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Research article

Analysis of existence and structure of solutions for Caputo and Grünwald–Letnikov fractional differential systems

Peng E[1], Weihong Zhou[1,2,*], Tingting Xu[1], Jie Cao[1,3], Yuxia Liu[1], Shangxi Li[1], Xueliang Zhou[1] and Wei Zhou[1]

[1] School of Mathematics and Computer Science, Yunnan Minzu University, Kunming, Yunnan, 650504, China

[2] Key Laboratory for the Structure and Evolution of Celestial Objects, Chinese Academy of Sciences, Kunming, Yunnan, 650011, China

[3] Center for Astrophysics and Great Bay Center of National Astronomical Data Center, Guangzhou University, Guangzhou, 510006, China

* **Correspondence:** Email: zwh@ymu.edu.cn.

Abstract: In this paper, we investigate the existence and structure of power-type solutions for Caputo fractional differential equation systems (CFDESs) and Grünwald–Letnikov fractional differential equation systems (GLFDESs). Building on the definitions of the Caputo fractional derivative (CFD) and the Grünwald–Letnikov fractional derivative (GLFD), we derive explicit expansion formulas for the fractional differential operators, construct joint coefficient-solution matrices for the considered systems, and, from these, obtain necessary and sufficient rank conditions for the existence of m-th order power solutions. On this basis, we further (1) establish equivalent criteria that guarantee the uniqueness of the degree of power solutions and (2) derive rank-based conditions for the existence of two, and more generally p, linearly independent power particular solutions with distinct degrees. Taken together, these results provide a unified matrix-based theoretical framework for analyzing the existence, uniqueness, and multiplicity of power-type solutions and the associated system structure of the two types of fractional differential systems. Two numerical examples are also provided to demonstrate the validity of the proposed results.

Keywords: Caputo fractional derivative; Grünwald–Letnikov fractional derivative; linearly independent solution; rank of coefficient matrices
Mathematics Subject Classification: 26A33, 34A08, 34A30, 34K37

1. Introduction

Fractional calculus extends the classical notions of differentiation and integration to non-integer orders and has become an effective tool for describing systems with memory, hereditary effects, and multi-scale dynamics. During the past decades, fractional-order models have been successfully applied in viscoelasticity, anomalous diffusion and transport, control theory, signal and image processing, mathematical biology, and finance, among many other fields. In these applications, fractional derivatives provide a more flexible description of long-range temporal and spatial interactions than their integer-order counterparts, and they often lead to models that fit experimental data more accurately.

Within this broad context, a large body of work has been devoted to the qualitative and quantitative analysis of fractional differential equations. For Caputo-type systems, recent contributions include the development of fractional Taylor formulas and power-series representations, refined existence and uniqueness results, stability analysis, and applications to complex dynamical behaviors and control problems. For example, Nuca et al. discuss fractional Taylor formulas in the Caputo setting, where the coefficients of the polynomial expansion are expressed in terms of fractional derivatives and Gamma-function weights [1]. Calatayud et al. present fixed-point results, a Cauchy–Kovalevskaya theorem for fractional power series, and Gronwall- and Nagumo-type uniqueness criteria; using Mikusiński operational calculus, they obtain global power-series representations and closed-form solutions for certain linear equations, together with refined bounds and stability properties [2]. Murillo-Arcila et al. analyze the dynamical behavior of the Caputo complex fractional derivative and prove that the associated operator is Devaney chaotic in the Mittag–Leffler Caputo space [3].

In the broader context of fractional calculus, several important fractional series, such as Mittag–Leffler-type expansions and α-exponential series, have been developed. Ortigueira et al. introduce a generalized power-series representation $f(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha_n}}{\Gamma(\alpha_n+1)}$ and applied it to the inversion of transfer functions in fractional-order signal processing [4]. This viewpoint further motivates our focus on power-type solutions and suggests that the matrix-based framework developed in this paper could, in future work, be extended to more general fractional series of this form. A number of works highlight the modeling strength of Caputo and Caputo-type derivatives in concrete applications. Omri et al. study stabilization for Ψ-Caputo fractional homogeneous polynomial systems and design stabilizing feedback laws via Lyapunov functions [5]. Bendrici et al. consider a nonlocal boundary-value problem for nonlinear fractional differential equations driven by a Ψ-Caputo operator and establish existence and Ulam–Hyers stability using Mönch's fixed-point theorem and measures of noncompactness [6]. AlAhmad develops an approach for solving nonlinear Caputo–Fabrizio fractional differential equations by exploiting exactness properties and integrating factors [7]. Ma et al. propose the m-ρ-Laplace transform and apply it to stability analysis of Caputo–Katugampola systems [8]. Larhrissi et al. introduce constrained fractional controllability for Caputo-type systems with Riemann–Liouville output derivatives and characterise optimal controls that keep the fractional derivative of the final state within prescribed bounds [9]. Das et al. formulate fractional optimal control problems governed by Caputo systems in Hilbert spaces, derive a Pontryagin maximum principle and Hamilton–Jacobi–Bellman equations, and show that the value function is a viscosity solution [10].

Further lines of research address stability, data-driven modeling, and boundary-value/inverse problems for Caputo-type and related fractional systems. An et al. investigate the asymptotic stability of impulsive fuzzy fractional dynamic systems described by Caputo derivatives in the short-memory sense, and they design linear feedback controllers based on Lyapunov's direct method and direct evaluation techniques involving Laplace transforms, Mittag–Leffler functions, and Gronwall–Bellman inequalities [11]. Yang et al. study data-driven modeling of discrete fractional chaotic systems by constructing a sparse identification framework, using iterative thresholding, and matrix perturbation theory to jointly determine the structure of the sparse matrix, the vector field, and the fractional orders, and to verify the approach on discrete fractional Lorenz and Chua systems [12]. Vu proves existence and uniqueness of mean-square solutions for Caputo fractional random boundary-value problems using fixed-point theorems, and introduces Ulam–Hyers and generalized Ulam–Hyers stability concepts, supported by illustrative examples [13]. Durdiev et al. consider initial-boundary value and inverse problems for a fourth-order equation with Caputo fractional derivatives; by expanding in eigenfunctions, they obtain a Fourier-series representation of the direct problem, and then establish existence and uniqueness for the inverse problems via integral-equation methods [14]. Fan et al. analyze a new system of fractional differential equations with integral boundary conditions, involving Caputo derivatives, integer-order derivatives, and Riemann integral boundary values; they obtain existence and uniqueness results via fixed-point theorems for increasing ϕ-(h, e)-concave operators and construct iterative schemes for approximating the unique solution [15]. Thai and Tuan systematically treat the asymptotic behavior of several classes of higher-order fractional differential equations with multiple terms by using properties of Caputo fractional differentiable functions, comparison principles, and spectral analysis based on integral representations of fundamental solutions [16]. Phung et al. study solvability issues for multi-term Caputo and Riemann–Liouville fractional oscillatory integro-differential equations [17]. Bouguetof et al. consider stochastic differential equations with fractional integrals driven by Riemann–Liouville multifractional Brownian motion and standard Brownian motion, derive approximate numerical solutions, and validate the results on a colon cancer chemotherapy effect model [18].

Alongside Caputo-type derivatives, the Grünwald–Letnikov (GL) derivative and its variants play a central role in discrete-time modeling and numerical approximation. Tenreiro Machado proposes a conceptual experiment that couples a bouncing ball model with the GL formulation, relating the restitution coefficient to GL coefficients [19]. Li and Wang examine numerical stability of time-fractional delay differential equations using GL approximations for the Caputo derivative, analysing stability regions and Mittag–Leffler stability [20]. Zuffi et al. study a near-field acoustic levitation system and investigate the influence of fractional orders on response profiles by approximating temporal and spatial fractional derivatives via Caputo and GL formulas [21]. Pawluszewicz investigates observability of discrete-time polynomial control systems described by GL h-type difference operators [22]. Gabriel et al. introduce non-square-integrable power-type signals, define their fractional derivatives, and analyse almost periodic signals and stationary stochastic processes together with associated correlations and generalized harmonic analysis [23]. Ashurov and Mukhiddinova consider systems of fractional-order partial differential equations and prove existence and uniqueness of classical solutions for initial-boundary value problems under suitable Lipschitz and growth conditions [24]. Al-Musalhi employs transmutation relations to solve fractional equations with variable coefficients involving general transmuted operators; by appropriate choices, a variety of known fractional operators, including weighted, tempered, and Hadamard-type derivatives, can be reduced to Caputo-type equations whose solutions are expressed via Riemann–Liouville integrals [25]. In particular, Hadamard-type derivatives are naturally suited to scale-invariant (multiplicative) systems.

These developments demonstrate both the modeling flexibility and the analytical richness of fractional differential equations. At the same time, they raise structural questions about the solution space of fractional systems with variable coefficients, especially those admitting polynomial or power-type solutions, which can serve as local approximations, benchmark solutions, or building blocks for spectral and Galerkin-type numerical schemes. In the integer-order setting, there has been sustained interest in characterizing polynomial particular solutions of linear homogeneous differential equations with variable polynomial coefficients via matrix-theoretic conditions.

More concretely, Hu and Li et al. [26] considered the second-order linear homogeneous differential equation with quadratic polynomial coefficients

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0,$$

where

$$P(x) = a_{12}x^2 + a_{11}x + a_{10}, \quad Q(x) = a_{22}x^2 + a_{21}x + a_{20}, \quad R(x) = a_{32}x^2 + a_{31}x + a_{30},$$

and all $a_{ij} \in \mathbb{R}$ are fixed real constants. Assuming a polynomial particular solution of degree $m \in \mathbb{N}$,

$$y(x) = A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0,$$

with unknown real coefficients $A_0, \ldots, A_m$, substitution into the equation leads to a homogeneous linear algebraic system for the vector of coefficients $(A_m, \ldots, A_0)^\top$. In this process, they constructed the matrix $\boldsymbol{T}_m$, a coefficient-related matrix tailored to the second-order equation structure, whose properties directly support the verification of the aforementioned sufficient conditions and lay a matrix-theoretic foundation for determining polynomial particular solutions of second-order systems:

$$\boldsymbol{T}_m = \begin{pmatrix} ma_{22} + a_{31} & a_{32} & 0 & \cdots & 0 \\ (m^2 - m)a_{12} + ma_{21} + a_{30} & & & & \\ (m^2 - m)a_{11} + ma_{20} & & \boldsymbol{T}_{m-1} & & \\ (m^2 - m)a_{10} & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix},$$

$$\boldsymbol{T}_1 = \begin{pmatrix} a_{22} + a_{31} & a_{32} \\ a_{21} + a_{30} & a_{31} \\ a_{20} & a_{30} \end{pmatrix},$$

where $\boldsymbol{T}_m$ is an $(m + 2) \times (m + 1)$ real matrix, and the equation $\boldsymbol{T}_m(A_m, \ldots, A_0)^\top = \boldsymbol{0}$ encodes the algebraic constraints on the coefficients of polynomial-type solutions. In particular, nontrivial polynomial solutions exist precisely when the rank of $\boldsymbol{T}_m$ satisfies suitable conditions.

