.

Definition 3.1. Let $\alpha > 0$ and $\gamma \in \mathbb{R}$. The exponential Orlicz space E_α consists of all Lebesgue measurable functions f on $\partial\mathbb{D}$ satisfying

$$\|f\|_{E_\alpha} = \inf\left\{\lambda > 0 : \frac{1}{2\pi}\int_0^{2\pi}(\exp((|f(e^{i\theta})|/\lambda)^\alpha) - 1)d\theta \le 1\right\} < \infty.$$

The exponential Orlicz space $\mathcal{E}_\alpha$ consists of all Lebesgue measurable functions f on $\mathbb{T}$ satisfying

$$\|f\|_{\mathcal{E}_\alpha} = \inf\left\{\lambda > 0 : \frac{1}{2\pi}\int_0^{2\pi}(\exp(\exp((|f(e^{i\theta})|/\lambda)^\alpha)) - e)d\theta \le 1\right\} < \infty.$$

The exponential Orlicz space $EL_{\alpha,\gamma}$ consists of all Lebesgue measurable functions f on $\mathbb{T}$ satisfying

$$\|f\|_{EL_{\alpha,\gamma}} = \inf\{\lambda > 0 : \rho(|f|/\lambda) \le 1\} < \infty,$$

where

$$\rho(f) = \frac{1}{2\pi}\int_0^{2\pi}(\exp((|f(e^{i\theta})|)^\alpha(1 + |\log(|f(e^{i\theta})|)|)^{\alpha/\gamma}) - 1)d\theta.$$

The function spaces E_α, $\mathcal{E}_\alpha$ and $EL_{\alpha,\gamma}$ had been used in [12, 17, 18] for the studies of the martingale inequalities, the probabilistic inequalities, and the mapping properties of the Bergman projections and the Berezin transforms.

It is easy to see that

$$L_\infty \hookrightarrow E_\alpha \hookrightarrow \mathcal{E}_\alpha. \tag{3.1}$$

We have the following characterizations of the exponential Orlicz spaces E_α, $\mathcal{E}_\alpha$, and $EL_{\alpha,\gamma}$.

Proposition 3.1. *Let $\alpha > 0$ and $\gamma \in \mathbb{R}$.*

(1) For any fixed $k_0 \in \mathbb{N}$, if $\sup_{p\in\mathbb{N},p\ge k_0} p^{-1/\alpha}\|f\|_{L_p} < \infty$, then $f \in E_\alpha$. Moreover, there exist constants $C_0, C_1 > 0$ such that for all $f \in E_\alpha$, we have

$$C_0\|f\|_{E_\alpha} \le \sup_{p\in\mathbb{N},p\ge k_0} p^{-1/\alpha}\|f\|_{L_p} \le C_1\|f\|_{E_\alpha}. \tag{3.2}$$

(2) For any fixed $k_0 \in \mathbb{N}$, if $\sup_{p\in\mathbb{N}\setminus\{1\},p\ge k_0}(e + \log p)^{-1/\alpha}\|f\|_{L_p} < \infty$, then $f \in \mathcal{E}_\alpha$. Moreover, there exist constants $C_0, C_1 > 0$ such that for all $f \in \mathcal{E}_\alpha$, we have

$$C_0\|f\|_{\mathcal{E}_\alpha} \le \sup_{p\in\mathbb{N}\setminus\{1\},p\ge k_0}(e + \log p)^{-1/\alpha}\|f\|_{L_p} \le C_1\|f\|_{\mathcal{E}_\alpha}. \tag{3.3}$$

(3) *For any fixed* $k_0 \in \mathbb{N}$, *if* $\sup_{p\in\mathbb{N}\setminus\{1\}, p\geq k_0} \frac{(e+\log p)^{1/\gamma}}{p^{1/\alpha}}\|f\|_{L_p} < \infty$, *then* $f \in EL_{\alpha,\gamma}$. *Moreover, there exist constants* $C_0, C_1 > 0$ *such that for all* $f \in EL_{\alpha,\gamma}$, *we have*

$$C_0\|f\|_{EL_{\alpha,\gamma}} \leq \sup_{p\in\mathbb{N}\setminus\{1\}, p\geq k_0} \frac{(e+\log p)^{1/\gamma}}{p^{1/\alpha}}\|f\|_{L_p} \leq C_1\|f\|_{EL_{\alpha,\gamma}}. \tag{3.4}$$

The reader is referred to [19, Corollary 3.2], [20, (2.1)] and [13, Chapter VI, Exercise 17] for the proofs of the above characterizations of exponential Orlicz spaces.

