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Research article

On the "good" Lie brackets related to a polynomial system

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Abstract: Small-time local controllability (STLC) at a point x_0 is a fundamental property of control systems, and is intimately connected to the local structure of their reachable sets. This study built upon the notion of a *tangent vector field* to the reachable set of a control system, a concept introduced by Hermes in [7], based on an idea of Krener (cf. [18]). The importance of this concept stemed from the fact that the set $E^+(x_0)$, consisting of all tangent vector fields to the reachable set at x_0, formed a convex cone. If the zero vector lay in the interior of this cone, the system is STLC at x_0. A long-standing open question concerns the precise characterization of the set $E^+(x_0)$. In this paper, we studied the Lie algebra generated by the drift term—a vector field homogeneous of degree two—and the constant vector fields of a polynomial control system. By applying the classical Campbell–Baker–Hausdorff formula from Lie group theory, along with symmetries inherent to the control system, we derived new elements of the set $E^+(x_0)$. Our results showed that certain "bad" Lie brackets (in the sense of Sussmann) do not obstruct the STLC property. As a corollary, we provided a sufficient condition for STLC.

Keywords: small-time local controllability; polynomial control systems
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1. Introduction

Controllability is a fundamental property of nonlinear control systems. The relationship between global and local controllability is explored in the work by Bacciotti and Stefani (cf. [3]) for a class of analytic control systems defined on an analytic manifold. One can study the small-time controllability of a control system at a given point, or the small-time attainability of a set (cf., for example, [5, 20]). Furthermore, controllability is closely linked to the continuity properties of the minimum time function (cf., for instance, [23, 25]), among other aspects.

Various approaches have been developed to study small-time local controllability (STLC) at a point, leading to different results under varying assumptions. General sufficient conditions for STLC have been established by Agrachev and Gamkrelidze [1], Sussmann [28, 29], Meng et al. [21] and others. The first necessary condition for STLC was proven by Sussmann in [28]. This condition was later generalized by Stefani in [26]. Kawski established in [9] a necessary condition that depends on the "size" of a symmetric interval within which the controls take their values. In [11] this condition was further generalized for the case of a non-symmetric interval. Additionally, a necessary condition for Lipschitz continuous control systems was proven in [13]. Moreover, a complete answer to the open problem on the small-time local controllability of Korteweg-de Vries system on all critical lengths is given in [22]. Additionally, both sufficient and necessary conditions have been presented for specific cases by Brunovský (cf. [4]), Jurdjevic and Kupka (cf. [8]), and others (cf. [17, 30]). Despite these efforts, a significant gap remains between the existing sufficient and necessary conditions for STLC.

An attempt to bridge this gap was made by Aguilar (cf. [2]), who analyzed a class of homogeneous control systems. The results obtained in [2] were later generalized in [14]. The present paper continues the line of investigation initiated in [14, 15], focusing on a class of polynomial control systems. To analyze STLC, we employ the set $E^+(x_0)$ of tangent vector fields to the reachable set at a fixed point x_0.

The set $E^+(x_0)$ was introduced by Hermes [7]. This concept was subsequently used in [30] for piecewise linear control systems and in [17] for switching linear systems. It was further extended in [16] to study the small-time attainability of a set. Other concepts of tangent vector fields can be found in [19, 27]. By applying the classical Campbell–Baker–Hausdorff (C-B-H) formula from Lie group theory together with certain symmetries intrinsic to the control system, we show that specific "bad" Lie brackets (in the sense of Sussmann) belong to the set $E^+(x_0)$ and therefore do not obstruct the STLC property. As a corollary, we obtain a sufficient condition for STLC.

The structure of the paper is as follows. In Section 2, we present the differential-geometrical framework employed in this study and state the main result, accompanied by a corollary and illustrative examples that demonstrate the applicability of our approach. Section 3 is devoted to the proof of the main theorem. Section 4 concludes the paper, and all technical details are collected in an Appendix.

2. Preliminaries and statement of the main result

First, we introduce some notations used throughout the exposition:

- $\mathrm{co}(S)$ denotes the convex hull of the elements in the subset S of $\mathbb{R}^n$,
- $\mathrm{span}(S)$ refers to the linear space generated by the elements of S,
- $\mathrm{cone}(S)$ represents the cone generated by the elements of S, and
- $\mathrm{rec}(C)$ is the largest linear space contained within the convex closed cone C in $\mathbb{R}^n$.

Let us consider the control system Σ in $\mathbb{R}^n$:

$$\dot{x}(t) = f(x(t)) + u(t), \quad u(t) \in U \cap \bar{\mathbf{B}}, \tag{2.1}$$

with $x(0) = 0$, where $U \subset \mathbb{R}^n$ is a closed convex cone, $\bar{\mathbf{B}}$ is the closed unit ball, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field with homogeneous quadratic components.

Remark 2.1. System (2.1) represents a *class* of nonlinear control systems in $\mathbb{R}^n$ with additive but *constrained* control input. Although the control enters linearly, the constraint $u(t) \in U \cap \bar{\mathbf{B}}$, where $U \subset \mathbb{R}^n$ is a closed convex cone, makes the analysis of reachability and controllability nontrivial. In particular, the directions of admissible control actions are restricted by U, so the system cannot in general be steered arbitrarily in $\mathbb{R}^n$. Moreover, the drift term $f(x)$ has homogeneous quadratic components, which introduces nonlinear coupling between the state variables and leads to nontrivial geometric properties of the attainable sets.

For each $T > 0$, we define the set of admissible controls:

$$\mathcal{U}_T := \left\{u : [0, T] \to \mathbb{R}^n \text{ measurable} \mid u(t) \in U \cap \bar{\mathbf{B}} \text{ a.e.}\right\}.$$

An admissible trajectory is any absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ satisfying (2.1) with $u \in \mathcal{U}_T$.

The reachable set from x_0 in time T is

$$\mathcal{R}(x_0, T) := \{x(T) \mid x \text{ is an admissible trajectory with } x(0) = x_0\}.$$

Definition 2.2. The system Σ is STLC at the origin if $\mathbf{0} \in \operatorname{int}\mathcal{R}(\mathbf{0}, T)$ for every $T > 0$.

We make use of the Lie bracket of smooth vector fields X and Y,

$$[X, Y](x) = Y'(x)X(x) - X'(x)Y(x),$$

as well as the exponential map $\operatorname{Exp}(tZ)(x_0)$, which denotes the time flow of the ODE $\dot{x}(\tau) = tZ(x(\tau))$, with $x(0) = x_0$ for $\tau = 1$.

Given $u \in U \cap \bar{\mathbf{B}}$, the vector field $x \mapsto f(x) + u$ belongs to the set $\mathcal{S}^+(0)$ defined below.

Definition 2.3. An analytic vector field Z belongs to $\mathcal{S}^+(0)$ if there exist a compact neighborhood Ω of the origin, and constants $K, T > 0$, such that

$$\operatorname{Exp}(tZ)(x) \in \mathcal{R}(x, Kt) \quad \forall x \in \Omega,\ t \in [0, T].$$

We use the notation $o(t^\alpha)$ and $O(t^\alpha)$ to denote families of analytic vector fields that satisfy standard asymptotic bounds as $t \to 0$. We write A^0 for families of vector fields $a(t, x)$, parametrized by t, such that $\|a(t, x)\| \le ct^\theta\|x\|$ for some constants $c, \theta > 0$. We also call a polynomial of the form

$$p(t) := a_1 t^{b_1} + \cdots + a_s t^{b_s},$$

where $a_i > 0$ and

$$0 < b_1 < b_2 < \cdots < b_s$$

for all i, a *positive polynomial*. The minimal positive number b_1 is called the *order* of the polynomial p, and it is denoted by $\operatorname{ord}(p)$.

Remark 2.4. Based on some classical results (see, e.g., Proposition 4.3 in [28] and Proposition 2.1 in [1], the analyticity of the vector fields ensures uniform convergence of the expansions on compact sets and stability under compositions of bounded length.

Definition 2.5. An analytic vector field Z belongs to the set $E^+(0)$ if there exist $\alpha > 0$, a compact neighborhood Ω of the origin, a time $T > 0$, a positive polynomial $p(t)$, and families $a(t) \in A^0$ and $o(t^\alpha)$ such that

$$\mathrm{Exp}(t^\alpha Z + a(t) + o(t^\alpha))(x) \in \mathcal{R}(x, p(t)) \quad \forall x \in \Omega,\ t \in [0, T].$$

One can similarly define the sets $E^+(x_0)$ and $\mathcal{S}^+(x_0)$, where x_0 is an arbitrary equilibrium point of the control system under consideration (cf., for example, [10, 16, 17]). The significance of the sets E^+ and $\mathcal{S}^+$ in analyzing the local properties of the reachable sets of a nonlinear control system is emphasized by the following lemmas:

Lemma 2.6. *If* $Z_1, \ldots, Z_k \in E^+(x_0)$ *and* $0 \in \operatorname{int} \operatorname{co}\{Z_i(x_0)\}$, *then* Σ *is STLC at* x_0.

Lemma 2.7. *The set* $E^+(x_0)$ *is a convex cone.*

Lemma 2.8. *Let* $A_1, A_2 \in E^+(x_0)$ *with* $A_1(x_0) + A_2(x_0) = 0$, *and* $B \in \mathcal{S}^+(x_0)$ *with* $B(x_0) = 0$. *Then* $[B, A_1], [B, A_2] \in E^+(x_0)$.

Lemmas 2.6 and 2.7 are proved in [17]. Lemma 2.8 is a corollary of Proposition 2.4 proven in [16].

Next, we briefly outline our approach, which will be developed in detail later. First, we introduce the following notation: For an arbitrary element $u \in U$, we denote by g_u the constant vector field defined by

$$g_u(x) := u \quad \text{for all } x \in \mathbb{R}^n.$$

Let x be an arbitrary point in $\mathbb{R}^n$, and let $u_1, u_2, \ldots, u_{k-1}$ and u_k be arbitrary elements of U. Also consider arbitrary positive real polynomials $\alpha_i(t)$ and $\beta_i(t)$, $i = 1, 2, \ldots, k$, for $t \in [0, t_0)$. In what follows, we assume that $t_0 > 0$ is sufficiently small; if necessary, we can decrease t_0 to satisfy the required assumptions. Without loss of generality, we assume that

$$\beta_i(t) u_i \in U \cap \bar{\mathbf{B}} \quad \text{for all } t \in [0, t_0) \text{ and each } i = 1, 2, \ldots, k.$$

Then the vector field $f + \beta_i(t) g_{u_i}$ is admissible for the control system (2.1). We define

$$f_i := f \quad \text{and} \quad g_i := g_{u_i}, \quad \text{for } i = 1, 2, \ldots, k.$$

Clearly, for each $i = 1, 2, \ldots, k$, we have that

$$\mathrm{Exp}\,(\alpha_i(t) f_i + \alpha_i(t)\beta_i(t) g_i) = \mathrm{Exp}\,(\alpha_i(t)\,(f_i + \beta_i(t) g_i))\,(x) \in \mathcal{R}\,(x, \alpha_i(t)) \quad \text{for all } t \in [0, t_0]. \tag{2.2}$$

With the vector fields f_i and g_i, we associate the polynomials $\alpha_i(t)$ and $\alpha_i(t)\beta_i(t)$, respectively, for each $i = 1, 2, \ldots, k$.

According to the C-B-H formula, there exists a Lie polynomial $\mathcal{S}(t)$ such that

$$\mathrm{Exp}(\mathcal{S}(t))(x) := \mathrm{Exp}\,(\alpha_1(t)(f_1 + \beta_1(t) g_1)) \circ \cdots \circ \mathrm{Exp}\,(\alpha_k(t)(f_k + \beta_k(t) g_k))\,(x), \quad t \in [0, t_0).$$

Taking into account (2.2), we obtain that

$$\mathrm{Exp}\,(\mathcal{S}(t))\,(x) \in \mathcal{R}\left(x, \sum_{i=1}^{k} \alpha_i(t)\right) \quad \text{for all } t \in [0, t_0].$$

Moreover, the C-B-H formula implies that each $\mathcal{S}(t)$, for $t \in [0, t_0)$, is a Lie series composed of Lie brackets Λ_j, $j = 1, 2, \ldots$, of the vector fields f_i and g_i, $i = 1, 2, \ldots, k$, i.e.,

$$\mathcal{S}(t) = \sum_{j=1}^{\infty} \lambda_j p_j(t) \Lambda_j,$$

where $\lambda_j \in \mathbb{R}$, and if

$$\Lambda_j = [h_1^j, [h_2^j, [\ldots [h_{s-1}^j, h_s^j] \ldots]]] \quad \text{with} \quad h_\mu^j \in \{f_i, g_i \mid i = 1, \ldots, k\},$$

then the corresponding polynomial $p_j(t)$ is given by

$$p_j(t) = p_1^j(t) \cdot p_2^j(t) \cdots p_s^j(t),$$

where $p_\mu^j(t)$ denotes the polynomial associated with the vector field h_μ^j, for $\mu = 1, 2, \ldots, s$.

Suppose that there exists a Lie bracket Λ_{i_0} such that $\Lambda_{i_0}(0) \neq \mathbf{0}$, and the order of the corresponding polynomial p_{i_0} is strictly less than the orders of the polynomials p_j associated with all other Lie brackets Λ_j that are non-vanishing at the point 0. Then, the Lie bracket Λ_{i_0} belongs to the set $E^+(0)$. This idea is illustrated by Lemma 2.10 below.

Moreover, using this idea, we select below appropriate elements $u_i \in U$, together with appropriate positive real polynomials $\alpha_i(t)$ and $\beta_i(t)$, $i = 1, 2, \ldots, k$. We then demonstrate, in detail, how one can construct elements of the set $E^+(0)$. Notably, some of these elements correspond to so-called "bad" Lie brackets in the sense of Sussmann (cf. [28, 29]).