Subsequently, building on the work of Hu et al. [26], Li and Jiang et al. [27] studied polynomial-type solutions of third-order linear homogeneous differential equations with quadratic polynomial coefficients

$$P(x)y'''(x) + Q(x)y''(x) + R(x)y'(x) + S(x)y(x) = 0,$$

where

$$P(x) = a_{12}x^2+a_{11}x+a_{10}, \quad Q(x) = a_{22}x^2+a_{21}x+a_{20}, \quad R(x) = a_{32}x^2+a_{31}x+a_{30}, \quad S(x) = a_{42}x^2+a_{41}x+a_{40},$$

and $a_{ij} \in \mathbb{R}$ are real constants. For an mth-degree polynomial solution

$$y(x) = K_m x^m + K_{m-1}x^{m-1} + \cdots + K_1 x + K_0,$$

with unknown real coefficients $K_0, \ldots, K_m$, repeated differentiation of x^k yields the falling-factorial factors

$$A_m^p := m(m-1)\cdots(m-p+1), \quad p \in \mathbb{N}, \qquad A_m^0 := 1.$$

Collecting the coefficients of like powers of x leads again to a homogeneous linear system for $(K_m, \ldots, K_0)^\top$, whose coefficient matrix is denoted by $\boldsymbol{G}_m$. The row-column relationships of $\boldsymbol{G}_m$ characterize the coupling between the equation coefficients and the structure of its solutions, enabling explicit derivation of polynomial particular solutions:

$$\boldsymbol{G}_m = \begin{pmatrix} ma_{32} + a_{41} & a_{42} & 0 & \cdots & 0 \\ A_m^2 a_{22} + ma_{31} + a_{40} & & & & \\ A_m^3 a_{12} + A_m^2 a_{21} + ma_{30} & & & & \\ A_m^3 a_{11} + A_m^2 a_{20} & & \boldsymbol{G}_{m-1} & & \\ A_m^3 a_{10} & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix},$$

$$\boldsymbol{G}_0 = \begin{pmatrix} a_{41} \\ a_{40} \end{pmatrix},$$

where $\boldsymbol{G}_m$ is an $(m+2) \times (m+1)$ real matrix. As in the second-order case, the rank of $\boldsymbol{G}_m$ provides necessary and sufficient conditions for the existence of nontrivial polynomial solutions of prescribed degree.

It is worth noting that although fourth-order variable-coefficient homogeneous differential equations are generally more complex than second- or third-order ones, they are unavoidable in certain applications. E et al. [28] further studied the solutions of fourth-order and nth-order linear homogeneous differential equations with polynomial coefficients and constructed a family of matrices $\boldsymbol{F}_m$ for these higher-order systems. In their setting, the fourth-order equation

$$P(x)y''''(x) + Q(x)y'''(x) + R(x)y''(x) + S(x)y'(x) + T(x)y(x) = 0$$

has polynomial coefficients of degree n, which can be written compactly as

$$\begin{pmatrix} P(x) \\ Q(x) \\ R(x) \\ S(x) \\ T(x) \end{pmatrix} = \begin{pmatrix} k_{1,n} & k_{1,n-1} & \cdots & k_{1,1} & k_{1,0} \\ k_{2,n} & k_{2,n-1} & \cdots & k_{2,1} & k_{2,0} \\ k_{3,n} & k_{3,n-1} & \cdots & k_{3,1} & k_{3,0} \\ k_{4,n} & k_{4,n-1} & \cdots & k_{4,1} & k_{4,0} \\ k_{5,n} & k_{5,n-1} & \cdots & k_{5,1} & k_{5,0} \end{pmatrix} \begin{pmatrix} x^n \\ x^{n-1} \\ \vdots \\ x \\ 1 \end{pmatrix},$$

where $k_{i,j} \in \mathbb{R}$ $(i = 1, \ldots, 5,\ j = 0, \ldots, n)$ are constant coefficients of the polynomials multiplying y'''', y''', y'', y', y, respectively. For a polynomial solution of degree m,

$$y(x) = K_m x^m + K_{m-1} x^{m-1} + \cdots + K_1 x + K_0,$$

substitution into the equation and collection of coefficients of $x^{m+n}, \ldots, x^0$ again produce a homogeneous linear system for $(K_m, \ldots, K_0)^\top$. The associated coefficient matrix is denoted by $\boldsymbol{F}_m$, and the falling factorials A_m^p appear from repeated differentiation of x^k as before. By means of sufficient conditions for the existence of solutions to such equations, they established a direct connection between the rank of $\boldsymbol{F}_m$ and the existence of polynomial particular solutions—specifically, the rank of $\boldsymbol{F}_m$ reflects the linear independence of coefficient constraints in higher-order equations, which in turn determines whether polynomial particular solutions can exist:

$$\boldsymbol{F}_m = \begin{bmatrix} mk_{4,n} + k_{5,n-1} & k_{5,n} & 0 & \cdots & 0 \\ A_m^2 k_{3,n} + mk_{4,n-1} + k_{5,n-2} & & & & \\ A_m^3 k_{2,n} + A_m^2 k_{3,n-1} + mk_{4,n-2} + k_{5,n-3} & & & & \\ A_m^4 k_{1,n} + A_m^3 k_{2,n-1} + A_m^2 k_{3,n-2} + mk_{4,n-3} + k_{5,n-4} & & \boldsymbol{F}_{m-1} & & \\ A_m^4 k_{1,n-1} + A_m^3 k_{2,n-2} + A_m^2 k_{3,n-3} + mk_{4,n-4} + k_{5,n-5} & & & & \\ \vdots & & & & \\ A_m^4 k_{1,4} + A_m^3 k_{2,3} + A_m^2 k_{3,2} + mk_{4,1} + k_{5,0} & & & & \\ A_m^4 k_{1,3} + A_m^3 k_{2,2} + A_m^2 k_{3,1} + mk_{4,0} & & & & \\ A_m^4 k_{1,2} + A_m^3 k_{2,1} + A_m^2 k_{3,0} & & & & \\ A_m^4 k_{1,1} + A_m^3 k_{2,0} & & & & \\ A_m^4 k_{1,0} & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix},$$

$$\boldsymbol{F}_1 = \begin{bmatrix} k_{4,n} + k_{5,n} & k_{5,n} \\ k_{4,n-1} + k_{5,n-1} & k_{5,n-1} \\ \vdots & \vdots \\ k_{4,1} + k_{5,1} & k_{5,1} \\ k_{4,0} & k_{5,0} \end{bmatrix},$$

where $\boldsymbol{F}_m$ is an $(m + n) \times (m + 1)$ real matrix assembling the coefficients of the linear system for $(K_m, \ldots, K_0)^\top$. Appropriate rank conditions on $\boldsymbol{F}_m$ are then equivalent to the existence of higher-order polynomial particular solutions.

The above works show that for integer-order systems, and to some extent for Riemann–Liouville fractional systems, matrix-based rank conditions provide a powerful and concise way to characterize the existence and structure of polynomial solutions. However, for linear CFDESs and GLFDESs with variable polynomial coefficients, the relationship between the rank of appropriately constructed coefficient matrices and the existence, uniqueness, and multiplicity of power solutions has not been systematically investigated. In modeling practice, Caputo and GL derivatives play a central role: Caputo-type operators are widely used in the formulation of fractional initial-value problems, while

GL derivatives offer a natural link to finite-difference schemes and discrete-time implementations. Consequently, it is natural to develop a unified framework that treats both operators in parallel, so that the structural properties of power-type solutions can be analysed consistently in continuous-time modelling and in discrete-time or numerical implementations. Moreover, both operators are closely related to the Riemann–Liouville derivative, which allows us to derive explicit algebraic representations of the operator terms used in this paper. Although many other fractional operators—such as Ψ-Caputo, tempered, and Hadamard-type derivatives—have been proposed and shown to be effective in specific applications [6–8, 25], their kernels typically lead to non-polynomial or more intricate nonlocal structures (in particular, Hadamard-type derivatives are especially suited to scale-invariant systems), which fall outside the matrix-based framework considered here.

Although fractional-order homogeneous differential equations with variable coefficients are generally more complex than their integer-order counterparts, they offer greater universality and flexibility for modelling memory and hereditary effects. In this paper, we focus on Caputo fractional differential equation systems (CFDESs) and Grünwald–Letnikov fractional differential equation systems (GLFDESs) with variable polynomial coefficients, and we study the existence and structure of their power-type solutions by combining fractional calculus with matrix theory. We derive explicit expansion formulas for each operator term, construct coefficient matrices that encode both the system coefficients and the unknown coefficients of power solutions, and establish necessary and sufficient rank conditions for the existence of mth-order power solutions. We also obtain equivalent conditions for the uniqueness of the order of such solutions and for the existence of arbitrarily many linearly independent power solutions of distinct orders. These results provide a unified framework for analysing power-type solutions of Caputo and Grünwald–Letnikov systems with variable polynomial coefficients and extend the matrix-based structural theory of polynomial solutions from integer-order equations to two of the most widely used fractional derivatives in applications.

Compared with the above studies, which mainly address existence, stability, controllability, or data-driven identification for specific Caputo- or GL-type models, the present work emphasises the structural relation between power-type solutions and the rank of coefficient matrices in variable-coefficient fractional systems. In the integer-order setting, Hu et al., Li et al., and E et al. established matrix-based criteria for polynomial particular solutions of second-, third-, and higher-order equations with polynomial coefficients by analysing the rank of the associated matrices T_m, G_m, and F_m [26–28]. Building on this line of research, we extend the matrix–solution correspondence to Caputo and Grünwald–Letnikov fractional systems. Under suitable low-rank or sparse assumptions on the coefficient matrices, we show that the existence, uniqueness of the degree, and multiplicity of power solutions can be completely characterised by explicit rank conditions, thereby generalising the integer-order results to a broad class of fractional systems with nonlocal kernels.

We also clarify the scope of the present work. The results obtained here are derived at the level of differential operators and power-type solutions and do not rely on any specific choice or interpretation of initial conditions. In particular, our rank conditions characterise when a given Caputo or Grünwald–Letnikov system admits power-type solutions, independently of how initial data are prescribed. There is an extensive discussion in the fractional-calculus literature about the physical meaning and correctness of certain initial conditions associated with Caputo-type operators; a detailed study of how our structural criteria interact with these issues for concrete initial-value problems is beyond the scope of this paper and will be addressed in future work.

The rest of this paper is organised as follows. Section 2 deals with CFDESs, recalling the relevant fractional operators, deriving the expansion formulas, and establishing the corresponding matrix constructions and rank conditions. Section 3 develops the analogous framework for GLFDESs. Section 4 presents numerical validations and discusses the implications, limitations, and possible extensions of the results.

2. Existence of solutions to Caputo fractional differential systems (CFDESs)

To analyze the existence of solutions for CFDESs, we briefly recall the classical Riemann–Liouville fractional integral and derivative and the Caputo fractional derivative, which will be used throughout the paper. These standard definitions are included here only to fix notation; more detailed discussions can be found in the monographs and survey articles on fractional calculus and in the references cited in Section 1.

