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Research article

A new formulation of Hardy-type dynamic inequalities on time scales

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Abstract: In this paper, we introduce a novel formulation of dynamic Hardy-type inequalities on a time scale, motivated by a recently-established convexity approach in the Haar measure. The classical Hardy inequality is refined so that the classical Lebesgue-measure constant is replaced by the sharp constant 1. We obtain time-scale analogues on finite intervals with best constants, and, for nonincreasing and nondecreasing functions, reversed inequalities with explicit weights described by incomplete β-functions. To establish our results, we employ two distinct time scales and apply the chain rule, together with the substitution rule, the derivative of inverse functions, and Fubini's theorem for delta integration. Our approach generalizes classical integral inequalities in the continuous setting, while yielding fundamentally new inequalities in the discrete setting. Furthermore, we explore the application of our results in the quantum case, demonstrating their broader relevance.

Keywords: Hardy-type inequalities; chain rule on time scales; Jensen's inequality
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1. Introduction

In 1920, Hardy [1] showed that if $p > 1$ and $\{a(n)\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p} \le \left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n). \tag{1.1}$$

In 1925, Hardy [2] provided a continuous analogue of (1.1), proving that if $p > 1$ and $g \ge 0$ is

p-integrable on $(0, \infty)$, then g is integrable on each finite interval $(0, x)$ for $x \in (0, \infty)$ and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\mathrm{d}t\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p(x)\mathrm{d}x. \tag{1.2}$$

In both the discrete and continuous cases, the constant $C = (p/(p-1))^p$ is sharp and cannot be replaced by a smaller value without losing its validity for certain sequences and functions.

In 1928, Hardy [3] generalized (1.2) and proved that if $p > 1$, $\alpha < p - 1$, and g is measurable and nonnegative for $x \in (0, \infty)$, then

$$\int_0^\infty x^\alpha \left(\frac{1}{x}\int_0^x g(t)\mathrm{d}t\right)^p \mathrm{d}x \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty x^\alpha g^p(x)\mathrm{d}x. \tag{1.3}$$

For nonincreasing functions, (1.3) holds with the inequality reversed and the constant $C = 1$.

By substituting $g(x) = f(x^{1-1/p})x^{-1/p}$ for $p > 1$, one can see that Hardy's inequality (1.2) is equivalent to

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \leq 1 \cdot \int_0^\infty f^p(x)\frac{\mathrm{d}x}{x} \tag{1.4}$$

for $p > 1$. As discussed in Persson et al. [4], this version (1.4) of Hardy's inequality in the Haar measure $\mathrm{d}x/x$ can be succinctly derived using Jensen's inequality and Fubini's theorem:

$$\begin{aligned}\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} &\leq \int_0^\infty \left(\frac{1}{x}\int_0^x f^p(t)\mathrm{d}t\right) \frac{\mathrm{d}x}{x} \\ &= \int_0^\infty f^p(y)\left(\int_y^\infty \frac{\mathrm{d}x}{x^2}\right)\mathrm{d}y \\ &= \int_0^\infty f^p(x)\frac{\mathrm{d}x}{x}.\end{aligned}$$

It follows that all three inequalities (1.2), (1.3), and (1.4) are equivalent, and, in fact, hold for $p < 0$ as well, since $\phi(u) = u^p$ remains convex for negative p. Moreover, (1.4) also holds with equality when $p = 1$, which gives a possibility to interpolate and analyze the mapping properties of the Hardy operator further.

For the case of a finite interval, Persson et al. [4] extended the basic (convexity) form of Hardy's inequality (1.4). They proved that if g is a nonnegative measurable function on $(0, \ell)$ for $0 < \ell \leq \infty$, then

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \leq 1 \cdot \int_0^\ell f^p(x)\left(1 - \frac{x}{\ell}\right)\frac{\mathrm{d}x}{x}, \tag{1.5}$$

provided either $p < 0$ or $p \geq 1$, and, if $p < 0$, that $f(x) > 0$ for $0 < x \leq \ell$ (see [4, Lemma 2.2]). They also showed that in the reversed case $0 < p < 1$,

$$\int_0^\ell x^{-p}\left(\int_0^x f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \geq 1 \cdot \int_0^\ell f^p(x)\left(1 - \frac{x}{\ell}\right)\frac{\mathrm{d}x}{x}. \tag{1.6}$$

In both (1.5) and (1.6), the constant $C = 1$ is sharp, just as it is for (1.4).