Since the polynomial vector field f is homogeneous of degree two, we have that $[g_u, [g_u, f]]$ is a constant vector field equal to $2f(u)$ for every point $x \in \mathbb{R}^n$. For simplicity, we denote the constant vector field $g_{f(u)}$ simply by $f(u)$.

We define the following sets:

$$G := \{g_u : u \in U \cap \bar{\mathbf{B}}\}, \quad G^{\pm} := \{g_u : \pm u \in U \cap \bar{\mathbf{B}}\}, \quad G_f := G \cup \{f\}. \tag{2.3}$$

Remark 2.9. Recall that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial vector field homogeneous of degree two, and that each element of G is a constant vector field. We denote by $\mathcal{L}$ the Lie algebra generated by the elements of G_f. Let Λ be a Lie bracket composed of elements from G_f, involving the vector field f exactly k times and elements of G exactly m times. Then, one can directly verify that Λ is either a homogeneous vector field of degree $k - m + 1$, or identically equal to zero.

In particular, if Λ is homogeneous of degree one, then $k = m$; that is, Λ has even length (where the **length** of a Lie bracket refers to the total number of vector field occurrences in Λ). Likewise, if Λ is homogeneous of degree zero, then Λ has odd length.

The proof of the next lemma is given in the Appendix:

Lemma 2.10. *Let $C \in G^{\pm}$ with $f(C) = 0$. Let $\alpha > 0$ and $\beta_i > 0$, $i = 1, 2$, be real numbers, and let*

$$Exp\left(A_i(\varepsilon) + \varepsilon^{\beta_i} B_i + O\left(\varepsilon^{\alpha + 2\beta_i}\right)\right)(x) \in \mathcal{R}(x, q_i(\varepsilon)) \quad \textit{for all } \varepsilon \in [0, \varepsilon_0], \tag{2.4}$$

and for all x belonging to a neighborhood of the origin, where:

- B_i, $i = 1, 2$, *are elements of* $\mathcal{L}$, *homogeneous of degree zero, with* $B_1 + B_2 = 0$,
- $A_i(\varepsilon) = p_i^0(\varepsilon)f + \sum_{j=1}^{k_i} p_i^j(\varepsilon)\Lambda_i^j$,
- Λ_i^j *are elements of* $\mathcal{L}$, *homogeneous of degree two,*
- p_i^j *and* q_i *are positive real polynomials, with*

$$\min\left\{\text{ord }\left(p_i^j\right)\ j = 0, 1, \dots, k_i,\ i = 1, 2\right\} \geq \alpha > 0,$$

- $\min\{\beta_1, \beta_2\} > \alpha$.

Then the Lie bracket $[B_1, [C_1, f]]$ *belongs to the set* $E^+(0)$.

Corollary 2.11. *Let* $C \in G^\pm$ *with* $f(C) = 0$. *Let* $\alpha > 0$ *and* $\beta_i > 0$, $i = 1, 2$, *be real numbers, and let*

$$Exp\left(A_i(\varepsilon) + \varepsilon^{\beta_i}B_i + O\left(\varepsilon^{\alpha+2\beta_i}\right)\right)(x) \in \mathcal{R}\left(x, q_i(\varepsilon)\right) \quad \text{for all } \varepsilon \in [0, \varepsilon_0], \tag{2.5}$$

and for all x belonging to a neighborhood of the origin, where:

- B_i, $i = 1, 2$, *are elements of* $\mathcal{L}$, *homogeneous of degree zero, with* $B_1 + B_2 = 0$ *and* $f(B_1) = 0$,
- $A_i(\varepsilon) = p_i^0(\varepsilon)f + \sum_{j=1}^{k_i} p_i^j(\varepsilon)\Lambda_i^j$,
- Λ_i^j *are elements of* $\mathcal{L}$, *homogeneous of degree two,*
- p_i^j *and* q_i *are positive real polynomials, with*

$$\min\left\{\text{ord }\left(p_i^j\right)\ j = 0, 1, \dots, k_i,\ i = 1, 2\right\} \geq \alpha > 0,$$

- $\min\{\beta_1, \beta_2\} \geq \alpha$.

Then the Lie bracket $[B_1, [C_1, f]]$ *belongs to the set* $E^+(0)$.

One of the most fruitful ideas in the geometric theory of nonlinear systems, particularly in establishing sufficient conditions for STLC, is the use of symmetries inherent in the control system under consideration. This is demonstrated in the proofs of the general sufficient controllability conditions derived in [28, 29]. We also emphasize that the so-called chronological calculus, developed by Agrachev and Gamkrelidze [1], is a powerful tool for representing the flows generated by control systems. More recently, new results concerning the properties of nonlinear systems have been presented in [6, 24].

To analyze the reachable sets of system (2.1), we follow the approach of [29], using compositions of admissible flows and their C-B-H representations. These compositions give rise to diffeomorphisms whose Lie series expansions involve Lie brackets endowed with symmetry-induced invariance properties.

The following Proposition 2.12 is proved in [29] (cf. Proposition 5.1). To formulate it, we introduce the following notions.

Let L be a finite-dimensional nilpotent Lie algebra over $\mathbb{R}$, and let G_L denote the corresponding connected, simply connected Lie group. Since the exponential map

$$\exp : L \to G_L$$

is a global diffeomorphism, any map $A : L \to L$ induces a unique map $\widetilde{A} : G_L \to G_L$ defined by

$$\widetilde{A}(\exp(z)) = \exp(A(z)), \qquad z \in L.$$

Proposition 2.12. *Let L be a finite-dimensional, nilpotent Lie algebra over $\mathbb{R}$, and let G_L be the corresponding connected, simply connected Lie group. Let Λ be a finite group of pseudoautomorphisms of L, and let $\tilde{\Lambda} := \{\tilde{\lambda} : \lambda \in \Lambda\}$ be the group of bijections of G_L induced by Λ. Let S be a nonempty subset of G_L which is closed under multiplication. Suppose that every $\tilde{\lambda} \in \tilde{\Lambda}$ maps S into S. Then, S contains an element s such that $\tilde{\lambda}(s) = s$ for all $\lambda \in \Lambda$.*

Let $\Lambda = [X_1, [X_2, \dots, [X_{k-1}, X_k] \dots]]$ be a Lie bracket of length k, where $X_i \in G_f$. We define the linear involutions:

$$\mathbb{A} f = f, \quad \mathbb{A} g_u = \begin{cases} -g_u & \text{if } g_u \in G^{\pm}, \\ g_u & \text{otherwise}, \end{cases} \qquad \mathbb{T}\Lambda := [X_k, [X_{k-1}, \dots, [X_2, X_1] \dots]],$$

extended linearly to any sum of Lie brackets. A bracket Λ is said to be *invariant* under $\mathbb{A}$ or $\mathbb{T}$ if $\mathbb{A}\Lambda = \Lambda$ or $\mathbb{T}\Lambda = \Lambda$, respectively. The maps $\mathbb{A}$, $\mathbb{T}$ satisfy $\mathbb{A}^2 = \mathbb{T}^2 = \mathbb{I}$ and commute.

Let $\Xi := \{\mathbb{I}, \mathbb{T}\}$ and $\Xi^{\pm} := \{\mathbb{I}, \mathbb{A}, \mathbb{T}, \mathbb{A}\mathbb{T}\}$. It is known (cf. Lemma 7.1 and Corollary 7.2 in [29]) that a Lie bracket is invariant under Ξ if and only if it is of odd length. Similarly, a Lie bracket is invariant under $\Xi^{\pm}$ if and only if it is of odd length and each element of $G^{\pm}$ appears an even number of times or not at all (cf. Theorem 7.3 in [29]).

Let $\Theta \in \{\Xi, \Xi^{\pm}\}$, and let $\mathcal{S}$ be the Lie series associated with a composition of admissible flows:

$$\mathrm{Exp}(\mathcal{S}) = \mathrm{Exp}\big(t_1(f + g_{u_1})\big) \circ \cdots \circ \mathrm{Exp}\big(t_\ell(f + g_{u_\ell})\big),$$

where $u_i \in U \cap \bar{\mathbf{B}}$ and $t_i > 0$ for $i = 1, \dots, \ell$. Then, for small $T := t_1 + \cdots + t_\ell$, we have

$$\mathrm{Exp}(\mathcal{S})(x) \in \mathcal{R}(x, T).$$

According to the C-B-H formula, the series $\mathcal{S}$ can be presented as

$$\mathcal{S} = \Sigma^{<k}_{\mathrm{inv}} + \Sigma^{<k}_{\mathrm{not\,inv}} + \Sigma^{\geq k},$$

where $\Sigma^{<k}_{\mathrm{inv}}$ consists of Lie brackets of length $< k$ that are invariant under the group Θ, $\Sigma^{<k}_{\mathrm{not\,inv}}$ consists of Lie brackets of length $< k$ that are not invariant under Θ, and $\Sigma^{\geq k}$ consists of brackets of length $\geq k$.

As a corollary of Proposition 2.12 and under the above assumptions on $\mathcal{S}$, we obtain the following result:

Corollary 2.13. *There exist a positive integer m and a time $T > 0$ such that, for any x in a compact neighborhood of the origin,*

$$\mathrm{Exp}(\bar{\mathcal{S}})(x) \in \mathcal{R}(x, mT), \quad \bar{\mathcal{S}} = m\,\Sigma^{<k}_{\mathrm{inv}} + \bar{\Sigma}^{<k}_{\mathrm{inv}} + \bar{\Sigma}^{\geq k},$$

where $\bar{\Sigma}^{<k}_{\mathrm{inv}}$ is a finite sum of Lie brackets invariant under Θ.

Remark 2.14. If $\Theta = \Xi$, then all invariant Lie brackets in $\bar{\mathcal{S}}$ have odd length. If $\Theta = \Xi^{\pm}$, then the invariant Lie brackets are either constant vector fields or vector fields that are homogeneous of degree at least two.

Remark 2.15. The proof of Proposition 2.12 (i.e., the proof of Proposition 5.1 in [29]) shows that the number m in Corollary 2.13 depends only on k and not on the specific choice of Lie brackets.

In order to formulate our main result, we define the following sets:

Step 0:

$$\mathcal{K}_0 = U, \quad \mathcal{M}_0 = \mathrm{rec}\, \mathcal{K}_0;$$

Step 1:

$$\mathcal{K}_1 = \mathrm{co}\left(\{f(u) : u \in \mathcal{M}_0\} \cup U\right), \quad \mathcal{M}_1 = \mathrm{rec}\, \mathcal{K}_1;$$

Step 2:

$$\mathcal{K}_2 = \mathrm{co}\left(\{f(u) : u \in \mathcal{M}_1\} \cup U\right);$$

$$\mathcal{L}_2 = \{u \in \mathcal{M}_1 : -f(u) \in \mathcal{K}_2\};$$

$$\mathcal{M}_2 = span\left(\{[g_u, [g_v, f]](0) : v \in \mathcal{M}_1, u \in \mathcal{L}_2\} \cup \mathcal{M}_1\right);$$

Step $s+1$**:** For $s = 2, 3, \ldots$, we define the sets $\mathcal{K}_{s+1}$, $\mathcal{L}_{s+1}$, and $\mathcal{M}_{s+1}$ recursively as follows:

$$\mathcal{K}_{s+1} = co\left(\mathcal{K}_2 \cup \mathcal{M}_s\right),$$

$$\mathcal{L}_{s+1} = \{u \in \mathcal{M}_1 : -f(u) \in \mathcal{K}_{s+1}\},$$

$$\mathcal{M}_{s+1} = span\left(\{[g_u, [g_v, f]](0) : v \in \mathcal{M}_1, u \in \mathcal{L}_{s+1}\} \cup \mathcal{M}_s\right).$$

Finally, we set $\kappa = \min\{s : \mathcal{M}_{s+1} = \mathcal{M}_s\}$. Clearly, $\kappa \leq n$.

Then the main result is the following:

Theorem 2.16. *The set* $\{g_u : u \in \mathcal{K}_\kappa\}$ *is a subset of* $E^+(0)$.

Corollary 2.17. *If the origin belongs to the interior of the set*

$$\{g_u : u \in \mathcal{K}_\kappa\},$$

then the control system (2.1) is small-time locally controllable at the origin.

Proof. Taking into account Theorem 2.16, we obtain that the set

$$\{g_u : u \in \mathcal{K}_\kappa\}$$

is a subset of $E^+(0)$. Applying Lemma 2.6, we complete the proof. □

As a corollary of Theorem 2.16, we obtain the main result in [14]:

Corollary 2.18. *If the origin belongs to the interior of the set*

$$\{g_u : u \in \mathcal{K}_2\},$$

then the control system (2.1) is small-time locally controllable at the origin.

To clarify the relation of Theorem 2.16 to the main result in [15], we define the sets

$$\mathcal{N}_1 = \{u \in \mathcal{M}_0 : -f(u) \in \mathcal{K}_1\};$$
$$\tilde{\mathcal{M}}_1 = \operatorname{span}\left(\{[g_u,[g_v,f]](0) : v \in \mathcal{M}_0, u \in \mathcal{N}_1\} \cup \mathcal{M}_0\right).$$

For $s = 1, 2, 3, \ldots$, we define the sets $\tilde{\mathcal{K}}_{s+1}$, $\mathcal{N}_{s+1}$, and $\tilde{\mathcal{M}}_{s+1}$ recursively as follows:

$$\tilde{\mathcal{K}}_{s+1} = \operatorname{cone}\left(\mathcal{K}_1 \cup \tilde{\mathcal{M}}_s\right),$$
$$\mathcal{N}_{s+1} = \{u \in \mathcal{M}_0 : -f(u) \in \tilde{\mathcal{K}}_{s+1}\},$$
$$\tilde{\mathcal{M}}_{s+1} = \operatorname{span}\left(\{[g_u,[g_v,f]](0) : v \in \mathcal{M}_0, u \in \mathcal{N}_{s+1}\} \cup \tilde{\mathcal{M}}_s\right).$$

One can directly verify the following inclusions:

$$\tilde{\mathcal{K}}_s \subseteq \mathcal{K}_s, \quad \tilde{\mathcal{M}}_s \subseteq \mathcal{M}_s, \quad s = 2, 3, \ldots,$$

where the sets $\mathcal{K}_s$ and $\mathcal{M}_s$ (for $s = 2, 3, \ldots$) are defined prior to the statement of Theorem 2.16.