We now define the exponential Hardy spaces.

Definition 3.2. Let $\alpha > 0$ and $\gamma \in \mathbb{R}$. The exponential Hardy space $H(E_\alpha)$ consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H(E_\alpha)} = \sup_{r\in(0,1)} \|f(r\cdot)\|_{E_\alpha} < \infty.$$

The exponential Hardy space $H(\mathcal{E}_\alpha)$ consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H(\mathcal{E}_\alpha)} = \sup_{r\in(0,1)} \|f(r\cdot)\|_{\mathcal{E}_\alpha} < \infty.$$

The exponential Hardy space $H(EL_{\alpha,\gamma})$ consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H(EL_{\alpha,\gamma})} = \sup_{r\in(0,1)} \|f(r\cdot)\|_{EL_{\alpha,\gamma}} < \infty.$$

The above definitions are well-defined, as the radial restriction $f(re^{i\theta})$ is measurable.

As the radial limits preserve embedding, in view of (3.1), we find that

$$H^\infty \hookrightarrow H(E_\alpha) \hookrightarrow H(\mathcal{E}_\alpha).$$

We recall the Nevanlinna class in the following.

Definition 3.3. Let $f \in H(\mathbb{D})$. We write $f \in N(\mathbb{D})$ if

$$\|f\|_{N(\mathbb{D})} = \sup_{r\in(0,1)} \frac{1}{2\pi}\int_{\partial\mathbb{D}} \log^+|f(re^{i\theta})|d\theta < \infty$$

where $\log^+(x) = \max(0, \log x)$, $x \in (0,\infty)$.

We see that for any $x > 0$,

$$\log^+(x) < e^x - 1, \text{ and } \log^+(x) < e^{x^\alpha(1+|\log(x)|^{\alpha/\gamma}} - 1.$$

We also have

$$\log^+(x) < e^{e^{x^\alpha}} - e$$

because $\alpha > 0$ gives

$$e^{e^{x^\alpha}} > e^{1+x^\alpha} = ee^{x^\alpha} > e(1+x^\alpha) > e + \log x, \quad x > 1.$$

Therefore,

$$H(E_\alpha), H(\mathcal{E}_\alpha), H(EL_{\alpha,\gamma}) \subset N(\mathbb{D}).$$

As a well-known fact from the Nevanlinna class $N(\mathbb{D})$ [13, Theorem 5.3], we have the following result for the exponential Hardy spaces.

Theorem 3.2. *Let* $\alpha > 0$ *and* $\gamma \in \mathbb{R}$. *For any* $\theta \in (0, 2\pi)$ *and* $f \in H(E_\alpha) \cup H(\mathcal{E}_\alpha) \cup H(EL_{\alpha,\gamma})$, *the nontangential boundary value* $\lim_{z\to e^{i\theta}} f(z)$ *exists a.e. and*

$$f(e^{i\theta}) = \lim_{z\to e^{i\theta}} f(z).$$

4. Cesáro operators and Cauchy transform

In this section, we establish the main results of this paper, the mapping properties of the Cesáro operators and Cauchy transform on the exponential Orlicz spaces and the exponential Hardy spaces.

We begin with the characterizations of the exponential Hardy spaces. The followings are the characterizations of the exponential Hardy spaces $H(E_\alpha)$, $H(\mathcal{E}_\alpha)$, and $H(EL_{\alpha,\gamma})$.

Proposition 4.1. *Let $\alpha > 0$, $\gamma \in \mathbb{R}$, and $k_0 \in \mathbb{N}$.*

(1) If $\sup_{p\in\mathbb{N},p\geq k_0} p^{-1/\alpha}\|f\|_{H^p} < \infty$, then $f \in H(E_\alpha)$. Moreover, there exist constants $C_0, C_1 > 0$ depending on α, k_0 such that for all $f \in H(E_\alpha)$, we have

$$C_0\|f\|_{H(E_\alpha)} \leq \sup_{p\in\mathbb{N},p\geq k_0} p^{-1/\alpha}\|f\|_{H^p} \leq C_1\|f\|_{H(E_\alpha)}. \tag{4.1}$$

(2) If $\sup_{p\in\mathbb{N},p\geq k_0}(e+\log p)^{-1/\alpha}\|f\|_{H^p} < \infty$, then $f \in H(\mathcal{E}_\alpha)$. Moreover, there exist constants $C_0, C_1 > 0$ depending on α, k_0 such that for all $f \in H(\mathcal{E}_\alpha)$, we have

$$C_0\|f\|_{H(\mathcal{E}_\alpha)} \leq \sup_{p\in\mathbb{N}\setminus\{1\},p\geq k_0} (e+\log p)^{-1/\alpha}\|f\|_{H^p} \leq C_1\|f\|_{H(\mathcal{E}_\alpha)}. \tag{4.2}$$

