AIMS Mathematics
https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 29650–29663.
DOI: 10.3934/math.20251303
Received: 29 September 2025
Revised: 27 November 2025
Accepted: 04 December 2025
Published: 16 December 2025

*Research article*

# Algebraic invariants of edge ideals from strong products of paths and cycles

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**Abstract:** This paper investigates algebraic invariants of edge ideals associated with families of graphs constructed as the strong product of a path or cycle with the complete graph $K_m$, namely $\mathscr{P}_\gamma = P_\gamma \boxtimes K_m$ and $\mathscr{C}_\gamma = C_\gamma \boxtimes K_m$. For an edge ideal $I(\mathbb{G}) \subset \mathfrak{R}$ in a polynomial ring over a field $\mathbb{K}$, we derive explicit combinatorial formulas for key homological and ring-theoretic invariants of the quotient ring $\mathfrak{R}/I(\mathbb{G})$. These include Castelnuovo-Mumford regularity, depth, Stanley depth, projective dimension, and Krull dimension. Furthermore, we characterize all Cohen–Macaulay graphs in defined families, providing a complete classification where $\mathfrak{R}/I(\mathbb{G})$ is Cohen–Macaulay.

**Keywords:** edge ideal; depth; projective dimension; stanley depth; Castelnuovo-Mumford regularity; Krull dimension; Cohen–Macaulay rings
**Mathematics Subject Classification:** 05C69, 05C70, 13D02, 13E40

## 1. Introduction

Let $\mathfrak{R} = \mathbb{K}[\mathfrak{c}_1, \ldots, \mathfrak{c}_\gamma]$ be a polynomial ring in $\gamma$ variables over a field $\mathbb{K}$, and $\mathbb{G}$ be a simple graph with vertex set $V(\mathbb{G}) = \{\mathfrak{c}_1, \ldots, \mathfrak{c}_\gamma\}$. The edge ideal of $\mathbb{G}$, denoted $I(\mathbb{G})$, is the ideal in $\mathfrak{R}$ generated by all monomials $\mathfrak{c}_i\mathfrak{c}_j$ corresponding to edges $\{\mathfrak{c}_i, \mathfrak{c}_j\} \in E(\mathbb{G})$. In other words, $I(\mathbb{G}) = (\mathfrak{c}_i\mathfrak{c}_j : \{\mathfrak{c}_i, \mathfrak{c}_j\} \in E(\mathbb{G}))$ is generated by monomials representing the edges of $\mathbb{G}$.

Let $\mathbb{F}$ be a finitely generated $\mathbb{Z}^\gamma$-graded $\mathfrak{R}$-module. Let $u \in \mathbb{F}$ be a homogeneous element and $Z \subseteq \{\mathfrak{c}_1, \ldots, \mathfrak{c}_\gamma\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements $uv$ with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$ if it is a free $\mathbb{K}[Z]$-module. Here, as usual, $|Z|$ denotes the number of elements of $Z$. A decomposition $\mathscr{D}$ of $\mathbb{F}$ as a finite direct sum of Stanley spaces is called a Stanley decomposition of $\mathbb{F}$. The minimum dimension of a Stanley space in $\mathscr{D}$ is called the Stanley depth of $\mathscr{D}$ and is denoted by sdepth($\mathscr{D}$). The quantity

$$\mathrm{sdepth}(\mathbb{F}) := \max\{\mathrm{sdepth}(\mathscr{D}) \mid \mathscr{D} \text{ is a Stanley decomposition of } \mathbb{F}\}$$

is called the Stanley depth of $\mathbb{F}$ [1].

We say that $\mathbb{F}$ satisfies Stanley's inequality if depth($\mathbb{F}$) ≤ sdepth($\mathbb{F}$). Stanley's conjecture proposes that a fundamental relationship between depth and Stanley depth holds for any module $\mathbb{F}$, namely that the Stanley depth is always at least the depth [1]. This was refuted in 2016 by Duval et al. [2]. For monomial ideals $I \subset J \subset \mathfrak{R}$, the Stanley depth sdepth($J/I$) is an invariant of a purely combinatorial nature [3]. Furthermore, it continues to exhibit properties that are analogous to those of classical depth; see references [4–9].

In combinatorial commutative algebra, invariants derived from minimal free resolutions play a central role. For a graded module $\mathbb{F}$ over a polynomial ring $\mathfrak{R}$ that admits a free resolution, the graded minimal free resolution is an exact sequence of the form

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} \mathfrak{R}(-j)^{\beta_{p,j}(\mathbb{F})} \longrightarrow \bigoplus_{j\in\mathbb{Z}} \mathfrak{R}(-j)^{\beta_{\gamma-1,j}(\mathbb{F})} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} \mathfrak{R}(-j)^{\beta_{0,j}(\mathbb{F})} \longrightarrow \mathbb{F} \longrightarrow 0.$$

The graded Betti numbers $\beta_{i,j}(\mathbb{F})$ count the number of minimal generators of degree $j$ in the $i$-th syzygy module. From these Betti numbers, two fundamental invariants are defined, namely the Castelnuovo–Mumford regularity

$$\mathrm{reg}(\mathbb{F}) = \max\{j - i : \beta_{i,j}(\mathbb{F}) \neq 0\},$$

and the projective dimension

$$\mathrm{pdim}(\mathbb{F}) = \max\{i : \beta_{i,j}(\mathbb{F}) \neq 0 \text{ for some } j\}.$$

The Castelnuovo-Mumford regularity reg($\mathbb{F}$) measures the maximum degree shift required throughout the minimal free resolution and bounds the degrees of the generators required at each step of the resolution. In contrast, the projective dimension pdim($\mathbb{F}$) measures the length of resolution, indicating the number of syzygy steps required. Together, these invariants provide complementary measures of the homological complexity of $\mathbb{F}$ and have been extensively studied for edge ideals and other monomial ideals [10–15].

A graph $\mathbb{G}$ is said to be Cohen–Macaulay if the quotient ring $\mathfrak{R}/I(\mathbb{G})$ is Cohen–Macaulay over every field $\mathbb{K}$. The graph is called unmixed if its edge ideal $I(\mathbb{G})$ is unmixed over every field $\mathbb{K}$, a condition equivalent to all minimal vertex covers of $\mathbb{G}$ having the same cardinality [16]. It is well-established that every Cohen–Macaulay graph is unmixed [17]. Furthermore, for chordal graphs, the Cohen–Macaulay property is known to be independent of the choice of field $\mathbb{K}$ [18].

In this article, we determine explicit combinatorial formulas for key homological invariants of rings $\mathfrak{R}/I(\mathscr{P}_\gamma)$ and $\mathfrak{R}/I(\mathscr{C}_\gamma)$, where $\mathscr{P}_\gamma = P_\gamma \boxtimes K_m$ and $\mathscr{C}_\gamma = C_\gamma \boxtimes K_m$ are the strong products of a path or cycle, respectively, with the complete graph $K_m$. Our results demonstrate a profound connection between the combinatorial structure of these graph classes and algebraic properties of their associated edge ideals. We acknowledge the use of CoCoA [19] and Macaulay2 [20].