As a corollary of Theorem 2.16, we also recover the main result of [15].

Corollary 2.19. *Let*

$$\tilde{\mathcal{M}}_{\kappa+1} = \tilde{\mathcal{M}}_\kappa.$$

If the origin belongs to the interior of the set

$$\{g_u : u \in \tilde{\mathcal{K}}_\kappa\},$$

then the control system (2.1) is small-time locally controllable at the origin.

Next, we present illustrative examples showing the applicability of the obtained results. The STLC property of the control system in the first example does not follow directly from Theorem 2.16. However, by applying ideas from the proofs of Lemma 2.10, Corollary 2.11, and Theorem 2.16, one can prove its small-time controllability at the origin.

Example 2.20. Let us consider the following control system Σ_1:

$$\begin{array}{lll}
\dot{x}_1(t) = u_1(t), & x_1(0) = 0, & u_1(t) \in [-1,1],\\
\dot{x}_2(t) = u_2(t), & x_2(0) = 0, & u_2(t) \in [-1,1],\\
\dot{x}_3(t) = u_3(t), & x_3(0) = 0, & u_3(t) \in [-1,1],\\
\dot{x}_4(t) = u_4(t), & x_4(0) = 0, & u_4(t) \in [-1,1],\\
\dot{x}_5(t) = x_1^2(t) - x_2^2(t), & x_5(0) = 0, & \\
\dot{x}_6(t) = x_3(t)x_5(t), & x_6(0) = 0, & \\
\dot{x}_7(t) = x_4(t)x_6(t), & x_7(0) = 0. &
\end{array}$$

We define:

$$\begin{aligned}
&x := (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T,\\
&f(x) := (0,0,0,0,x_1^2 - x_2^2, x_3x_5, x_4x_6)^T,\\
&g_1(x) := (\pm 1,0,0,0,0,0,0)^T,\\
&g_2(x) := (0,\pm 1,0,0,0,0,0)^T,\\
&g_3(x) := (0,0,\pm 1,0,0,0,0)^T,\\
&g_4(x) := (0,0,0,\pm 1,0,0,0)^T.
\end{aligned}$$

One can directly verify that:

$$\mathcal{K}_0 = \mathcal{M}_0 = \{x \in \mathbb{R}^7 : x_5 = x_6 = x_7 = 0\},$$

$$\mathcal{K}_1 = \mathcal{M}_1 = \{x \in \mathbb{R}^7 : x_6 = 0, x_7 = 0\},$$

$$\mathcal{K}_2 = \{x \in \mathbb{R}^7 : x_7 = 0\},$$

$$\mathcal{L}_2 = \mathcal{M}_1,\ \mathcal{M}_2 = \mathcal{K}_2 = \mathcal{K}_3 = \mathcal{L}_3 = \mathcal{M}_3.$$

Thus, by applying Theorem 2.16, we cannot conclude anything regarding the STLC of the system Σ_1 at the origin.

On the other hand, one can directly verify (using the identity (A.9) from the proof of Theorem 2.16) that there exists positive integer m and real numbers c_i, i=1,2, such that

$$\mathrm{Exp}\left(4m\varepsilon^{\gamma} f + \frac{2m\varepsilon^{3\gamma}}{3} f(g_i) + c_i\varepsilon^{5\gamma}\Lambda_i + O_i(\varepsilon^{7\gamma})\right)(x) \in \mathcal{R}(x, 4m\varepsilon^{\gamma}), x \in \Omega_0, \varepsilon \in (0, \varepsilon_0),$$

where Λ_i is a sum of Lie brackets of f and g_i, $i = 1, 2$, of length five, that are invariant with respect to $\Xi^{\pm}$, and hence, Λ_i is sum of Lie brackets each of them is homogeneous of degree at least two. In this example, and taking into account Corollary 2.13, one can verify that the term $O_i(\varepsilon^{7\gamma})$ is a sum of Lie brackets of f and g_i, $i = 1, 2$, where each addend is homogeneous of degree at least two. Moreover, $f(g_1) + f(g_2) = 0$ and $f(f(g_1)) = 0$. By applying Lemma 2.10 with $\alpha = \gamma > 0, \beta_1 = \beta_2 := 3\gamma > \alpha$, $B_1 = f(g_1)$, $B_2 = f(g_2)$ and $C = g_3$, we obtain that

$$h_i := [f(g_i), [g_3, f]], \quad i = 1, 2,$$

belong to the set $E^+(0)$. Next, one can prove (following the approach proposed in [12]) that

$$[h_i, [g_4, f]], i = 1, 2,$$

also belong to the set $E^+(0)$. Applying Lemma 2.6, we conclude that the system Σ_1 is small-time locally controllable at the origin. Let us note that the main results presented in [14, 15] are not applicable to this example.

Again, we cannot decide, using Theorem 2.16, whether the control system in the next example is small-time local controllable at the origin. However, a direct calculation now shows that this system is not STLC:

Example 2.21. Let us consider the following control system Σ_2:

$$\begin{array}{lll}
\dot{x}_1(t) = u_1(t), & x_1(0) = 0, & u_1(t) \in [-1, 1], \\
\dot{x}_2(t) = u_2(t), & x_2(0) = 0, & u_2(t) \in [-1, 1], \\
\dot{x}_3(t) = u_3(t), & x_3(0) = 0, & u_3(t) \in [-1, 1], \\
\dot{x}_4(t) = u_4(t), & x_4(0) = 0, & u_4(t) \in [-1, 1], \\
\dot{x}_5(t) = x_1^2(t) - x_2^2(t), & x_5(0) = 0, & \\
\dot{x}_6(t) = x_3^2(t) - x_4^2(t), & x_6(0) = 0, & \\
\dot{x}_7(t) = x_5(t)x_6(t), & x_7(0) = 0, & \\
\dot{x}_8(t) = x_5^2(t) - x_7^2(t), & x_8(0) = 0. &
\end{array}$$

We define:

$$
\begin{aligned}
&x := (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T,\\
&f(x) := (0, 0, 0, 0, x_1^2 - x_2^2, x_3^2 - x_4^2, x_5x_6, x_5^2 - x_7^2)^T,\\
&g_1(x) := (\pm 1, 0, 0, 0, 0, 0, 0, 0)^T,\\
&g_2(x) := (0, \pm 1, 0, 0, 0, 0, 0, 0)^T,\\
&g_3(x) := (0, 0, \pm 1, 0, 0, 0, 0, 0)^T,\\
&g_4(x) := (0, 0, 0, \pm 1, 0, 0, 0, 0)^T.
\end{aligned}
$$

One can directly verify that:

$$
\mathcal{K}_0 = \mathcal{M}_0 = \{x \in \mathbb{R}^8 : x_5 = x_6 = x_7 = x_8 = 0\},
$$

$$
\mathcal{K}_1 = \mathcal{M}_1 = \{x \in \mathbb{R}^8 : x_7 = x_8 = 0\},
$$

$$
\mathcal{K}_2 = \{x \in \mathbb{R}^8 : x_7 = \alpha\beta,\ x_8 = \beta^2,\ \alpha, \beta \in \mathbb{R}\},
$$

$$
\mathcal{L}_2 = \mathcal{M}_1,
$$

$$
\mathcal{M}_2 = \{x \in \mathbb{R}^8 : x_8 = 0\}, \quad \mathcal{K}_3 = \{x \in \mathbb{R}^8 : x_8 \geq 0\},
$$

$$
\mathcal{L}_3 = \mathcal{L}_2, \quad \mathcal{M}_3 = \mathcal{M}_2.
$$

Thus, applying Theorem 2.16, we cannot conclude anything regarding the STLC of the system Σ_2 at the origin.

In fact, the control system Σ_2 is **not** small-time locally controllable at the origin. Indeed, let $u_i : [0, T] \to [-1, 1]$ for $i = 1, 2, 3, 4$ be arbitrary admissible controls, and let $x_1(t), \ldots, x_8(t)$ be the corresponding trajectory over $t \in [0, T]$. Since the controls are bounded, there exists a constant $C > 0$ such that:

$$
|x_i(t)| \leq Ct, \quad \text{for all } i = 1, \ldots, 8, \text{ and } t \in [0, T].
$$

Applying Hŏlder's inequality, we obtain:

$$
\left(\int_0^t x_5(s)x_6(s)\, ds\right)^2 \leq \left(\int_0^t x_5^2(s)\, ds\right)\left(\int_0^t x_6^2(s)\, ds\right) \leq \frac{C^2t^3}{3} \int_0^t x_5^2(s)\, ds. \tag{1}
$$

Then, for each sufficiently small $T > 0$, we have:

$$
x_8(T) = \int_0^T \left(x_5^2(t) - x_7^2(t)\right) dt = \int_0^T \left(x_5^2(t) - \left(\int_0^t x_5(s)x_6(s)\, ds\right)^2\right) dt.
$$

Using inequality (1), it follows that:

$$
x_8(T) \geq \left(1 - \frac{C^2T^4}{12}\right) \int_0^T x_5^2(t)\, dt \geq 0.
$$

Hence, any point with a negative x_8-coordinate does **not** belong to the reachable set of the system Σ_2. This confirms that Σ_2 is not small-time locally controllable at the origin.

3. Proof of Theorem 2.16

First, we remind the reader that there exists a compact neighborhood Ω_0 of the origin and a positive real number $T_0 > 0$ such that each trajectory x of Σ starting from a point $x_0 \in \Omega_0$, and corresponding to some admissible control, is well-defined on the interval $[0, T]$ with $T \leq T_0$, and remains within Ω.

Let $\mathbb{N}$ be the set of all positive integers. We define the following sets:

$$\mathcal{U} := \left\{ u : (0, \varepsilon_u) \to U \cap \bar{\mathbf{B}} \mid u(\varepsilon) = \sum_{i=1}^{m} \varepsilon^{\alpha_i} u_i,\ u_i \in U,\ \alpha_i > 0,\ i = 1, \ldots, m,\ m \in \mathbb{N} \right\},$$

$$\mathcal{U}^{\pm} := \left\{ u : (0, \varepsilon_u) \to (\text{rec } U) \cap \bar{\mathbf{B}} \mid u(\varepsilon) = \sum_{i=1}^{m} \varepsilon^{\alpha_i} u_i,\ u_i \in \text{rec } U,\ \alpha_i > 0,\ i = 1, \ldots, m,\ m \in \mathbb{N} \right\},$$

and finally,

$$\mathcal{U}_1 := \left\{ u : (0, \varepsilon_u) \to \sum_{i=1}^{m} \varepsilon^{\alpha_i} u_i \mid u_i \in \mathcal{M}_1,\ \alpha_i > 0,\ m \in \mathbb{N} \right\}, \quad \text{where} \quad \varepsilon_u \in (0, 1).$$

Without loss of generality we may assume that $\kappa \geq 2$, where $\kappa = \min\{s : \mathcal{M}_{s+1} = \mathcal{M}_s\}$. Let us fix the reals α, β, and γ such that

$$1 < \alpha < \frac{\gamma}{4} \quad \text{and} \quad 1 < 2^{\kappa-1}\beta < 2^{\kappa+1}\beta < \alpha. \tag{3.1}$$

Remark 3.1. The positive triples (α, β, γ) are explicitly defined via the inequalities (3.1). The non-emptiness of the set of feasible triples (α, β, γ) can be easily verified from these inequalities. Moreover, these inequalities ensure that the retained Lie terms dominate the remainders (for detailed, see pages 4 and 5).

Let $\mu := 2^{\kappa}\beta$, and for each $s = 1, \ldots, \kappa$, we set

$$\mu_1 := 2^{\kappa-1}\beta, \quad \mu_2 := 2^{\kappa-1}\left(1 + \frac{1}{2}\right)\beta, \quad \mu_s := 2^{\kappa-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{s-1}}\right)\beta.$$

Clearly, the inequalities (3.1) imply that

$$1 < \mu_1 < \mu_2 < \cdots < \mu_\kappa < \mu \quad \text{and} \quad 2\mu < \alpha. \tag{3.2}$$

Next we prove that for each elements p and q of the set $\{1, \ldots, \kappa\}$ with $p < q$ the following inequality holds true

$$\mu_p + \mu \leq 2\mu_q. \tag{3.3}$$

Indeed, we have that

$$\begin{aligned} 2\mu_q - \mu - \mu_p &= 2^{\kappa}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{q-1}}\right)\beta - 2^{\kappa}\beta - 2^{\kappa-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-1}}\right)\beta \\ &= 2^{\kappa}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{q-1}}\right)\beta - 2^{\kappa}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-1}} + \frac{1}{2^{p}}\right)\beta \geq 0. \end{aligned}$$

Let us fix an integer number p from the set $\{1, 2, \ldots, \kappa\}$ and an arbitrary Lie bracket $\left[g_{u_p}, \left[g_{v_p}, f\right]\right]$ with $\left[g_{u_p}, \left[g_{v_p}, f\right]\right](0) \in \mathcal{M}_p$. Then Lemma A.5 (cf. the Appendix) implies the existence of a real $\varepsilon_{u_p v_p} \in (0, 1)$, elements $v_i \in \mathcal{M}_1$ and reals $\delta_i \geq 0$, $i = 1, \ldots, \bar{\rho}$, such that

$$(0, \varepsilon_{u_p v_p}) \ni \varepsilon \to \varepsilon^{\mu+\mu_p} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} \sum_{i=1}^{\bar{\rho}} \varepsilon^{\delta_i} f(v_i)$$

is **an admissible sum of Lie brackets of length seven**, i.e., there exist $u_0 \in \mathcal{U}$, $u_\alpha \in \mathcal{U}^{\pm}$, $\alpha = 1, \ldots, s_1$, and $u_\alpha \in \mathcal{U}_1$, $\alpha = s_1 + 1, \ldots, s$, such that for each $\varepsilon \in (0, \varepsilon_{u_p v_p})$ we have that

$$\varepsilon^{\mu+\mu_p} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} \sum_{i=1}^{\bar{\rho}} \varepsilon^{\delta_i} f(v_i) = g_{u_0(\varepsilon)} + \sum_{\alpha=1}^{s} f(u_\alpha(\varepsilon)). \tag{3.4}$$