Furthermore, Persson et al. [4] derived the reversed form of (1.5) by introducing a new parameter $\alpha > 0$. In particular,

$$\int_0^\ell x^{-\alpha}\left(\int_0^x f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \geq \frac{p}{\alpha}\int_0^\ell x^{p-\alpha} f^p(x)\left(1-\left(\frac{x}{\ell}\right)^\alpha\right)\frac{\mathrm{d}x}{x} \tag{1.7}$$

holds provided $p \geq 1$, $0 < \alpha < p$, and f is a nonnegative, nonincreasing function on $(0, \ell)$, $0 < \ell \leq \infty$ (see [4, Theorem 3.2]). They also showed that

$$\int_0^\ell x^{-\alpha}\left(\int_0^x f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \geq \frac{p}{\alpha}\int_0^\ell x^{p-\alpha} f^p(x)T(x)\frac{\mathrm{d}x}{x}, \tag{1.8}$$

provided $\alpha \geq p > 0$ and f is a nonnegative, nondecreasing function on $(0, \ell)$ for $0 < \ell \leq \infty$, where

$$T(x) = \alpha\beta_{\frac{x}{\ell}}(\alpha - p + 1, p) \quad \text{and} \quad \beta_\lambda(u, v) = \int_\lambda^1 t^{u-1}(1-t)^{v-1}\mathrm{d}t \quad \text{for} \quad 0 \leq \lambda < 1,$$

see [4, Theorem 3.3]. Note that β_0 is the usual β-function $\beta(u, v)$. Lastly, the authors of [4] showed that if f is a nonnegative and nonincreasing function on (ℓ, ∞) for $0 \leq \ell < \infty$, $\alpha > 0$, and $p \geq 1$, then

$$\int_\ell^\infty x^{\alpha}\left(\int_x^\infty f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \geq \frac{p}{\alpha}\int_\ell^\infty x^{p+\alpha} f^p(x)T_0(x)\frac{\mathrm{d}x}{x}, \tag{1.9}$$

where $T_0(x) = \alpha\beta_{\ell/x}(\alpha, p)$ for $x \geq \ell$ (see [4, Theorem 3.5]). As in the case $\alpha = p$, the constant $C = p/\alpha$ is sharp in all three inequalities (1.7), (1.8), and (1.9).

In the last few decades, many researchers have addressed the problem of extending classical continuous results to a broader framework that unifies both discrete and continuous cases. Since Hilger's seminal paper [5], the idea of unification of continuous and discrete analysis has attracted substantial attention. The theory of time scales provides a unified approach by defining a time scale $\mathbb{T}$ to be any closed, nonempty subset of the real numbers with the subspace topology from $\mathbb{R}$. Within this framework, dynamic inequalities generalize both differential and difference inequalities. In particular, choosing a specific time scale determines the type of inequalities addressed, thereby allowing various discrete analogues of classical continuous results to be formulated.

Notably, considerable attention has been directed towards dynamic inequalities on time scales, as evidenced by works such as [6–11] (on Hardy inequalities), [12, 13] (on inequalities involving monotone functions), and [14] (on Littlewood inequalities), and, in particular, the monograph by Agarwal et al. [15].

In [16], Saker and O'Regan established a time-scale analogue of (1.3). Specifically, they showed that if $p, \gamma > 1$, $f \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, $a \geq 0$, and $\Lambda(t) = \int_a^t f(s)\Delta s$ for any $t \in [a,\infty)_{\mathbb{T}}$, then

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^p}{(\sigma(t)-a)^\gamma}\Delta t \leq \left(\frac{p}{\gamma-1}\right)^p \int_a^\infty \frac{(\sigma(t)-a)^{\gamma(p-1)}}{(t-a)^{p(\gamma-1)}} f^p(t)\Delta t. \tag{1.10}$$

When $\mathbb{T} = \mathbb{R}$, (1.10) reduces to the classical integral inequality (1.3), while $\mathbb{T} = \mathbb{N}$ yields its discrete analogue, and $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ recovers the corresponding formula in quantum calculus.