(3) If $\sup_{p\in\mathbb{N}\setminus\{1\},p\geq k_0} \frac{(e+\log p)^{1/\gamma}}{p^{1/\alpha}}\|f\|_{H^p} < \infty$, then $f \in H(EL_{\alpha,\gamma})$, Moreover, there exist constants $C_0, C_1 >$ 0 depending on α, k_0, γ such that for all $f \in H(EL_{\alpha,\gamma})$, we have

$$C_0\|f\|_{H(EL_{\alpha,\gamma})} \leq \sup_{p\in\mathbb{N}\setminus\{1\},p\geq k_0} \frac{(e+\log p)^{1/\gamma}}{p^{1/\alpha}}\|f\|_{H^p} \leq C_1\|f\|_{H(EL_{\alpha,\gamma})}. \tag{4.3}$$

Proof. As the proofs for (4.2) and (4.3) are similar to the proof of (4.1), for simplicity, we just present the proof for (4.1).

For any $f \in H(E_\alpha)$ and $r \in (0,1)$, (3.2) assures that there exist C_0, C_1 independent of r such that

$$C_0\|f(r\cdot)\|_{E_\alpha} \leq \sup_{p\in\mathbb{N},p\geq k_0} p^{-1/\alpha}\|f(r\cdot)\|_{L_p} \leq C_1\|f(r\cdot)\|_{E_\alpha}.$$

By taking the supremum over $r \in (0,1)$, we have

$$D\sup_{r\in(0,1)}\|f(r\cdot)\|_{E_\alpha} \leq \sup_{r\in(0,1)}\sup_{p\in\mathbb{N},p\geq k_0} p^{-1/\alpha}\|f(r\cdot)\|_{L_p} \leq C\sup_{r\in(0,1)}\|f(r\cdot)\|_{E_\alpha}$$

for some $C, D > 0$ because $f(r\cdot) \in L^p$ guarantees that we can swap the supremum over r and the supremum over p. As

$$\sup_{r\in(0,1)}\sup_{p\in\mathbb{N},p\geq k_0} p^{-1/\alpha}\|f(r\cdot)\|_{L_p} = \sup_{p\in\mathbb{N},p\geq k_0} p^{-1/\alpha}\sup_{r\in(0,1)}\|f(r\cdot)\|_{L_p} = \sup_{p\in\mathbb{N},p\geq k_0} p^{-1/\alpha}\|f\|_{H^p},$$

we find that

$$C_0\|f\|_{H(E_\alpha)} \leq \sup_{p\in\mathbb{N},p\geq k_0} p^{-1/\alpha}\|f\|_{H^p} \leq C_1\|f\|_{H(E_\alpha)}$$

for some $C_0, C_1 > 0$. □

We establish the mapping properties of the Cesáro operator on the exponential Hardy spaces in the following.

Theorem 4.2. *Let $\alpha > 0$.*

(1) The Cesáro operator $C : H^\infty \to H(E_1)$ is bounded.
(2) The Cesáro operator $C : H(E_\alpha) \to H(E_{\frac{\alpha}{\alpha+1}})$ is bounded.
(3) The Cesáro operator $C : H(\mathcal{E}_\alpha) \to H(EL_{1,-\alpha})$ is bounded.

Proof. Let $f \in H^\infty$. We have $f \in H^p$ for all $p \in (1, \infty)$; therefore, Cf is well defined.

For any $r \in (0, 1)$ and fixed $k_0 \in \mathbb{N}$, we find that $\sup_{p\in\mathbb{N},p\geq k_0} \|f(r\cdot)\|_{L_p} = \|f(r\cdot)\|_{L_\infty}$. Consequently,

$$\begin{aligned}\sup_{p\in\mathbb{N},p\geq k_0} \|f\|_{H^p} &= \sup_{p\in\mathbb{N},p\geq k_0} \sup_{r\in(0,1)} \|f(r\cdot)\|_{L_p} = \sup_{r\in(0,1)} \sup_{p\in\mathbb{N},p\geq k_0} \|f(r\cdot)\|_{L_p} \\ &= \sup_{r\in(0,1)} \|f(r\cdot)\|_{L_\infty} = \|f\|_{H^\infty}. \end{aligned} \tag{4.4}$$