According to Lemma A.4 (cf. the Appendix), there exist positive numbers q_0, q_1^i, and q_2^i, for $i = 1, \ldots, s$, a real number $\bar{\varepsilon} \in (0, 1)$, and a family of vector fields $a(\cdot) \in A^0$ (the set A^0 of parameterized families of analytic vector fields is defined before the set $E^+(0)$) such that for each $x \in \Omega_0$ and each $\varepsilon \in (0, \bar{\varepsilon})$, we have that $\mathcal{V}(\varepsilon)$ is an admissible flow, where

$$\begin{aligned}\mathcal{V}(\varepsilon)(x) = \operatorname{Exp}&\left(\left(\varepsilon^{6\gamma+3} + q_0 \varepsilon^{2\gamma+1} + \sum_{i=s_1+1}^{s} \left(q_1^i \varepsilon^{3\gamma} + q_2^i \varepsilon^{\gamma}\right)\right) f\right.\\ &\left. + \varepsilon^{6\gamma+3\alpha} \left(g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f(u_i(\varepsilon))\right) + a(\varepsilon) + O\left(\varepsilon^{6\gamma+4\alpha}\right)\right)(x)\\ &\in \mathcal{R}\left(x, \varepsilon^{6\gamma+3\alpha} + q_0 \varepsilon^{2\gamma+\alpha} + \sum_{i=s_1+1}^{s} \left(q_1^i \varepsilon^{3\gamma} + q_2^i \varepsilon^{\gamma}\right)\right).\end{aligned} \tag{3.5}$$

Taking into account (3.4), we obtain that

$$\begin{aligned}\mathcal{V}(\varepsilon)(x) = Exp&\left(\left(\varepsilon^{6\gamma+3} + q_0 \varepsilon^{2\gamma+1} + \sum_{i=s_1+1}^{s_2} \left(q_1^i \varepsilon^{3\gamma} + q_2^i \varepsilon^{\gamma}\right)\right) f\right.\\ &\left. + \varepsilon^{6\gamma+3\alpha} \left(\varepsilon^{\mu+\mu_p} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} \sum_{i=1}^{\bar{\rho}} \varepsilon^{\delta_i} f(v_i)\right) + a(\varepsilon) + O\left(\varepsilon^{6\gamma+4\alpha}\right)\right)(x).\end{aligned} \tag{3.6}$$

Our choice of μ_p and μ (cf. the inequalities (3.2) and (3.3)) implies the inequalities $\mu_p < \mu$ and $2\mu < \alpha$. From here we obtain that

$$6\gamma + 3\alpha + \mu_p + \mu < 6\gamma + 3\alpha + 2\mu < 6\gamma + 4\alpha.$$

Taking into account that f and $a(\varepsilon)$ vanish at the origin for each $\varepsilon \in (0, \bar{\varepsilon}_{u_p v_p})$, we obtain that $\left[g_{u_p}, \left[g_{v_p}, f\right]\right] \in E^+(0)$. This completes the proof of Theorem 2.16.

4. Conclusions

We apply the general differential-geometrical approach proposed by Hermes (cf. [7]) to study the local properties of the reachable sets of smooth control systems. This approach was later employed in [17, 30] to characterize the STLC property for a class of piecewise linear control systems and a class of switching control systems, respectively.

In the present paper, we extend this approach to the study of reachable sets for a class of polynomial control systems whose dynamics are determined by a convex compact set and by a polynomial drift term, a polynomial vector field that is homogeneous of degree two. The main result shows that certain "bad" Lie brackets (in the sense of Sussmann, cf. [28, 29]) are not obstructions to the STLC property and, in fact, yield tangent vector fields to the reachable set of the considered control system.

As a corollary, we derive a new sufficient condition for STLC. This condition generalizes the results previously obtained in [14, 15]. Finally, two examples are provided to illustrate the applicability of the results.

We emphasize that the main results of this paper rely critically on the quadratic homogeneity of the drift term f. In particular, the recursive cone construction and the analysis of Lie brackets use the specific scaling properties of quadratic vector fields to ensure the dominance of certain terms and the control of remainders. While the overall strategy may provide insight for more general analytic drifts, extending the sufficient conditions for STLC beyond the quadratic case remains an open problem. Thus, the current results are strictly valid for systems with quadratic homogeneous drifts, and any generalization to higher-degree or non-homogeneous analytic drifts would require additional techniques and careful analysis.

Author contributions

The contributions of the first author include methodology, conceptualization, and writing – original draft. The contributions of the second author include validation, writing – original draft, and writing – review and editing. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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Appendix

Proof of Lemma 2.10. According to the assumptions of Lemma 2.10, we have that $C \in G^{\pm}$ with $f(C) = 0$. Also, for $i = 1, 2$, we have

$$\operatorname{Exp}\left(A_i(\varepsilon) + \varepsilon^{\beta_i} B_i + O\left(\varepsilon^{\alpha+2\beta_i}\right)\right)(x) \in \mathcal{R}(x, q_i(\varepsilon)) \quad \text{for all } \varepsilon \in [0, \varepsilon_0], \tag{A.1}$$

and for all x belonging to a neighborhood of the origin, where:

- B_i, $i = 1, 2$, are elements of $\mathcal{L}$, homogeneous of degree zero, with $B_1 + B_2 = 0$,
- $A_i(\varepsilon) = p_i^0(\varepsilon) f + \sum_{j=1}^{k_i} p_i^j(\varepsilon) \Lambda_i^j$,
- Λ_i^j are elements of $\mathcal{L}$, homogeneous of degree two,
- p_i^j and q_i are positive real polynomials, with

$$\min\left\{\text{ord }\left(p_i^j\right)\ j = 0, 1, \ldots, k_i,\ i = 1, 2\right\} \geq \alpha > 0,$$

- $\min\{\beta_1, \beta_2\} > \alpha$.

Without loss of generality, we may assume that $\beta_1 = \beta_2 = \beta$. Indeed, let us assume that $\beta_1 < \beta_2$. Then we set $\beta := \beta_2$ and $\varepsilon := \varepsilon^{\beta_2/\beta_1}$ in (A.1) for $i = 1$, and obtain that

$$\mathrm{Exp}\left(\tilde{A}_1(\varepsilon) + \varepsilon^{\beta} B_1 + \tilde{O}_1\left(\varepsilon^{\alpha+2\beta}\right)\right)(x) \in \mathcal{R}\left(x, \tilde{q}_1(\varepsilon)\right) \quad \text{for all } \varepsilon \in [0, \varepsilon_0^{\beta_1/\beta_2}],$$

where $\tilde{A}_1(\varepsilon) := A_1(\varepsilon^{\beta_2/\beta_1})$, $\tilde{O}_1(\varepsilon^{\alpha+2\beta}) := O_1(\varepsilon^{\alpha\beta/\beta_1+2\beta})$ and $\tilde{q}_1(\varepsilon) := q_1(\varepsilon^{\beta_2/\beta_1})$. One can proceed similarly if $\beta_1 > \beta_2$.

Since $C \in G^{\pm}$ and $\beta > \alpha$, we fix arbitrary $\delta \in (\alpha, \beta)$ and $\hat{\alpha} \in (\alpha/2, \alpha)$ such that $\delta + \alpha > \hat{\alpha} + \beta$. Possibly after reducing $\varepsilon_0 > 0$, we have (because $\delta > \alpha > 0$) that

$$\mathrm{Exp}\left(\varepsilon^{\alpha} f \pm \varepsilon^{\delta} C\right)(x) \in \mathcal{R}\left(x, \varepsilon^{\alpha}\right) \quad \text{for all } \varepsilon \in [0, \varepsilon_0] \text{ and all } x \in \Omega_0. \tag{A.2}$$

Because $\hat{\alpha} \in (\alpha/2, \alpha)$, for $i = 1, 2$, the ratio

$$\frac{O_i(\varepsilon^{\alpha+2\beta})}{\varepsilon^{\hat{\alpha}+2\beta}} = \frac{O_i(\varepsilon^{\alpha+2\beta})}{\varepsilon^{\alpha+2\beta}} \cdot \frac{\varepsilon^{\alpha+2\beta}}{\varepsilon^{\hat{\alpha}+2\beta}}$$

tends to zero as $\varepsilon \to 0$, uniformly with respect to $x \in \Omega_0$. For this reason, we write $O_i\left(\varepsilon^{\alpha+2\beta}\right)$ as $o_i\left(\varepsilon^{\hat{\alpha}+2\beta}\right)$, for $i = 1, 2$.

From this point, we may assume (after possibly decreasing $\varepsilon_0 > 0$ and Ω_0) that

$$\mathcal{M}(\varepsilon)(x) := \mathcal{M}^1(\varepsilon) \circ \mathrm{Exp}\left(\varepsilon^{\hat{\alpha}} f\right) \circ \mathcal{M}^2(\varepsilon)(x) \in \mathcal{R}\left(x, \varepsilon^{\hat{\alpha}} + 2\varepsilon^{\alpha} + q_1(\varepsilon) + q_2(\varepsilon)\right) \tag{A.3}$$

for all $\varepsilon \in [0, \varepsilon_0]$ and all $x \in \Omega_0$, and

$$\mathcal{M}^i(\varepsilon) := \mathrm{Exp}\left(A_i(\varepsilon) + \varepsilon^{\beta} B_i + o(\varepsilon^{\hat{\alpha}+2\beta})\right) \circ \mathrm{Exp}\left(\varepsilon^{\alpha} f + \varepsilon^{\delta} C_i\right), \tag{A.4}$$

with $C_i := (-1)^{i+1} C$, $i = 1, 2$. Applying the C-B-H formula, we obtain that

$$\begin{aligned}
\mathcal{M}^i(\varepsilon) = \mathrm{Exp}\Bigg(& A_i(\varepsilon) + \varepsilon^{\alpha} f + \varepsilon^{\beta} B_i + \varepsilon^{\delta} C_i + \frac{1}{2}\varepsilon^{\alpha}[A_i(\varepsilon), f] + \frac{\varepsilon^{\delta}}{2}[A_i(\varepsilon), C_i] \\
& + \frac{\varepsilon^{\alpha+\beta}}{2}[B_i, f] + \frac{\varepsilon^{\beta+\delta}}{2}[B_i, C_i] + \frac{\varepsilon^{\alpha}}{12}[A_i(\varepsilon), [A_i(\varepsilon), f]] + \frac{\varepsilon^{\delta}}{12}[A_i(\varepsilon), [A_i(\varepsilon), C_i]] \\
& + \frac{\varepsilon^{\alpha+\beta}}{12}[A_i(\varepsilon), [B_i, f]] + \frac{\varepsilon^{\beta+\delta}}{12}[A_i(\varepsilon), [B_i, C_i]] + \frac{\varepsilon^{\alpha+\beta}}{12}[B_i, [A_i(\varepsilon), f]] + \frac{\varepsilon^{\beta+\delta}}{12}[B_i, [A_i(\varepsilon), C_i]] \\
& + \frac{\varepsilon^{\alpha+2\beta}}{12}[B_i, [B_i, f]] + \frac{\varepsilon^{2\beta+\delta}}{12}[B_i, [B_i, C_i]] + \frac{\varepsilon^{2\alpha}}{12}[f, [f, A_i(\varepsilon)]] + \frac{\varepsilon^{2\alpha+\beta}}{12}[f, [f, B_i]] \\
& + \frac{\varepsilon^{\alpha+\delta}}{12}[f, [C_i, A_i(\varepsilon)]] + \frac{\varepsilon^{\alpha+\beta+\delta}}{12}[f, [C_i, B_i]] + \frac{\varepsilon^{\alpha+\delta}}{12}[C_i, [f, A_1(\varepsilon)]] \\
& + \frac{\varepsilon^{\alpha+\beta+\delta}}{12}[C_i, [f, B_i]] + \frac{\varepsilon^{2\delta}}{12}[C_i, [C_i, A_i(\varepsilon)]] + \frac{\varepsilon^{\beta+2\delta}}{12}[C_i, [C_i, B_i]] + O\left(\varepsilon^{4\alpha}\right) + o_i\left(\varepsilon^{\hat{\alpha}+2\beta}\right)\Bigg) \\
& \qquad \text{(using the inequality } \alpha + \delta > \hat{\alpha} + \beta\text{)} \\
= Exp\Bigg(& \varepsilon^{\beta} B_i + \varepsilon^{\delta} C_i + \frac{\varepsilon^{\alpha+\beta}}{2}[B_i, f] + \frac{\varepsilon^{\delta}}{2}[A_i(\varepsilon), C_i] + a_i^2(\varepsilon) + o_i\left(\varepsilon^{\hat{\alpha}+\beta+\delta}\right)\Bigg),
\end{aligned} \tag{A.5}$$

where $a_i^2(\varepsilon) = A_i(\varepsilon) + \varepsilon^{\alpha} f$ plus a finite sum of Lie brackets of $A_i(\varepsilon)$, $\varepsilon^{\alpha} f$, $\varepsilon^{\beta} B_i$, and $\varepsilon^{\delta} C_i$, each of which is homogeneous of degree greater than two. Also, we have used the inequalities $\mathrm{ord}(p_i^j) \geq \alpha$ for each $j = 1, \ldots, k_i$, $i = 1, 2$.