We conclude this introduction by outlining the structure of the paper. Section 2 reviews essential preliminary concepts and established results on graph products and monomial ideals. Section 3 presents our study of algebraic invariants, including a characterization of Cohen–Macaulay graphs.

Our results on Castelnuovo–Mumford regularity are developed in Section 4. Section 5 presents concluding remarks on the research. Finally, Section 6 discusses open questions and potential generalizations of our work.

## 2. Notation and preliminaries

This section gives a brief overview of the graph-theoretic and algebraic concepts that are central to this work. The strong product of graphs $\mathbb{G}$ and $\mathbb{H}$, denoted $\mathbb{G} \boxtimes \mathbb{H}$, is a graph whose vertex set is $V(\mathbb{G} \boxtimes \mathbb{H}) = V(\mathbb{G}) \times V(\mathbb{H})$ and for $(\mathfrak{c}, \mathfrak{d}), (\mathfrak{e}, \mathfrak{f}) \in V(\mathbb{G} \boxtimes \mathbb{H})$, $\{(\mathfrak{c}, \mathfrak{d}), (\mathfrak{e}, \mathfrak{f})\} \in E(\mathbb{G} \boxtimes \mathbb{H})$, whenever $\{\mathfrak{c}, \mathfrak{e}\} \in E(\mathbb{G})$ and $\mathfrak{d} = \mathfrak{f}$, or $\{\mathfrak{d}, \mathfrak{f}\} \in E(\mathbb{H})$ and $\mathfrak{c} = \mathfrak{e}$, or $\{\mathfrak{c}, \mathfrak{e}\} \in E(\mathbb{G})$ and $\{\mathfrak{d}, \mathfrak{f}\} \in E(\mathbb{H})$ [21].

Define $\mathscr{P}_\gamma := P_\gamma \boxtimes K_m$ for $\gamma \geq 1$, and $\mathscr{C}_\gamma := C_\gamma \boxtimes K_m$ for $\gamma \geq 3$. Let $V(\mathscr{P}_\gamma) = V(\mathscr{C}_\gamma) = \{\mathfrak{a}_i^1, \ldots, \mathfrak{a}_i^m \mid 1 \leq i \leq \gamma\}$ be the vertex set for both families of graphs. In $\mathscr{P}_\gamma$, the vertices are arranged in $m$ horizontal layers. In contrast, $\mathscr{C}_\gamma$ arranges them in $m$ concentric layers (see Figure 1). Clearly, $|V(\mathscr{P}_\gamma)| = |V(\mathscr{C}_\gamma)| = m\gamma$.

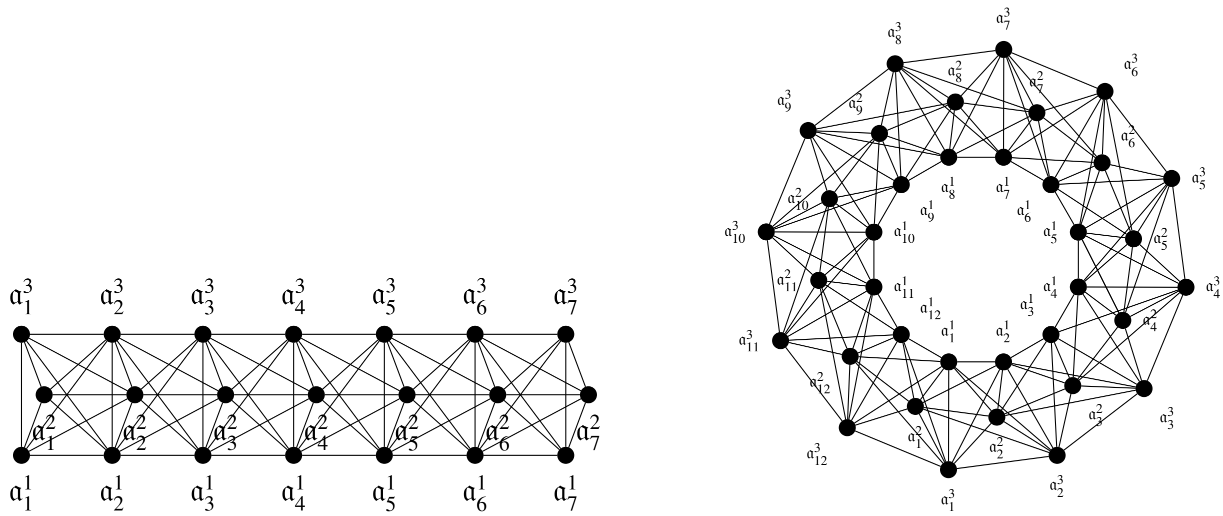

**Figure 1.** Vertex labeling of $\mathscr{P}_7 = P_7 \boxtimes K_3$ (left) and $\mathscr{C}_{12} = C_{12} \boxtimes K_3$ (right).

An independent set of a graph $\mathbb{G}$ is a subset $\mathbb{X}$ of the vertex set $V(\mathbb{G})$ such that for any two distinct vertices $u, v \in \mathbb{X}$, the edge $\{u, v\} \notin E(\mathbb{G})$. The independence number of $\mathbb{G}$, denoted by $\alpha(\mathbb{G})$, is the maximum cardinality of an independent set in $\mathbb{G}$. That is,

$$\alpha(\mathbb{G}) = \max\{|\mathbb{X}| : \mathbb{X} \subseteq V(\mathbb{G}) \text{ is an independent set}\}.$$

**Lemma 2.1** ( [22, Lemma 1])**.** *If* $J = I(\mathbb{G})$*, then* $\dim(\mathfrak{R}/I) = \alpha(\mathbb{G})$.

**Lemma 2.2** ( [23, Corollary 3])**.** *For a complete graph* $K_m$ *on* $m \geq 1$ *vertices,*

$$\alpha(\mathbb{G} \boxtimes K_m) = \alpha(\mathbb{G}).$$

A subgraph $\mathbb{H}$ of $\mathbb{G}$ is an induced subgraph if every edge of $\mathbb{G}$ having its endpoints in $V(\mathbb{H})$ is also an edge of $\mathbb{H}$. Equivalently, $\mathbb{H}$ is induced if $E(\mathbb{H})$ consists of all edges of $\mathbb{G}$ whose endpoints belong to $V(\mathbb{H})$. If $T \subseteq V(\mathbb{G})$, the induced subgraph of $\mathbb{G}$ with vertex set $T$ is denoted by $\mathbb{G}[T]$. A matching is a set of edges, no two of which share a vertex. An induced matching is a matching $\mathbb{M}$ for which $\mathbb{G}[V(\mathbb{M})]$ contains exactly the edges of $\mathbb{M}$. The induced matching number of graph $\mathbb{G}$ is the invariant $\mathrm{im}(\mathbb{G})$ given by

$$\mathrm{im}(\mathbb{G}) = \max\{|\mathbb{M}| : \mathbb{M} \subseteq E(\mathbb{G}) \text{ is an induced matching}\}.$$

**Lemma 2.3** ( [24, Corollary 6.9]). *Let $\mathbb{G}$ be a chordal graph and $J = I(\mathbb{G})$; then* $\mathrm{reg}(\mathfrak{R}/J) = im(\mathbb{G})$.