Motivation. In their recent work [4], Persson et al. formulated the Hardy inequality (1.4) via convexity in the natural Haar measure $\mathrm{d}x/x$, replacing the classical Lebesgue setting. In this form, the Hardy average becomes a Jensen mean, the sharp constant is $C = 1$, the endpoint $p = 1$ holds with equality, and the range $0 < p < 1$ follows by concavity. On finite intervals, the same argument yields explicit weights with the sharp constants (see (1.5) and (1.6)). On the cones of nonincreasing/nondecreasing functions, it gives the reversed inequalities (1.7), (1.8), and (1.9) with the optimal factor p/α, and with the associated weights expressed via incomplete β-functions $\beta_\lambda(u, v)$.

In this paper, we apply time-scale theory to transfer the continuous, Haar-measure, convexity-based formulation of Hardy's inequality on time scales. Our approach involves utilizing the chain rule on a time scale $\mathbb{T}$ alongside another chain rule linking time scales $\mathbb{T}$ and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$, where $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing. Additionally, we explore the substitution rule, the derivative of inverse functions, and Fubini's theorem. In this way, we extend the integral inequalities (1.5)–(1.9) to time scales.

2. Auxiliary lemmas

A time scale $\mathbb{T}$ is any arbitrary nonempty closed subset of reals. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and the graininess function $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$. Here, we do not repeat further background on time scales, see the monograph of Bohner and Peterson [17] for a comprehensive overview of the theory. We only state a few basic theorems needed in Section 3.

Theorem 2.1 (see [17, Theorem 1.76]). *If $f^\Delta \geq 0$ ($f^\Delta \leq 0$), then f is nondecreasing (nonincreasing).*

Theorem 2.2 (chain rule, see [17, Theorem 1.87]). *Assume $\eta : \mathbb{R} \to \mathbb{R}$ is continuous, $\eta : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then, there exists c in the real interval $[t, \sigma(t)]$ with*

$$(\varphi \circ \eta)^\Delta(t) = \varphi'(\eta(c))\eta^\Delta(t). \tag{2.1}$$

Theorem 2.3 (see [17, Theorem 1.77]). *If $a, b \in \mathbb{T}$ and $\varphi, \eta \in C_{\mathrm{rd}}([a, b]_\mathbb{T}, \mathbb{R})$, then the integration by parts rule is given by*

$$\int_a^b \varphi(t)\eta^\Delta(t)\Delta t = [\varphi(t)\eta(t)]_a^b - \int_a^b \varphi^\Delta(t)\eta^\sigma(t)\Delta t. \tag{2.2}$$

Theorem 2.4 (see [17, Theorem 1.75]). *If $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and $t \in \mathbb{T}$, then*

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t). \tag{2.3}$$

Theorem 2.5 (Jensen's inequality, see [17, Theorem 6.17]). *Let $a, b \in \mathbb{T}$, $c, d \in \mathbb{R}$, $g \in C_{\mathrm{rd}}([a, b]_\mathbb{T}, (c, d))$, and $F \in C((c, d), \mathbb{R})$ be convex. Then,*

$$F\left(\frac{\int_a^b g(t)\Delta t}{b - a}\right) \leq \frac{\int_a^b F(g(t))\Delta t}{b - a}. \tag{2.4}$$

When F is concave, (2.4) *holds with reversed sign.*

Let $n \in \mathbb{N}$ be fixed. For each $i = 1, 2, \ldots, n$, let $\mathbb{T}_i$ denote a time scale, and

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : \ t_i \in \mathbb{T}_i, 1 \leq i \leq n\}$$

denote an n-dimensional time scale. Let μ_Δ be the σ-additive Lebesgue Δ-measure on Λ^n and F be the family of Δ-measurable subsets of Λ^n. Let $E \in F$ and (E, F, μ_Δ) be a time-scale measure space. Then, for a Δ-measurable function $f : E \to \mathbb{R}$, a corresponding Δ-integral of f over E will be denoted by

$$\int_E f(t_1, \ldots, t_n)\Delta_1 t_1 \ldots \Delta_n t_n, \quad \int_E f(t)\Delta t, \quad \int_E f \mathrm{d}\mu_\Delta \quad \text{or} \quad \int_E f(t)\mathrm{d}\mu_\Delta t.$$