Note that the vector fields B_i, and C_i, $i = 1, 2$, are homogeneous of degree zero, and hence $[B_i, B_j] \equiv 0$, $[B_i, C_j] \equiv 0$ and $[C_i, C_j] \equiv 0$, $i = 1, 2$; $j = 1, 2$. Also, the identity

$$[C_i, [B_i, f]] + [B_i, [f, C_i]] + [f, [C_i, B_i]] = 0,$$

implies that $[C_i, [B_i, f]] = [B_i, [C_i, f]]$.

Taking into account (A.5) and that $B_1 + B_2 = 0$, we apply again the C-B-H formula and obtain that

$$\begin{aligned} \mathcal{M}(\varepsilon)(x) &= \mathcal{M}^1(\varepsilon) \circ \operatorname{Exp}\left(\varepsilon^{\hat{\alpha}} f\right) \circ \mathcal{M}^2(\varepsilon)(x) \\ &= \operatorname{Exp}\left(\frac{1}{12}[\varepsilon^{\beta} B_1 + \varepsilon^{\delta} C_1, [\varepsilon^{\beta} B_1 + \varepsilon^{\delta} C_1, f]]\right. \\ &\quad \left. + \frac{1}{12}[\varepsilon^{\beta} B_2 + \varepsilon^{\delta} C_2, [\varepsilon^{\beta} B_2 + \varepsilon^{\delta} C_2, f]] + a(\varepsilon) + \tilde{o}\left(\varepsilon^{\hat{\alpha}+\beta+\delta}\right)\right), \end{aligned}$$

where $a(\varepsilon)$ is a finite sum of Lie brackets that vanish at zero. This equality follows from a straightforward calculation based on the C-B-H formula. In the computation, we use the relations $B_1 + B_2 = 0$ and $C_1 + C_2 = 0$, as well as the fact that the Lie bracket of two vector fields that vanish at the origin also vanishes at the origin.

Since $[C_1, [C_1, f]] = 2f(C_1) = 0$, $B_2 = -B_1$, $C_2 = -C_1$, and $[C_i, [B_i, f]] = [B_i, [C_i, f]]$ for $i = 1, 2$, the previously written equality together with (A.3) implies that

$$\mathcal{M}(\varepsilon)(x) = \operatorname{Exp}\left(\frac{1}{3}\varepsilon^{\hat{\alpha}+\beta+\delta}[B_1, [C_1, f]] + a(\varepsilon) + o\left(\varepsilon^{\hat{\alpha}+\beta+\delta}\right)\right) \in \mathcal{R}\left(x, \varepsilon^{\hat{\alpha}} + 2\varepsilon^{\alpha} + q_1(\varepsilon) + q_2(\varepsilon)\right)$$

for all $x \in \Omega$ and for each $\varepsilon \in [0, \varepsilon_0)$. Hence, we obtain that $[B_1, [C_1, f]] \in E^+(0)$. This completes the proof of Lemma 2.10. □

Proof of Corollary 2.11. Assume that $\alpha = \beta_1 = \beta_2$. The proof of Corollary 2.11 follows the same steps as the proof of Lemma 2.10, with the only difference being the choice of δ: we set $\delta := \alpha$. Continuing as in the proof of Lemma 2.10, we obtain that

$$\begin{aligned} \mathcal{M}(\varepsilon)(x) &= \operatorname{Exp}\left(\frac{1}{3}\varepsilon^{\hat{\alpha}+2\alpha}[B_1, [C_1, f]] + \frac{\varepsilon^{\hat{\alpha}+2\alpha}}{6}[B_1, [B_1, f]] + a(\varepsilon) + \tilde{o}\left(\varepsilon^{\hat{\alpha}+2\alpha}\right)\right) \\ &\in \mathcal{R}\left(x, \varepsilon^{\hat{\alpha}} + 2\varepsilon^{\alpha} + q_1(\varepsilon) + q_2(\varepsilon)\right) \end{aligned}$$

for all $x \in \Omega$ and all $\varepsilon \in [0, \varepsilon_0)$. Since

$$[B_1, [B_1, f]] = 2f(B_1) = 0,$$

it follows that $[B_1, [C_1, f]] \in E^+(0)$. This completes the proof of Corollary 2.11. □

We recall that $\gamma > 1$, and the sets $\mathcal{U}$, $\mathcal{U}^{\pm}$, and $\mathcal{U}_1$ are defined in Section 3. Furthermore, there exist a compact neighborhood Ω_0 of the origin and a constant $T_0 > 0$ such that every trajectory x of the system Σ, starting from a point $x_0 \in \Omega_0$ and corresponding to some admissible control, is well-defined on the interval $[0, T]$ for all $T \leq T_0$, and remains within Ω.

For notational convenience, we define the function $\tau : \mathbb{N} \to \mathbb{R}_+$ by

$$\tau(\eta) := \frac{T_0}{\eta}.$$

Let u be an arbitrary element of $\mathcal{U}^\pm$, and let $\varepsilon \in (0, \bar{\varepsilon}_u)$, where $\bar{\varepsilon}_u := \min(\varepsilon_u, \tau(4))$. Applying the equality $-g_{u(\varepsilon)} = g_{-u(\varepsilon)}$ and the C-B-H formula, we obtain

$$\begin{aligned} &\mathrm{Exp}\,(\varepsilon^\gamma(f \pm g_{u(\varepsilon)})) \circ \mathrm{Exp}\,(\varepsilon^\gamma(f \mp g_{u(\varepsilon)})) \\ &= \mathrm{Exp}\left(2\varepsilon^\gamma f \pm \varepsilon^{2\gamma}[g_{u(\varepsilon)}, f] + \frac{\varepsilon^{3\gamma}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + O^\pm(\varepsilon^{4\gamma})\right). \end{aligned} \tag{A.6}$$

Our choice of ε implies that for each $x \in \Omega_0$, the trajectory $\mathcal{P}_u(\varepsilon)(x)$ is well-defined, where

$$\begin{aligned} \mathcal{P}_u(\varepsilon)(x) &:= \mathrm{Exp}\,(\varepsilon^\gamma(f + g_{u(\varepsilon)})) \circ \mathrm{Exp}\,(\varepsilon^\gamma(f - g_{u(\varepsilon)})) \\ &\circ \mathrm{Exp}\,(\varepsilon^\gamma(f - g_{u(\varepsilon)})) \circ \mathrm{Exp}\,(\varepsilon^\gamma(f + g_{u(\varepsilon)}))\,(x) \in \mathcal{R}(x, 4\varepsilon^\gamma). \end{aligned} \tag{A.7}$$

Taking into account Eq (A.6) and the C-B-H formula, we obtain

$$\begin{aligned} \mathcal{P}_u(\varepsilon) = \mathrm{Exp}&\left(2\varepsilon^\gamma f + \varepsilon^{2\gamma}[g_{u(\varepsilon)}, f] + \frac{\varepsilon^{3\gamma}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + O^+(\varepsilon^{4\gamma})\right) \\ \circ\, \mathrm{Exp}&\left(2\varepsilon^\gamma f - \varepsilon^{2\gamma}[g_{u(\varepsilon)}, f] + \frac{\varepsilon^{3\gamma}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + O^-(\varepsilon^{4\gamma})\right). \end{aligned}$$

Thus, we have

$$\mathcal{P}_u(\varepsilon) = \mathrm{Exp}\left(4\varepsilon^\gamma f + \frac{2\varepsilon^{3\gamma}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + 2\varepsilon^{3\gamma}[f, [f, g_{u(\varepsilon)}]] + O(\varepsilon^{4\gamma})\right). \tag{A.8}$$

Applying Corollary 2.13 with $\Xi^\pm$, we obtain that there exists a positive integer m and an admissible flow $\hat{\mathcal{P}}_u(\varepsilon)$, for $\varepsilon \in (0, \bar{\varepsilon}_u^m)$ with $\bar{\varepsilon}_u^m := \min(\{\varepsilon_u, \tau(4m)\})$, such that

$$\begin{aligned} \hat{\mathcal{P}}_u(\varepsilon)(x) = \mathrm{Exp}&\left(4m\varepsilon^\gamma f + m\frac{2\varepsilon^{3\gamma}}{3}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]]\right. \\ &\left. + \varepsilon^{5\gamma}\Lambda_{u(\varepsilon)} + O(\varepsilon^{7\gamma})\right)(x) \in \mathcal{R}(x, 4m\varepsilon^\gamma), \quad x \in \Omega_0,\ \varepsilon \in (0, \bar{\varepsilon}_u^m), \end{aligned} \tag{A.9}$$

where $\Lambda_{u(\varepsilon)}$ is a finite sum of Lie brackets of length five of f and $g_{u(\varepsilon)}$ that are invariant with respect to $\Xi^\pm$. Taking into account Remark 2.14, we obtain that $\Lambda_{u(\varepsilon)}$ is a finite sum of Lie brackets that are homogeneous of degree at least two, and hence vanish at the origin.

Remark A.1. Note that the proof of Proposition 5.1 from [29] implies that the positive integer m does not depend on the particular choice of the element u from $\mathcal{U}^\pm$.

We choose arbitrary elements $u_0 \in \mathcal{U}$ and $u_1, \ldots, u_s$ from $\mathcal{U}^{\pm}$, set $\hat{u} := (u_0, u_1, \ldots, u_s), \varepsilon_{\hat{u}} = \min\{\varepsilon_{u_i}, i = 0, 1, \ldots, s\} > 0$,

$$\bar{\varepsilon}_{\hat{u}} := \min\{\varepsilon_{\hat{u}}, \tau(2m(2s+1)\}$$

and consider the function

$$(0, \bar{\varepsilon}_{\hat{u}}) \ni \varepsilon \to g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f(u_i(\varepsilon)) .$$

We call this function **an admissible sum of Lie brackets of length three**. Then, taking into account (A.7)–(A.9), we obtain that for each $x \in \Omega_0$ and for each $\varepsilon \in (0, \bar{\varepsilon}_{\hat{u}})$ we have that $Q_{\hat{u}}(\varepsilon)(x)$ is an admissible flow, where

$$\begin{aligned} Q_{\hat{u}}(\varepsilon)(x) := Exp\left(\frac{4m}{3}\varepsilon^{3\gamma}\left(f + g_{u_0(\varepsilon)}\right)\right) \circ \hat{\mathcal{P}}_{u_1}(\varepsilon) \circ \ldots \\ \circ \hat{\mathcal{P}}_{u_s}(\varepsilon)(x) \in \mathcal{R}\left(x, \frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right). \end{aligned} \tag{A.10}$$

According to the C-B-H, we have that

$$Q_{\hat{u}}(\varepsilon)(x) = Exp\left(\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right) f + \frac{4m}{3}\varepsilon^{3\gamma}\left(g_{u_0(\varepsilon)} + \frac{1}{2}\sum_{i=1}^{s}[g_{u_i(\varepsilon)}, [g_{u_i(\varepsilon)}, f]]\right) + O^s(\varepsilon^{4\gamma})\right).$$

Applying Corollary 2.13 with $\Xi^{\pm}$, we obtain that there exists a positive integer p such that for each $\varepsilon \in (0, \bar{\varepsilon}^p_{\hat{u}})$ with $\bar{\varepsilon}^p_{\hat{u}} := \min\{\bar{\varepsilon}_{\hat{u}}, \tau(2pm(2s+1))\}$ and for each $x \in \Omega_0$ it is well defined the admissible flow

$$\begin{aligned} \hat{Q}_{\hat{u}}(\varepsilon)(x) := Exp\Bigg(p\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right) f \\ + \frac{4mp}{3}\varepsilon^{3\gamma}\left(g_{u_0(\varepsilon)} + \frac{1}{2}\sum_{i=1}^{s}[g_{u_i(\varepsilon)}, [g_{u_i(\varepsilon)}, f]]\right) + a(\varepsilon) + O^s(\varepsilon^{7\gamma})\Bigg)(x) \\ = Exp\Bigg(p\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right) f + \frac{4mp}{3}\varepsilon^{3\gamma}\left(g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f(u_i(\varepsilon))\right) \\ + a(\varepsilon) + O^s(\varepsilon^{7\gamma})\Bigg)(x) \in \mathcal{R}\left(x, p\frac{4m}{3}\varepsilon^{3\gamma} + 4mps\varepsilon^{\gamma}\right), \end{aligned}$$

where $a(\varepsilon)$ is a finite sum of Lie brackets of length five of f and $g_{u_i(\varepsilon)}$, $i = 0, 1, \ldots, s$, that are invariant with respect to $\Xi^{\pm}$. Taking into account Remark 2.14, we obtain that $a(\varepsilon)$ is a finite sum of Lie brackets that are homogeneous of degree at least two, and hence, vanishing at the origin. In such a way we have proved the following:

Lemma A.2. *Let*

$$(0, \varepsilon_{\hat{u}}) \ni \varepsilon \to g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f(u_i(\varepsilon)) ,$$

be an arbitrary admissible sum of Lie brackets of length three where $u_0 \in \mathcal{U}$ and $u_1, \ldots, u_s$ belong to $\mathcal{U}^{\pm}$,

$$\hat{u} := (u_0, u_1, \ldots, u_s) \text{ and } \varepsilon_{\hat{u}} = \min\{\varepsilon_{u_i}, i = 0, 1, \ldots, s\}.$$

Then there exists an admissible flow $\hat{Q}_{\hat{u}}(\varepsilon)$, $\varepsilon \in (0, \bar{\varepsilon}_{\hat{u}})$ with

$$\bar{\varepsilon}^p_{\hat{u}} := \min\{\bar{\varepsilon}_{\hat{u}}, \tau(2pm(s+1))\}$$

such that for each $\varepsilon \in (0, \bar{\varepsilon}^p_{\hat{u}})$ and each $x \in \Omega_0$ we have that

$$\hat{Q}_{\hat{u}}(\varepsilon) := Exp\left(p\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right)f + \frac{4mp}{3}\varepsilon^{3\gamma}\left(g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f(u_i(\varepsilon))\right)\right.$$
$$\left. + a(\varepsilon) + O^s(\varepsilon^{7\gamma})\right) \in \mathcal{R}\left(x, p\frac{4m}{3}\varepsilon^{3\gamma} + 4mps\varepsilon^{\gamma}\right), \tag{A.11}$$

where $a(\varepsilon)$ is a finite sum of Lie brackets of length five of f and $g_{u_i(\varepsilon)}$, $i = 1, \ldots, s$, homogeneous of degree at least two, and thus vanishing at the origin.