Definition 1. *For a function $f(t)$ and a positive real number $p > 0$, the Riemann-Liouville fractional integral of order p from the lower limit a to t is defined as:*

$$ {}_{a}^{RL}D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau, $$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2. *For a function $f(t)$ and a real number p satisfying $n-1 \le p < n$ (where $n \in \mathbb{N}^+$), the Riemann-Liouville fractional derivative of order p from a to t is given by:*

$$ {}_{a}^{RL}D_t^{p} f(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{p+1-n}} d\tau. $$

Definition 3. *For a function $f(t)$ and a real number α with $n-1 \le \alpha < n$ ($n \in \mathbb{N}^+$), the Caputo fractional derivative of order α from 0 to t is defined as:*

$$ {}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. $$

Definitions 1–3 coincide with the usual Riemann–Liouville fractional integral/derivative and the Caputo fractional derivative in the fractional calculus literature; they are recalled here for completeness and do not constitute new results.

We consider a class of linear CFDESs with specific structures, focusing on the existence of power-form solutions. The system is formulated as follows:

$$ \begin{cases} A_n D^{\alpha_n} y + A_{n-1} D^{\alpha_{n-1}} y + \cdots + A_1 D^{\alpha_1} y + A_0 D^{\alpha_0} y = 0, \\ A = TX, \\ y = K_m x^m + K_{m-1} x^{m-1} + \cdots + K_1 x \end{cases} \tag{1-1} $$

where: $\alpha_k = \alpha + k$ for $k = 0, 1, \cdots, n$ ($\alpha > 0$ and $\alpha \notin \mathbb{N}^+$), where the coefficients of the differential equation are

$$\begin{pmatrix} A_n \\ A_{n-1} \\ \vdots \\ A_1 \\ A_0 \end{pmatrix} = \begin{bmatrix} t_{1,s} & t_{1,s-1} & \cdots & t_{1,1} & t_{1,0} \\ t_{2,s} & t_{2,s-1} & \cdots & t_{2,1} & t_{2,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n,s} & t_{n,s-1} & \cdots & t_{n,1} & t_{n,0} \\ t_{n+1,s} & t_{n+1,s-1} & \cdots & t_{n+1,1} & t_{n+1,0} \end{bmatrix} \begin{pmatrix} x^s \\ x^{s-1} \\ \vdots \\ x \\ 1 \end{pmatrix},$$

y denotes the power-form solution to be determined, with $K_m, K_{m-1}, \cdots, K_1$ as undetermined coefficients, and $0 \leq s \leq n \leq m$.

Here we use the shorthand notation

$$A := (A_n, A_{n-1}, \ldots, A_1, A_0)^\top, \qquad T := (t_{i,j})_{1\leq i\leq n+1,\ 0\leq j\leq s}, \qquad X := (x^s, x^{s-1}, \ldots, x, 1)^\top,$$

where each $A_k = A_k(x)$ is a polynomial in x of degree at most s, T is an $(n+1)\times(s+1)$ real coefficient matrix, and X collects the monomials in x. Thus, the compact relation $A = TX$ is equivalent, componentwise, to

$$A_k(x) = \sum_{j=0}^{s} t_{k+1,j}\, x^j, \qquad k = 0, 1, \ldots, n,$$

that is, each coefficient $A_k(x)$ of the differential operator in (1-1) is a polynomial in x with coefficients given by the entries of T. The unknown function is sought in the power form

$$y(x) = K_m x^m + K_{m-1} x^{m-1} + \cdots + K_1 x,$$

where the constant matrices K_i are to be determined.

In this paper we focus on systems whose derivative orders are of the form $\alpha + k$ ($k = 0, 1, \ldots, n$), rather than allowing a completely arbitrary sequence $\{\alpha_k\}$. This choice is motivated by two reasons. First, when the orders increase by integers, the action of the Caputo and Grünwald–Letnikov operators on power functions $x^{\alpha+j}$ again produces powers with integer shifts and simple gamma factors, which lead to explicit and tractable matrix representations of the operator terms. Second, the equidistant structure $\alpha + k$ allows a direct extension of existing matrix-based results for integer-order systems and Riemann–Liouville fractional systems to the Caputo and Grünwald–Letnikov setting. Treating the fully general case of non-equidistant orders $\{\alpha_k\}$ would require more involved bookkeeping of the exponents and gamma factors, and is therefore left for future work.

To establish the existence conditions for power solutions of system 1-1, we derive the following core theorems, focusing on the relationship between the rank of coefficient matrices and the existence of solutions.

Lemma 1. *Let A be an $n \times m$ matrix, and let X be an m-dimensional column vector. There then exists a non-zero solution $X^* \neq 0$ to the system of homogeneous linear equations $AX = 0$, and the sufficient and necessary condition that its first component is not zero is:*

$$rank(\hat{A}) = rank(A),$$

where $\widehat{A}$ is the submatrix of A minus the first column.

Proof. **Sufficiency:** If $rank(\hat{A}) = rank(A)$, there exists a non-trivial solution $X^* \neq 0$.

Since $rank(A)$ is the maximum number of linearly independent columns in A.
Given that

$$rank(\hat{A}) = rank(A),$$

the first column a_1 of A can be linearly represented by the columns of $\hat{A}$. Otherwise, $rank(A) = rank(\hat{A}) + 1$.
Let $a_1 = c_2a_2 + c_3a_3 + \cdots + c_ma_m$, then:

$$(-1)a_1 + c_2a_2 + \cdots + c_ma_m = 0.$$

Let $X^* = (-1\ c_2\ c_3\ \cdots\ c_m)^T$, substituting into AX:

$$AX^* = (-1)a_1 + c_2a_2 + \cdots + c_ma_m = 0.$$

Obviously, X^* is a solution to $AX = 0$, and the first component $x_1^* = -1 \neq 0$, so X^* is a non-trivial solution.
Necessity: If there exists a non trivial solution X^* with $x_1^* \neq 0$, then $rank(\hat{A}) = rank(A)$.

Let $X^* = (x_1^*\ x_2^*\ \cdots\ x_m^*)^T$ be a non trivial solution to $AX = 0$ with $x_1^* \neq 0$, then:

$$AX^* = x_1^*a_1 + x_2^*a_2 + \cdots + x_m^*a_m = 0.$$

Since $x_1^* \neq 0$, we have:

$$a_1 = -\frac{x_2^*}{x_1^*}a_2 - \cdots - \frac{x_m^*}{x_1^*}a_m.$$

That is, the first column a_1 of A can be linearly represented by the last $m - 1$ columns, so the maximum number of linearly independent columns in A is the same as that in $\hat{A}$.
Hence,

$$rank(\hat{A}) = rank(A).$$

This completes the proof.

Proof completed (PC).

□

Theorem 1. *For $\alpha > 0$ and $n - 1 < \alpha < n$ ($n \in \mathbb{N}^+$), the Riemann-Liouville fractional derivative of a function $f(x)$ can be expressed in terms of its Caputo fractional derivative as:*

$${}_a^{RL}D_x^{\alpha} f(x) = {}_a^C D_x^{\alpha} f(x) + \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a).$$

Remark 1. *This relation is classical in fractional calculus. Therefore, we recall it here without proof.*

Theorem 2. *In the CFDESs, the general term formula of the operator term $A_k\, {}_a^C D_k^{\alpha_k}$ is:*

$$A_k\, {}_a^C D_k^{\alpha_k} y = \sum_{i=k+1}^{m} \sum_{j=0}^{s} K_i t_{n+1-k,j} \frac{\Gamma(i+1)}{\Gamma(i+1-(\alpha+k))} x^{i+j-(\alpha+k)}, \quad (k = 0, 1, \cdots, n).$$

Proof. For the power function $f(x) = x^p$, according to the definition of the CFD:

$$ {}_{a}^{C}D_{k}^{\alpha_k} x^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}, & p > \alpha - 1 \\ 0, & p < \alpha - 1 \end{cases}. $$

Given $y = K_m x^m + K_{m-1} x^{m-1} + \cdots + K_1 x$, apply ${}_{0}^{C}D_{t}^{\alpha_k}$ ($\alpha_k = \alpha + k$) to each term $K_i x^i$ of y:

When $i \geq \alpha + k$, ${}_{0}^{C}D_{t}^{\alpha_k}(K_i x^i) = K_i \cdot \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha_k)} x^{i-\alpha_k}$.

When $i \leq \alpha + k$, ${}_{0}^{C}D_{t}^{\alpha_k}(K_i x^i) = 0$.

Thus, the CFD of y can be expressed as:

$$ {}_{0}^{C}D_{t}^{\alpha_k} y = \sum_{i=\alpha+k}^{m} K_i \cdot \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha_k)} x^{i-\alpha_k}. $$

Combine $0 \leq s \leq n \leq m$ to screen out non zero differential terms, and multiply $A_k = \sum_{j=0}^{s} t_{k+1,j} x^j$ by ${}_{0}^{C}D_{t}^{\alpha_k} y$:

$$ \begin{aligned} A_k\, {}_{0}^{C}D_{t}^{\alpha_k} y &= \left(\sum_{j=0}^{s} t_{k+1,j} x^j \right) \cdot \left(\sum_{i=\alpha+k}^{m} K_i \cdot \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha_k)} x^{i-\alpha_k} \right) \\ &= \sum_{j=0}^{s} \sum_{i=\alpha+k}^{m} \left[t_{k+1,j} \cdot K_i \cdot \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha_k)} \right] x^{j+i-\alpha_k} \\ &= \sum_{i=k+1}^{m} \sum_{j=0}^{s} K_i t_{n+1-k,j} \frac{\Gamma(i+1)}{\Gamma(i+1-(\alpha+k))} x^{i+j-(\alpha+k)}. \end{aligned} $$

This proves the general term formula.

PC.