Theorem 2.6 (Fubini's theorem, see [18, Theorem 1.1]). *Let (X, M, μ_Δ) and (Y, L, λ_Δ) be two finite-dimensional time-scale measure spaces. If $f : X \times Y \to \mathbb{R}$ is a Δ-integrable function, and the function $\phi(y) = \int_X f(x, y)\Delta x$ is defined for a.e. $y \in Y$ and $\psi(x) = \int_Y f(x, y)\Delta y$ for a.e. $x \in X$, then ϕ is λ_Δ-integrable on Y, ψ is μ_Δ-integrable on X, and*

$$\int_X \Delta x \int_Y f(x, y)\Delta y = \int_Y \Delta y \int_X f(x, y)\Delta x. \tag{2.5}$$

Before we present the next theorem, we shall define the set $\mathbb{T}^\kappa$, which is given by

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Theorem 2.7 (chain rule, see [17, Theorem 1.93]). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing, $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale, and $\omega : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^\Delta(t)$ and $\omega^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(\omega \circ \nu)^\Delta = \left(\omega^{\tilde{\Delta}} \circ \nu\right) \nu^\Delta. \tag{2.6}$$

Here, $\omega^{\tilde{\Delta}}$ is the delta derivative of the function ω on the time scale $\tilde{\mathbb{T}}$.

Theorem 2.8 (derivative of the inverse, see [17, Theorem 1.97]). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Then,*

$$\frac{1}{\nu^\Delta} = (\nu^{-1})^{\tilde{\Delta}} \circ \nu \tag{2.7}$$

at points where ν^Δ is different from zero.

Throughout this paper, we assume that all integrals under consideration are well defined. In what follows, we present several auxiliary lemmas on time scales that will be used to prove our main results. In particular, the first lemma provides a foundation for extending inequalities (1.7)–(1.9) to the time-scale setting.

Lemma 2.1. *Let $a, b \in \mathbb{T}$, $f \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R}^+)$, and $p \geq 1$. If f is nonincreasing, then*

$$\left(\int_a^b f(y)\Delta y\right)^p \geq p \int_a^b (y - a)^{p-1} f^p(y)\Delta y \tag{2.8}$$

and

$$\left(\int_a^\infty f(y)\Delta y\right)^p \geq p \int_a^\infty (y - a)^{p-1} f^p(y)\Delta y. \tag{2.9}$$

Also, if f is nondecreasing, then

$$\left(\int_a^b f(y)\Delta y\right)^p \geq p \int_a^b (b - \sigma(y))^{p-1} f^p(y)\Delta y. \tag{2.10}$$

Proof. Denote

$$F(x) = \int_a^x f(y)\Delta y \quad \text{and} \quad \Omega(t) = \int_t^b f(y)\Delta y.$$

Now we are prepared to prove (2.8). Employing the chain rule (2.1) with $\eta(x) = F(x)$ and $\varphi(t) = t^p$, we get

$$(\varphi \circ \eta)^{\Delta}(x) = \varphi'(\nu(\eta))\eta^{\Delta}(x) = pF^{p-1}(c)F^{\Delta}(x) \quad \text{for some} \quad c \in [x, \sigma(x)]. \tag{2.11}$$

Since $F^{\Delta}(x) = f(x) \geq 0$, the function F is nondecreasing on $[a,b]_{\mathbb{T}}$ and we have $F^{p-1}(c) \geq F^{p-1}(x)$ for $c \geq x$ and $p \geq 1$. Then, (2.11) gives

$$(\varphi \circ \eta)^{\Delta}(x) \geq pF^{p-1}(x)f(x).$$

Therefore,

$$(\varphi \circ \eta)(b) - (\varphi \circ \eta)(a) = \int_a^b (\varphi \circ \eta)^{\Delta}(x)\Delta x \geq p \int_a^b F^{p-1}(x)f(x)\Delta x,$$

which implies

$$\left(\int_a^b f(y)\Delta y\right)^p \geq p \int_a^b \left(\int_a^x f(y)\Delta y\right)^{p-1} f(x)\Delta x. \tag{2.12}$$

Since f is nonincreasing on $[a,b]_{\mathbb{T}}$, we observe that $\int_a^x f(y)\Delta y \geq (x-a)f(x)$ and (2.12) becomes

$$\left(\int_a^b f(y)\Delta y\right)^p \geq p \int_a^b (x-a)^{p-1} f^p(x)\Delta x,$$

which is the desired (2.8).

By using a similar technique as before, one can get

$$\left(\int_a^{\infty} f(y)\Delta y\right)^p \geq p \int_a^{\infty} (y-a)^{p-1} f^p(y)\Delta y,$$

which represents (2.9).