Let us fix an arbitrary element $u \in \mathcal{U}_1$. According to the definition of the set $\mathcal{U}_1$, we have that

$$(0, \varepsilon_u) \ni \varepsilon \to u(\varepsilon) = \sum_{i=1}^{m_0} \varepsilon^{\alpha_i} u_i, \quad \text{with } \alpha_i > 1 \text{ and } u_i \in \mathcal{M}_1, i = 1, \ldots m_0.$$

The definition of $\mathcal{M}_1$ imply that each $u_i = u_{i0} + \sum_{j=1}^{m_i} f(u_{ij})$ with $u_{i0} \in U$ and $u_{ij} \in \mathcal{M}_0$, $j = 1, \ldots, m_i$, $i = 1, \ldots, m_0$. Then

$$u(\varepsilon) = \sum_{i=1}^{m_0} \varepsilon^{\alpha_i}\left(u_{i0} + \sum_{j=1}^{m_i} f(u_{ij})\right) = \sum_{i=1}^{m_0} \varepsilon^{\alpha_i} u_{i0} + \sum_{i=1}^{m_0}\sum_{j=1}^{m_i} f(\varepsilon^{\alpha_i/2} u_{ij})$$
(after renumbering these sums can be written as follows)
$$= \hat{u}_0(\varepsilon) + \sum_{i=1}^{m} f(\varepsilon^{\alpha_i/2}\bar{u}_i) = \hat{u}_0(\varepsilon) + \sum_{i=1}^{m} f(\hat{u}_i(\varepsilon)),$$

where $\hat{u}_0(\varepsilon) := \sum_{i=1}^{m_0} \varepsilon^{\alpha_i} u_{i0}$, $\hat{u}_i(\varepsilon) := \varepsilon^{\alpha_i/2}\bar{u}_i$. We set $\hat{u} := (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_m)$. Without loss of generality we may think that $\varepsilon_{\hat{u}} := \min\{\hat{u}_i, i = 0, 1, \ldots, m\} \in (0, 1)$ is so small that for each $\varepsilon \in (0, \varepsilon_{\hat{u}})$ we have that

$$\sum_{i=1}^{m_0} \varepsilon^{\alpha_i} u_{i0} \in U \cap \bar{\mathbf{B}} \text{ and } \varepsilon^{\alpha_i/2}\bar{u}_i \in \mathcal{M}_0 \cap \bar{\mathbf{B}}, i = 1, \ldots m. \tag{A.12}$$

Applying Lemma A.2, we obtain that there exists a positive integer p^+ such that for each point $x \in \Omega_0$ and each $\varepsilon \in (0, \bar{\varepsilon}^{p^+}_{\hat{u}})$ it is satisfied

$$Exp\left(p^+\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right)f + p^+\frac{4m}{3}\varepsilon^{3\gamma}\left(g_{u_0(\varepsilon)} + \sum_{i=1}^{m} f(\hat{u}_i(\varepsilon))\right) + a^+(\varepsilon) + O^+(\varepsilon^{7\gamma})\right)(x)$$
$$= Exp\left(p^+\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right)f + p^+\frac{4m}{3}\varepsilon^{3\gamma}u(\varepsilon) + a^+(\varepsilon) + O^+(\varepsilon^{7\gamma})\right)(x)$$
$$\in \mathcal{R}\left(x, p^+\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right)\right), \tag{A.13}$$

where $a^{+}(\varepsilon)$ is a finite sum of Lie brackets of f and $g_{u_i(\varepsilon)}$, $i = 0, 1, \ldots, s$ of length five that are invariant with respect to $\Xi^{\pm}$. Taking into account Remark 2.14, we obtain that $a(\varepsilon)$ is a finite sum of Lie brackets that are homogeneous of degree at least two, and hence, vanishing at the origin.

By the following change of the parameter

$$\varepsilon := \varepsilon\left(\frac{p^{+}4m}{3}\right)^{\frac{1}{3\gamma}},$$

it follows from (A.13) that there exists positive reals σ^{+} and $\bar{\varepsilon}_{\hat{u}}^{\sigma^{+}}$ such that for each point $x \in \Omega_0$ and each $\varepsilon \in (0, \bar{\varepsilon}_{\hat{u}}^{\sigma^{+}})$ the following inclusion is satisfied

$$Exp\left((\varepsilon^{3\gamma} + \sigma^{+}\varepsilon^{\gamma})f + \varepsilon^{3\gamma}u(\varepsilon) + \hat{a}^{+}(\varepsilon) + \hat{O}^{+}(\varepsilon^{7\gamma})\right)(x) \in \mathcal{R}\left(x, \varepsilon^{3\gamma} + \sigma^{+}\varepsilon^{\gamma}\right), \tag{A.14}$$

where $\hat{a}^{+}(\varepsilon) \in A^{0}$.

According to the definition of $\mathcal{M}_1$, we have that

$$-u_i = \tilde{u}_{i0} + \sum_{j=1}^{l_i} f(\tilde{u}_{ij}), i = 1, ..., m_0,$$

where each $\tilde{u}_{i0} \in U$ and each $\tilde{u}_{ij} \in \mathcal{M}_0$. Hence,

$$-u(\varepsilon) = -\sum_{i=1}^{m_0} \varepsilon^{\alpha_i} u_i = \sum_{i=1}^{m_0} \varepsilon^{\alpha_i}\left(\tilde{u}_{i0} + \sum_{j=1}^{l_i} f(\tilde{u}_{ij})\right) = \sum_{i=1}^{m_0} \tilde{u}_{i0}(\varepsilon) + \sum_{i=1}^{m_0}\sum_{j=1}^{l_i} f(\varepsilon^{\alpha_i/2}\tilde{u}_{ij}) = \tilde{u}_0(\varepsilon) + \sum_{i=1}^{\tilde{m}} f(\tilde{u}_i(\varepsilon)), \tag{A.15}$$

where $\tilde{u}_0(\varepsilon) = \sum_{i=1}^{m_0} \tilde{u}_{i0}(\varepsilon)$ with $\tilde{u}_{i0}(\varepsilon) := \varepsilon^{\alpha_i}\tilde{u}_{i0}$ and $\tilde{u}_i(\varepsilon)$, $i = 1, \ldots .\tilde{m}$, are determined by the equality

$$\sum_{i=1}^{\tilde{m}} f\left(\tilde{u}_i(\varepsilon)\right) = \sum_{i=1}^{m}\sum_{j=1}^{l_i} f(\varepsilon^{\alpha_i/2}\tilde{u}_{ij}). \tag{A.16}$$

Without loss of generality we may think that $\varepsilon_{\hat{u}}$ is so small that for each $\varepsilon \in (0, \varepsilon_{\hat{u}})$ we have that

$$\sum_{i=1}^{m_0} \varepsilon^{\alpha_i}\tilde{u}_{i0} \in U \cap \tilde{\mathbf{B}} \text{ and } \varepsilon^{\alpha_i/2}\tilde{u}_i \in \mathcal{M}_0 \cap \bar{\mathbf{B}}, i = 1, \ldots m. \tag{A.17}$$

Applying Lemma A.2, we obtain that there exists a positive integer p^{-} such that for each point $x \in \Omega$ and each $\varepsilon \in (0, \bar{\varepsilon}_{\hat{u}}^{p^{-}})$ it is satisfied

$$\begin{aligned} &Exp\left(p^{-}\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right)f + p^{-}\frac{4m}{3}\varepsilon^{3\gamma}\left(g_{\tilde{u}_0(\varepsilon)} + \sum_{i=1}^{\tilde{m}} f\left(\tilde{u}_i(\varepsilon)\right)\right)\right. \\ &\left. + a^{-}(\varepsilon) + O^{s}(\varepsilon^{7\gamma})\right)(x) = Exp\left(p^{-}\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right)f\right. \\ &\left. -p^{-}\frac{4m}{3}\varepsilon^{3\gamma}u(\varepsilon) + a^{-}(\varepsilon) + O^{-}(\varepsilon^{7\gamma})\right)(x) \in \mathcal{R}\left(x, p^{-}\left(\frac{4m}{3}\varepsilon^{3\gamma} + 4ms\varepsilon^{\gamma}\right)\right), \end{aligned} \tag{A.18}$$

where $a^{-}(\varepsilon)$ is a finite sum of Lie brackets of f and $g_{u_i(\varepsilon)}$, $i = 0, 1, \ldots, s$, of length five that are invariant with respect to $\Xi^{\pm}$. Taking into account Remark 2.14, we obtain that $a(\varepsilon)$ is a finite sum of Lie brackets that are homogeneous of degree at least two, and hence, vanishing at the origin.

By the following change of the parameter

$$\varepsilon := \varepsilon\left(\frac{p^{-}4m}{3}\right)^{\frac{1}{3\gamma}},$$

it follows from (A.18) that there exists positive reals σ^{-} and $\bar{\varepsilon}_{\hat{u}}^{\sigma^{-}}$ such that for each point $x \in \Omega_0$ and each $\varepsilon \in (0, \bar{\varepsilon}_{\hat{u}}^{\sigma^{-}})$ the following inclusion is satisfied

$$Exp\left((\varepsilon^{3\gamma} + \sigma^{-}\varepsilon^{\gamma})f + \varepsilon^{3\gamma}u(\varepsilon) + \hat{a}^{-}(\varepsilon) + \hat{O}^{-}(\varepsilon^{7\gamma})\right)(x) \in \mathcal{R}\left(x, \varepsilon^{3\gamma} + \sigma^{-}\varepsilon^{\gamma}\right), \tag{A.19}$$

where $\hat{a}^{-}(\varepsilon) \in A^0$.

Using (A.14) and (A.19), we obtain that

$$\begin{aligned} &Exp\left((\varepsilon^{3\gamma} + \sigma^{+}\varepsilon^{\gamma})f + \varepsilon^{3\gamma}u(\varepsilon) + \hat{a}^{+}(\varepsilon) + \hat{O}^{+}(\varepsilon^{7\gamma})\right) \circ Exp\left(\varepsilon^{3\alpha}f\right) \\ &\circ Exp\left((\varepsilon^{3\gamma} + \sigma^{-}\varepsilon^{\gamma})f + \varepsilon^{3\gamma}u(\varepsilon) + \hat{a}^{-}(\varepsilon) + \hat{O}^{-}(\varepsilon^{7\gamma})\right)(x) \\ &\in \mathcal{R}\left(x, 2\varepsilon^{3\gamma} + (\sigma^{+} + \sigma^{-})\varepsilon^{\gamma} + \varepsilon^{3\alpha}\right) \end{aligned} \tag{A.20}$$

for each $x \in \Omega_0$ and for each $\varepsilon \in (0, \bar{\varepsilon}_{\hat{u}}^{\sigma^{+},\sigma^{-}})$ with

$$\bar{\varepsilon}_{\hat{u}}^{\sigma^{+},\sigma^{-}} := \min\{\varepsilon_{\hat{u}}, \tau(3 + \sigma^{+} + \sigma^{-})\}.$$

Applying Corollary 2.13 with Ξ, we obtain that there exists a positive integer σ^{*} and a real $\tilde{\varepsilon}_u$ such that for each point $x \in \Omega$ and each $\varepsilon \in (0, \tilde{\varepsilon}_u)$ it is satisfied

$$\begin{aligned} \mathcal{S}_u^{*}(\varepsilon) &:= Exp\left(\sigma^{*}(2\varepsilon^{3\gamma} + (\sigma^{+} + \sigma^{-})\varepsilon^{\gamma} + \varepsilon^{3\alpha})f + \sigma^{*}\frac{\varepsilon^{6\gamma+3\alpha}}{12}[g_{u(\varepsilon)}, [g_{u(\varepsilon)}, f]] + \tilde{a}^{u}(\varepsilon) + \tilde{O}^{u}\left(\varepsilon^{6\gamma+4\alpha}\right)\right) \\ &= Exp\left(\sigma^{*}(2\varepsilon^{3\gamma} + (\sigma^{+} + \sigma^{-})\varepsilon^{\gamma} + \varepsilon^{3\alpha})f + \sigma^{*}\frac{\varepsilon^{6\gamma+3\alpha}}{6}f(u(\varepsilon)) + \tilde{a}^{u}(\varepsilon) + \tilde{O}^{u}\left(\varepsilon^{6\gamma+4\alpha}\right)\right) \\ &\in \mathcal{R}\left(x, \sigma^{*}(2\varepsilon^{3\gamma} + (\sigma^{+} + \sigma^{-})\varepsilon^{\gamma} + \varepsilon^{3\alpha})\right) \end{aligned} \tag{A.21}$$

where $\tilde{a}^{u}(\varepsilon)$ is a finite sum of Lie brackets (obtained from the C-B-H formula) that are invariant with respect to Ξ (and, hence, of odd length) and such that $g_{u(\varepsilon)}$ appears in them at most one time (according to the choice of the reals α and γ in (3.1)). Taking into account Remark 2.9, we obtain that $a(\varepsilon)$ is a finite sum of Lie brackets that are homogeneous of degree at least two, and hence, vanishing at the origin.