**Lemma 2.4.** *For a short exact sequence $0 \to \mathbb{F}_2 \to \mathbb{F}_1 \to \mathbb{F}_3 \to 0$ of $\mathbb{Z}^{\gamma}$-graded $\mathfrak{R}$-modules:*

$$\mathrm{depth}(\mathbb{F}_1) \geq \min\{\mathrm{depth}(\mathbb{F}_2), \mathrm{depth}(\mathbb{F}_3)\},$$

*and*

$$\mathrm{sdepth}(\mathbb{F}_1) \geq \min\{\mathrm{sdepth}(\mathbb{F}_2), \mathrm{sdepth}(\mathbb{F}_3)\}.$$

The first inequality follows from [25, Proposition 1.2.9] and the second from [5, Lemma 2.2].

**Lemma 2.5.** *Let $\bar{\mathfrak{R}} = \mathfrak{R} \otimes_{\mathbb{K}} \mathbb{K}[\mathfrak{c}_{\gamma+1}]$ be a polynomial ring of variables $\gamma + 1$ and $I$ be a monomial ideal of $\mathfrak{R}$. If $U = \bar{\mathfrak{R}}/I$ and $V = \mathfrak{R}/I$, then:*

- ***a.*** $\mathrm{depth}(U) = \mathrm{depth}(V) + 1$.
- ***b.*** $\mathrm{sdepth}(U) = \mathrm{sdepth}(V) + 1$.
- ***c.*** $\mathrm{reg}(U) = \mathrm{reg}(V)$.

The results for depth and Stanley depth are consequences of [3, Lemma 3.6], with the result for regularity following from [26, Lemma 3.5].

For $1 \leq n < \gamma$, if $\mathfrak{R} = \mathbb{K}[\mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{c}_{n+1}, \ldots, \mathfrak{c}_\gamma]$ with $J_1 \subset \mathfrak{R}_1 = \mathbb{K}[\mathfrak{c}_1, \ldots, \mathfrak{c}_n]$ and $J_2 \subset \mathfrak{R}_2 = \mathbb{K}[\mathfrak{c}_{n+1}, \ldots, \mathfrak{c}_\gamma]$ are monomial ideals, then $\mathfrak{R}/(J_1 + J_2) \cong \mathfrak{R}_1/J_1 \otimes_{\mathbb{K}} \mathfrak{R}_2/J_2$ by [17, Proposition 2.2.20]. Thus, $\mathrm{depth}(\mathfrak{R}/(J_1+J_2)) = \mathrm{depth}(\mathfrak{R}_1/J_1 \otimes_{\mathbb{K}} \mathfrak{R}_2/J_2)$ and $\mathrm{sdepth}(\mathfrak{R}/(J_1+J_2)) = \mathrm{sdepth}(\mathfrak{R}_1/J_1 \otimes_{\mathbb{K}} \mathfrak{R}_2/J_2)$. The following lemma is obtained by applying [17, Proposition 2.2.21] for depth, and [5, Theorem 3.1] for Stanley's depth.

**Lemma 2.6.** *Let $J_1 \subset \mathfrak{R}_1 = \mathbb{K}[\mathfrak{c}_1, \ldots, \mathfrak{c}_n]$, $J_2 \subset \mathfrak{R}_2 = \mathbb{K}[\mathfrak{c}_{n+1}, \ldots, \mathfrak{c}_\gamma]$ be monomial ideals and $\mathfrak{R} = \mathbb{K}[\mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{c}_{n+1}, \ldots, \mathfrak{c}_\gamma]$, for $1 \leq n < \gamma$. Then* $\mathrm{depth}_{\mathfrak{R}}(\mathfrak{R}_1/J_1 \otimes_{\mathbb{K}} \mathfrak{R}_2/J_2) = \mathrm{depth}_{\mathfrak{R}}(\mathfrak{R}/(J_1 + J_2)) = \mathrm{depth}_{\mathfrak{R}_1}(\mathfrak{R}_1/J_1) + \mathrm{depth}_{\mathfrak{R}_2}(\mathfrak{R}_2/J_2)$*, and* $\mathrm{sdepth}_{\mathfrak{R}}(\mathfrak{R}_1/J_1 \otimes_{\mathbb{K}} \mathfrak{R}_2/J_2) \geq \mathrm{sdepth}_{\mathfrak{R}_1}(R_1/J_1) + \mathrm{sdepth}_{\mathfrak{R}_2}(\mathfrak{R}_2/J_2)$.

**Lemma 2.7.** *Let $I$ be a monomial ideal of $\mathfrak{R}$ and $\mathfrak{c} \in \mathfrak{R}$ such that $\mathfrak{c} \notin I$. If $U = \mathfrak{R}/(I : \mathfrak{c})$ and $V = \mathfrak{R}/I$, then* $\mathrm{depth}(U) \geq \mathrm{depth}(V)$ *and* $\mathrm{sdepth}(U) \geq \mathrm{sdepth}(V)$.

The first inequality follows from [5, Corollary 1.3] and the second from [7, Proposition 2.7].

**Lemma 2.8** ( [27, Lemma 8]). *For $1 \leq n < \gamma$, let $\mathfrak{R} = \mathbb{K}[\mathfrak{c}_1, \ldots, \mathfrak{c}_n, \mathfrak{c}_{n+1}, \ldots, \mathfrak{c}_\gamma]$ with $J_1 \subset \mathfrak{R}_1 = \mathbb{K}[\mathfrak{c}_1, \ldots, \mathfrak{c}_n]$, and $J_2 \subset \mathfrak{R}_2 = \mathbb{K}[\mathfrak{c}_{n+1}, \ldots, \mathfrak{c}_\gamma]$ be edge ideals, then*

$$\mathrm{reg}(\mathfrak{R}/(J_1 + J_2)) = \mathrm{reg}(\mathfrak{R}_1/J_1) + \mathrm{reg}(\mathfrak{R}_2/J_2).$$

**Lemma 2.9** ( [28, Corollary 20.19]). *Let $\mathfrak{c} \in \mathfrak{R}$ and the monomial ideal $I \subset \mathfrak{R}$. If* $\mathrm{reg}(\mathfrak{R}/(I : \mathfrak{c})) < \mathrm{reg}(\mathfrak{R}/(I, \mathfrak{c}))$*, then* $\mathrm{reg}(\mathfrak{R}/I) = \mathrm{reg}(\mathfrak{R}/(I, \mathfrak{c}))$.