According to Theorem 2.1, we find that

$$p^{-1}\|Cf\|_{H^p} \leq K\|f\|_{H^p} \tag{4.5}$$

for some $K > 0$ independent of p and f. By taking the supremum over $p \in \mathbb{N}$ with $p \geq k_0$ on both sides of the above inequality and using (4.4), we obtain

$$\sup_{p\in\mathbb{N},p\geq k_0} p^{-1}\|Cf\|_{H^p} \leq K \sup_{p\in\mathbb{N},p\geq k_0} \|f\|_{H^p} = K\|f\|_{H^\infty}.$$

Thus, $\sup_{p\in\mathbb{N},p\geq k_0} p^{-1}\|Cf\|_{H^p} < \infty$, Item (1) of Proposition 4.1 yields $Cf \in H(E_1)$ and

$$\|Cf\|_{H(E_1)} \leq K\|f\|_{H^\infty}$$

for some $K > 0$.

Let $f \in H(E_\alpha)$. As $H(E_\alpha) \hookrightarrow H^p$ for all $p \in (1, \infty)$, Cf is well defined. By multiplying $p^{-\frac{1}{\alpha}}$ on both sides of (4.5), we find that

$$p^{-\frac{1}{\alpha}-1}\|Cf\|_{H^p} \leq Kp^{-\frac{1}{\alpha}}\|f\|_{H^p}$$

for some $K > 0$ independent of p and f. By taking the supremum over $p \in \mathbb{N}$ with $p \geq k_0$ on both sides of the above inequality, (4.1) assures that

$$\sup_{p\in\mathbb{N},p\geq k_0} p^{-\frac{1}{\alpha}-1}\|Cf\|_{H^p} \leq K \sup_{p\in\mathbb{N},p\geq k_0} p^{-\frac{1}{\alpha}}\|f\|_{H^p} = K\|f\|_{H(E_\alpha)}.$$

Therefore, Item (1) of Proposition 4.1 gives $Cf \in H(E_{\frac{\alpha}{\alpha+1}})$ and

$$\|Cf\|_{H(E_{\frac{\alpha}{\alpha+1}})} \leq K\|f\|_{H(E_\alpha)}$$

for some $K > 0$.

Similarly, for any $f \in H(\mathcal{E}_\alpha)$, as $H(\mathcal{E}_\alpha) \hookrightarrow H^p$ for all $p \in (1, \infty)$, Cf is well defined. By multiplying $(e + \log p)^{-1/\alpha}$ on both sides of (4.5), we have

$$\frac{(e + \log p)^{-1/\alpha}}{p}\|Cf\|_{H^p} \leq K(e + \log p)^{-1/\alpha}\|f\|_{H^p}$$

for some $K > 0$ independent of p and f.

By taking the supremum over $p \in \mathbb{N}\backslash\{1\}$ with $p \geq k_0$ on both sides of the above inequality, (4.2) guarantees that

$$\sup_{p\in\mathbb{N},p\geq k_0} \frac{(e+\log p)^{-1/\alpha}}{p}\|Cf\|_{H^p} \leq K \sup_{p\in\mathbb{N},p\geq k_0} (e+\log p)^{-1/\alpha}\|f\|_{H^p}$$
$$= K\|f\|_{H(\mathcal{E}_\alpha)}.$$

Consequently, Item (3) of Proposition 4.1 gives $Cf \in H(EL_{1,-\alpha})$ and

$$\|Cf\|_{H(EL_{1,-\alpha})} \leq K\|f\|_{H(\mathcal{E}_\alpha)}$$

for some $K > 0$. □

We also have the mapping properties of the Cauchy transform on the exponential Hardy spaces.

Theorem 4.3. *Let $\alpha > 0$.*

(1) The Cauchy transform $C : L_\infty \to H(E_1)$ is bounded.
(2) The Cauchy transform $C : E_{\frac{\alpha}{\alpha+1}} \to H(E_\alpha)$ is bounded.
(3) The Cauchy transform $C : EL_{1,-\alpha} \to H(\mathcal{E}_\alpha)$ is bounded.

Since $\lim_{p\to\infty} \frac{\pi/p}{\sin(\pi/p)} = 1$, there is a $k_0 \in \mathbb{N}$ such that $\csc\frac{\pi}{p} \leq \frac{1}{\pi}p$ when $p > k_0$. Consequently, the above results follow from Theorem 2.2 and Proposition 4.1.

5. Conclusions

This paper introduces the exponential Hardy spaces. These exponential Hardy spaces provide applications to the boundedness of the Cauchy transform and the Cesáro operator.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

Kwok-Pun Ho is an editorial board member for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article.

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