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Research article

Leighton-Hille-Kneser-type oscillation criteria for Emden-Fowler neutral delay differential equations

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Abstract: Emden-Fowler differential equations have numerous applications in physics and engineering. In this article, we extended the classical Leighton, Hille, and Kneser types oscillation criteria for second-order linear differential equations to the second-order Emden-Fowler neutral delay differential equation

$$\left(r(t)\,|z'(t)|^{\alpha-1}\,z'(t)\right)' + q(t)\,|y(\sigma(t))|^{\beta}\,\mathrm{sgn}\,y(\sigma(t)) = 0, \quad t \geq t_0,$$

where $z(t) = y(t) + p(t)y(\tau(t))$. The criteria obtained extended and improved some well-known results reported in the literature. Moreover, several examples are provided to show the significance of the new findings.

Keywords: Emden-Fowler differential equation; neutral differential equation; delayed argument; second-order; Riccati method; Leighton-Hille-Kneser-type oscillation criteria
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1. Introduction

In this article, we consider the oscillation of the second-order Emden-Fowler neutral delay differential equation

$$\left(r(t)\,|z'(t)|^{\alpha-1}\,z'(t)\right)' + q(t)\,|y(\sigma(t))|^{\beta}\,\mathrm{sgn}y(\sigma(t)) = 0,\quad t \geq t_0, \tag{1.1}$$

where $z(t) = y(t) + p(t)y(\tau(t))$. We assume without further mention that

($\mathbf{H}_1$) α and β are positive constants;

($\mathbf{H}_2$) $p, q \in C([t_0,\infty),[0,\infty))$, $0 \leq p(t) \leq p_0 < 1$, and $q(t)$ is not eventually zero on $[t^*,\infty)$ for $t^* \geq t_0$;

($\mathbf{H}_3$) $\tau, \sigma \in C([t_0,\infty),\mathbb{R})$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

($\mathbf{H}_4$) $r \in C^1([t_0,\infty),(0,\infty))$ satisfies

$$R(t,t_0) = \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)ds$$

and $\lim_{t\to\infty} R(t,t_0) = \infty$.

A function $y(t) \in C^1([T_x,\infty),\mathbb{R})$, $T_x \geq t_0$, is called a solution of (1.1) if it has the property $r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. We only consider the nontrivial solutions of (1.1) which ensure $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (1.1) is said to be oscillatory if it has an arbitrarily large zero point on $[T_x,\infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

One of the main reasons for this study is that half-linear differential equations and neutral differential equations arise in natural sciences and engineering; see, e.g., the papers [6,7,11,13,25,32], respectively. In particular, (1.1) and its particular cases have numerous applications in mathematical, theoretical, and chemical physics; see, e.g., the papers [5, 24, 30] for the oscillation of Emden-Fowler differential equations, [4, 22, 23] for the oscillatory behavior of Emden-Fowler differential equations with a linear neutral term, and [9, 31] for the oscillatory properties of Emden-Fowler differential equations with a sublinear neutral term. The main aim of this article is to establish the classical Leighton-Hille-Kneser-type oscillation criteria for (1.1), which differ from those obtained in the cited papers and improve related results reported in [3, 13, 28].

1.1. Related work

Oscillation and delay phenomena arise in various models from real world applications; see, e.g., the papers [20, 21] for models from mathematical biology and physics where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. It has been shown that the increasing interest in oscillatory properties of solutions to different classes of second-order differential and functional differential equations; see, e.g., the monographs [1,2], the papers [8,10,28] for the oscillation of delay differential equations, the papers [12,26,27] for the oscillatory behavior of neutral differential equations, and the references cited therein. In the remaining part of this section, we briefly introduce related results which motivated our investigation. There are many research studies concerning different cases of (1.1), for instance,

$$(r(t)y'(t))' + q(t)y(t) = 0, \tag{1.2}$$

$$\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + q(t)\left|y(\sigma(t))\right|^{\alpha-1}y(\sigma(t)) = 0, \tag{1.3}$$

$$(r(t)(y'(t))^{\alpha})' + q(t)y^{\beta}(\sigma(t)) = 0, \tag{1.4}$$

$$(y(t) + p(t)y(t-\tau))'' + q(t)y(t-\sigma) = 0, \tag{1.5}$$

$$\left(r(t)\left[y(t) + p(t)y(\tau(t))\right]'\right)' + q(t)\left|y(\sigma(t))\right|^{\beta}\operatorname{sgn}y(\sigma(t)) = 0. \tag{1.6}$$

For the linear equation (1.2), Leighton [19] proved the following oscillation result:
Theorem A. *[19, Leighton-type oscillation criterion] Assume that*

$$\int_{t_0}^{\infty}\frac{1}{r(t)}dt = \int_{t_0}^{\infty}q(t)dt = \infty. \tag{1.7}$$

Then (1.2) is oscillatory.

Grammatikopoulos [14] extended and improved the Leighton-type oscillation criterion and obtained
Theorem B. *[14] Assuming that* $0 \le p(t) < 1,\ q(t) \ge 0$, *and*

$$\int_{t_0}^{\infty}q(t)[1-p(t-\sigma)]dt = \infty,$$

then (1.5) is oscillatory.

In 1995, Kusano and Wang [18] studied the half-linear delay differential equation (1.4) with $\alpha = \beta$ and presented the following result:
Theorem C. *[18, Theorem 2] Let* $\alpha = \beta$ *in (1.4) and assume that*

$$\liminf_{t\to\infty} R^{\alpha}(\sigma(t), t_0)\int_{t}^{\infty}q(s)ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}.$$

Then (1.4) is oscillatory.

We mention here that Theorem C generalizes the following famous Hille-type oscillation criterion:
Theorem D. *[15, Hille-type oscillation criterion] Let* $r(t) = 1$ *in (1.2) and assume that*

$$\liminf_{t\to\infty} t\int_{t}^{\infty}q(s)ds > \frac{1}{4}.$$

Then (1.2) is oscillatory.

Kneser [17] considered (1.2) with $r(t) = 1$, and another famous oscillation criterion is given as follows:
Theorem E. *[17, Kneser-type oscillation criterion] Let* $r(t) = 1$ *in (1.2) and assume that*

$$\liminf_{t\to\infty} t^2 q(t) > \frac{1}{4}.$$

Then (1.2) is oscillatory.