**Lemma 2.10** ( [29, Lemma 3.1]). *Let $\alpha \geq 2$ be an integer, and consider $\{\mathbb{D}_j : 1 \leq j \leq \alpha\}$ and $\{\mathbb{F}_i : 0 \leq i \leq \alpha\}$ be a sequence of $\mathbb{Z}^\gamma$-graded $\mathfrak{R}$-modules, and consider the following chain of short exact sequences:*

$$0 \longrightarrow \mathbb{D}_1 \longrightarrow \mathbb{F}_0 \longrightarrow \mathbb{F}_1 \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{D}_2 \longrightarrow \mathbb{F}_1 \longrightarrow \mathbb{F}_2 \longrightarrow 0,$$
$$\vdots$$
$$0 \longrightarrow \mathbb{D}_{\alpha-1} \longrightarrow \mathbb{F}_{\alpha-2} \longrightarrow \mathbb{F}_{\alpha-1} \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{D}_{\alpha} \longrightarrow \mathbb{F}_{\alpha-1} \longrightarrow \mathbb{F}_{\alpha} \longrightarrow 0.$$

***a.*** *If* $\operatorname{depth} \mathbb{D}_\alpha \leq \operatorname{depth} \mathbb{F}_\alpha$ *and* $\operatorname{depth} \mathbb{D}_{j-1} \leq \operatorname{depth} \mathbb{D}_j$*, for all* $2 \leq j \leq \alpha$*, then* $\operatorname{depth} \mathbb{F}_0 = \operatorname{depth} \mathbb{D}_1$.

***b.*** *If* $\operatorname{depth} \mathbb{F}_\alpha \leq \operatorname{depth} \mathbb{D}_\alpha$ *and* $\operatorname{depth} \mathbb{D}_{j-1} \leq \operatorname{depth} \mathbb{D}_j$*, for all* $2 \leq j \leq \alpha$*, then by applying Lemma 2.4 and Lemma 2.7 on the chain of short exact sequences, we obtain*

$$\operatorname{depth} \mathbb{F}_\alpha \leq \operatorname{depth} \mathbb{F}_0 \leq \operatorname{depth} \mathbb{D}_1.$$

***c.*** *If* $\operatorname{reg} \mathbb{D}_\alpha < \operatorname{reg} \mathbb{F}_\alpha$ *and* $\operatorname{reg} \mathbb{D}_{j-1} \leq \operatorname{reg} \mathbb{D}_j$*, for all* $2 \leq j \leq \alpha$*, then by applying Lemma 2.9 on the chain of short exact sequences, we get*

$$\operatorname{reg} \mathbb{F}_0 = \operatorname{reg} \mathbb{F}_\alpha.$$

## 3. Depth, projective dimension, stanley depth, Krull dimension, and Cohen–Macaulay graphs

We determine the depth, projective dimension, Stanley depth, and Krull dimension of $\mathfrak{R}/J$ for $\mathscr{P}_\gamma$, extend these results to $\mathscr{C}_\gamma$ and characterize all Cohen–Macaulay graphs within defined families.

**Theorem 3.1.** *Let $\mathfrak{R} = \mathbb{K}[V(\mathscr{P}_\gamma)]$ and $J = I(\mathscr{P}_\gamma)$. If $\gamma \geq 1$, then*

$$\operatorname{depth}(\mathfrak{R}/J) = \operatorname{sdepth}(\mathfrak{R}/J) = \left\lceil \frac{\gamma}{3} \right\rceil.$$

*Proof.* We begin by proving the result for depth. For $\gamma \leq 2$, $\mathscr{P}_1 \cong K_m$ and $\mathscr{P}_2 \cong K_{2m}$ which complements our result. For $\gamma = 3$, define $J_i := (J_{i-1}, \mathfrak{a}_2^i)$ for $1 \leq i \leq m$, with $J_0 = J$; we obtain the following chain of sequences:

$$0 \to \mathfrak{R}/(J : \mathfrak{a}_2^1) \to \mathfrak{R}/J \to \mathfrak{R}/J_1 \to 0,$$
$$0 \to \mathfrak{R}/(J_1 : \mathfrak{a}_2^2) \to \mathfrak{R}/J_1 \to \mathfrak{R}/J_2 \to 0,$$
$$\vdots$$
$$0 \to \mathfrak{R}/(J_{m-2} : \mathfrak{a}_2^{m-1}) \to \mathfrak{R}/J_{m-2} \to \mathfrak{R}/J_{m-1} \to 0,$$
$$0 \to \mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^{m}) \to \mathfrak{R}/J_{m-1} \to \mathfrak{R}/J_{m} \to 0.$$

The following isomorphisms hold:

$$\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m) \cong \mathbb{K}[\mathfrak{a}_2^m],$$
$$\mathfrak{R}/J_m \cong \mathbb{K}[V(\mathscr{P}_1)]/J(\mathscr{P}_1) \otimes_{\mathbb{K}} \mathbb{K}[V(K_m)]/I(K_m).$$

Using Lemma 2.6 implies

$$\begin{aligned}\operatorname{depth}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m)) &= 1,\\ \operatorname{depth}(\mathfrak{R}/J_m) &= 2.\end{aligned}$$

Since $\operatorname{depth}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m)) \leq \operatorname{depth}(\mathfrak{R}/J_m)$ and $\operatorname{depth}(\mathfrak{R}/(J_{i-1} : \mathfrak{a}_2^i)) = 1$, by Lemma 2.10(a) , we get

$$\operatorname{depth}(\mathfrak{R}/J) = 1 = \left\lceil \frac{\gamma}{3} \right\rceil.$$

For $\gamma \geq 4$, the proof proceeds by induction and construction of short exact sequences. As before, defining $J_i := (J_{i-1}, \mathfrak{a}_{\gamma-1}^i)$ for $1 \leq i \leq m$, with $J_0 = J$, we obtain the following chain of sequences:

$$\begin{aligned}
&0 \to \mathfrak{R}/(J : \mathfrak{a}_{\gamma-1}^1) \to \mathfrak{R}/J \to \mathfrak{R}/J_1 \to 0,\\
&0 \to \mathfrak{R}/(J_1 : \mathfrak{a}_{\gamma-1}^2) \to \mathfrak{R}/J_1 \to \mathfrak{R}/J_2 \to 0,\\
&\qquad\qquad\vdots\\
&0 \to \mathfrak{R}/(J_{m-2} : \mathfrak{a}_{\gamma-1}^{m-1}) \to \mathfrak{R}/J_{m-2} \to \mathfrak{R}/J_{m-1} \to 0,\\
&0 \to \mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^m) \to \mathfrak{R}/J_{m-1} \to \mathfrak{R}/J_m \to 0.
\end{aligned}$$

The following isomorphisms hold:

$$\begin{aligned}\mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^m) &\cong \mathbb{K}[V(\mathscr{P}_{\gamma-3})]/J(\mathscr{P}_{\gamma-3}) \otimes_{\mathbb{K}} \mathbb{K}[\mathfrak{a}_{\gamma-1}^m],\\ \mathfrak{R}/J_m &\cong \mathbb{K}[V(\mathscr{P}_{\gamma-2})]/J(\mathscr{P}_{\gamma-2}) \otimes_{\mathbb{K}} \mathbb{K}[V(K_m)]/I(K_m).\end{aligned}$$