□

Theorem 3. *The necessary and sufficient condition for the existence of an m-th order power solution in the CFDESs* (1-1) *is:*

$$ t_{n+1,s} = 0, \quad rank(\hat{C_m}) = rank(C_m), $$

where C_m is an $(m+s) \times (m-n+1)$ matrix, with the specific form as follows:

$$
C_m = \begin{bmatrix}
\sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+1))} & t_{n+1,s}\frac{\Gamma(m)}{\Gamma(m-\alpha)} & 0 & \cdots & 0 \\
\sum_{j=s-2}^{s} t_{(n-1)+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+2))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{s} t_{s+1-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j+n-s))} & & & & \\
\vdots & & & C_{m-1} & \\
\sum_{j=0}^{s-1} t_{s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j+n-s+1))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{0} t_{1-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+n))} & & & & \\
0 & & & & \\
\vdots & & & & \\
0 & & & &
\end{bmatrix},
$$

$$
C_n = \begin{bmatrix}
\sum_{j=1}^{s} t_{n+s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(a+j-s+1))} & t_{n+1,s}\frac{\Gamma(n)}{\Gamma(n-a)} & 0 & \cdots & 0 \\
\sum_{j=s-z}^{s} t_{(n-1)+s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(a+j-s+2))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(a+j))} & & & & \\
\sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(a+1))} & & & & \\
\vdots & & & C_{n-1} & \\
\sum_{j=0}^{s} t_{s+1-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+j+n-s))} & & & & \\
\sum_{j=0}^{s} t_{s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+j+n-s+1))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{0} t_{s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+n))} & & & &
\end{bmatrix},
$$

$$
C_s = \begin{bmatrix}
\sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(a+j-s+1))} & t_{n+1,s}\frac{\Gamma(s)}{\Gamma(s-a)} \\
\sum_{j=s-2}^{s} t_{(n-1)+s-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(a+j-s+2))} & \\
\vdots & \\
\sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(a+j))} & C_{s-1} \\
\sum_{j=0}^{s-1} t_{(n-s)+s-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(a+j+1))} & \\
\vdots & \\
\sum_{j=0}^{0} t_{(n-s+1)-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(\alpha+s))} &
\end{bmatrix},
$$

$$
C_1 = \begin{bmatrix}
\sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(2)}{\Gamma(2-(\alpha+j-s+1))} \\
\sum_{j=s-2}^{s-1} t_{(n-1)+s-j,j}\frac{\Gamma(2)}{\Gamma(2-(\alpha+j-s+2))} \\
\vdots \\
\sum_{j=0}^{1} t_{(n-s+1)+s-j,j}\frac{\Gamma(2)}{\Gamma(2-(\alpha+j))} \\
\sum_{j=0}^{0} t_{(n-s)+s-j,j}\frac{\Gamma(2)}{\Gamma(2-(\alpha+j+1))}
\end{bmatrix}.
$$

Proof. See Appendix A. □

Theorem 4. *When $t_{n,s} \neq 0$, the following conditions are equivalent:*

1. *The order of the power solution of the CFDESs* (1-1) *is unique;*
2. $m = -\frac{t_{n+1,s-1}}{t_{n,s}} + \alpha \in [n, +\infty) \cap \mathbb{Z}$, *and* $\text{rank}(\overline{C_m}) = m - n$, *where* $\overline{C_m}$ *is the submatrix obtained by removing the first row of* C_m.

Note: When $m = n = 0$, $K_0 = 0$ or the system is equivalent to $A_0 = 0$.

Proof. See Appendix B. □

Theorem 5. *When $t_{n-1,s} \neq 0$, the following conditions are equivalent:*

1. *The CFDESs* (1-1) *has two distinct-order power special solutions;*
2. $t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0$, *and* $c(x) = \sum_{j=s-2}^{s} t_{(n-1)+s-j,j} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+2))}$ *has two distinct positive roots* m_1, m_2 $(m_1 \neq m_2)$, *while satisfying* $\text{rank}(C_{m_1}) = \text{rank}(C_{m_1-1})$ *and* $\text{rank}(C_{m_2}) = \text{rank}(C_{m_2-1})$, *where* C_m *is an* $(m+s) \times (m-n+1)$ *matrix.*

Proof. Let $m_1 < m_2$ without loss of generality. From Theorems 3 and 4, the prerequisites for the CFDESs (1-1) to have two distinct-order power special solutions are:

1. $t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0$, otherwise, two power solutions of different orders cannot exist;
2. Rank$(C_{m_1}) = \text{rank}(C_{m_1-1})$, $\text{rank}(C_{m_2}) = \text{rank}(C_{m_2-1})$.

Hence, $t_{n+1,s-1} = 0$. Substitute $t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0$ into the matrix rank condition and simplify to get:

Obviously, for $C_m = (c_{ij})$, we have $c_{ii} = 0$, $c_{i,i+1} = 0$ for $i = n, n+1, \ldots, m$, and

$$\text{rank}(C_{m_1-1}) = \text{rank}(\bar{C}_{m_1}), \text{rank}(C_{m_2-1}) = \text{rank}(\bar{C}_{m_2}).$$

Then from $\text{rank}(C_{m_1}) = \text{rank}(\bar{C}_{m_1}), \text{rank}(C_{m_2}) = \text{rank}(\bar{C}_{m_2})$, we get:

$$\text{rank}(C_{m_1}) = \text{rank}(C_{m_1-1}),$$

$$\text{rank}(C_{m_2}) = \text{rank}(C_{m_2-1}),$$

$$t_{n-2,s}\frac{\Gamma(m_i+1)}{\Gamma(m_i-1-\alpha)} + t_{n-1,s-1}\frac{\Gamma(m_i+1)}{\Gamma(m_i-\alpha)} + t_{n,s-2}\frac{\Gamma(m_i+1)}{\Gamma(m_i+1-\alpha)} = 0 \quad (i = 1, 2).$$

That is, $c(m_1) = 0$ and $c(m_2) = 0$. Since $c(x)$ is a quadratic function, m_1 and m_2 are two distinct positive roots of $c(x)$, which proves that condition 1 is equivalent to condition 2.

PC. □

Corollary 1. *When $t_{n-1,s} \neq 0$, the following conditions are equivalent:*

1. *The CFDESs* (1-1) *has two linearly independent special solutions;*

2. $t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0$,
$c(x) = \sum_{j=s-2}^{s} t_{(n-1)+s-j,j} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+2))}$ *has two distinct positive roots* m_1, m_2 $(m_1 \neq m_2)$*;*
$rank(C_{m_1}) = rank(C_{m_1-1})$ *and* $rank(C_{m_2}) = rank(C_{m_2-1})$*, where* C_m *is an* $(m+s)\times(m-n+1)$ *matrix.*

Theorem 6. *When* $t_{n-p+1,s} \neq 0$*, the following conditions are equivalent:*

1. *The CFDESs* (1-1) *has p distinct-order power special solutions;*
2. $t_{(n-p+1)+s-j,z} = 0$ *(*$z = \max\{s-p, 0\}$*),*

$$
\begin{aligned}
& t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0, t_{n+s-j,j} = 0, (j = s-1, s) \\
& t_{(n-1)+s-j,j} = 0, (j = s-2, s-1, s) \\
& \vdots \\
& t_{(n-p+2)+s-j,j} = 0 (j = s-p+1, \cdots, s)
\end{aligned}
$$

$c_p(x) = \sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+p))}$ *has p positive roots* $m_1, m_2 \cdots m_p$*, where* $z_1 = \max\{s-p, 0\}$*,*
$z_2 = \min\{s+n-p, s\}$*,*
and $rank(C_{m_i-p+1}) = rank(C_{m_i})$*,* $(i = 1 \cdots p)$*,*
where C_m *is an* $(m+s)\times(m-n+1)$ *matrix.*

Proof. Let $m_1 < m_2 < \cdots < m_p$,

$$
\begin{aligned}
& t_{n+s-j,j} = 0, \quad (j = s-1, s) \\
& t_{(n-1)+s-j,j} = 0, \quad (j = s-2, s-1, s) \\
& \vdots \\
& t_{(n-p+2)+s-j,j} = 0 \quad (j = s-p+1, \ldots, s).
\end{aligned}
$$

For $C_m = (r_{ij})$, we have $c_{p,1} = 0$, that is

$$t_{(n-p+1)+s-j,z} = 0, \quad z = \max\{s-p, 0\}.$$

Obviously, for $C_m = (r_{ij})$, we have $c_{ii} = 0$, $c_{i,i+1} = 0$ for $i = n, n+1, \ldots, m$.
And

$$\mathrm{rank}(C_{m_i-p+1}) = \mathrm{rank}(\bar{C}_{m_i}), \quad (i = 1 \cdots p),$$

from which we get:

$$\mathrm{rank}(C_{m_i}) = \mathrm{rank}(\bar{C}_{m_i}), \quad (i = 1 \cdots p),$$

$$\sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(m_i+1)}{\Gamma(x+1-(\alpha+j-s+p))} = 0,$$

$$\sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(m_i+1)}{\Gamma(x+1-(\alpha+j-s+p))} = 0,$$

where $z_1 = \max\{s-p,0\}$, $z_2 = \min\{s+n-p, s\}$,

$$c_p(x) = \sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+p))},$$

$m_1, m_2 \cdots, m_p$ are exactly the p distinct positive roots of $c_p(x)$., which satisfy the differential equation when substituted. This proves that condition 1 is equivalent to condition 2.

PC.

□

Corollary 2. *When $t_{n-p+1,s} \neq 0$, the following conditions are equivalent:*

1. *The CFDESs* (1-1) *has p linearly independent special solutions;*
2. $t_{(n-p+1)+s-j,z} = 0$ *($z = \max\{s-p,0\}$),*

$$\begin{aligned} &t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0, t_{n+s-j,j} = 0, (j = s-1, s) \\ &t_{(n-1)+s-j,j} = 0, (j = s-2, s-1, s) \\ &\qquad\vdots \\ &t_{(n-p+2)+s-j,j} = 0 (j = s-p+1, \cdots, s) \end{aligned}$$

$c_p(x) = \sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+p))}$ *has p positive roots* $m_1, m_2 \cdots m_p$, *where* $z_1 = \max\{s-p,0\}$, $z_2 = \min\{s+n-p, s\}$,
and rank(C_{m_i-p+1}) = *rank*(C_{m_i}), $(i = 1 \cdots p)$,
where C_m is an $(m+s)\times(m-n+1)$ matrix.

3. Solutions under the Grünwald–Letnikov fractional derivative (GLFD)

The limit definition of the integer-order derivative reads

$$f^{(n)}(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(t-kh),$$

where $\binom{n}{k}$ denotes the binomial coefficient and h is the step size.

Definition 4. *Let f be a real-valued function defined on $(-\infty, b)$, and let $\alpha > 0$. The left-sided Grünwald–Letnikov fractional derivative of order α at a point $x \in (-\infty, b)$ is defined by*

$${}_{a}^{GL}D_t^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kh)$$

whenever the limit exists. In this paper we are mainly interested in the behavior of such derivatives near $x = 0$; restricting the sum to those terms with $x - kh$ in the domain of f is understood, where $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$, *and* $\binom{\alpha}{0} = 1$.

Consider the following GLFDESs, and explore the existence of its power solutions:

$$\begin{cases} B_n D^{\alpha_n} y + B_{n-1} D^{\alpha_{n-1}} y + \cdots + B_1 D^{\alpha_1} y + B_0 D^{\alpha_0} y = 0, \quad \alpha_k = \alpha + k \; (k = 0, 1, \cdots, n) \\ B = TX \\ y = T_m x^m + T_{m-1} x^{m-1} + \cdots + T_1 x + T_0 \end{cases} \tag{2-1}$$

where $\alpha_k = \alpha + k$ for $k = 0, 1, \cdots, n$ ($\alpha > 0$ and $\alpha \notin \mathbb{N}^+$), where the coefficients of the differential equation are:

$$\begin{pmatrix} B_n \\ B_{n-1} \\ \vdots \\ B_1 \\ B_0 \end{pmatrix} = \begin{bmatrix} t_{1,s} & t_{1,s-1} & \cdots & t_{1,1} & t_{1,0} \\ t_{2,s} & t_{2,s-1} & \cdots & t_{2,1} & t_{2,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n,s} & t_{n,s-1} & \cdots & t_{n,1} & t_{n,0} \\ t_{n+1,s} & t_{n+1,s-1} & \cdots & t_{n+1,1} & t_{n+1,0} \end{bmatrix} \begin{pmatrix} x^s \\ x^{s-1} \\ \vdots \\ x \\ 1 \end{pmatrix}.$$

y denotes the power-form solution to be determined, with $T_m, T_{m-1}, \cdots, T_0$ as undetermined coefficients and $0 \leq s \leq n \leq m$.