Recently, Jadlovská and Džurina [16] studied the half-linear delay differential equation (1.3), and improved the Kneser-type theorem as follows:

Theorem F. *[16, Theorem 1] Assume that*

$$\liminf_{t\to\infty} r^{\frac{1}{\alpha}}(t)R^{\alpha}(\sigma(t),t_0)\,R(t,t_0)q(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

Then (1.3) is oscillatory.

In 2006, Sun and Meng [29] obtained the following oscillation result by using the Riccati transformation.

Theorem G. *[29, Theorem 2.1] Assume that*

$$\int^{\infty}\left[R^{\alpha}(\sigma(t),t_0)\,q(t)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\sigma'(t)}{R(\sigma(t),t_0)r^{\frac{1}{\alpha}}(\sigma(t))}\right]dt=\infty.$$

Then (1.3) is oscillatory.

Now, we shall use some generalized Riccati inequalities to extend the classical Leighton-Hille-Kneser-type oscillation criteria (for second-order linear differential equations) to the Emden-Fowler neutral differential equation (1.1). To the best of our knowledge, very little is known regarding the oscillation of (1.1). The applicability and effectiveness of our theorems are illustrated by carefully selected examples. The rest of the paper is organized as follows. In Section 2, we state and prove our main results. In Section 3, we present eight examples to illustrate our results. In Section 4, we give the conclusion of this article.

2. Main results

Without loss of generality, we only deal with the positive solution of (1.1) in the proofs of our results. We also assume that the following inequalities containing the variable t hold for all sufficiently large t if there is no special note.

Lemma 1. *[4, Lemma 3] Assume that $y(t)$ is an eventually positive solution of (1.1). Then*

$$z(t)>0,\ \ z'(t)>0,\ \ \textit{and}\ \ (r(t)(z'(t))^{\alpha})'\le 0.$$

Now we have the following theorem which extends Theorems A and B.

Theorem 1. *Assume that*

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)ds=\int_{t_0}^{\infty} q(s)ds=\infty. \tag{2.1}$$

Then (1.1) is oscillatory.

Proof. Assume that (1.1) has an eventually positive solution $y(t)$. By Lemma 1, (1.1) can be written as

$$(r(t)(z'(t))^{\alpha})'+q(t)y^{\beta}(\sigma(t))=0,\ \ t\ge t_1\ge t_0. \tag{2.2}$$

Since $z(t) = y(t) + p(t)y(\tau(t))$ and $z'(t) > 0$, then

$$y(t) \geq (1 - p(t))\, z(t) \geq (1 - p_0)z(t).$$

It follows from (2.2) that

$$(r(t)(z'(t))^{\alpha})' \leq -(1 - p_0)^{\beta} q(t) z^{\beta}(\sigma(t)). \tag{2.3}$$

Define

$$u(t) = \frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(\sigma(t))}, \quad t \geq t_1.$$

Then $u(t) > 0, t \geq t_1$, and

$$u'(t) \leq -(1 - p_0)^{\beta} q(t) - \beta\sigma'(t)\frac{r(t)(z'(t))^{\alpha} z'(\sigma(t))}{z^{\beta+1}(\sigma(t))},$$

and so

$$u'(t) \leq -(1 - p_0)^{\beta} q(t), \quad t \geq t_1.$$

Integrating the above inequality from t_1 to t, we obtain

$$u(t) \leq u(t_1) - (1 - p_0)^{\beta} \int_{t_1}^{t} q(s)ds.$$

Letting $t \to \infty$ in the latter inequality, from the hypothesis (2.1), we obtain a contradiction with $u(t) > 0$. The proof is complete. □

Corollary 1. *Assuming that (1.7) holds, then (1.6) is oscillatory.*

Another crucial lemma is stated as follows.

Lemma 2. *Let $y(t)$ be a positive solution of (1.1). If*

$$\int^{\infty} \left(\frac{1}{r(s)} \int_{s}^{\infty} q(u)du \right)^{\frac{1}{\alpha}} ds = \infty, \tag{2.4}$$

then $\lim\limits_{t\to\infty} z(t) = \infty$.

Proof. Letting $y(t)$ be a positive solution of (1.1), there exists a $t_1 \geq t_0$ such that $z(t) > 0$ and $z(\sigma(t)) > 0$ for $t \geq t_1$. From Lemma 1 and Theorem 1, we see that $z'(t) > 0$ and (2.3) holds, and thus

$$(r(t)(z'(t))^{\alpha})' + (1 - p_0)^{\beta} q(t) z^{\beta}(\sigma(t)) \leq 0, \quad t \geq t_1. \tag{2.5}$$

If $z(t)$ is bounded, then there exist positive constants c_1 and c_2 such that (notice that $z'(t) > 0$)

$$0 < c_1 \leq z(t) \leq c_2, \quad c_1 \leq z(\sigma(t)) \leq c_2. \tag{2.6}$$

Integrating the inequality (2.5) from t to ∞, we obtain

$$r(t)(z'(t))^{\alpha} \geq \int_{t}^{\infty} (1-p_0)^{\beta} q(s) z^{\beta}(\sigma(s)) ds.$$

Hence

$$z'(t) \geq \left(\frac{(1-p_0)^{\beta}}{r(t)} \int_{t}^{\infty} q(s) z^{\beta}(\sigma(s)) ds \right)^{\frac{1}{\alpha}}, \quad t \geq t_1.$$

Integrating the above inequality from t_1 to t, we obtain

$$z(t) \geq (1-p_0)^{\frac{\beta}{\alpha}} \int_{t_1}^{t} \left(\frac{1}{r(s)} \int_{s}^{\infty} q(u) z^{\beta}(\sigma(u)) du \right)^{\frac{1}{\alpha}} ds.$$

Using (2.6), we have

$$c_2 \geq z(t) \geq c_1^{\frac{\beta}{\alpha}} (1-p_0)^{\frac{\beta}{\alpha}} \int_{t_1}^{t} \left(\frac{1}{r(s)} \int_{s}^{\infty} q(u) du \right)^{\frac{1}{\alpha}} ds,$$

which contradicts (2.4). Therefore, $z(t)$ is boundless. Noting that $z'(t) > 0$, we have $\lim_{t\to\infty} z(t) = \infty$. The proof is complete. □

In the following section, we use the following notation for a compact presentation of our criteria:

$$R(t) = R(t, t_*) \quad \text{and} \quad Q(t) = (1-p_0)^{\beta} q(t).$$

Theorem 2. *Equation (1.1) is oscillatory provided that one of the following conditions holds:*
(i) $\alpha = \beta$ *and*

$$\int^{\infty} \left(R^{\alpha}(\sigma(t)) Q(t) - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{\sigma'(t)}{R(\sigma(t)) r^{\frac{1}{\alpha}}(\sigma(t))} \right) dt = \infty; \tag{2.7}$$