Using the induction hypothesis along with Lemmas 2.5 and 2.6 implies

$$\begin{aligned}\operatorname{depth}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^m)) &= \left\lceil \frac{\gamma-3}{3} \right\rceil + 1 = \left\lceil \frac{\gamma}{3} \right\rceil,\\ \operatorname{depth}(\mathfrak{R}/J_m) &= \left\lceil \frac{\gamma-2}{3} \right\rceil + 1 = \left\lceil \frac{\gamma+1}{3} \right\rceil.\end{aligned}$$

Since $\operatorname{depth}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^m)) \leq \operatorname{depth}(\mathfrak{R}/J_m)$ and $\operatorname{depth}(\mathfrak{R}/(J_{i-1} : \mathfrak{a}_{\gamma-1}^i)) = \left\lceil \frac{\gamma}{3} \right\rceil$, by Lemma 2.10(a), we get

$$\operatorname{depth}(\mathfrak{R}/J) = \left\lceil \frac{\gamma}{3} \right\rceil.$$

The Stanley depth follows identically by replacing depth with Stanley depth throughout the proof. □

**Corollary 3.2.** *Let* $\mathfrak{R} = \mathbb{K}[V(\mathscr{P}_\gamma)]$ *and* $J = I(\mathscr{P}_\gamma)$. *If* $\gamma \geq 1$, *then*

$$\operatorname{pdim}(\mathfrak{R}/J) = m\gamma - \left\lceil \frac{\gamma}{3} \right\rceil.$$

*Proof.* The corollary follows directly from [25, Theorems 1.3.3] and Theorem 3.1. □

**Theorem 3.3.** *Let* $\mathfrak{R} = \mathbb{K}[V(\mathscr{P}_\gamma)]$ *and* $J = I(\mathscr{P}_\gamma)$. *If* $\gamma \geq 1$, *then*

$$\dim(\mathfrak{R}/J) = \left\lceil \frac{\gamma}{2} \right\rceil.$$

*Proof.* Lemma 2.2 and $\mathscr{P}_\gamma = P_\gamma \boxtimes K_m$ imply $\alpha(\mathscr{P}_\gamma) = \lceil \gamma/2 \rceil$. Lemma 2.1 then completes the proof. □

**Theorem 3.4.** *Let* $\mathfrak{R} = \mathbb{K}[V(\mathscr{C}_\gamma)]$ *and* $J = I(\mathscr{C}_\gamma)$*. If* $\gamma \geq 3$*, then*

$$\left\lceil \frac{\gamma-1}{3} \right\rceil \leq \operatorname{depth}(\mathfrak{R}/J), \operatorname{sdepth}(\mathfrak{R}/J) \leq \left\lceil \frac{\gamma}{3} \right\rceil.$$

*Proof.* We begin by proving the result for depth. For $\gamma = 3$, define $J_i := (J_{i-1}, \mathfrak{a}_2^i)$ for $1 \leq i \leq m$, with $J_0 = J$; we obtain the following chain of sequences:

$$\begin{aligned}
&0 \to \mathfrak{R}/(J : \mathfrak{a}_2^1) \to \mathfrak{R}/J \to \mathfrak{R}/J_1 \to 0,\\
&0 \to \mathfrak{R}/(J_1 : \mathfrak{a}_2^2) \to \mathfrak{R}/J_1 \to \mathfrak{R}/J_2 \to 0,\\
&\qquad\qquad\vdots\\
&0 \to \mathfrak{R}/(J_{m-2} : \mathfrak{a}_2^{m-1}) \to \mathfrak{R}/J_{m-2} \to \mathfrak{R}/J_{m-1} \to 0,\\
&0 \to \mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m) \to \mathfrak{R}/J_{m-1} \to \mathfrak{R}/J_m \to 0.
\end{aligned}$$

The following isomorphisms hold:

$$\begin{aligned}
\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m) &\cong \mathbb{K}[\mathfrak{a}_2^m],\\
\mathfrak{R}/J_m &\cong \mathbb{K}[V(\mathscr{P}_2)]/J(\mathscr{P}_2).
\end{aligned}$$

Using Lemma 2.6 implies

$$\begin{aligned}
\operatorname{depth}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m)) &= 1,\\
\operatorname{depth}(\mathfrak{R}/J_m) &= 1.
\end{aligned}$$

Since $\operatorname{depth}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m)) \leq \operatorname{depth}(\mathfrak{R}/J_m)$ and $\operatorname{depth}(\mathfrak{R}/(J_{i-1} : \mathfrak{a}_2^i)) = 1$, by Lemma 2.10(a), we get

$$\operatorname{depth}(\mathfrak{R}/J) = 1.$$

For $\gamma \geq 4$, defining $J_i := (J_{i-1}, \mathfrak{a}_{\gamma-1}^i)$ for $1 \leq i \leq m$ as before, with $J_0 = J$; we obtain the following chain of sequences:

$$\begin{aligned}
&0 \to \mathfrak{R}/(J : \mathfrak{a}_{\gamma-1}^1) \to \mathfrak{R}/J \to \mathfrak{R}/J_1 \to 0,\\
&0 \to \mathfrak{R}/(J_1 : \mathfrak{a}_{\gamma-1}^2) \to \mathfrak{R}/J_1 \to \mathfrak{R}/J_2 \to 0,\\
&\qquad\qquad\vdots\\
&0 \to \mathfrak{R}/(J_{m-2} : \mathfrak{a}_{\gamma-1}^{m-1}) \to \mathfrak{R}/J_{m-2} \to \mathfrak{R}/J_{m-1} \to 0,\\
&0 \to \mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^m) \to \mathfrak{R}/J_{m-1} \to \mathfrak{R}/J_m \to 0.
\end{aligned}$$

The following isomorphisms hold:

$$\begin{aligned}
\mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^m) &\cong \mathbb{K}[V(\mathscr{P}_{\gamma-3})]/J(\mathscr{P}_{\gamma-3}) \otimes_{\mathbb{K}} \mathbb{K}[\mathfrak{a}_{\gamma-1}^m],\\
\mathfrak{R}/J_m &\cong \mathbb{K}[V(\mathscr{P}_{\gamma-1})]/J(\mathscr{P}_{\gamma-1}).
\end{aligned}$$

Using Theorem 3.1 along with Lemmas 2.5 and 2.6 implies

$$\operatorname{depth}(\mathfrak{R}/(J_{m-1}:\mathfrak{a}^{m}_{\gamma-1})) = \left\lceil\frac{\gamma-3}{3}\right\rceil + 1 = \left\lceil\frac{\gamma}{3}\right\rceil,$$
$$\operatorname{depth}(\mathfrak{R}/J_m) = \left\lceil\frac{\gamma-1}{3}\right\rceil.$$

Since $\operatorname{depth}(\mathfrak{R}/J_m) \leq \operatorname{depth}(\mathfrak{R}/(J_{m-1}:\mathfrak{a}^{m}_{\gamma-1}))$ and $\operatorname{depth}(\mathfrak{R}/(J_{i-1}:\mathfrak{a}^{i}_{\gamma-1})) = \left\lceil\frac{\gamma}{3}\right\rceil$, Lemma 2.10(b) concludes that

$$\left\lceil\frac{\gamma-1}{3}\right\rceil \leq \operatorname{depth}(\mathfrak{R}/J) \leq \left\lceil\frac{\gamma}{3}\right\rceil.$$