This and the inclusion (A.21) imply the existence of positive reals q_0, q_1 and q_2 such that for each $x \in \Omega_0$ and for each $\varepsilon \in (0, \tilde{\varepsilon})$ there exists an admissible flow $\mathcal{F}_u(\varepsilon)$ such that

$$\begin{aligned} \mathcal{F}_u(\varepsilon) &:= \mathrm{Exp}\ \left(\left(q_0\varepsilon^{3\gamma} + q_1\varepsilon^{\gamma} + q_2\varepsilon^{3\alpha}\right)f + \varepsilon^{6\gamma+3\alpha}f(u(\varepsilon)) + a^{u}(\varepsilon) + O^{u}\left(\varepsilon^{6\gamma+4\alpha}\right)\right) \\ &\in \mathcal{R}\left(x, q_0\varepsilon^{3\gamma} + q_1\varepsilon^{\gamma} + q_2\varepsilon^{3\alpha}\right), \end{aligned}$$

where $a^{u}(\varepsilon)$ is a family of analytic vector fields parameterized by ε, homogeneous of degree at least two and thus vanishing at the origin. Thus we have proved the following:

Lemma A.3. *For each $u \in \mathcal{U}_1$ with $\varepsilon_u \in (0,1)$ that satisfies the relations (A.12) and (A.17) there exists an admissible flow $\mathcal{F}_u(\varepsilon)$, $\varepsilon \in (0, \tilde{\varepsilon}_u)$, such that*

$$\mathcal{F}_u(\varepsilon) := Exp\ \left(\left(q_0\varepsilon^{3\gamma} + q_1\varepsilon^{\gamma} + q_2\varepsilon^{3\alpha}\right) f + \varepsilon^{6\gamma+3\alpha} f(u(\varepsilon)) + a^u(\varepsilon) + O^u\left(\varepsilon^{6\gamma+4\alpha}\right)\right)$$

$$\text{and } \mathcal{F}_u(\varepsilon)(x) \in \mathcal{R}\left(x, q_0\varepsilon^{3\gamma} + q_1\varepsilon^{\gamma} + q_2\varepsilon^{3\alpha}\right), x \in \Omega_0, \varepsilon \in (0, \tilde{\varepsilon}_u), \tag{A.22}$$

where $a^u(\varepsilon)$ is a family of analytic vector fields parameterized by ε, homogeneous of degree at least two, and hence, vanishing at the origin.

We choose arbitrary elements $u_0 \in \mathcal{U}$ and $u_1, \ldots, u_s$ from $\mathcal{U}^{\pm} \cup \mathcal{U}_1$, set $\bar{\varepsilon} = \min\{\varepsilon_{u_i}, i = 0, 1, \ldots, s\} > 0$ and consider the function

$$(0, \bar{\varepsilon}) \ni \varepsilon \to g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f\left(u_i(\varepsilon)\right).$$

We call this function **an admissible sum of Lie brackets of length seven**.

Let

$$(0, \bar{\varepsilon}) \ni \varepsilon \to g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f\left(u_i(\varepsilon)\right)$$

be an admissible sum of Lie brackets of length seven. Without loss of generality we may think that

$$(0, \bar{\varepsilon}) \ni \varepsilon \to g_{u_0(\varepsilon)} + \sum_{i=1}^{s_1} f\left(u_i(\varepsilon)\right) + \sum_{i=s_1+1}^{s} f\left(u_i(\varepsilon)\right),$$

where $u_i \in \mathcal{U}^{\pm}$ for $i = 1, \ldots, s_1$, and $u_i \in \mathcal{U}_1$ for $i = s_1 + 1, \ldots, s$. We set $\hat{u} := (u_0, u_1, \ldots, u_{s_1})$. Also, we may think that the real $\bar{\varepsilon} > 0$ is so small that for each $\varepsilon \in (0, \bar{\varepsilon})$ and for each $x \in \Omega_0$ the following admissible flow is well defined

$$\mathcal{V}(\varepsilon) := Q_{\hat{u}}(\varepsilon^{\frac{2\gamma+\alpha}{\gamma}}(4mp/3)^{-1/(3\gamma)}) \circ \mathcal{F}_{u_{s_1+1}}(\varepsilon) \circ \cdots \circ \mathcal{F}_{u_{s_2}}(\varepsilon).$$

Taking into account Lemmas A.2 and A.3, we obtain existence of positive real numbers p_0, q_0^i, q_1^i and q_2^i, $i = s_1 + 1, \ldots, s$, such that

$$\mathcal{V}(\varepsilon) = Exp\left(\left(\varepsilon^{6\gamma+3\alpha} + p_0\varepsilon^{2\gamma+\alpha}\right) f + \varepsilon^{6\gamma+3\alpha}\left(g_{u_0(\varepsilon)} + \sum_{i=1}^{s_1} f\left(u_i(\varepsilon)\right)\right) + a^{s_1}(\varepsilon) + O^{s_1}(\varepsilon^{6\gamma+4\alpha})\right)$$

$$\circ \text{Exp}\ \left(\left(q_0^{s_1+1}\varepsilon^{3\gamma} + q_1^{s_1+1}\varepsilon^{\gamma} + q_2^{s_1+1}\varepsilon^{3\alpha}\right) f + \varepsilon^{6\gamma+3\alpha} f(u(\varepsilon)) + a^{u_{s_1+1}}(\varepsilon) + O^{u_{s_1+1}}\left(\varepsilon^{6\gamma+4\alpha}\right)\right)$$

$$\circ \cdots \circ \text{Exp}\ \left(\left(q_0^{s}\varepsilon^{3\gamma} + q_1^{s}\varepsilon^{\gamma} + q_2^{s}\varepsilon^{3\alpha}\right) f + \varepsilon^{6\gamma+3\alpha} f(u(\varepsilon)) + a^{u_s}(\varepsilon) + O^{u_s}\left(\varepsilon^{6\gamma+4\alpha}\right)\right).$$

Applying the C-B-H formula, we obtain that

$$\mathcal{V}(\varepsilon) = Exp\left(\left(\varepsilon^{6\gamma+3} + q_0\varepsilon^{2\gamma+1} + \sum_{i=s_1+1}^{s}\left(q_0^{s_1+1}\varepsilon^{3\gamma} + q_1^{s_1+1}\varepsilon^{\gamma} + q_2^{s_1+1}\varepsilon^{3\alpha}\right)\right) f\right.$$

$$\left. + \varepsilon^{6\gamma+3\alpha}\left(g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f\left(u_i(\varepsilon)\right)\right) + a(\varepsilon) + O\left(\varepsilon^{6\gamma+4\alpha}\right)\right),$$

where $a(\varepsilon)$ is a family of analytic vector fields parameterized by ε that are homogeneous of degree at least two and vanishing at the origin. Moreover, for each $x \in \Omega_0$ and $\varepsilon \in (0, \bar{\varepsilon})$ the following inclusion holds true

$$\mathcal{V}(\varepsilon)(x) \in \mathcal{R}\left(x, \varepsilon^{6\gamma+3\alpha} + q_0\varepsilon^{2\gamma+\alpha} + \sum_{i=s_1+1}^{s}\left(q_0^{s_1+1}\varepsilon^{3\gamma} + q_1^{s_1+1}\varepsilon^{\gamma} + q_2^{s_1+1}\varepsilon^{3\alpha}\right)\right).$$

In this way we have proved the following:

Lemma A.4. *If*

$$(0, \varepsilon) \ni \varepsilon \to g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f\left(u_i(\varepsilon)\right)$$

is an arbitrary admissible sum of Lie brackets of length seven with $u_0 \in \mathcal{U}$, $u_i \in \mathcal{U}^{\pm}$ for $i = 1, \ldots, s_1$, and $u_i \in \mathcal{U}_1$ for $i = s_1 + 1, \ldots, s$, then there exist a real $\bar{\varepsilon} > 0$ and an admissible flow

$$\mathcal{V}(\varepsilon) = Exp\left(\left(\varepsilon^{6\gamma+3} + q_0\varepsilon^{2\gamma+1} + \sum_{i=s_1+1}^{s}\left(q_0^{s_1+1}\varepsilon^{3\gamma} + q_1^{s_1+1}\varepsilon^{\gamma} + q_2^{s_1+1}\varepsilon^{3\alpha}\right)\right)f\right.$$
$$\left. + \varepsilon^{6\gamma+3\alpha}\left(g_{u_0(\varepsilon)} + \sum_{i=1}^{s} f\left(u_i(\varepsilon)\right)\right) + a(\varepsilon) + O\left(\varepsilon^{6\gamma+4\alpha}\right)\right),$$

with $a(\varepsilon)$ belonging to A^0, and such that for each $x \in \Omega_0$ and each $\varepsilon \in (0, \bar{\varepsilon})$ the following inclusion holds true $\mathcal{V}(\varepsilon)(x) \in$

$$\in \mathcal{R}\left(x, \varepsilon^{6\gamma+3\alpha} + q_0\varepsilon^{2\gamma+\alpha} + \sum_{i=s_1+1}^{s}\left(q_0^{s_1+1}\varepsilon^{3\gamma} + q_1^{s_1+1}\varepsilon^{\gamma} + q_2^{s_1+1}\varepsilon^{3\alpha}\right)\right). \tag{A.23}$$

As a reminder, the real numbers α, β and γ satisfy the following inequalities $1 < \alpha < \gamma/4$ and $1 < 2^{\kappa-1}\beta < 2^{\kappa+1}\beta < \alpha$. Moreover, $\mu := 2^{\kappa}\beta$, $\mu_1 := 2^{\kappa-1}\beta, \mu_2 := 2^{\kappa-1}\left(1 + \frac{1}{2}\right)\beta, \mu_s := 2^{\kappa-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{s-1}}\right)\beta$, for each $s = 1, \ldots, \kappa$. Also, we have shown that for each elements p and q of the set $\{1, \ldots, \kappa\}$ with $p < q$ the following inequality holds true:

$$\mu_p + \mu \leq 2\mu_q.$$

Next, we prove the following:

Lemma A.5. *Let $\kappa \geq 2$. Then for each positive integer $q \in \{2, \ldots, \kappa\}$ and for each Lie bracket $\left[g_{u_q}, \left[g_{v_q}, f\right]\right]$ with $\left[g_{u_q}, \left[g_{v_q}, f\right]\right](0) \in \mathcal{M}_q$ there exist $\varepsilon_{u_q v_q} \in (0, 1)$, elements $v_{qi} \in \mathcal{M}_1$ and reals $\delta_{qi} \geq 0$, $i = 1, \ldots, \bar{q}$, such that the function*

$$(0, \varepsilon_{u_q v_q}) \ni \varepsilon \to \varepsilon^{\mu_q+\mu}\left[g_{u_q}, \left[g_{v_q}, f\right]\right] + \varepsilon^{2\mu}\sum_{i=1}^{\bar{q}} \varepsilon^{\delta_{qi}} f(v_{qi}) \tag{A.24}$$

is an admissible sum of Lie brackets of length seven, i.e., there exists $u_0 \in \mathcal{U}$ and $u_\alpha \in \mathcal{U}^{\pm} \cup \mathcal{U}_1$, $\alpha = 1, \ldots, s$, such that

$$\varepsilon^{\mu_q+\mu}\left[g_{u_q}, \left[g_{v_q}, f\right]\right] + \varepsilon^{2\mu}\sum_{i=1}^{\bar{q}} \varepsilon^{\delta_{qi}} f(v_{qi}) = g_{u_0(\varepsilon)} + \sum_{\alpha=1}^{s} f(u_\alpha(\varepsilon))$$

for each $\varepsilon \in (0, \varepsilon_{u_q v_q})$.

Proof of Lemma A5. The proof will be done by induction. First, we show that the claim holds true for $q = 2$. Indeed, let $[g_{u_1}, [g_{v_1}, f]]$ be a Lie bracket with $[g_{u_1}, [g_{v_1}, f]](0) \in \mathcal{M}_2$. Then $v_1 \in \mathcal{M}_1, u_1 \in \mathcal{L}_2$. According to the definition of $\mathcal{L}_2$, $u_1 \in M_1$ and

$$-f(u_1) = u_{1,0} + \sum_{j=1}^{p_1} f(u_{1j}) \tag{A.25}$$

where $u_{1,0} \in U$ and $u_{1j} \in \mathcal{M}_1,\ j = 1, \ldots, p_1$.

Because u_1 and v_1 belong to $\mathcal{M}_1$, we have that

$$u_1 = u_{10} + \sum_{i=1}^{a_1} f(u_{1i}) \text{ with } u_{10} \in U, u_{1i} \in \mathcal{M}_0, i = 1, \ldots, a_1,$$

and

$$v_1 = v_{10} + \sum_{i=1}^{b_1} f(v_{1i}) \text{ with } v_{10} \in U, v_{1i} \in \mathcal{M}_0, i = 1, \ldots, b_1.$$

Then

$$\varepsilon^{\mu_1} u_1 + \varepsilon^{\mu} v_1 = \varepsilon^{\mu_1} u_{10} + \varepsilon^{\mu} v_{10} + \sum_{i=1}^{a_1} f(\varepsilon^{\mu_1/2} u_{1i}) + \sum_{i=1}^{b_1} f(\varepsilon^{\mu/2} v_{1i}). \tag{A.26}$$

Clearly, there exists $\varepsilon_{u_1 v_1} \in (0, 1)$ such that for each $\varepsilon \in (0, \varepsilon_{u_1 v_1})$ we have that $\varepsilon^{\mu_1} u_{10} + \varepsilon^{\mu} v_{10} \in U \cap \bar{\mathbf{B}}$ and the elements $\varepsilon^{\mu_1/2} u_{1i}, i = 1, \ldots, a_1$, and $\varepsilon^{\mu/2} v_{1i}, i = 1, \ldots, b_1$, belong to $\mathcal{M}_0 \cap \bar{\mathbf{B}}$. Moreover, we have that

$$f(\varepsilon^{\mu_1} u_1 + \varepsilon^{\mu} v_1) + \varepsilon^{2\mu_1} g_{u_{1,0}} + \sum_{j=1}^{p_1} f(\varepsilon^{\mu_1} u_{1j}) = \varepsilon^{2\mu_1} f(u_1) + \varepsilon^{\mu_1+\mu} [g_{u_1}, [g_{v_1}, f]] + \varepsilon^{2\mu} f(v_1) + \varepsilon^{2\mu_1} g_{u_{1,0}} + \varepsilon^{2\mu_1} \sum_{j=1}^{p_1} f(u_{1j})$$

Applying (A.25) we obtain that

$$\varepsilon^{\mu_1+\mu} [g_{u_1}, [g_{v_1}, f]] + \varepsilon^{2\mu} f(v_1) = \varepsilon^{2\mu_1} g_{u_{1,0}} + f(\varepsilon^{\mu_1} u_1 + \varepsilon^{\mu} v_1) + \sum_{j=1}^{p_1} f(\varepsilon^{\mu_1} u_{1j}). \tag{A.27}$$

Clearly,

$$(0, \varepsilon_{u_1 v_1}) \in \varepsilon \to \varepsilon^{2\mu_1} g_{u_{1,0}} + f(\varepsilon^{\mu_1} u_1 + \varepsilon^{\mu} v_1) + \sum_{j=1}^{p_1} f(\varepsilon^{\mu_1} u_{1j})$$

is an admissible sum of Lie brackets of length seven. Hence, according to (A.27), the function

$$(0, \varepsilon_{u_1 v_1}) \in \varepsilon \to \varepsilon^{\mu_1+\mu} [g_{u_1}, [g_{v_1}, f]] + \varepsilon^{2\mu} f(v_1)$$

is also an admissible sum of Lie brackets of length seven. So, we obtain that Lemma A.5 holds true for $q = 2$.