For the Grünwald–Letnikov fractional differential system (2-1), we use the analogous shorthand notation

$$B := (B_n, B_{n-1}, \ldots, B_1, B_0)^\top, \qquad T := (t_{i,j})_{1 \leq i \leq n+1,\ 0 \leq j \leq s}, \qquad X := (x^s, x^{s-1}, \ldots, x, 1)^\top,$$

where each $B_k = B_k(x)$ is a polynomial in x of degree at most s, T is an $(n+1) \times (s+1)$ real coefficient matrix, and X collects the monomials in x. Thus, the compact relation $B = TX$ is equivalent, componentwise, to

$$B_k(x) = \sum_{j=0}^{s} t_{k+1,j}\, x^j, \qquad k = 0, 1, \ldots, n,$$

that is, each coefficient $B_k(x)$ of the Grünwald–Letnikov fractional differential operator in (2-1) is a polynomial in x with coefficients given by the entries of T. As in the Caputo case, the unknown function is sought in the power form

$$y(x) = K_m x^m + K_{m-1} x^{m-1} + \cdots + K_1 x,$$

where the constant matrices K_i are to be determined.

Theorem 7. *For any positive integer m and fractional order α, we have:*

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} k^m = (-1)^m \alpha(\alpha - 1) \cdots (\alpha - m + 1).$$

Proof. Prove by mathematical induction: Assume the formula holds when $n = m$, i.e.:

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} k^m = (-1)^m \alpha(\alpha - 1) \cdots (\alpha - m + 1).$$

When $n = m + 1$: Consider the $(m + 1)$-th order finite difference $\Delta^{m+1} f(0)$ of function $f(k) = k^m$. According to the properties of finite differences, it is related to the generalized binomial coefficients. Using the Stirling number expansion of power functions:

$$k^n = \sum_{i=0}^{n} S(n, i) k^{\underline{i}},$$

where $k^{\underline{i}} = k(k-1)\cdots(k-i+1)$ is the "falling factorial". Substitute it into the left-hand side summation:

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} k^n = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \sum_{i=0}^{n} S(n, i) k^{\underline{i}}.$$

Exchange the order of summation, and use the product property of generalized binomial coefficients and falling factorials $\binom{\alpha}{k} k^{\underline{i}} = \binom{\alpha-i}{k-i} \alpha^{\underline{i}}$:

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} k^n = \sum_{i=0}^{n} S(n, i) \alpha^{\underline{i}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha - i}{k - i}.$$

Let $j = k - i$ (then $\binom{\alpha-i}{j} = 0$ when $j < 0$), and simplify the inner summation:

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha - i}{k - i} = (-1)^i \sum_{j=0}^{\infty} (-1)^j \binom{\alpha - i}{j} = (-1)^i (1 - 1)^{\alpha - i}.$$

When $i < \alpha$, $(1-1)^{\alpha-i} = 0$; only when $i = n$, $\alpha^{\underline{n}} = \alpha(\alpha-1)\cdots(\alpha-n+1)$ and $S(n, n) = 1$. Therefore:

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} k^n = (-1)^n \alpha(\alpha - 1)\cdots(\alpha - n + 1).$$

This proves the lemma.

□

Theorem 8. *In the GLFDESs, the operator term admits the following general formula:*

$$B_k \, {}_a^{\mathrm{GL}}D_k^{\alpha_k} y = \sum_{i=k+1}^{m} \sum_{j=0}^{s} T_i \, t_{n+1-k,j} \, \frac{\Gamma(i + 1)}{\Gamma(i + 1 - (\alpha + k))} \, x^{i+j-(\alpha+k)}, \qquad (k = 0, 1, \ldots, n),$$

where $\alpha_k = \alpha + k$. Assume $m > -1$ and $m > \alpha_k - 1$. Here $\Gamma(\cdot)$ denotes the Gamma function.

Proof. We first consider the GLFD of a power function.

Definition of the GLFD:

$${}_0^{\mathrm{GL}}D_x^{\alpha} f(x) = \lim_{h \to 0^+} \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - kh).$$

Substitute $f(x) = x^m$:

$${}_0^{\mathrm{GL}}D_x^{\alpha} x^m = \lim_{h \to 0^+} \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} (x - kh)^m.$$

Apply the binomial theorem and exchange the order of summation:

$$(x-kh)^m = \sum_{n=0}^{m} \binom{m}{n} x^{m-n} (-kh)^n,$$

$$ {}_{0}^{\mathrm{GL}}D_x^{\alpha} x^m = \lim_{h\to 0^+} \sum_{n=0}^{m} \binom{m}{n} (-1)^n x^{m-n}\, h^{n-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} k^n.$$

Coefficient identity:

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} k^n = (-1)^n\, \alpha(\alpha-1)\cdots(\alpha-n+1) = (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)}.$$

Using $\binom{m}{n} = \dfrac{\Gamma(m+1)}{\Gamma(n+1)\Gamma(m-n+1)}$, and noting that as $h \to 0^+$: If $n > \alpha$, then $h^{n-\alpha} \to 0$; if $n < \alpha$, then $h^{n-\alpha} \to \infty$, one obtains via analytic continuation the closed form for the GLFD of a power:

$$ {}_{0}^{\mathrm{GL}}D_x^{\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\, x^{m-\alpha}.$$

Linearity and substitution of the target function: Let $y = \sum_{i=1}^{m} T_i x^i, \quad \alpha_k = \alpha + k$, then

$$ {}_{0}^{\mathrm{GL}}D_x^{\alpha_k} y = \sum_{i=k+1}^{m} T_i\, \frac{\Gamma(i+1)}{\Gamma(i+1-(\alpha+k))}\, x^{i-(\alpha+k)}.$$

Multiply by the polynomial operator B_k: Let $B_k = \sum_{j=0}^{s} t_{n+1-k,j}\, x^j$, then

$$B_k \cdot {}_{0}^{\mathrm{GL}}D_x^{\alpha_k} y = \Big(\sum_{j=0}^{s} t_{n+1-k,j} x^j\Big)\Big(\sum_{i=k+1}^{m} T_i \frac{\Gamma(i+1)}{\Gamma(i+1-(\alpha+k))} x^{i-(\alpha+k)}\Big) = \sum_{i=k+1}^{m} \sum_{j=0}^{s} T_i\, t_{n+1-k,j}\, \frac{\Gamma(i+1)}{\Gamma(i+1-(\alpha+k))}\, x^{i+j-(\alpha+k)}.$$

Replacing the lower limit 0 by a general a only changes the subscript of the GL operator, yielding the stated formula under $m > -1$ and $m > \alpha_k - 1$.

PC.

□

Theorem 9. *The necessary and sufficient condition for the existence of an m-th order power solution in the GLFDESs* (2-1) *is:*

$$t_{n+1,s} = 0, \quad rank(\hat{G_m}) = rank(G_m),$$

where G_m is an $(m+s)\times(m-n+1)$ matrix, with the specific form as follows:

$$G_m = \left[\begin{array}{ccccc}
\sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+1))} & t_{n+1,s}\frac{\Gamma(m)}{\Gamma(m-\alpha)} & 0 & \cdots & 0 \\
\sum_{j=s-2}^{s} t_{(n-1)+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+2))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{s} t_{s+1-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j+n-s))} & & & & \\
\vdots & & & G_{m-1} & \\
\sum_{j=0}^{s-1} t_{s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j+n-s+1))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{0} t_{1-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+n))} & & & & \\
0 & & & & \\
\vdots & & & & \\
0 & & & &
\end{array}\right],$$

$$G_n = \left[\begin{array}{ccccc}
\sum_{j=1}^{s} t_{n+s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(a+j-s+1))} & t_{n+1,s}\frac{\Gamma(n)}{\Gamma(n-a)} & 0 & \cdots & 0 \\
\sum_{j=s-z}^{s} t_{(n-1)+s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(a+j-s+2))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(a+j))} & & & & \\
\sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(a+1))} & & & & \\
\vdots & & & G_{n-1} & \\
\sum_{j=0}^{s} t_{s+1-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+j+n-s))} & & & & \\
\sum_{j=0}^{s} t_{s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+j+n-s+1))} & & & & \\
\vdots & & & & \\
\sum_{j=0}^{0} t_{s-j,j}\frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+n))} & & & &
\end{array}\right],$$

$$G_s = \left[\begin{array}{cc}
\sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(a+j-s+1))} & t_{n+1,s}\frac{\Gamma(s)}{\Gamma(s-a)} \\
\sum_{j=s-2}^{s} t_{(n-1)+s-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(a+j-s+2))} & \\
\vdots & \\
\sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(a+j))} & G_{s-1} \\
\sum_{j=0}^{s-1} t_{(n-s)+s-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(a+j+1))} & \\
\vdots & \\
\sum_{j=0}^{0} t_{(n-s+1)-j,j}\frac{\Gamma(s+1)}{\Gamma(s+1-(\alpha+s))} &
\end{array}\right],$$

$$G_1 = \left[\begin{array}{c}
\sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(2)}{\Gamma(2-(\alpha+j-s+1))} \\
\sum_{j=s-2}^{s-1} t_{(n-1)+s-j,j}\frac{\Gamma(2)}{\Gamma(2-(\alpha+j-s+2))} \\
\vdots \\
\sum_{j=0}^{1} t_{(n-s+1)+s-j,j}\frac{\Gamma(2)}{\Gamma(2-(\alpha+j))} \\
\sum_{j=0}^{0} t_{(n-s)+s-j,j}\frac{\Gamma(2)}{\Gamma(2-(\alpha+j+1))}
\end{array}\right].$$

Proof. The argument is completely analogous to that of Theorem 3 for the Caputo case: One replaces the Caputo derivatives with the Grünwald–Letnikov ones and uses the corresponding expansion formulas. We therefore omit the details. □

Theorem 10. *When $t_{n,s} \neq 0$, the following conditions are equivalent:*

1. *The order of the power solution of the GLFDES* (2-1) *is unique;*
2. $m = -\frac{t_{n+1,s-1}}{t_{n,s}} + \alpha \in [n, +\infty) \cap \mathbb{Z}$*, and rank*$(\overline{G_m}) = m - n$*, where* $\overline{G_m}$ *is the submatrix obtained by removing the first row of* G_m*.*

Note: When $m = n = 0$*,* $T_0 = 0$ *or the system is equivalent to* $B_0 = 0$*.*

Proof. The argument is completely analogous to that of Theorem 4 for the Caputo case: One replaces the Caputo derivatives with the Grünwald–Letnikov ones and uses the corresponding expansion formulas. We therefore omit the details.

□

Theorem 11. *When $t_{n-1,s} \neq 0$, the following conditions are equivalent:*

1. *The GLFDESs* (2-1) *has two distinct-order power special solutions;*
2. $t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0$,
   $g(x) = \sum_{j=s-2}^{s} t_{(n-1)+s-j,j} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+2))}$ *has two distinct positive roots* m_1, m_2 $(m_1 \neq m_2)$*,*
   while satisfying rank$(G_{m_1}) =$ *rank*(G_{m_1-1}) *and rank*$(G_{m_2}) =$ *rank*(G_{m_2-1})*, where* G_m *is an* $(m+s) \times (m-n+1)$ *matrix.*

Proof. The argument is completely analogous to that of Theorem 5 for the Caputo case: One replaces the Caputo derivatives with the Grünwald–Letnikov ones and uses the corresponding expansion formulas. We therefore omit the details.