Finally, we establish (2.10). Employing the chain rule (2.1) with $\eta(t) = \Omega(t)$ and $\varphi(x) = x^p$, we obtain

$$(\varphi \circ \eta)^{\Delta}(t) = \varphi'(\eta(c))\eta^{\Delta}(t) = p\Omega^{p-1}(c)\Omega^{\Delta}(t) \quad \text{for some} \quad c \in [t, \sigma(t)].$$

Since $\Omega^{\Delta}(t) = -f(t) \leq 0$ so that the function Ω is nonincreasing on $[a,b]_{\mathbb{T}}$, we have, for $c \leq \sigma(t)$, that $\Omega(c) \geq \Omega(\sigma(t))$, and thus $-(\varphi \circ \eta)^{\Delta}(t) \geq p\Omega^{p-1}(\sigma(t))f(t)$. So,

$$p \int_a^b \Omega^{p-1}(\sigma(t))f(t)\Delta t \leq -\int_a^b (\varphi \circ \eta)^{\Delta}(t)\Delta t = (\varphi \circ \eta)(a) - (\varphi \circ \eta)(b) = \left(\int_a^b f(y)\Delta y\right)^p. \tag{2.13}$$

Using that f is nondecreasing, (2.13) becomes

$$\begin{aligned} \left(\int_a^b f(y)\Delta y\right)^p &\geq p \int_a^b \left(\int_{\sigma(t)}^b f(y)\Delta y\right)^{p-1} f(t)\Delta t \\ &\geq p \int_a^b (b-\sigma(t))^{p-1} f^{p-1}(\sigma(t))f(t)\Delta t \\ &\geq p \int_a^b (b-\sigma(t))^{p-1} f^p(t)\Delta t, \end{aligned}$$

which shows (2.10) and completes the proof.

The following lemma will be used to prove the time-scale versions of (1.9) and of the reversed form (1.7).

Lemma 2.2. *Assume that $a, x \in \mathbb{T}$, $p \geq 1$, and $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^+)$ is nondecreasing. Then,*

$$\left(\int_a^x f(s)\Delta s\right)^p \leq p\int_a^x (\sigma(t)-a)^{p-1} f^p(t)\Delta t. \tag{2.14}$$

Proof. Denote $F(t) = \int_a^t f(s)\Delta s$. Employing the chain rule (2.1) with $\eta(t) = F(t)$ and $\varphi(x) = x^p$, we have

$$(\varphi \circ \eta)^{\Delta}(t) = \varphi'(\eta(c))\eta^{\Delta}(t) = pF^{p-1}(c)F^{\Delta}(t) \quad \text{for some} \quad c \in [t, \sigma(t)]. \tag{2.15}$$

Since $F^{\Delta}(t) = f(t) \geq 0$, the function F is nondecreasing, and (2.15) becomes

$$(\varphi \circ \eta)^{\Delta}(t) \leq pF^{p-1}(\sigma(t))f(t) = p\left(\int_a^{\sigma(t)} f(s)\Delta s\right)^{p-1} f(t). \tag{2.16}$$

Applying (2.3), since f is nondecreasing, we get

$$\begin{aligned}\int_a^{\sigma(t)} f(s)\Delta s &= \int_a^t f(s)\Delta s + \int_t^{\sigma(t)} f(s)\Delta s\\ &= \int_a^t f(s)\Delta s + \mu(t)f(t)\\ &\leq (t-a)f(t) + \mu(t)f(t)\\ &= (\sigma(t)-a)f(t).\end{aligned}$$

Thus, (2.16) gives

$$(\varphi \circ \eta)^{\Delta}(t) \leq p(\sigma(t)-a)^{p-1} f^p(t).$$

Integrating the last inequality over t from a to x, we obtain

$$(\varphi \circ \eta)(x) - (\varphi \circ \eta)(a) = \int_a^x (\varphi \circ \eta)^{\Delta}(t)\Delta t \leq p\int_a^x (\sigma(t)-a)^{p-1} f^p(t)\Delta t,$$

and, therefore,

$$\left(\int_a^x f(s)\Delta s\right)^p \leq p\int_a^x (\sigma(t)-a)^{p-1} f^p(t)\Delta t,$$

which is (2.14). The proof is complete.

The next lemma will be established for two different time scales $\mathbb{T}$ and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$, which are used to prove (1.8) on time scales.