The Stanley depth follows identically by replacing depth with Stanley depth throughout the proof. □

**Remark 3.5.** *Let $\mathfrak{R} = \mathbb{K}[V(\mathscr{C}_\gamma)]$ and $J = I(\mathscr{C}_\gamma)$. If $\gamma \geq 3$ and $\gamma \equiv 0, 2(mod3)$, then*

$$\operatorname{depth}(\mathfrak{R}/J) = \left\lceil\frac{\gamma}{3}\right\rceil.$$

**Corollary 3.6.** *Let $\mathfrak{R} = \mathbb{K}[V(\mathscr{C}_\gamma)]$, $J = I(\mathscr{C}_\gamma)$, and $\gamma \geq 3$; then:*

***a.*** *If $\gamma \equiv 0, 2(mod3)$, then*

$$\operatorname{pdim}(\mathfrak{R}/J) = m\gamma - \left\lceil\frac{\gamma}{3}\right\rceil.$$

***b.*** *If $\gamma \equiv 1(mod3)$, then*

$$m\gamma - \left\lceil\frac{\gamma}{3}\right\rceil \leq \operatorname{pdim}(\mathfrak{R}/J) \leq m\gamma - \left\lceil\frac{\gamma-1}{3}\right\rceil.$$

*Proof.* The corollary is immediate from [25, Theorem 1.3.3] and Theorem 3.4. □

**Theorem 3.7.** *Let $\mathfrak{R} = \mathbb{K}[V(\mathscr{C}_\gamma)]$ and $J = I(\mathscr{C}_\gamma)$. If $\gamma \geq 3$, then*

$$\dim(\mathfrak{R}/J) = \left\lfloor\frac{\gamma}{2}\right\rfloor.$$

*Proof.* Lemma 2.2 and $\mathscr{C}_\gamma = C_\gamma \boxtimes K_m$ imply $\alpha(\mathscr{C}_\gamma) = \lfloor\gamma/2\rfloor$. Lemma 2.1 then completes the proof. □

**Theorem 3.8.** *For $\gamma \geq 1$, a graph $\mathscr{P}_\gamma$ is Cohen–Macaulay if and only if $\gamma = 1$, $\gamma = 2$, or $\gamma = 4$.*

*Proof.* The result is immediate from Theorems 3.1 and 3.3. □

**Theorem 3.9.** *For $\gamma \geq 3$, a graph $\mathscr{C}_\gamma$ is Cohen–Macaulay if and only if $\gamma = 3$ or $\gamma = 5$.*

*Proof.* The result is immediate from Theorems 3.4 and 3.7. □

## 4. Castelnuovo-Mumford regularity

The Castelnuovo-Mumford regularity of $\mathfrak{R}/J$ is determined for edge ideals of $\mathscr{P}_\gamma$ and $\mathscr{C}_\gamma$, where $\mathscr{P}_\gamma = P_\gamma \boxtimes K_m$ and $\mathscr{C}_\gamma = C_\gamma \boxtimes K_m$.

**Theorem 4.1.** *Let* $\mathfrak{R} = \mathbb{K}[V(\mathscr{P}_\gamma)]$ *and* $J = I(\mathscr{P}_\gamma)$*. If* $\gamma \geq 1$*, then*

$$\operatorname{reg}(\mathfrak{R}/J) = \left\lceil \frac{\gamma}{2} \right\rceil.$$

*Proof.* For $\gamma \leq 2$, the graph is complete, that is, $\mathscr{P}_1 \cong K_m$ and $\mathscr{P}_2 \cong K_{2m}$, which complements our main result. For $\gamma = 3$, define $J_i := (J_{i-1}, \mathfrak{a}_2^i)$ for $1 \leq i \leq m$, with $J_0 = J$; we obtain the following chain of sequences:

$$\begin{gathered}
0 \to \mathfrak{R}/(J : \mathfrak{a}_2^1) \to \mathfrak{R}/J \to \mathfrak{R}/J_1 \to 0,\\
0 \to \mathfrak{R}/(J_1 : \mathfrak{a}_2^2) \to \mathfrak{R}/J_1 \to \mathfrak{R}/J_2 \to 0,\\
\vdots\\
0 \to \mathfrak{R}/(J_{m-2} : \mathfrak{a}_2^{m-1}) \to \mathfrak{R}/J_{m-2} \to \mathfrak{R}/J_{m-1} \to 0,\\
0 \to \mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^{m}) \to \mathfrak{R}/J_{m-1} \to \mathfrak{R}/J_m \to 0.
\end{gathered}$$

The following isomorphisms hold:

$$\begin{aligned}
\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m) &\cong \mathbb{K}[\mathfrak{a}_2^m],\\
\mathfrak{R}/J_m &\cong \mathbb{K}[V(\mathscr{P}_1)]/J(\mathscr{P}_1) \otimes_{\mathbb{K}} \mathbb{K}[V(K_m)]/I(K_m).
\end{aligned}$$

Using Lemma 2.6 implies

$$\begin{aligned}
\operatorname{reg}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m)) &= 1,\\
\operatorname{reg}(\mathfrak{R}/J_m) &= 2.
\end{aligned}$$

Since $\operatorname{reg}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_2^m)) < \operatorname{reg}(\mathfrak{R}/J_m)$ and $\operatorname{reg}(\mathfrak{R}/(J_{i-1} : \mathfrak{a}_2^i)) = 1$, by Lemma 2.10(a), we get

$$\operatorname{reg}(\mathfrak{R}/J) = 2 = \left\lceil \frac{\gamma}{2} \right\rceil.$$

For $\gamma \geq 4$, the proof proceeds by induction and construction of short exact sequences. Define $J_i := (J_{i-1}, \mathfrak{a}_{\gamma-1}^i)$ for $1 \leq i \leq m$, with $J_0 = J$; we obtain the following chain of sequences:

$$\begin{gathered}
0 \to \mathfrak{R}/(J : \mathfrak{a}_{\gamma-1}^1) \to \mathfrak{R}/J \to \mathfrak{R}/J_1 \to 0,\\
0 \to \mathfrak{R}/(J_1 : \mathfrak{a}_{\gamma-1}^2) \to \mathfrak{R}/J_1 \to \mathfrak{R}/J_2 \to 0,\\
\vdots\\
0 \to \mathfrak{R}/(J_{m-2} : \mathfrak{a}_{\gamma-1}^{m-1}) \to \mathfrak{R}/J_{m-2} \to \mathfrak{R}/J_{m-1} \to 0,\\
0 \to \mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^{m}) \to \mathfrak{R}/J_{m-1} \to \mathfrak{R}/J_m \to 0.
\end{gathered}$$