□

Corollary 3. *When $t_{n-1,s} \neq 0$, the following conditions are equivalent:*

1. *The GLFDESs* (2-1) *has two linearly independent special solutions;*
2. $t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0$,
   $g(x) = \sum_{j=s-2}^{s} t_{(n-1)+s-j,j} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+2))}$ *has two distinct positive roots* m_1, m_2 $(m_1 \neq m_2)$*;*
   rank$(G_{m_1}) =$ *rank*(G_{m_1-1}) *and rank*$(G_{m_2}) =$ *rank*(G_{m_2-1})*, where* G_m *is an* $(m+s) \times (m-n+1)$ *matrix.*

Theorem 12. *When $t_{n-p+1,s} \neq 0$, the following conditions are equivalent:*

1. *The GLFDESs* (2-1) *has p distinct-order power special solutions;*
2. $t_{(n-p+1)+s-j,z} = 0$ $(z = \max\{s-p, 0\})$,

$$
\begin{aligned}
&t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0, t_{n+s-j,j} = 0, (j = s-1, s) \\
&t_{(n-1)+s-j,j} = 0, (j = s-2, s-1, s) \\
&\qquad\qquad \vdots \\
&t_{(n-p+2)+s-j,j} = 0 (j = s-p+1, \cdots, s)
\end{aligned}
$$

$g_p(x) = \sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+p))}$ has p positive roots $m_1, m_2 \cdots m_p$, where $z_1 = \max\{s-p, 0\}$, $z_2 = \min\{s+n-p, s\}$
and rank(G_{m_i-p+1}) = rank(G_{m_i}), $(i = 1 \cdots p)$,
where G_m is an $(m+s) \times (m-n+1)$ matrix.

Proof. The argument is completely analogous to that of Theorem 6 for the Caputo case: One replaces the Caputo derivatives with the Grünwald–Letnikov ones and uses the corresponding expansion formulas. We therefore omit the details.

□

Corollary 4. *When $t_{n-p+1,s} \neq 0$, the following conditions are equivalent:*

1. *The GLFDESs (2-1) has p linearly independent special solutions;*
2. *$t_{(n-p+1)+s-j,z} = 0$ ($z = \max\{s-p, 0\}$),*

$$
\begin{aligned}
&t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0, t_{n+s-j,j} = 0, (j = s-1, s)\\
&t_{(n-1)+s-j,j} = 0, (j = s-2, s-1, s)\\
&\qquad\qquad\vdots\\
&t_{(n-p+2)+s-j,j} = 0 (j = s-p+1, \cdots, s)
\end{aligned}
$$

$g_p(x) = \sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+p))}$ has p positive roots $m_1, m_2 \cdots m_p$, where $z_1 = \max\{s-p, 0\}$, $z_2 = \min\{s+n-p, s\}$
and rank(G_{m_i-p+1}) = rank(G_{m_i}), $(i = 1 \cdots p)$,
where G_m is an $(m+s) \times (m-n+1)$ matrix.

4. Numerical validation

In this section we present two numerical examples that illustrate the theoretical results for CFDESs and GLFDESs. In both cases we choose systems that admit explicit power-type solutions and verify, on a finite grid, that the operator terms cancel out so that the residual $L[y](x)$ remains numerically negligible.

4.1. Example 1

To substantiate the theoretical results, we consider the Caputo operator system. From the rank conditions, the system admits two linearly independent polynomial solutions y_1, y_2. For each candidate $y_i(x)$, we evaluate the corresponding Caputo operator terms and verify that the resulting expression indeed satisfies the governing differential system, thereby confirming the validity of the derived rank solution correspondence.

We first consider a three-term Caputo fractional differential equation :

$$35 {}_0^C D_x^{0.5} y(x) - 20\, x\, {}_0^C D_x^{1.5} y(x) + 4\, x^2\, {}_0^C D_x^{2.5} y(x) = 0. \tag{4-1}$$

For monomials x^p with $p > \mu - 1$, the Caputo derivative admits the known closed form

$$ {}_{0}^{C}D_x^{\mu} x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} x^{p-\mu}. \tag{4-2}$$

Using (4-2) with $p = 3, 4$ and $\mu \in \{0.5, 1.5, 2.5\}$, a direct substitution shows that

$$L[x^3](x) = 35\,{}_{0}^{C}D_x^{0.5} x^3 - 20\,x\,{}_{0}^{C}D_x^{1.5} x^3 + 4\,x^2\,{}_{0}^{C}D_x^{2.5} x^3 \equiv 0,$$

and

$$L[x^4](x) = 35\,{}_{0}^{C}D_x^{0.5} x^4 - 20\,x\,{}_{0}^{C}D_x^{1.5} x^4 + 4\,x^2\,{}_{0}^{C}D_x^{2.5} x^4 \equiv 0.$$

Hence any linear combination of x^3 and x^4 is an exact solution of (4-1). In particular, we obtain two linearly independent polynomial solutions

$$y_1(x) = x^3 + 2x^4, \qquad y_2(x) = 3x^3 - x^4. \tag{4-3}$$

To complement the theoretical result, we perform a numerical validation on $(0, 100]$ by evaluating, for each $y_i(x)$, the three operator terms

$$35\,{}_{0}^{C}D_x^{0.5} y_i(x), \quad -20\,x\,{}_{0}^{C}D_x^{1.5} y_i(x), \quad 4\,x^2\,{}_{0}^{C}D_x^{2.5} y_i(x),$$

and plotting their sum

$$L[y_i](x) = 35\,{}_{0}^{C}D_x^{0.5} y_i(x) - 20\,x\,{}_{0}^{C}D_x^{1.5} y_i(x) + 4\,x^2\,{}_{0}^{C}D_x^{2.5} y_i(x)$$

as the residual curve, together with $y_i(x)$ for scale reference. The residuals remain numerically close to zero on $(0, 100]$, which illustrates the term-by-term cancellation predicted by the theory and verifies that the polynomials in (4-3) indeed solve the Caputo system (4-1).

4.2. Example 2

We next consider the Grünwald–Letnikov (GL) counterpart of the above system, using the same coefficients and the same polynomial solutions, but replacing the Caputo derivatives with GL derivatives:

$$35{}_{0}^{C}D_x^{0.5} y(x) - 20\,x\,{}_{0}^{C}D_x^{1.5} y(x) + 4\,x^2\,{}_{0}^{C}D_x^{2.5} y(x) = 0. \tag{4-4}$$

For monomials x^p and $\mu \in \{0.5, 1.5, 2.5\}$, the GL derivative admits the same closed-form expression as in the Caputo case:

$$ {}_{0}^{GL}D_x^{\mu} x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} x^{p-\mu}, \qquad p > \mu - 1. \tag{4-5}$$

In particular, we again have

$$y_1(x) = x^3 + 2x^4, \qquad y_2(x) = 3x^3 - x^4 \tag{4-6}$$

as two linearly independent polynomial solutions of the GL system (4-4).

The numerical validation is performed in the same manner as for the Caputo system: For each $y_i(x)$ in (4-6), we evaluate the three GL operator terms

$$35\,{}_{0}^{GL}D_x^{0.5} y_i(x), \quad -20\,x\,{}_{0}^{GL}D_x^{1.5} y_i(x), \quad 4\,x^2\,{}_{0}^{GL}D_x^{2.5} y_i(x),$$

plot their sum $L[y_i](x)$ as the residual, and overlay $y_i(x)$ for scale reference. The residual curves remain close to zero on $(0, 100]$, confirming the exact cancellation among the GL terms and numerically verifying that the polynomials (4-5) are indeed solutions of the system (4-4).

4.3. *Comparison between the Caputo and GL numerical validations*

By construction, the Caputo system (4-1) and the GL system (4-4) share the same polynomial operator coefficients and the same power-type solutions. In the present examples, both the Caputo and GL derivatives acting on monomials are evaluated via the closed-form Gamma-function formulas (4-2) and (4-5), rather than by discrete finite-difference approximations. As a consequence, the per-term responses and residual curves in Figures 1 and 2 exhibit the same cancellation pattern and are numerically almost indistinguishable. This agreement provides a concrete illustration of the structural parallelism between the Caputo and GL based formulations.

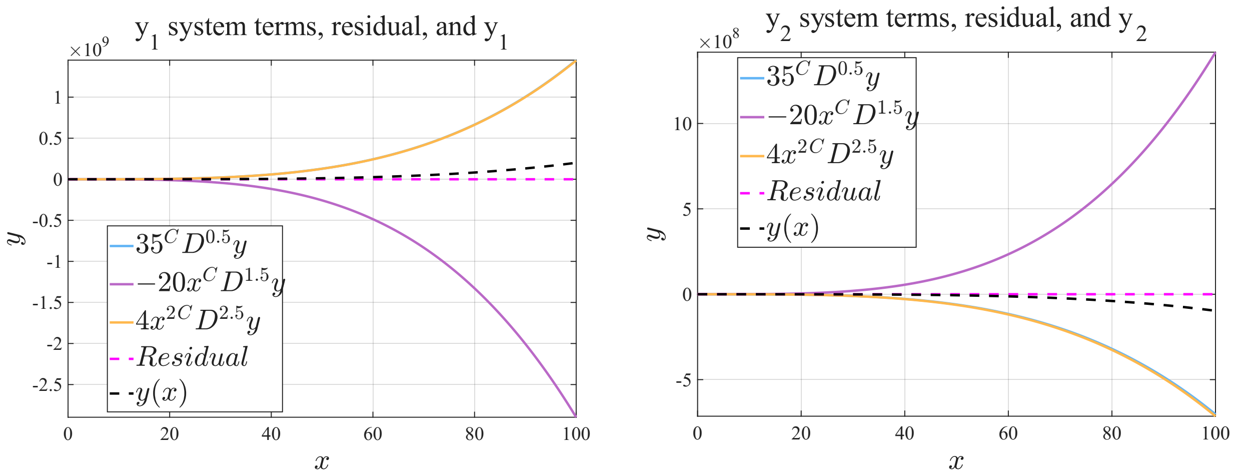

Figure 1. Per-term responses, residual, and solution overlays for the Caputo system (4-1). For each candidate $y_i(x)$ in (4-3), the three colored curves represent the operator terms $35\,{}_0^C D_x^{0.5} y_i(x)$, $-20x\,{}_0^C D_x^{1.5} y_i(x)$, and $4x^2\,{}_0^C D_x^{2.5} y_i(x)$. The pink dashed curve is the residual $L[y_i](x)$, and the black dashed curve is $y_i(x)$.