(ii) $\alpha < \beta$, *(2.4) and (2.7) hold;*
(iii) $\alpha > \beta$, *and for all constants* $K > 0$,

$$\int^{\infty} \left(R^{\beta}(\sigma(t)) Q(t) - \left(\frac{\beta}{\beta+1} \right)^{\beta+1} \frac{K\sigma'(t)}{R(\sigma(t)) r^{\frac{1}{\alpha}}(\sigma(t))} \right) dt = \infty. \tag{2.8}$$

Proof. Let $y(t)$ be a positive solution of (1.1). From Lemma 1, we have

$$z(t) > 0, \;\; z'(t) > 0, \;\; \left(r(t)(z'(t))^{\alpha}\right)' \leq 0, \;\; t \geq t_1 \geq t_0,$$

which together with the assumption $\sigma(t) \leq t$ implies that $r(t)(z'(t))^{\alpha} \leq r(\sigma(t))(z'(\sigma(t)))^{\alpha}$. Hence

$$\frac{z'(\sigma(t))}{z'(t)} \geq \left(\frac{r(t)}{r(\sigma(t))} \right)^{\frac{1}{\alpha}}, \; t \geq t_1. \tag{2.9}$$

Let $\rho(t) \in C^1([t_0, \infty), (0, \infty))$, $\rho'(t) \geq 0$ and define

$$v(t) = \rho(t)\frac{r(t)\,(z'(t))^{\alpha}}{z^{\beta}(\sigma(t))}, \quad t \geq t_1. \tag{2.10}$$

Subsequently, $v(t) > 0$. Using (2.3), (2.9) and (2.10), we get

$$v'(t) \leq -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}v(t) - \beta\sigma'(t)\rho(t)r(t)\left(\frac{r(t)}{r(\sigma(t))}\right)^{\frac{1}{\alpha}} \frac{(z'(t))^{\alpha+1}}{z^{\beta+1}(\sigma(t))}. \tag{2.11}$$

Now we will discuss this inequality in three cases.
Case (i). Assume $\alpha = \beta$. It follows from (2.10) and (2.11) that

$$v'(t) \leq -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}v(t) - \frac{\alpha\sigma'(t)}{(\rho(t)r(\sigma(t)))^{\frac{1}{\alpha}}}v^{\frac{\alpha+1}{\alpha}}(t). \tag{2.12}$$

Let $\mu = v(t), A = \frac{\rho'(t)}{\rho(t)}$, and $B = \frac{\alpha\sigma'(t)}{(\rho(t)r(\sigma(t)))^{\frac{1}{\alpha}}}$. Note that $A \geq 0,\ B > 0$, and $\mu > 0$. Hence by (2.12) and the following inequality (see [32])

$$A\mu - B\mu^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{A^{\alpha+1}}{B^{\alpha}}, \tag{2.13}$$

we get

$$v'(t) \leq -\rho(t)Q(t) + \frac{r(\sigma(t))(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\rho(t)\sigma'(t))^{\alpha}}.$$

Letting $\rho(t) = R^{\alpha}(\sigma(t))$ in the above inequality, we have

$$v'(t) \leq -R^{\alpha}(\sigma(t))Q(t) + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(t)}{R(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))}. \tag{2.14}$$

Integrating both sides of (2.14) from t_1 to ∞, we obtain

$$\int_{t_1}^{\infty}\left(R^{\alpha}(\sigma(s))Q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\sigma'(s)}{R(\sigma(s))r^{\frac{1}{\alpha}}(\sigma(s))}\right)ds \leq v(t_1),$$

which contradicts the condition (2.7).
Case (ii). Suppose $\alpha < \beta$. By (2.11), we obtain

$$v'(t) \leq -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}v(t) - \frac{\alpha\sigma'(t)\,[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}}}{(\rho(t)r(\sigma(t)))^{\frac{1}{\alpha}}}v^{\frac{\alpha+1}{\alpha}}(t).$$

By Lemma 2, there exists a $t_2 \geq t_1$ such that

$$[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} \geq 1$$

for all sufficiently large $t \geq t_2$. Therefore, (2.12) holds. Similarly to the case $\alpha = \beta$, we get a contradiction with (2.7).
Case (iii). Assume now $\alpha > \beta$. It follows from (2.11) that

$$\begin{aligned} v'(t) \leq & -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}v(t) \\ & -\frac{\beta\sigma'(t)}{(\rho(t)r(t))^{\frac{1}{\beta}}}\left(\frac{r(t)}{r(\sigma(t))}\right)^{\frac{1}{\alpha}}[z'(t)]^{\frac{\beta-\alpha}{\beta}}v^{\frac{\beta+1}{\beta}}(t). \end{aligned} \tag{2.15}$$

Write

$$[z'(t)]^{\frac{\beta-\alpha}{\beta}} = \frac{[r(t)(z'(t))^{\alpha}]^{\frac{1}{\alpha}-\frac{1}{\beta}}}{(r(t))^{\frac{1}{\alpha}-\frac{1}{\beta}}}.$$

Noting that $[r(t)(z'(t))^{\alpha}]^{\frac{1}{\alpha}-\frac{1}{\beta}}$ is an increasing function, there exists a $t_3 \geq t_2$ and a constant $m > 0$ such that

$$[r(t)(z'(t))^{\alpha}]^{\frac{1}{\alpha}-\frac{1}{\beta}} \geq m > 0, \quad t \geq t_3.$$

Hence, (2.15) yields

$$v'(t) \leq -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}v(t) - \frac{\beta m\sigma'(t)}{\rho^{\frac{1}{\beta}}(t)r^{\frac{1}{\alpha}}(\sigma(t))}v^{\frac{\beta+1}{\beta}}(t).$$

By (2.13) we get

$$v'(t) \leq -\rho(t)Q(t) + \frac{r^{\frac{\beta}{\alpha}}(\sigma(t))\,(\rho'(t))^{\beta+1}}{(\beta+1)^{\beta+1}m^{\beta}\,(\rho(t)\sigma'(t))^{\beta}}.$$

Letting $\rho(t) = R^{\beta}(\sigma(t))$ in the above inequality, we have

$$v'(t) \leq -R^{\beta}(t)Q(t) + \frac{\beta^{\beta+1}\sigma'(t)}{m^{\beta}\,(\beta+1)^{\beta+1}\,R(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))}.$$

Integrating the latter inequality, we obtain

$$\int_{t_3}^{\infty}\left(R^{\beta}(\sigma(t))Q(t) - \left(\frac{\beta}{\beta+1}\right)^{\beta+1}\frac{K\sigma'(t)}{R(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))}\right)dt \leq v(t_3),$$

where $K = \frac{1}{m^{\beta}}$, which contradicts condition (2.8). The proof is complete. □

Remark 1. *Theorem 2 can be applied to the nonlinear neutral differential equation (1.1) and generalizes Theorem G which deals with (1.3).*

The following Hille-type oscillation criterion of (1.1) can be derived from the conditions (2.7) and (2.8) of Theorem 2.