The following isomorphisms hold:

$$\mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^m) \cong \mathbb{K}[V(\mathscr{P}_{\gamma-3})]/J(\mathscr{P}_{\gamma-3}) \otimes_{\mathbb{K}} \mathbb{K}[\mathfrak{a}_{\gamma-1}^m],$$

$$\mathfrak{R}/J_m \cong \mathbb{K}[V(\mathscr{P}_{\gamma-2})]/J(\mathscr{P}_{\gamma-2}) \otimes_{\mathbb{K}} \mathbb{K}[V(K_m)]/I(K_m).$$

Using the induction hypothesis along with Lemmas 2.5(c) and 2.8 implies

$$\operatorname{reg}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}^m_{\gamma-1})) = \left\lceil \frac{\gamma-3}{2} \right\rceil,$$
$$\operatorname{reg}(\mathfrak{R}/J_m) = \left\lceil \frac{\gamma-2}{2} \right\rceil + 1 = \left\lceil \frac{\gamma}{2} \right\rceil.$$

Since $\operatorname{reg}(\mathfrak{R}/J_m) < \operatorname{reg}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}^m_{\gamma-1}))$ and $\operatorname{reg}(\mathfrak{R}/(J_{i-1} : \mathfrak{a}^i_{\gamma-1})) = \left\lceil \frac{\gamma-3}{2} \right\rceil$, Lemma 2.10(c) concludes that

$$\operatorname{reg}(\mathfrak{R}/J) = \left\lceil \frac{\gamma}{2} \right\rceil.$$

□

**Theorem 4.2.** *Let* $\mathfrak{R} = \mathbb{K}[V(\mathscr{C}_\gamma)]$ *and* $J = I(\mathscr{C}_\gamma)$. *If* $\gamma \geq 3$, *then*

$$\operatorname{reg}(\mathfrak{R}/J) = \left\lceil \frac{\gamma-1}{2} \right\rceil.$$

*Proof.* For $\gamma = 3$, define $J_i := (J_{i-1}, \mathfrak{a}^i_2)$ for $1 \leq i \leq m$, with $J_0 = J$; we obtain the following chain of sequences

$$0 \to \mathfrak{R}/(J : \mathfrak{a}^1_2) \to \mathfrak{R}/J \to \mathfrak{R}/J_1 \to 0,$$
$$0 \to \mathfrak{R}/(J_1 : \mathfrak{a}^2_2) \to \mathfrak{R}/J_1 \to \mathfrak{R}/J_2 \to 0,$$
$$\vdots$$
$$0 \to \mathfrak{R}/(J_{m-2} : \mathfrak{a}^{m-1}_2) \to \mathfrak{R}/J_{m-2} \to \mathfrak{R}/J_{m-1} \to 0,$$
$$0 \to \mathfrak{R}/(J_{m-1} : \mathfrak{a}^m_2) \to \mathfrak{R}/J_{m-1} \to \mathfrak{R}/J_m \to 0.$$

The following isomorphisms hold:

$$\mathfrak{R}/(J_{m-1} : \mathfrak{a}^m_2) \cong \mathbb{K}[\mathfrak{a}^m_2],$$
$$\mathfrak{R}/J_m \cong \mathbb{K}[V(\mathscr{P}_2)]/J(\mathscr{P}_2).$$

Using Lemma 2.6 and Theorem 4.1 implies

$$\operatorname{reg}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}^m_2)) = 0,$$
$$\operatorname{reg}(\mathfrak{R}/J_m) = 1.$$

Since $\operatorname{reg}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}^m_2)) < \operatorname{reg}(\mathfrak{R}/J_m)$ and $\operatorname{reg}(\mathfrak{R}/(J_{i-1} : \mathfrak{a}^i_2)) = 1$, by Lemma 2.10(c), we get

$$\operatorname{reg}(\mathfrak{R}/J) = 1 = \left\lceil \frac{\gamma-1}{2} \right\rceil.$$

For $\gamma \geq 4$, the proof proceeds by induction and construction of short exact sequences. Define $J_i := (J_{i-1}, \mathfrak{a}^i_{\gamma-1})$ for $1 \leq i \leq m$, with $J_0 = J$; we obtain the following chain of sequences:

$$0 \to \mathfrak{R}/(J : \mathfrak{a}^1_{\gamma-1}) \to \mathfrak{R}/J \to \mathfrak{R}/J_1 \to 0,$$

$$0 \to \mathfrak{R}/(J_1 : \mathfrak{a}_{\gamma-1}^2) \to \mathfrak{R}/J_1 \to \mathfrak{R}/J_2 \to 0,$$
$$\vdots$$
$$0 \to \mathfrak{R}/(J_{m-2} : \mathfrak{a}_{\gamma-1}^{m-1}) \to \mathfrak{R}/J_{m-2} \to \mathfrak{R}/J_{m-1} \to 0,$$
$$0 \to \mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^{m}) \to \mathfrak{R}/J_{m-1} \to \mathfrak{R}/J_m \to 0.$$

The following isomorphisms hold:

$$\mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^{m}) \cong \mathbb{K}[V(\mathscr{P}_{\gamma-3})]/J(\mathscr{P}_{\gamma-3}) \otimes_{\mathbb{K}} \mathbb{K}[\mathfrak{a}_{\gamma-1}^{m}],$$
$$\mathfrak{R}/J_m \cong \mathbb{K}[V(\mathscr{P}_{\gamma-1})]/J(\mathscr{P}_{\gamma-1}).$$

Using Theorem 4.1 along with Lemmas 2.5(c) and 2.8 implies

$$\operatorname{reg}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^{m})) = \left\lceil \frac{\gamma-3}{2} \right\rceil,$$
$$\operatorname{reg}(\mathfrak{R}/J_m) = \left\lceil \frac{\gamma-1}{2} \right\rceil.$$

Since $\operatorname{reg}(\mathfrak{R}/J_m) < \operatorname{reg}(\mathfrak{R}/(J_{m-1} : \mathfrak{a}_{\gamma-1}^{m}))$ and $\operatorname{reg}(\mathfrak{R}/(J_{i-1} : \mathfrak{a}_{\gamma-1}^{i})) = \left\lceil \frac{\gamma-3}{2} \right\rceil$, Lemma 2.10(c) concludes that

$$\operatorname{reg}(\mathfrak{R}/J) = \left\lceil \frac{\gamma-1}{2} \right\rceil.$$

□

## 5. Concluding remarks

This paper provides a comprehensive analysis of the homological algebra of edge ideals for strong products $\mathscr{P}_\gamma = P_\gamma \boxtimes K_m$ and $\mathscr{C}_\gamma = C_\gamma \boxtimes K_m$. For these graph classes, we have derived explicit combinatorial formulas that precisely determine several key invariants of the quotient ring $\mathfrak{R}/I(\mathbb{G})$, notably including depth, projective dimension, Stanley depth, Krull dimension, and Castelnuovo-Mumford regularity.

A central achievement is the complete characterization of the Cohen–Macaulay property within these families. Our results demonstrate how strong product operation with a complete graph $K_m$ imposes a specific combinatorial structure that determines the resulting algebraic properties. Unlike Cartesian or direct products, the strong product's denser connectivity imposes a rich combinatorial constraint that is directly reflected in these homological invariants. The precise nature of our formulas provides a foundational reference point for investigating the homological properties of edge ideals derived from more sophisticated graph constructions.