Let us assume that the Lemma A.5 holds true for each positive integer r satisfying the inequality $r \leq q$ for some positive integer $q < \kappa$. We prove that it holds true also for $p := q + 1$.

Indeed, let us fix a Lie bracket $\left[g_{u_p},\left[g_{v_p},f\right]\right]$ with $\left[g_{u_p},\left[g_{v_p},f\right]\right](0)\in\mathcal{M}_p\setminus\mathcal{M}_q$. Then $v_p\in\mathcal{M}_1, u_p\in\mathcal{L}_p$. According to the definition of $\mathcal{L}_p$, we have that

$$-f(u_p)=u_{\alpha_p}+\sum_{\beta_p=1}^{\bar\beta_p}f(u_{\beta_p})+\sum_{\delta_p=1}^{\bar\delta_p}f(u_{\delta_p})+\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma p}}\left[g_{u_{\gamma_p j_p}},\left[g_{v_{\gamma_p j_p}},f\right]\right](0), \tag{A.28}$$

where $u_{\alpha_p}\in U$, $u_{\beta_p}\in\mathcal{M}_0$, $u_{\delta_p}\in\mathcal{M}_1$ and each $\left[g_{u_{\gamma_p j_p}},\left[g_{v_{\gamma_p j_p}},f\right]\right](0)$ belongs to $\mathcal{M}_{\gamma_p}$ with $\gamma_p<p$. Clearly, there exists $\varepsilon_{u_p^0v_p^0}\in(0,1)$ such that for each $\varepsilon\in(0,\varepsilon_{u_p^0v_p^0})$ the sum $\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p$ can be present as a sum (analogously to the equality (A.26))

$$\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p=\varepsilon^{\mu_p}u_{p0}+\varepsilon^{\mu}v_{p0}+\sum_{i=1}^{a_p}f(\varepsilon^{\mu_p/2}u_{pi})+\sum_{i=1}^{b_p}f(\varepsilon^{\mu/2}v_{pi}), \tag{A.29}$$

where $\varepsilon^{\mu_p}u_{p0}+\varepsilon^{\mu}v_{p0}\in U\cap\bar{\mathbf{B}}$, and $\varepsilon^{\mu_p/2}u_{pi}, i=1,\dots,a_p$, and $\varepsilon^{\mu/2}v_{pi}, i=1,\dots,b_p$, belong to $\mathcal{M}_0\cap\bar{\mathbf{B}}$. Moreover,

$$f\left(\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p\right)=\varepsilon^{2\mu_p}f\left(u_p\right)+\varepsilon^{\mu_p+\mu}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}f\left(v_p\right).$$

We add

$$\varepsilon^{2\mu_p}g_{u_{\alpha_p}}+\varepsilon^{2\mu_p}\sum_{\beta_p=1}^{\bar\beta_p}f(u_{\beta_p})+\varepsilon^{2\mu_p}\sum_{\delta_p=1}^{\bar\delta_p}f(u_{\delta_p})+\varepsilon^{2\mu_p}\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma p}}\left[g_{u_{\gamma_p j_p}},\left[g_{v_{\gamma_p j_p}},f\right]\right]$$

to both sides of this equality and obtain

$$\begin{aligned}f\left(\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p\right)+\varepsilon^{2\mu_p}g_{u_{\alpha_p}}+\varepsilon^{2\mu_p}\sum_{\beta_p=1}^{\bar\beta_p}f(u_{\beta_p})+\varepsilon^{2\mu_p}\sum_{\delta_p=1}^{\bar\delta_p}f(u_{\delta_p})+\varepsilon^{2\mu_p}\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma p}}\left[g_{u_{\gamma_p j_p}},\left[g_{v_{\gamma_p j_p}},f\right]\right]\\=\varepsilon^{2\mu_p}f\left(u_p\right)+\varepsilon^{\mu_p+\mu}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}f\left(v_p\right)\\+\varepsilon^{2\mu_p}g_{u_{\alpha_p}}+\varepsilon^{2\mu_p}\sum_{\beta_p=1}^{\bar\beta_p}f(u_{\beta_p})+\varepsilon^{2\mu_p}\sum_{\delta_p=1}^{\bar\delta_p}f(u_{\delta_p})+\varepsilon^{2\mu_p}\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma p}}\left[g_{u_{\gamma_p j_p}},\left[g_{v_{\gamma_p j_p}},f\right]\right].\end{aligned}$$

Taking into account (A.28), we obtain that

$$\begin{aligned}f\left(\varepsilon^{\mu_p}u_p+\varepsilon^{\mu}v_p\right)+\varepsilon^{2\mu_p}g_{u_{\alpha_p}}+\varepsilon^{2\mu_p}\sum_{\beta_p=1}^{\beta_p}f(u_{\bar\beta_p})+\varepsilon^{2\mu_p}\sum_{\delta_p=1}^{\bar\delta_p}(f(u_{\delta_p})\\+\varepsilon^{2\mu_p}\sum_{\gamma_p=1}^{\bar\gamma_p}\sum_{j_p=1}^{\bar j_{\gamma p}}\left[g_{u_{\gamma_p j_p}},\left[g_{v_{\gamma_p j_p}},f\right]\right]=\varepsilon^{\mu_p+\mu}\left[g_{u_p},\left[g_{v_p},f\right]\right]+\varepsilon^{2\mu}f\left(v_p\right).\end{aligned} \tag{A.30}$$

According to the inductive assumption, for each multi-index $\gamma_p j_p$ there exists $\varepsilon_{\gamma_p j_p}\in(0,1)$ and a function

$$(0,\varepsilon_{\gamma_p j_p})\ni\varepsilon\to\varepsilon^{2\mu}\sum_{k_p=1}^{p_{\gamma_p j_p}}\varepsilon^{\delta_{\gamma_p j_p k_p}}f(v_{\gamma_p j_p k_p}),$$

(here each $v_{\gamma_p j_p k_p} \in \mathcal{M}_1$ and each real $\delta_{\gamma_p j_p k_p} \geq 0$) such that the function

$$(0, \varepsilon_{\gamma_p j_p}) \ni \varepsilon \to \varepsilon^{\mu+\mu_{\gamma_p}} \left[g_{u_{\gamma_p j_p}}, \left[g_{v_{\gamma_p j_p}}, f\right]\right] + \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p})$$

is an admissible sum of Lie brackets, i.e., for each $\varepsilon \in (0, \varepsilon_{\gamma_p j_p})$ we have that

$$\varepsilon^{\mu+\mu_{\gamma_p}} \left[g_{u_{\gamma_p j_p}}, \left[g_{v_{\gamma_p j_p}}, f\right]\right] + \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}) = g_{u^0_{\gamma_p j_p}(\varepsilon)} + \sum_{i_{\gamma_p j_p}=1}^{\bar{i}_{\gamma_p j_p}} f\left(u_{i_{\gamma_p j_p}}(\varepsilon)\right),$$

where $u^0_{\gamma_p j_p} \in \mathcal{U}$ and $u_{i_{\gamma_p j_p}} \in \mathcal{U}^{\pm} \cup \mathcal{U}_1$ for each $i_{\gamma_p j_p} = 1, \ldots, \bar{i}_{\gamma_p j_p}$.

Taking this into account and setting

$$\varepsilon_{u^1_p v^1_p} = \min\{\varepsilon_{u^0_p v^0_p}, \varepsilon_{\gamma_p j_p}, j_p = 1, \ldots, \bar{j}_{\gamma_p}, \gamma_p = 1, \ldots, \bar{\gamma}_p\} > 0,$$

we obtain from (A.30) that for each $\varepsilon \in (0, \varepsilon_{u^1_p v^1_p})$ the following equality holds true

$$\begin{aligned}
& f\left(\varepsilon^{\mu_p} u_p + \varepsilon^{\mu} v_p\right) + \varepsilon^{2\mu_p} g_{u_{\alpha_p}} + \varepsilon^{2\mu_p} \sum_{\beta_p=1}^{\bar{\beta}_p} f(u_{\beta_p}) + \varepsilon^{2\mu_p} \sum_{\delta_p=1}^{\bar{\delta}_p} f(u_{\delta_p}) \\
& + \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu_{\gamma_p}-\mu} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \varepsilon^{\mu_{\gamma_p}+\mu} \left[g_{u_{\gamma_p j_p}}, \left[g_{v_{\gamma_p j_p}}, f\right]\right] + \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu_{\gamma_p}-\mu} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}) \\
& = \varepsilon^{\mu_p+\mu} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} f\left(v_p\right) + \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \varepsilon^{\mu_p+\mu} \left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} f\left(v_p\right) + \varepsilon^{2\mu} \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}) \\
& = f\left(\varepsilon^{\mu_p} u_p + \varepsilon^{\mu} v_p\right) + \varepsilon^{2\mu_p} g_{u_{\alpha_p}} + \sum_{\beta_p=1}^{\bar{\beta}_p} f(\varepsilon^{\mu_p} u_{\beta_p}) + \sum_{\delta_p=1}^{\bar{\delta}_p} (f(\varepsilon^{\mu_p} u_{\delta_p}) \\
& \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_p} \left(\varepsilon^{\mu+\mu_{\gamma_p}} \left[g_{u_{\gamma_p}}, \left[g_{v_{\gamma_p}}, f\right]\right] + \varepsilon^{2\mu} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p})\right) \\
& = f\left(\varepsilon^{\mu_p} u_p + \varepsilon^{\mu} v_p\right) + \varepsilon^{2\mu_p} g_{u_{\alpha_p}} + \sum_{\beta_p=1}^{\bar{\beta}_p} f(\varepsilon^{\mu_p} u_{\beta_p}) + \sum_{\delta_p=1}^{\bar{\delta}_p} (f(\varepsilon^{\mu_p} u_{\delta_p}) \qquad (A.31) \\
& + \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \left(g_{u^0_{\gamma_p j_p}(\varepsilon)} + \sum_{i_{\gamma_p j_p}=1}^{\bar{i}_{\gamma_p j_p}} f\left(u_{i_{\gamma_p j_p}}(\varepsilon)\right)\right).
\end{aligned}$$

Clearly there exists $\varepsilon_{u_p v_p} \in (0, \varepsilon_{u_p^1 v_p^1})$ such that the sum

$$u_p^0(\varepsilon) := \varepsilon^{2\mu_p} u_{\alpha_p} + \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} u^0_{\gamma_p j_p}(\varepsilon)$$

belongs to the set $\mathcal{U}$ for each $\varepsilon \in (0, \varepsilon_{u_p v_p})$. Then (A.31) can be written as follows:

$$\varepsilon^{\mu_p+\mu}\left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} f\left(v_p\right) + \varepsilon^{2\mu} \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p})$$

$$= g_{u_p^0(\varepsilon)} + f\left(\varepsilon^{\mu_p} u_p + \varepsilon^{\mu} v_p\right) + \sum_{\beta_p=1}^{\bar{\beta}_p} f(\varepsilon^{\mu_p} u_{\beta_p}) + \sum_{\delta_p=1}^{\bar{\delta}_p} f(\varepsilon^{2\mu_p} u_{\delta_p}) + \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \sum_{i_{\gamma_p j_p}=1}^{\bar{i}_{\gamma_p j_p}} f\left(u_{i_{\gamma_p j_p}}(\varepsilon)\right).$$

Because $2\mu_p - \mu - \mu_{\gamma_p} \geq 0$, the last equality implies that the function $(0, \varepsilon_{u_p v_p}) \ni \varepsilon \to \Lambda(\varepsilon)$, where

$$\Lambda(\varepsilon) = \varepsilon^{\mu_p+\mu}\left[g_{u_p}, \left[g_{v_p}, f\right]\right] + \varepsilon^{2\mu} f\left(v_p\right) + \varepsilon^{2\mu} \sum_{\gamma_p=1}^{\bar{\gamma}_p} \varepsilon^{2\mu_p-\mu-\mu_{\gamma_p}} \sum_{j_p=1}^{\bar{j}_{\gamma_p}} \sum_{k_p=1}^{p_{\gamma_p j_p}} \varepsilon^{\delta_{\gamma_p j_p k_p}} f(v_{\gamma_p j_p k_p}),$$

is also an admissible sum of Lie brackets. Hence, the inductive assumption holds true for the Lie bracket $\left[g_{u_{\gamma_p}}, \left[g_{v_{\gamma_p}}, f\right]\right]$. Thus, we can conclude that the inductive assumption holds true also for $p := q + 1$. Therefore, the inductive assumption holds true for each $p \in \{1, 2, \ldots, \kappa\}$. This completes the proof of Lemma A.5.