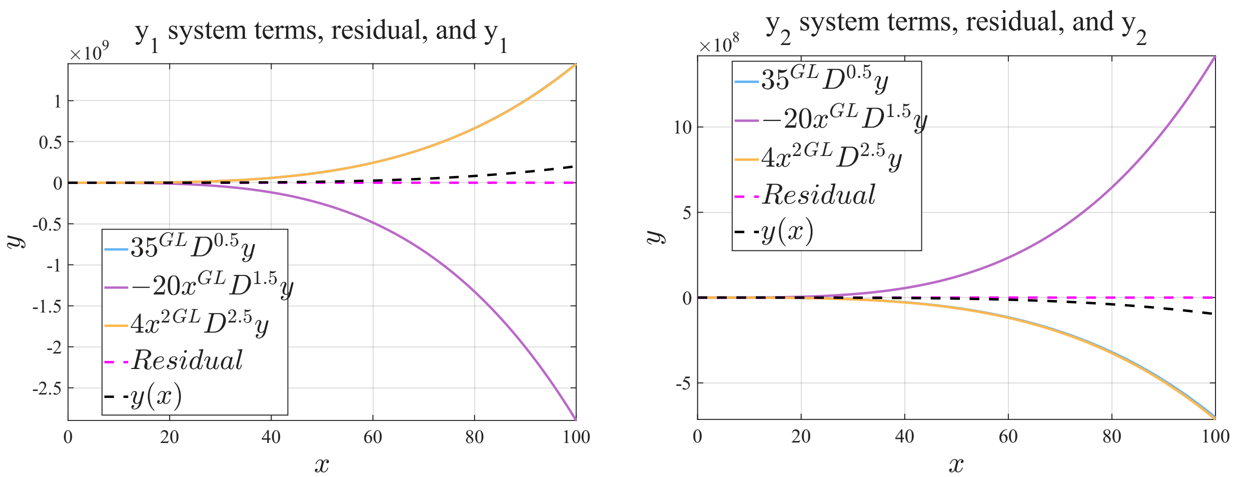

Figure 2. Per-term responses, residual, and solution overlays for the Grünwald–Letnikov system (4-4). For each candidate $y_i(x)$ in (4-6), the three colored curves represent the operator terms $35\,{}_0^{GL} D_x^{0.5} y_i(x)$, $-20x\,{}_0^{GL} D_x^{1.5} y_i(x)$, and $4x^2\,{}_0^{GL} D_x^{2.5} y_i(x)$. The pink dashed curve is the residual $L[y_i](x)$, and the black dashed curve is $y_i(x)$.

5. Discussion and conclusions

In this work, we utilize the lemmas regarding the necessary and sufficient conditions for the existence of solutions to differential equations and the properties of fractional differential equations, along with matrix theory, to characterize the coupling relationship between variable coefficients and fractional derivatives. We investigate the connection between the existence of power solutions (of dimension m) and n-dimensional fractional differential equation systems with variable coefficients based on CFD and GLFD. Compared with previous studies, this work is the first to extend conclusions to CFDESs and GLFDESs, and expands the number of linearly independent solutions that can be determined to an arbitrary count. The main results are as follows:

1. The general term formula for the operator term of the CFDESs (1-1) is derived:

$$A_k\, {}_a^C D_k^{\alpha_k} y = \sum_{i=k+1}^{m} \sum_{j=0}^{s} K_i t_{n+1-k,j} \frac{\Gamma(i+1)}{\Gamma(i+1-(\alpha+k))} x^{i+j-(\alpha+k)}, \quad (k = 0, 1, \cdots, n).$$

2. The relationship between the general term formulas of the operator terms for the CFDESs and the Riemann–Liouville fractional differential systems is proven:

$${}_a^{\mathrm{RL}} D_x^{\alpha} f(x) = {}_a^C D_x^{\alpha} f(x) + \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a).$$

3. The necessary and sufficient conditions for the existence of an m-degree power solution in the CFDESs (1-1) are proven:

$$t_{n+1,s} = 0, \quad \mathrm{rank}\left(\hat{C}_m\right) = \mathrm{rank}\left(C_m\right).$$

4. The necessary and sufficient conditions for the CFDESs (1-1) to have only one power solution and two distinct-degree power solutions are proven.
5. The necessary and sufficient conditions for the CFDESs (1-1) to have any p distinct-degree power solutions are proven: When $t_{n-p+1,s} \neq 0$,

$$t_{(n-p+1)+s-j,z} = 0 \quad (z = \max\{s-p, 0\}),$$

$$t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0, \quad t_{n+s-j,j} = 0 \quad (j = s-1, s),$$

$$t_{(n-1)+s-j,j} = 0 \quad (j = s-2, s-1, s),$$

$$t_{(n-p+2)+s-j,j} = 0 \quad (j = s-p+1, \cdots, s),$$

the function

$$h_p(x) = \sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+p))}$$

has p positive roots $m_1, m_2, \cdots, m_p$, and

$$\mathrm{rank}\left(C_{m_i-p+1}\right) = \mathrm{rank}\left(C_{m_i}\right) \quad (i = 1, \cdots, p),$$

where C_m is an $(m+s) \times (m-n+1)$ matrix.

6. The general term formula for the operator term of the GLFDESs (2-1) is derived:

$$B_k\, {}_a^{\mathrm{GL}}D_k^{\alpha_k} y = \sum_{i=k+1}^{m} \sum_{j=0}^{s} T_i t_{n+1-k,j} \frac{\Gamma(i+1)}{\Gamma(i+1-(\alpha+k))} x^{i+j-(\alpha+k)}, \quad (k=0,1,\cdots,n).$$

7. The necessary and sufficient conditions for the existence of an m-degree power solution in the GLFDESs (2-1) are proven:

$$t_{n+1,s} = 0, \quad \operatorname{rank}\left(\hat{G}_m\right) = \operatorname{rank}\left(G_m\right),$$

   where G_m is an $(m+s) \times (m-n+1)$ matrix.
8. The necessary and sufficient conditions for the GLFDESs (2-1) to have only one power solution and two distinct-degree power solutions are proven.
9. The necessary and sufficient conditions for the GLFDESs (2-1) to have any p distinct-degree power solutions are proven: When $t_{n-p+1,s} \neq 0$,

$$t_{(n-p+1)+s-j,z} = 0 \quad (z = \max\{s-p, 0\}),$$

$$t_{n+1,s} = t_{n,s} = t_{n+1,s-1} = 0, \quad t_{n+s-j,j} = 0 \quad (j = s-1, s),$$
$$t_{(n-1)+s-j,j} = 0 \quad (j = s-2, s-1, s),$$
$$t_{(n-p+2)+s-j,j} = 0 \quad (j = s-p+1, \cdots, s),$$

   the function

$$h_p(x) = \sum_{j=z_1}^{z_2} t_{(n-p)+s-j,z_1} \frac{\Gamma(x+1)}{\Gamma(x+1-(\alpha+j-s+p))}$$

   has p positive roots $m_1, m_2, \cdots, m_p$, and

$$\operatorname{rank}\left(G_{m_i-p+1}\right) = \operatorname{rank}\left(G_{m_i}\right) \quad (i = 1, \cdots, p),$$

   where G_m is an $(m+s) \times (m-n+1)$ matrix.
10. The differences and connections in the existence of solutions between the Riemann–Liouville fractional differential system and the CFDESs/GLFDESs are compared.

These results not only expand the category of solvable differential equation systems but also establish a connection between the rank of coefficient matrices and the existence of solutions under different fractional differential definitions. Compared with previous studies on Riemann–Liouville fractional differential systems, this paper builds a bridge between matrix theory and fractional calculus under the frameworks of Caputo and Grünwald–Letnikov fractional differential equation systems. It provides a new way of thinking for investigating the existence of solutions to fractional differential equation systems and offers a systematic and unified analytical framework for the study of fractional differential equation systems.

From an applied viewpoint, the Caputo and Grünwald–Letnikov fractional systems studied in this paper cover, after suitable linearization or separation of variables, a variety of linear subsystems arising in viscoelasticity, anomalous diffusion, electric circuits, and discrete-time control. In such settings, power-type solutions describe the leading-order temporal or spatial profiles of the system response and can be used as local approximations, benchmark solutions for validating numerical schemes, or basic modes in spectral and Galerkin-type methods. The rank conditions obtained in this paper therefore provide simple algebraic criteria to decide whether a given fractional model admits such modes, which is useful for model design, stability analysis, and parameter identification in real-world applications.

From the viewpoint of novelty, the main contributions of this paper can be summarized as follows. First, we extend the matrix-based characterization of polynomial solutions, which was previously available mainly for integer-order fractional systems [26–28], to Caputo and Grünwald–Letnikov fractional differential systems with variable polynomial coefficients. Second, by explicitly constructing the coefficient matrices C_m, C_s, C_1 for CFDESs and G_m, G_s, G_1 for GLFDESs, we derive necessary and sufficient rank conditions not only for the existence of an m-th order power solution, but also for the uniqueness of its order and for the existence of two or arbitrary p distinct-degree power solutions. Third, we treat Caputo and Grünwald–Letnikov systems in a unified framework and compare the corresponding rank conditions and solution structures, thereby clarifying the similarities and differences between these two widely used fractional derivatives. To the best of our knowledge, such a systematic structural analysis for Caputo/GL fractional differential systems with variable coefficients has not been reported in the existing literature. Fourth, to complement the theoretical developments, Section 4 presents two numerical validation examples—one for a Caputo system and one for the corresponding Grünwald–Letnikov system—and shows that the per-term responses and residual curves obtained from the Gamma-function-based formulas exhibit the expected cancellation patterns and are numerically almost indistinguishable, thereby confirming the operator expansions and power-type solutions derived in the paper.

Despite these advances, the present analysis has several limitations. It is restricted to linear fractional differential systems with a specific algebraic structure and to power-type solutions. Nonlinear extensions, nonhomogeneous forcing terms, and other classes of solutions (such as general series, periodic, or fractal-type solutions) are not treated, which limits the range of systems to which the current results directly apply. Moreover, the rank conditions are obtained under low-rank or sparse assumptions on the coefficient matrices; whether similar explicit criteria can be established for higher-rank or dense matrices remains an open question. Even with the basic numerical validations in Section 4, the paper still primarily provides a structural and theoretical framework: We do not yet undertake systematic numerical simulations, benchmark case studies, or sensitivity and uncertainty analyses. In particular, for more complicated systems, the numerical computation of power solutions, the influence of parameter variations, and the impact of noise and model uncertainties on the solvability conditions remain to be explored, leaving a gap between the present theory and comprehensive real-world applications.

Although the Caputo and Grünwald–Letnikov derivatives are closely related through their common Riemann–Liouville foundation, our analysis shows that the resulting coefficient matrices involve different Gamma factors and discretization structures. From a modeling perspective, Caputo-based systems are more directly connected with classical initial-value formulations, while GL-based systems are better suited for finite-difference and discrete-time realizations. A systematic numerical comparison of the two frameworks—in which the same polynomial-coefficient system is treated under both derivatives, and the corresponding power-type solutions and numerical approximations are contrasted—would further clarify these differences. Designing and implementing such numerical experiments, beyond the basic validation examples of Section 4, is an important direction for future work. Several directions for future research arise from these observations. First, we plan to investigate the existence and uniqueness of power, generalized series, and numerical solutions for Caputo/GL fractional differential systems with nonlinear terms and nonhomogeneous right-hand sides, thereby formulating and analyzing more general models beyond the current linear setting. Inspired by generalized power-series representations of fractional systems, it is natural to extend our analysis from pure power solutions of the form $x^{\alpha+k}$ to finite or infinite series

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^{\alpha_n}}{\Gamma(\alpha_n + 1)}$$

with suitably chosen exponents α_n; such an extension could reduce the number of Gamma factors appearing in computations and reveal richer families of particular solutions. Developing a corresponding matrix-based rank theory for these generalized series is an interesting topic for future work. Second, we will study fractional differential systems whose coefficient matrices have more general structures and derive flexible rank-type conditions—possibly using tools from matrix analysis, probability, and perturbation theory—to assess the existence and multiplicity of solutions in higher-rank and dense cases. Third, we aim to design efficient numerical algorithms to compute power and other special solutions, to perform parameter-sensitivity and uncertainty analyses, and to validate the theoretical criteria through simulations in representative application scenarios.