Theorem 3. *(Hille-type oscillation criterion) Assume that one of the following conditions holds:*
(i) $\alpha = \beta$ and

$$\liminf_{t\to\infty} R^{\alpha}\left(\sigma(t)\right) \int_t^{\infty} Q(s)ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}; \tag{2.16}$$

(ii) $\alpha < \beta$, (2.4) and (2.16) hold;
(iii) $\alpha > \beta$, and

$$\liminf_{t\to\infty} R^{\alpha}(\sigma(t)) \int_t^{\infty} Q(s)ds > \frac{K\beta^{\beta}}{(\beta+1)^{\beta+1}}, \tag{2.17}$$

for all constants $K > 0$. Then (1.1) is oscillatory.

Proof. **Case (i).** Let $\alpha = \beta$. Assuming that (2.7) does not hold, then there exists a $t_1 \geq t_0$ such that for any $\varepsilon > 0$ and $t \geq t_1$, it holds

$$\int_t^{\infty} \left(R^{\alpha}(\sigma(s))Q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(s)}{R(\sigma(s))r^{\frac{1}{\alpha}}(\sigma(s))} \right) ds < \varepsilon.$$

Since $R(\sigma(t))$ is an increasing function, we conclude that

$$R^{\alpha}(\sigma(t)) \int_t^{\infty} \left(Q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(s)}{R^{\alpha+1}(\sigma(s))r^{\frac{1}{\alpha}}(\sigma(s))} \right) ds < \varepsilon,$$

or

$$R^{\alpha}(\sigma(t)) \int_t^{\infty} \left(Q(s) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \left(\frac{1}{R^{\alpha}(\sigma(s))}\right)' \right) ds < \varepsilon.$$

Then, for any $\varepsilon > 0$, it holds

$$R^{\alpha}(\sigma(t)) \int_t^{\infty} Q(s)ds < \varepsilon + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}},$$

which contradicts (2.16).
Case (ii). Assume $\alpha < \beta$. Proceeding as in the proof of Theorem 2, from (2.16), we can see that (2.7) holds. The remaining steps are similar to Case (i) in Theorem 3. Then (1.1) is oscillatory.
Case (iii). Suppose now $\alpha > \beta$. Assuming that (2.8) does not hold, then there exists a $t_2 \geq t_1$ such that for any $\varepsilon > 0$ and $t \geq t_2$, it holds

$$\int_t^{\infty} \left(R^{\beta}(\sigma(s))Q(s) - \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \frac{K\sigma'(s)}{R(\sigma(s))r^{\frac{1}{\alpha}}(\sigma(s))} \right) ds < \varepsilon.$$

It follows that

$$R^{\beta}(\sigma(t)) \int_t^{\infty} \left(Q(s) - \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \frac{K\sigma'(s)}{R^{\beta+1}(\sigma(s))r^{\frac{1}{\alpha}}(\sigma(s))} \right) ds < \varepsilon,$$

or

$$R^{\beta}(\sigma(t)) \int_t^{\infty} \left(Q(s) + \frac{\beta^{\beta}K}{(\beta+1)^{\beta+1}} \left(\frac{1}{R^{\beta}(\sigma(s))}\right)' \right) ds < \varepsilon.$$

Then, for any $\varepsilon > 0$, it holds

$$R^{\beta}(\sigma(t))\int_t^{\infty} Q(s)ds < \varepsilon + \frac{K\beta^{\beta}}{(\beta+1)^{\beta+1}},$$

which contradicts (2.17). The proof is complete. □

Remark 2. *Theorem 3 generalizes the result of [18, Theorem C]. It extends the Hille-type oscillation criterion of the half-linear delay equation to the Hille-type criterion of the nonlinear neutral delay equation. The application range of the Hille-type oscillation criterion is extended.*

Lemma 3. *Assuming that $y(t)$ is an eventually positive solution of (1.1), then*

$$\left(\frac{z(t)}{R(t)}\right)' \leq 0. \tag{2.18}$$

Proof. Let $y(t)$ be a positive solution of (1.1). From Lemma 1, we can see that $r^{\frac{1}{\alpha}}(t)z'(t)$ is nonincreasing, then

$$r^{\frac{1}{\alpha}}(s)z'(s) \geq r^{\frac{1}{\alpha}}(t)z'(t), \quad s \in [t_1, t].$$

It follows that

$$z(t) = z(t_1) + \int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)r^{\frac{1}{\alpha}}(s)z'(s)ds \geq r^{\frac{1}{\alpha}}(t)z'(t)R(t),$$

which implies that (2.18) holds. The proof is complete. □

Lemma 4. *Supposing that $\alpha > \beta$, then there exists a constant $c > 0$ such that for all sufficiently large t, it holds*

$$z^{\beta-\alpha}(t) \geq cR^{\beta-\alpha}(t).$$

Proof. From Lemma 1, we see that (2.1) holds. Then we have

$$r(t)(z'(t))^{\alpha} \leq r(t_1)(z'(t_1))^{\alpha} = M_1.$$

Therefore,

$$z(t) \leq z(t_1) + M_1^{\frac{1}{\alpha}}R(t, t_1). \tag{2.19}$$

From ($\mathbf{H}_4$), there exist sufficiently large constants $N > 0$ and $t_N > t_1$ such that

$$R(t, t_1) > N, \quad t \geq t_N.$$

It follows from (2.19) that

$$z(t) \leq z(t_1)\frac{R(t,t_1)}{N} + M_1^{\frac{1}{\alpha}}R(t,t_1) = \left(\frac{z(t_1)}{N} + M_1^{\frac{1}{\alpha}}\right)R(t,t_1) = M_2R(t,t_1).$$

Hence for $\alpha > \beta$, we obtain

$$z^{\beta-\alpha}(t) \geq cR^{\beta-\alpha}(t),$$

where $c = M_2^{\beta-\alpha}$. The proof is complete. □

Theorem 4. *Assume that one of the following conditions holds:*
(i) $\alpha = \beta$ and

$$\int^{\infty}\left(R^{\alpha}(\sigma(t))Q(t) - \frac{\alpha_1}{R(t)r^{\frac{1}{\alpha}}(t)}\right)dt = \infty; \tag{2.20}$$

(ii) $\alpha < \beta$, (2.4) and (2.20) hold;
(iii) $\alpha > \beta$, and

$$\int^{\infty}\left(R^{\beta}(\sigma(t))Q(t) - \frac{M\alpha_1}{R(t)r^{\frac{1}{\alpha}}(t)}\right)dt = \infty \tag{2.21}$$

for all constants $M > 0$, where $\alpha_1 = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$. Then (1.1) is oscillatory.