Lemma 2.3. *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $g \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$, then*

$$\int_a^b g(t)\Delta t = \int_{\nu(a)}^{\nu(b)} \left(g \circ \nu^{-1}\right)(s)\left(\nu^{-1}\right)^{\tilde{\Delta}}(s)\tilde{\Delta}s. \tag{2.17}$$

Proof. Since g is an rd-continuous function, it possesses an antiderivative G with $G^{\Delta} = g$ and

$$\begin{aligned}\int_a^b g(t)\Delta t &= \int_a^b G^{\Delta}(t)\Delta t = G(b) - G(a) \\ &= \left(G \circ \nu^{-1}\right)(\nu(b)) - \left(G \circ \nu^{-1}\right)(\nu(a)) \\ &= \int_{\nu(a)}^{\nu(b)} \left(G \circ \nu^{-1}\right)^{\tilde{\Delta}}(s)\tilde{\Delta}s.\end{aligned} \tag{2.18}$$

Applying (2.6) with $\omega = G \circ \nu^{-1}$, we observe that

$$G^{\Delta}(t) = \left(G \circ \nu^{-1}\right)^{\tilde{\Delta}}(\nu(t))\nu^{\Delta}(t).$$

Then, by using (2.7), we get

$$G^{\Delta}(t)\left(\nu^{-1}\right)^{\tilde{\Delta}}(\nu(t)) = \left(G \circ \nu^{-1}\right)^{\tilde{\Delta}}(\nu(t)). \tag{2.19}$$

Since

$$G^{\Delta}(t)\left(\nu^{-1}\right)^{\tilde{\Delta}}(\nu(t)) = g(t)\left(\nu^{-1}\right)^{\tilde{\Delta}}(\nu(t)) = \left(g \circ \nu^{-1}\right)(\nu(t))\left(\nu^{-1}\right)^{\tilde{\Delta}}(\nu(t)),$$

we have from (2.19) that

$$\left(g \circ \nu^{-1}\right)(\nu(t))\left(\nu^{-1}\right)^{\tilde{\Delta}}(\nu(t)) = \left(G \circ \nu^{-1}\right)^{\tilde{\Delta}}(\nu(t)),$$

i.e.,

$$\left(g \circ \nu^{-1}\right)(s)\left(\nu^{-1}\right)^{\tilde{\Delta}}(s) = \left(G \circ \nu^{-1}\right)^{\tilde{\Delta}}(s), \quad s = \nu(t) \in \tilde{\mathbb{T}}. \tag{2.20}$$

Substituting (2.20) into (2.18), we obtain

$$\int_a^b g(t)\Delta t = \int_{\nu(a)}^{\nu(b)} \left(G \circ \nu^{-1}\right)^{\tilde{\Delta}}(s)\tilde{\Delta}s = \int_{\nu(a)}^{\nu(b)} \left(g \circ \nu^{-1}\right)(s)\left(\nu^{-1}\right)^{\tilde{\Delta}}(s)\tilde{\Delta}s,$$

which is (2.17). The proof is complete.

3. Main results

Now we are prepared to state and prove our results. The first result we start with is the time scales version of (1.5), which gives a new reformulation of the Hardy-type inequality valid in the case $p = 1$.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale with $a, \ell \in \mathbb{T}$, $f \in C_{\text{rd}}([a, \ell]_{\mathbb{T}}, \mathbb{R}^+)$, and either $p \geq 1$ or $p < 0$. Then,*

$$\int_a^{\ell} \left(\frac{\int_a^x f(y)\Delta y}{x - a}\right)^p \frac{\Delta x}{\sigma(x) - a} \leq \int_a^{\ell} \left(1 - \frac{\sigma(x) - a}{\ell - a}\right) f^p(x) \frac{\Delta x}{\sigma(x) - a}, \tag{3.1}$$

where $f(x) > 0$ if $p < 0$.