## 6. Open problems and future research

The research presented herein naturally gives rise to several compelling research directions.

- **Strong product of arbitrary graphs**: A fundamental challenge and a broader goal is to find general bounds or formulas for $\operatorname{reg}(I(\mathbb{G} \boxtimes \mathbb{H}))$ for arbitrary graphs $\mathbb{G}$ and $\mathbb{H}$. Can the regularity be expressed in terms of the regularities and other combinatorial invariants of the factor graphs $\mathbb{G}$ and $\mathbb{H}$?

- **Extended algebraic properties**: Investigating algebraic properties beyond the Cohen–Macaulay condition presents a natural direction. Promising research problems include characterizing when these edge ideals admit linear resolutions, studying the behavior of symbolic powers, and exploring connections to other homological invariants.

## Author contributions

Ahtsham ul Haq: Writing–original draft, Data curation, Methodology, Validation, Investigation, Conceptualization. Muhammad Usman Rashid: Writing – review & editing, Validation, Investigation, Formal analysis, Conceptualization. Muhammad Ishaq: Review & editing, Validation, Supervision, Conceptualization.

All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgements

The authors gratefully acknowledge the anonymous referees for insightful comments and constructive suggestions that have greatly improved the clarity and quality of this manuscript.

## Conflict of interest

All authors declare no conflicts of interest in this paper.

## References

1. R. P. Stanley, Linear Diophantine equations and local cohomology, *Invent. Math.*, **68** (1982), 175–193. https://doi.org/10.1007/BF01394054

2. A. M. Duval, B. Goeckner, C. J. Klivans, J. L. Martin, A non-partitionable Cohen–Macaulay simplicial complex, *Adv. Math.*, **299** (2016), 381–395. https://doi.org/10.1016/j.aim.2016.05.011

3. J. Herzog, M. Vladoiu, X. Zheng, How to compute the Stanley depth of a monomial ideal, *J. Algebra*, **322** (2009), 3151–3169. https://doi.org/10.1016/j.jalgebra.2008.01.006

4. A. Alipour, A. Tehranian, Depth and Stanley depth of edge ideals of star graphs, *Int. J. Appl. Math. Stat.*, **56** (2017), 63–69.

5. A. Rauf, Depth and Stanley depth of multigraded modules, *Commun. Algebra*, **38** (2010), 773–784. https://doi.org/10.1080/00927870902829056

6. L. Fouli, S. Morey, A lower bound for depths of powers of edge ideals, *J. Algebr. Comb.*, **42** (2015), 829–848. https://doi.org/10.1007/s10801-015-0604-3
7. M. Cimpoeas, Several inequalities regarding Stanley depth, *Rom. J. Math. Comput. Sci.*, **2** (2012), 28–40.
8. M. Ishaq, Values and bounds for the Stanley depth, *Carpathian J. Math.*, **27** (2011), 217–224.
9. S. A. S. Fakhari, On the Stanley depth of powers of edge ideals, *J. Algebra*, **489** (2017), 463–474. https://doi.org/10.1016/j.jalgebra.2017.07.011
10. G. Kalai, R. Meshulam, Intersections of Leray complexes and regularity of monomial ideals, *J. Comb. Theory A*, **113** (2006), 1586–1592. https://doi.org/10.1016/j.jcta.2006.01.005
11. H. Dao, C. Huneke, J. Schweig, Bounds on the regularity and projective dimension of ideals associated to graphs, *J. Algebr. Comb.*, **38** (2013), 37–55. https://doi.org/10.1007/s10801-012-0391-z
12. M. E. Uribe-Paczka, A. Van Tuyl, The regularity of some families of circulant graphs, *Mathematics*, **7** (2019), 657. https://doi.org/10.3390/math7070657
13. R. R. Bouchat, Free resolutions of some edge ideals of simple graphs, *J. Commut. Algebra*, **2** (2010), 1–35. https://doi.org/10.1216/JCA-2010-2-1-1
14. S. Faridi, B. Hersey, Resolutions of monomial ideals of projective dimension 1, *Commun. Algebra*, **45** (2017), 5453–5464. https://doi.org/10.1080/00927872.2017.1313422
15. X. Zheng, Resolutions of facet ideals, *Commun. Algebra*, **32** (2004), 2301–2324. https://doi.org/10.1081/AGB-120037222
16. M. Barile, A. Macchia, A note on Cohen–Macaulay graphs, *Comm. Algebra*, **44** (2016), 1473–1477. https://doi.org/10.1080/00927872.2015.1027361
17. R. H. Villarreal, *Monomial algebras*, 2Eds, Chapman and Hall/CRC, 2015. https://doi.org/10.1201/b18224
18. J. Herzog, T. Hibi, X. Zheng, Cohen–Macaulay chordal graphs, *J. Comb. Theory A*, **113** (2006), 911–916. https://doi.org/10.1016/j.jcta.2005.08.007
19. CoCoATeam, CoCoA: A system for doing computations in commutative algebra, Available from: `http://cocoa.dima.unige.it/`.
20. D. R. Grayson, M. E. Stillman, Macaulay2: A software system for research in algebraic geometry, Available from: `https://www.macaulay2.com/`.
21. R. Hammack, W. Imrich, S. Klavar, *Handbook of product graphs*, 2Eds, CRC Press, 2011. https://doi.org/10.1201/b10959
22. A. Hirano, K. Matsuda, Matching numbers and dimension edge ideals, *Graph. Combinator.*, **37** (2021), 761–774. https://doi.org/10.1007/s00373-021-02277-x
23. A. Vesel, The independence number of strong product of cycles, *Comput. Math. Appl.*, **36** (1998), 9–21. https://doi.org/10.1016/S0898-1221(98)00169-2
24. H. T. Hà, A. Van Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, *J. Algebr. Comb.*, **27** (2008), 215–245. https://doi.org/10.1007/s10801-007-0079-y

25. W. Bruns, H. J. Herzog, *Cohen–Macaulay rings*, 2Eds, Cambridge University Press, 1998. https://doi.org/10.1017/CBO9780511608681

26. S. Morey, R. H. Villarreal, Edge ideals: Algebraic and combinatorial properties, In: *Progress in commutative algebra 1: Combinatorics and homology*, 2012, 85–126. https://doi.org/10.1515/9783110250404.85

27. R. Woodroofe, Matchings, coverings, and Castelnuovo-Mumford regularity, *J. Commut. Algebra*, **6** (2014), 287–304. https://doi.org/10.1216/JCA-2014-6-2-287

28. D. Eisenbud, *Commutative algebra: With a view toward algebraic geometry*, Springer, 2013.

29. Z. Iqbal, M. Ishaq, Depth and Stanley depth of the edge ideals of the powers of paths and cycles, *Analele ştiinţifice ale Universităţii "Ovidius" Constanţa. Seria Matematică*, **27** (2019), 113–135. https://doi.org/10.2478/auom-2019-0037