Proof. Let $y(t)$ be a positive solution of (1.1). It follows from Lemma 3 that (2.18) holds, then we have

$$\frac{z(\sigma(t))}{z(t)} \geq \frac{R(\sigma(t))}{R(t)}, \quad t \geq t_1. \tag{2.22}$$

Let $\rho(t) \in C^1([t_0, \infty), (0, \infty))$, $\rho'(t) \geq 0$ and define

$$w(t) = \rho(t)\frac{r(t)\,(z'(t))^{\alpha}}{z^{\alpha}(t)}, \quad t \geq t_1. \tag{2.23}$$

Then $w(t) > 0$. Using (2.3), (2.22), and (2.23), we have

$$\begin{aligned} w'(t) &\leq -\rho(t)Q(t)\frac{z^{\beta}(\sigma(t))}{z^{\alpha}(t)} + \frac{\rho'(t)}{\rho(t)}w(t) - \alpha\rho(t)r(t)\left(\frac{z'(t)}{z(t)}\right)^{\alpha+1} \\ &\leq -\rho(t)Q(t)\frac{z^{\beta}(\sigma(t))}{z^{\alpha}(t)} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}w^{\frac{\alpha+1}{\alpha}}(t). \end{aligned}$$

Using inequality (2.13) in the above inequality, we have

$$w'(t) \leq -\rho(t)Q(t)\frac{z^{\beta}(\sigma(t))}{z^{\alpha}(t)} + \frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}. \tag{2.24}$$

Case (i). Suppose $\alpha = \beta$. Setting $\rho(t) = R^{\alpha}(t)$ and noting that (2.22) holds, then (2.24) yields

$$w'(t) \leq -R^{\alpha}(\sigma(t))Q(t) + \frac{\alpha_1}{R(t)r^{\frac{1}{\alpha}}(t)}, \quad t \geq t_1, \tag{2.25}$$

where $\alpha_1 = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$. Integrating both sides of (2.25), we obtain

$$\int_{t_1}^{\infty}\left(R^{\alpha}(\sigma(s))Q(s) - \frac{\alpha_1}{R(s)r^{\frac{1}{\alpha}}(s)}\right)ds \leq w(t_1),$$

which contradicts (2.20).
Case (ii). Assume $\alpha < \beta$. It follows from Lemma 2 that (2.4) holds, then $\lim_{t\to\infty} z(t) = \infty$. Therefore, there exists a sufficiently large $t_2 \geq t_1$ such that $z^{\beta-\alpha}(\sigma(t)) \geq 1$ for all $t \geq t_2$. Noting that (2.22) holds, then we obtain

$$\left(\frac{z(\sigma(t))}{z(t)}\right)^{\alpha} \geq \left(\frac{R(\sigma(t))}{R(t)}\right)^{\alpha}.$$

It follows from (2.24) that

$$w'(t) \leq -\rho(t)Q(t)\left(\frac{R(\sigma(t))}{R(t)}\right)^{\alpha} + \frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}.$$

Setting $\rho(t) = R^{\alpha}(t)$ in the above inequality, we can see that (2.25) holds. The rest of the proof is similar to that of Case (i).
Case (iii). Let $\alpha > \beta$. Using Lemma 4 in (2.24) we have $z^{\beta-\alpha}(\sigma(t)) \geq cR^{\beta-\alpha}(\sigma(t))$, where $c > 0$ is a constant. Hence

$$w'(t) \leq -c\rho(t)Q(t)R^{\beta-\alpha}(\sigma(t))\left(\frac{R(\sigma(t))}{R(t)}\right)^{\alpha} + \frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}. \tag{2.26}$$

Setting $\rho(t) = R^{\alpha}(t)$ in (2.26) yields

$$w'(t) \leq -cR^{\beta}(\sigma(t))Q(t) + \frac{\alpha_1}{R(t)r^{\frac{1}{\alpha}}(t)}, \quad t \geq t_2. \tag{2.27}$$

Integrating both sides of (2.27), we have

$$\int_{t_2}^{\infty}\left(R^{\beta}(\sigma(s))Q(s) - \frac{M\alpha_1}{R(s)r^{\frac{1}{\alpha}}(s)}\right)ds \leq Mw(t_2),$$

where $M = \frac{1}{c}$, which contradicts (2.21). The proof is complete. □

Applying Theorem 4 we deduce the following Kneser-type oscillation criterion for (1.1).

Theorem 5. *(Kneser-type oscillation criterion) Assume that one of the following conditions holds:*
(i) $\alpha = \beta$ *and*

$$\liminf_{t\to\infty} r^{\frac{1}{\alpha}}(t)R(t)R^{\alpha}(\sigma(t))Q(t) > \alpha_1; \tag{2.28}$$

(ii) $\alpha < \beta$, *(2.4) and (2.28) hold;*
(iii) $\alpha > \beta$, *and*

$$\liminf_{t\to\infty} r^{\frac{1}{\alpha}}(t)R(t)R^{\beta}(\sigma(t))Q(t) > M\alpha_1 \tag{2.29}$$

for all constants $M > 0$, *where* $\alpha_1 = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$. *Then (1.1) is oscillatory.*

Proof. We prove only the case $\alpha = \beta$, the others are similar. Assuming that (2.28) holds, there exists a sufficiently large $T \ge t_0$ such that

$$r^{\frac{1}{\alpha}}(t)R(t)R^{\alpha}(\sigma(t))Q(t) \ge \alpha_1 + \varepsilon \tag{2.30}$$

for any small $\varepsilon > 0$ and all $t \ge T$. Dividing both sides of (2.30) by $R(t)r^{\frac{1}{\alpha}}(t)$ yields

$$R^{\alpha}(\sigma(t))Q(t) - \frac{\alpha_1}{R(t)r^{\frac{1}{\alpha}}(t)} > \frac{\varepsilon}{R(t)r^{\frac{1}{\alpha}}(t)}. \tag{2.31}$$

It follows from (2.31) that (2.20) holds, then (1.1) is oscillatory. The proof is complete. □

3. Examples

In this section, we provide some examples to illustrate our main results.