In addition, we will apply the theoretical conclusions to specific practical problems: For example, using the Caputo fractional differential system to analyze the stress–strain relationship of viscoelastic materials, and using the GL fractional differential system for anomaly detection in signal processing and image processing, to verify the effectiveness of the theoretical conclusions in practice. We will consider uncertainty factors in practical problems, introduce methods such as robust control, stochastic analysis, and fuzzy mathematics, and study the stability and robustness of solutions to fractional differential systems in uncertain environments. Combining the needs of practical applications, we will study the application of fractional differential systems in emerging fields—such as using fractional differential systems to construct memory units for deep learning models, analyzing fractional dynamic behaviors in quantum systems, and establishing fractional prediction models for financial market fluctuations—to promote the interdisciplinary development of fractional calculus theory.

We emphasize that the results in this paper are derived at the level of differential operators and power-type solutions, and do not rely on any specific choice or interpretation of initial conditions. In particular, our rank conditions characterize when a given Caputo or Grünwald–Letnikov system admits power-type solutions, independently of how initial data are prescribed. There is an extensive discussion in the fractional-calculus literature about the physical meaning and correctness of certain initial conditions associated with Caputo-type operators. A careful study of how our structural criteria interact with these issues for concrete initial-value problems is beyond the scope of this work and will be considered in future research.

Appendix A: Proof of Theorem 3

Substitute $y = K_m x^m + K_{m-1}x^{m-1} + \cdots + K_1 x$ into the CFDESs (1-1), and expand to get:

$$\begin{aligned} & A_n D^{\alpha_n} y + A_{n-1} D^{\alpha_{n-1}} y + \cdots + A_1 D^{\alpha_1} y + A_0 D^{\alpha_0} y \\ = & K_m t_{n+1,s} \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} x^{m+s-\alpha} + \left(x^{m+s-(\alpha+1)} \; x^{m+s-(\alpha+2)} \; \cdots \; x^{1-\alpha} \; x^{-\alpha} \right) C_m \begin{pmatrix} K_m \\ K_{m-1} \\ \vdots \\ K_2 \\ K_1 \end{pmatrix} \end{aligned}$$

where C_m is an $(m+s) \times m$ matrix.

$$C_m = \left[\begin{array}{ccccc} \sum_{j=s-1}^{s} t_{n+s-j,j} \frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+1))} & t_{n+1,s} \frac{\Gamma(m)}{\Gamma(m-\alpha)} & 0 & \cdots & 0 \\ \sum_{j=s-2}^{s} t_{(n-1)+s-j,j} \frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+2))} & & & & \\ \vdots & & & & \\ \sum_{j=0}^{s} t_{(n-s+1)+s-j,j} \frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j))} & & & & \\ \vdots & & & & \\ \sum_{j=0}^{s} t_{s+1-j,j} \frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j+n-s))} & & & & \\ \vdots & & & C_{m-1} & \\ \sum_{j=0}^{s-1} t_{s-j,j} \frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j+n-s+1))} & & & & \\ \vdots & & & & \\ \sum_{j=0}^{0} t_{1-j,j} \frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+n))} & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{array} \right],$$

$$C_n = \left[\begin{array}{ccccc} \sum_{j=1}^{s} t_{n+s-j,j} \frac{\Gamma(n+1)}{\Gamma(n+1-(a+j-s+1))} & t_{n+1,s} \frac{\Gamma(n)}{\Gamma(n-a)} & 0 & \cdots & 0 \\ \sum_{j=s-z}^{s} t_{(n-1)+s-j,j} \frac{\Gamma(n+1)}{\Gamma(n+1-(a+j-s+2))} & & & & \\ \vdots & & & & \\ \sum_{j=0}^{s} t_{(n-s+1)+s-j,j} \frac{\Gamma(n+1)}{\Gamma(n+1-(a+j))} & & & & \\ \sum_{j=0}^{s} t_{(n-s+1)+s-j,j} \frac{\Gamma(n+1)}{\Gamma(n+1-(a+1))} & & & & \\ \vdots & & & C_{n-1} & \\ \sum_{j=0}^{s} t_{s+1-j,j} \frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+j+n-s))} & & & & \\ \sum_{j=0}^{s} t_{s-j,j} \frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+j+n-s+1))} & & & & \\ \vdots & & & & \\ \sum_{j=0}^{0} t_{s-j,j} \frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha+n))} & & & & \end{array} \right],$$

$$C_s = \begin{bmatrix} \sum_{j=s-1}^{s} t_{n+s-j,j} \frac{\Gamma(s+1)}{\Gamma(s+1-(a+j-s+1))} & t_{n+1,s} \frac{\Gamma(s)}{\Gamma(s-a)} \\ \sum_{j=s-2}^{s} t_{(n-1)+s-j,j} \frac{\Gamma(s+1)}{\Gamma(s+1-(a+j-s+2))} & \\ \vdots & \\ \sum_{j=0}^{s} t_{(n-s+1)+s-j,j} \frac{\Gamma(s+1)}{\Gamma(s+1-(a+j))} & C_{s-1} \\ \sum_{j=0}^{s-1} t_{(n-s)+s-j,j} \frac{\Gamma(s+1)}{\Gamma(s+1-(a+j+1))} & \\ \vdots & \\ \sum_{j=0}^{0} t_{(n-s+1)-j,j} \frac{\Gamma(s+1)}{\Gamma(s+1-(\alpha+s))} & \end{bmatrix},$$

$$C_1 = \begin{bmatrix} \sum_{j=s-1}^{s} t_{n+s-j,j} \frac{\Gamma(2)}{\Gamma(2-(\alpha+j-s+1))} \\ \sum_{j=s-2}^{s-1} t_{(n-1)+s-j,j} \frac{\Gamma(2)}{\Gamma(2-(\alpha+j-s+2))} \\ \vdots \\ \sum_{j=0}^{1} t_{(n-s+1)+s-j,j} \frac{\Gamma(2)}{\Gamma(2-(\alpha+j))} \\ \sum_{j=0}^{0} t_{(n-s)+s-j,j} \frac{\Gamma(2)}{\Gamma(2-(\alpha+j+1))} \end{bmatrix}.$$

It follows from the lemma that, for the equation to hold, the following conditions must be satisfied:

1. The highest-order term coefficient $t_{n+1,s} = 0$;
2. The matrix equation $\begin{pmatrix} x^{m+s-(\alpha+1)} & \cdots & x^{-\alpha} \end{pmatrix} C_m \begin{pmatrix} K_m \\ \vdots \\ K_1 \end{pmatrix} = 0$ has a solution, which is equivalent to the rank of the augmented matrix $\hat{C_m}$ being equal to the rank of the coefficient matrix C_m, i.e., $\text{rank}(\hat{C_m}) = \text{rank}(C_m)$.

In conclusion, the system (1-1) has an m-th order power solution if and only if $t_{n+1,s} = 0$ and $\text{rank}(\hat{C_m}) = \text{rank}(C_m)$.

PC.

Appendix B: Proof of Theorem 4

From Theorem 3, the necessary and sufficient condition for the CFDESs (1-1) to have an m-th order power solution is $t_{n+1,s} = 0$ and $\text{rank}(\hat{C_m}) = \text{rank}(C_m)$, where C_m is an $(m+s)\times m$ matrix:

$$C_m = \begin{bmatrix} \sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+1))} & t_{n+1,s}\frac{\Gamma(m)}{\Gamma(m-\alpha)} \; 0 \; \cdots \; 0 \\ \sum_{j=s-2}^{s} t_{(n-1)+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+2))} & \\ \vdots & \\ \sum_{j=0}^{s} t_{(n-s+1)+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j))} & \\ \vdots & \\ \sum_{j=0}^{s} t_{s+1-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j+n-s))} & \\ \vdots & C_{m-1} \\ \sum_{j=0}^{s-1} t_{s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j+n-s+1))} & \\ \vdots & \\ \sum_{j=0}^{0} t_{1-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+n))} & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}.$$

Analyzing $\text{rank}(\hat{C_m}) = \text{rank}(C_m)$, we get:

$$\sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(m+1)}{\Gamma(m+1-(\alpha+j-s+1))} = 0.$$

Substitute $t_{n,s} \neq 0$ and simplify:

$$t_{n,s}\frac{\Gamma(m+1)}{\Gamma(m-\alpha)} + t_{n+1,s-1}\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} = 0 \implies m = -\frac{t_{n+1,s-1}}{t_{n,s}} + \alpha.$$

$$\begin{aligned} & t_{n,s}\frac{\Gamma(m+1)}{\Gamma(m-\alpha)} + t_{n+1,s-1}\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} = 0, \ (\Gamma(m-\alpha)\neq 0, \Gamma(m+1-\alpha)\neq 0, \Gamma(m+1)\neq 0) \\ \Rightarrow & t_{n,s}\Gamma(m+1)\Gamma(m+1-\alpha) + t_{n+1,s-1}\Gamma(m+1)\Gamma(m-\alpha) = 0, \\ \Rightarrow & t_{n,s}\Gamma(m+1-\alpha) + t_{n+1,s-1}\Gamma(m-\alpha) = 0, \\ \Rightarrow & m = -\frac{t_{n+1,s-1}}{t_{n,s}} + \alpha \ \in [n,+\infty)\cap\mathbb{Z}. \end{aligned}$$

It is also required that $m \in [n,+\infty)\cap\mathbb{Z}$. In addition, the diagonal elements of C_m must satisfy $\sum_{j=s-1}^{s} t_{n+s-j,j}\frac{\Gamma(l+1)}{\Gamma(l+1-(\alpha+j-s+1))} \neq 0$ ($l \in [n, m-1]\cap\mathbb{Z}$), otherwise $\text{rank}(\hat{C_m}) \neq \text{rank}(C_m)$.

At this point, the rank of $\overline{C_m}$ is $m-n$, which proves that condition 1 is equivalent to condition 2.

PC.

Author contributions

P.E. responsible for formula derivation, programming, and paper writing; W.H.Z. guided crucial steps from theoretical hypothesis to conclusion derivation; T.T.X. provided technical support for the research; J.C. in charge of literature research and data analysis; Y.X.L. participated in the formulation of the research plan; S.X.L. assisted in verifying theoretical results and supplementing references; X.L.Z. contributed to revising the manuscript structure and optimizing mathematical expressions; W.Z. collaborated in discussing key conclusions and finalizing the manuscript. All authors reviewed the manuscript and approved the final version.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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