Proof. Applying Jensen's inequality (2.4) with the convex function $F(x) = x^p$, we see that

$$\left(\frac{\int_a^x f(y)\Delta y}{x-a}\right)^p \leq \frac{\int_a^x f^p(y)\Delta y}{x-a}.$$

Then,

$$\int_a^\ell \left(\frac{\int_a^x f(y)\Delta y}{x-a}\right)^p \frac{\Delta x}{\sigma(x)-a} \leq \int_a^\ell \frac{1}{(x-a)(\sigma(x)-a)}\left(\int_a^x f^p(y)\Delta y\right)\Delta x. \tag{3.2}$$

Employing the integration by parts rule (2.2) on the right-hand side of (3.2) with

$$\varphi(x) = \int_a^x f^p(y)\Delta y \quad \text{and} \quad \eta(x) = -\int_x^\ell \frac{\Delta y}{(y-a)(\sigma(y)-a)},$$

we have

$$\begin{aligned}
&\int_a^\ell \frac{1}{(x-a)(\sigma(x)-a)}\left(\int_a^x f^p(y)\Delta y\right)\Delta x \\
&= \int_a^\ell \varphi(x)\eta^\Delta(x)\Delta x = \varphi(\ell)\eta(\ell) - \varphi(a)\eta(a) - \int_a^\ell \varphi^\Delta(x)\eta(\sigma(x))\Delta x \\
&= -\int_a^\ell \varphi^\Delta(x)\eta(\sigma(x))\Delta x \\
&= \int_a^\ell \left(\int_{\sigma(x)}^\ell \frac{\Delta y}{(y-a)(\sigma(y)-a)}\right) f^p(x)\Delta x,
\end{aligned}$$

and hence (3.2) becomes

$$\int_a^\ell \left(\frac{\int_a^x f(y)\Delta y}{x-a}\right)^p \frac{\Delta x}{\sigma(x)-a} \leq \int_a^\ell \left(\int_{\sigma(x)}^\ell \frac{\Delta y}{(y-a)(\sigma(y)-a)}\right) f^p(x)\Delta x. \tag{3.3}$$

Note that

$$h(y) = -\frac{1}{y-a} \quad \text{implies} \quad h^\Delta(y) = \frac{1}{(y-a)(\sigma(y)-a)},$$

and therefore,

$$\begin{aligned}
\int_{\sigma(x)}^\ell \frac{1}{(y-a)(\sigma(y)-a)}\Delta y &= \int_{\sigma(x)}^\ell h^\Delta(y)\Delta y = h(\ell) - h(\sigma(x)) \\
&= \frac{1}{\sigma(x)-a} - \frac{1}{\ell-a} = \frac{1}{\sigma(x)-a}\left(1 - \frac{\sigma(x)-a}{\ell-a}\right).
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) yields (3.1).

Corollary 3.1. *Let* $\mathbb{T} = \mathbb{R}$ *and* $a = 0$*. Then,* (3.1) *reduces to* (1.5)*, proved by Persson et al. [4].*

Corollary 3.2. *Let* $\mathbb{T} = \mathbb{N}_0$*,* $a, \ell \in \mathbb{N}_0$*, and* f *be a positive sequence. Then,*

$$\sum_{x=a}^{\ell-1} \frac{1}{x+1-a}\left(\frac{\sum_{y=a}^{x-1} f(y)}{x-a}\right)^p \leq \sum_{x=a}^{\ell-1} \frac{1}{x+1-a}\left(1 - \frac{x+1-a}{\ell-a}\right) f^p(x), \tag{3.5}$$

provided either $p \geq 1$ *or* $p < 0$*.*

Corollary 3.3. *Let $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, $a = q^m$, $\ell = q^n$, $m, n \in \mathbb{N}_0$, and f be a positive sequence. Then,*

$$\sum_{k=m}^{n-1}\left(\frac{\sum_{s=m}^{k-1}(q-1)q^s f(q^s)}{q^k - q^m}\right)^p \frac{q^k}{q^{k+1}-q^m} \leq \sum_{k=m}^{n-1}\left(1 - \frac{q^{k+1}-q^m}{q^n - q^m}\right) f^p(q^k)\frac{q^k}{q^{k+1}-q^m}, \tag{3.6}$$

provided either $p \geq 1$ or $p < 0$.

To illustrate our results achieved so far, we provide two examples (the cases $p < 0$ and $p \geq 1$, respectively) for each of Corollaries 3.1, 3.2, and 3.3.