Example 1. *[12, Example 2.11] For $t > 5$ and $\alpha > 0$, consider the Emden-Fowler neutral differential equation*

$$\left(\frac{1}{\sqrt{t}}\left(y(t) + \frac{1}{\sqrt{t-1}}y(t-1)\right)'\right)' + \frac{t^{\alpha+1}(2+\cos t)}{(t-2)^{\alpha}}\left|y\left(\frac{t}{3}\right)\right|^{\alpha} \operatorname{sgn} y\left(\frac{t}{3}\right) = 0. \tag{3.1}$$

Observe that $p(t) = \frac{1}{\sqrt{t-1}} \le \frac{1}{2}$, $r(t) = \frac{1}{\sqrt{t}}$, and $q(t) = \frac{t^{\alpha+1}(2+\cos t)}{(t-2)^{\alpha}}$. Clearly, the condition (1.7) is satisfied. Hence, by Corollary 1, (3.1) is oscillatory.

Example 2. *[28, Example 2] Consider the nonlinear delay differential equation*

$$\left((y'(t))^{\frac{1}{3}}\right)' + t^{\lambda}y^{\frac{7}{3}}(t-2) = 0. \tag{3.2}$$

Clearly, when $\lambda \in [-1, \infty)$, the condition (2.1) of Theorem 1 is satisfied. Hence, (3.2) is oscillatory. This example has also been studied by Santra et al. [28, Example 2]. According to [28, Theorem 2], every solution of (3.2) is oscillatory or converges to zero if $\lambda = 1$.

Example 3. *[27, Eq (42)] Consider the Euler neutral equation*

$$\left(y(t) + \frac{1}{2}y(\tau_0 t)\right)'' + \frac{q_0}{t^2}y(\lambda t) = 0, \tag{3.3}$$

where $q_0 > 0$ and $\lambda, \tau_0 \in (0, 1)$.

We will use Theorem 3 to show that (3.3) is oscillatory. Note that $r(t) = 1$, $p(t) = \frac{1}{2}$, and $q(t) = \frac{q_0}{t^2}$. Thus we have

$$\liminf_{t\to\infty} R^{\alpha}(\sigma(t)) \int_t^{\infty} Q(s)ds = \liminf_{t\to\infty} \lambda t \frac{q_0}{2t} > \frac{1}{4}$$

when $q_0 > 0$ and $\lambda \in (0, 1)$. Then by Theorem 3, we conclude that (3.3) is oscillatory if $q_0 > \frac{1}{2\lambda}$.

Remark 3. *If we choose $\lambda = \frac{1}{3}$, then by Theorem 3, (3.3) is oscillatory if $q_0 > 1.5$. According to [27, Corollary 1], (3.3) is oscillatory if*

$$\lambda\frac{q_0}{2}\left(1+\frac{1}{2}\lambda q_0\right)\ln\frac{1}{\lambda} > \frac{1}{e},$$

that is, $q_0 > 1.588$. Consequently, Theorem 3 improves the results of [27, Corollary 1].

Example 4. *[13, Example 1] Consider the second-order neutral differential equation*

$$((z'(t))^{\alpha})' + \frac{q_0}{t^{\alpha+1}}y^{\alpha}(\lambda t) = 0, \tag{3.4}$$

where $z(t) = y(t) + p_0 y(\tau(t))$, $p_0 \in [0,1)$, $\lambda \in (0,1)$, $q_0 > 0$, $\tau(t) \le t$, and α is a ratio of odd positive integers.

It follows from (2.16) that

$$\liminf_{t\to\infty} R^{\alpha}(\sigma(t))\int_t^{\infty} Q(s)ds = \liminf_{t\to\infty}(\lambda t)^{\alpha}\frac{q_0(1-p_0)^{\alpha}}{\alpha t^{\alpha}}.$$

Then by Theorem 3, we see that (3.4) is oscillatory if

$$\lambda^{\alpha} q_0(1-p_0)^{\alpha} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{3.5}$$

However, according to [13, Corollary 1], (3.4) is oscillatory if

$$\lambda^{\alpha}(1-p_0)^{\alpha} q_0\frac{(\alpha+\lambda^{\alpha}(1-p_0)^{\alpha}q_0)^{\alpha}}{\alpha^{\alpha}}\ln\frac{1}{\lambda} > \frac{1}{e}. \tag{3.6}$$

Remark 4. *If we choose $\alpha = 1$, $p_0 = \frac{1}{2}$, $\lambda = \frac{1}{e}$, and $\tau(t) = t-1$, then (3.4) reads*

$$\left(y(t)+\frac{1}{2}y(t-1)\right)'' + \frac{q_0}{t^2}y\left(\frac{t}{e}\right) = 0. \tag{3.7}$$

From (3.5), we can see that (3.7) is oscillatory if

$$q_0 > \frac{e}{2} \approx 1.35914. \tag{3.8}$$

Now, according to (3.6), (3.7) is oscillatory if

$$q_0 > \sqrt{e^2+4e} - e \approx 1.55515. \tag{3.9}$$

Obviously, the condition (3.8) is superior to (3.9). In addition, in (1.1), $\alpha \ne \beta$. Consequently, Theorem 3 improves the result of [13, Corollary 1].

Example 5. *[3, Example 3] Consider the half-linear delay equation*

$$\left(t^{\alpha}\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + \frac{c}{t\left(\ln(ct)\right)^{\alpha+1}}\left|y\left(\frac{t}{2}\right)\right|^{\alpha-1}y\left(\frac{t}{2}\right) = 0, \quad t \ge 1, \tag{3.10}$$

where $c > 0$, $\alpha > 0$.

In this example, we can find that

$$R(t) = R(t, t_0) = \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)ds = \int_{1}^{t} \frac{1}{s}ds = \ln t$$

and

$$\begin{aligned}\liminf_{t\to\infty} R^{\alpha}(\sigma(t)) \int_t^{\infty} q(s)ds &= \liminf_{t\to\infty} \left(\ln \frac{t}{2}\right)^{\alpha} \int_t^{\infty} \frac{c}{s(\ln(cs))^{\alpha+1}}ds \\ &= \liminf_{t\to\infty} \frac{c\left(\ln \frac{t}{2}\right)^{\alpha}}{\alpha(\ln(ct))^{\alpha}} = \frac{c}{\alpha}.\end{aligned}$$

Then by Theorem 3(i), from (2.16), we can see that (3.10) is oscillatory if

$$c > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

And according to [3, Theorem 2.3(I_1)], (3.10) is oscillatory if

$$c > 2^{\alpha}\alpha.$$

Consequently, Theorem 3 improves [3, Theorem 2.3].

Example 6. *[28, Example 1] Consider the Emden-Fowler delay equation*

$$\left(e^{-t}\left(y'(t)\right)^{\frac{11}{3}}\right)' + \frac{1}{(t+1)^{\xi}} y^{\frac{1}{3}}(t-2) = 0, \quad t \geq 2. \tag{3.11}$$

Comparing (3.11) with (1.4), we can see that $\alpha = \frac{11}{3}$, $\beta = \frac{1}{3}$, $r(t) = e^{-t}$, $q(t) = \frac{1}{(t+1)^{\xi}}$, and $\sigma(t) = t-2$. According to [28, Example 1], every solution of (3.11) is oscillatory or converges to zero only for $\xi = 1$.

Now we use Theorem 5 to work through this example, we need to verify the conditions (2.4) and (2.29). It follows from

$$\int^{\infty} \left(\frac{1}{r(t)} \int_t^{\infty} q(s)ds\right)^{\frac{1}{\alpha}} dt = \int^{\infty} \left(e^t \int_t^{\infty} \frac{1}{(1+s)^{\xi}}ds\right)^{\frac{1}{\alpha}} dt = \infty$$

that (2.4) holds. To verify condition (2.29), we can see that $r^{-\frac{1}{\alpha}}(t) = e^{\frac{3t}{11}}$, $R(t) = R(t, t_0) = \frac{11}{3}\left(e^{\frac{3t}{11}} - e^{\frac{6}{11}}\right)$, and $R^{\beta}(\sigma(t)) = \left[\frac{11}{3}\left(e^{\frac{3(t-2)}{11}} - e^{\frac{6}{11}}\right)\right]^{\frac{1}{3}}$. Therefore, $\liminf_{t\to\infty} r^{\frac{1}{\alpha}} R(t) R^{\beta}(\sigma(t)) Q(t) = \infty$, as a result, condition (2.29) is satisfied. Then by Theorem 5, (3.11) is oscillatory for all $\xi \in \mathbb{R}$.

Therefore, Theorem 5 improves [28, Theorem 1].

Example 7. *[13, Eq (22)] Consider the half-linear delay equation*

$$\left((y'(t))^{\frac{1}{3}}\right)' + \frac{q_0}{t^{\frac{4}{3}}} y^{\frac{1}{3}}(0.9t) = 0, \quad t \geq 1. \tag{3.12}$$

In this example, $\alpha = \frac{1}{3}$, $r(t) = 1$, $q(t) = \frac{q_0}{t^{\frac{4}{3}}}$, and $\sigma(t) = 0.9t$. Then we have

$$\begin{aligned}\liminf_{t\to\infty} r^{\frac{1}{\alpha}} R(t)R^{\alpha}(\sigma(t))Q(t) &= \liminf_{t\to\infty}(t-1)(0.9t-1)^{\frac{1}{3}}\frac{q_0}{t^{\frac{4}{3}}}\\ &= \sqrt[3]{0.9}q_0 > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} = \frac{\sqrt[3]{0.25}}{4},\end{aligned}$$

which shows that condition (2.28) is satisfied. By Theorem 5, we find that (3.12) is oscillatory if $q_0 > 0.35143$. However, according to [13, Theorem 3], (3.12) is oscillatory when $q_0 > 1.92916$. Therefore, Theorem 5 improves the result of [13, Theorem 3].

Example 8. *Consider the Emden-Fowler neutral delay differential equation*

$$\left(t^c \left|z'(t)\right|^{\alpha-1} z'(t)\right)' + \frac{1}{t(\ln t)^{1+\frac{c}{2}}}\left|y\left(\frac{t}{2}\right)\right|^{\beta} \operatorname{sgn} y\left(\frac{t}{2}\right) = 0, \quad t \geq t_0, \tag{3.13}$$

where $\alpha > 0,\ \beta > 0,\ c = \min\{\alpha,\beta\}$ *and* $z(t) = y(t) + \frac{1}{2}y(t-1)$.

We claim that this equation satisfies conditions of Theorem 5. First, in (3.13), we can see that $r(t) = t^c$, $p(t) = \frac{1}{2}$, $q(t) = \frac{1}{t(\ln t)^{1+\frac{c}{2}}}$, $\sigma(t) = \frac{t}{2}$, and $\tau(t) = t-1$. Then we obtain

$$\int^{\infty}\left(\frac{1}{r(t)}\int_t^{\infty} q(s)ds\right)^{\frac{1}{\alpha}} dt = \int^{\infty}\left(\frac{2}{\alpha}\right)^{\frac{1}{\alpha}} \frac{1}{t\ln^{\frac{1}{2}} t}dt = \infty.$$

Hence the condition (2.4) is satisfied. Note that

$$\liminf_{t\to\infty} r^{\frac{1}{\alpha}}(t)R(t)R^c(\sigma(t))\,Q(t) = \liminf_{t\to\infty} t\left(\ln\frac{t}{t_0}\right)\left(\ln\frac{t}{2t_0}\right)^c \frac{1}{t(\ln t)^{1+\frac{c}{2}}} = \infty.$$

Hence (2.28) and (2.29) are satisfied. Therefore, by Theorem 5, (3.13) is oscillatory for any $\alpha > 0$ and $\beta > 0$.

4. Conclusions

In this article, we use the Riccati transformation and several inequality techniques to establish some new Leighton-Hille-Kneser-type oscillation criteria for the nonlinear neutral delay differential equation (1.1). Our Leighton-type oscillation criterion can be used to deal with the canonical equation (1.1) when q is integral. Theorem 3 extends [18, Hille-type criterion] to the nonlinear neutral equation (1.1). Theorem 5 extends the [16, Kneser-type criterion] to (1.1). Those examples given in Section 3 show that our results improve some well-known results reported recently in the literature. For future research one could consider studying third-order or higher-order Emden-Fowler neutral delay differential equations.

Author contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations,* Dordrecht: Springer, 2002. https://doi.org/10.1007/978-94-017-2515-6_1

2. R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order dynamic equations,* London: CRC Press, 2002. https://doi.org/10.4324/9780203222898

3. R. P. Agarwal, S. R. Grace, D. O'Regan, On the oscillation of second order functional differential equations, *Adv. Math. Sci. Appl.*, **12** (2002), 257–272.

4. B. Baculíková, J. Džurina, Oscillation theorems for second-order nonlinear neutral differential equations, *Comput. Math. Appl.*, **62** (2011), 4472–4478. https://doi.org/10.1016/j.camwa.2011.10.024

5. M. Bartušek, Z. Došlá, M. Marini, Oscillatory solutions of Emden-Fowler type differential equation, *Electron. J. Qual. Theo. Differ. Equ.*, **54** (2021), 1–17.

6. M. Bohner, T. S. Hassan, T. Li, Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments, *Indagat. Math.*, **29** (2018), 548–560. https://doi.org/10.1016/j.indag.2017.10.006

7. G. E. Chatzarakis, S. R. Grace, I. Jadlovská, On the sharp oscillation criteria for half-linear second-order differential equations with several delay arguments, *Appl. Math. Comput.*, **397** (2021), 125915. https://doi.org/10.1016/j.amc.2020.125915

8. G. E. Chatzarakis, I. Jadlovská, Improved oscillation results for second-order half-linear delay differential equations, *Hacet. J. Math. Stat.*, **48** (2019), 170–179.

9. J. Džurina, S. R. Grace, I. Jadlovská, T. Li, Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term, *Math. Nachr.*, **293** (2020), 910–922. https://doi.org/10.1002/mana.201800196

10. J. Džurina, I. Jadlovská, A sharp oscillation result for second-order half-linear noncanonical delay differential equations, *Electron. J. Qual. Theo. Differ. Equ.*, **46** (2020), 1–14.

11. O. Došlý, P. Řehák, *Half-linear differential equations*, Elsevier, **202** (2005).

12. L. Erbe, T. S. Hassan, A. Peterson, Oscillation of second order neutral delay differential equations, *Adv. Dyn. Syst. Appl.*, **3** (2008), 53–71.

13. S. R. Grace, J. Džurina, I. Jadlovská, T. Li, An improved approach for studying oscillation of second-order neutral delay differential equations, *J. Inequal. Appl.*, **193** (2018), 1–13. https://doi.org/10.1186/s13660-018-1767-y

14. M. K. Grammatikopoulos, G. Ladas, A. Meimaridou, Oscillations of second order neutral delay differential equations, *Rad. Mat.*, **1** (1985), 267–274.

15. E. Hille, Non-Oscillation theorems, *T. Am. Math. Soc.*, **64** (1948), 234–252. https://doi.org/10.1090/S0002-9947-1948-0027925-7

16. I. Jadlovská, J. Džurina, Kneser-type oscillation criteria for second order half-linear delay differential equations, *Appl. Math. Comput.*, **380** (2020), 125289. https://doi.org/10.1016/j.amc.2020.125289

17. A. Kneser, Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen, *Math. Ann.*, **42** (1893), 409–435. https://doi.org/10.1007/BF01444165

18. T. Kusano, J. Wang, Oscillation properties of half-linear functional differential equations of second order, *Hiroshima Math. J.*, **25** (1995), 371–385. https://doi.org/10.32917/hmj/1206127717

19. W. Leighton, The detection of the oscillation of solutions of a second order linear differential equation, *Duke Math. J.*, **17** (1950), 57–61. https://doi.org/10.1215/S0012-7094-50-01707-8

20. T. Li, D. A. Soba, A. Columbu, G. Viglialoro, Dissipative gradient nonlinearities prevent δ-formations in local and nonlocal attraction-repulsion chemotaxis models, *Stud. Appl. Math.*, **154** (2025), e70018. https://doi.org/10.1111/sapm.70018

21. T. Li, S. Frassu, G. Viglialoro, Combining effects ensuring boundedness in an attraction-repulsion chemotaxis model with production and consumption, *Z. Angew. Math. Phys.*, **74** (2023), 109. https://doi.org/10.1007/s00033-023-01976-0

22. T. Li, Y. V. Rogovchenko, Oscillation of second-order neutral differential equations, *Math. Nachr.*, **288** (2015), 1150–1162. https://doi.org/10.1002/mana.201300029

23. T. Li, Y. V. Rogovchenko, Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations, *Monatsh. Math.*, **184** (2017), 489–500. https://doi.org/10.1007/s00605-017-1039-9

24. T. Li, Y. V. Rogovchenko, On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations, *Appl. Math. Lett.*, **67** (2017), 53–59. https://doi.org/10.1016/j.aml.2016.11.007

25. T. Li, Y. V. Rogovchenko, On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations, *Appl. Math. Lett.*, **105** (2020), 106293. https://doi.org/10.1016/j.aml.2020.106293

26. O. Moaaz, New criteria for oscillation of nonlinear neutral differential equations, *Adv. Differ. Equ.*, **2019** (2019), 484. https://doi.org/10.1186/s13662-019-2418-4

27. O. Moaaz, R. A. El-Nabulsi, W. Muhsin, O. Bazighifan, Improved oscillation criteria for 2nd-order neutral differential equations with distributed deviating arguments, *Mathematics*, **8** (2020), 849. https://doi.org/10.3390/math8050849
28. S. S. Santra, O. Bazighifan, H. Ahmad, Y. M. Chu, Second-order differential equation: Oscillation theorems and applications, *Math. Probl. Eng.*, **2020** (2020), 8820066. https://doi.org/10.1155/2020/8820066
29. Y. G. Sun, F. W. Meng, Note on the paper of Džurina and Staroulakis, *Appl. Math. Comput.*, **174** (2006), 1634–1641. https://doi.org/10.1016/j.amc.2005.07.008
30. J. S. W. Wong, On the generalized Emden-Fowler equation, *SIAM Rev.*, **17** (1975), 339–360. https://doi.org/10.1137/1017036
31. Y. Wu, Y. Yu, J. Xiao, Z. Jiao, Oscillatory behaviour of a class of second order Emden-Fowler differential equations with a sublinear neutral term, *Appl. Math. Sci. Eng.*, **31** (2023), 2246098. https://doi.org/10.1080/27690911.2023.2246098
32. S. Zhang, Q. Wang, Oscillation of second-order nonlinear neutral dynamic equations on time scales, *Appl. Math. Comput.*, **216** (2010), 2837–2848. https://doi.org/10.1016/j.amc.2010.03.134