Example 3.1. If $\mathbb{T} = \mathbb{R}$, $a = 0$, $\ell = 1$, $p = -1$, and $f(x) = x^{\frac{-1}{3}}$, then

$$\int_a^\ell \left(\frac{\int_a^x f(y)\Delta y}{x-a}\right)^p \frac{\Delta x}{\sigma(x)-a} = \int_0^1 \left(\frac{\int_0^x y^{\frac{-1}{3}}\mathrm{d}y}{x}\right)^{-1} \frac{\mathrm{d}x}{x} = 2$$

and

$$\int_a^\ell \left(1 - \frac{\sigma(x)-a}{\ell - a}\right) f^p(x)\frac{\Delta x}{\sigma(x)-a} = \int_0^1 (1-x)x^{\frac{1}{3}}\frac{\mathrm{d}x}{x} = \frac{9}{4}.$$

Hence, (3.1) is fulfilled.

Example 3.2. If $\mathbb{T} = \mathbb{R}$, $a = 0$, $\ell = 1$, $p = 2$, and $f(x) = x$, then

$$\int_a^\ell \left(\frac{\int_a^x f(y)\Delta y}{x-a}\right)^p \frac{\Delta x}{\sigma(x)-a} = \int_0^1 \left(\frac{\int_0^x f(y)\mathrm{d}y}{x}\right)^p \frac{\mathrm{d}x}{x} = \int_0^1 \left(\frac{\int_0^x y\mathrm{d}y}{x}\right)^2 \frac{\mathrm{d}x}{x} = \frac{1}{8}$$

and

$$\int_a^\ell \left(1 - \frac{\sigma(x)-a}{\ell-a}\right) f^p(x)\frac{\Delta x}{\sigma(x)-a} = \int_0^1 (1-x)f^p(x)\frac{\mathrm{d}x}{x} = \int_0^1 (1-x)x\mathrm{d}x = \frac{1}{6}.$$

Hence, (3.1) is fulfilled.

Example 3.3. If $\mathbb{T} = \mathbb{N}$, $a = 1$, $\ell = 7$, $p = -1$, and $f(y) = \frac{1}{y(y+1)}$, then $\sum_{y=1}^{x-1} \frac{1}{y(y+1)} = \frac{x-1}{x}$ and

$$\sum_{x=a}^{\ell-1} \frac{1}{x+1-a}\left(\frac{\sum_{y=a}^{x-1} f(y)}{x-a}\right)^p = \sum_{x=1}^{6}\frac{1}{x}\left(\frac{\sum_{y=1}^{x-1}\frac{1}{y(y+1)}}{x-1}\right)^{-1} = \sum_{x=1}^{6} 1 = 6.$$

Moreover,

$$\sum_{x=a}^{\ell-1}\frac{1}{x+1-a}\left(1 - \frac{x+1-a}{\ell-a}\right)f^p(x) = \sum_{x=1}^{6}\left[(x+1)\left(1-\frac{x}{6}\right)\right] = 8.3333.$$

Therefore, (3.5) is fulfilled.

Example 3.4. If $\mathbb{T} = \mathbb{N}_0$, $a = 0$, $\ell = 7$, $p = 2$, and $f(x) = x$, then $\sum_{y=0}^{x-1} y = \frac{1}{2}x(x-1)$ and

$$\sum_{x=a}^{\ell-1}\frac{1}{x+1-a}\left(\frac{\sum_{y=a}^{x-1} f(y)}{x-a}\right)^p = \sum_{x=0}^{6}\frac{1}{x+1}\left(\frac{\sum_{y=0}^{x-1} y}{x}\right)^2 = \sum_{x=0}^{6}\left[\frac{1}{x+1}\left(\frac{1}{2}(x-1)\right)^2\right] = 2.5925.$$

Moreover,

$$\sum_{x=a}^{\ell-1}\frac{1}{x+1-a}\left(1-\frac{x+1-a}{\ell-a}\right)f^p(x) = \sum_{x=0}^{6}\left[\frac{x^2}{x+1}\left(1-\frac{x+1}{7}\right)\right] = 3.59.$$

Therefore, (3.5) is fulfilled.

Example 3.5. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q = 2$, $m = 0$, $n = 3$, $p = -6$, and $f(x) = \frac{1}{\sqrt{x}}$, then

$$\sum_{s=m}^{k-1} q^s f(q^s) = \sum_{s=m}^{k-1} q^{s/2} = \frac{q^{k/2} - q^{m/2}}{q^{1/2} - 1},$$

so that

$$\frac{(q-1)\sum_{s=m}^{k-1} q^s f(q^s)}{q^k - q^m} = \frac{q^{1/2} + 1}{q^{k/2} + q^{m/2}}.$$

Thus, the left-hand side of (3.6) becomes

$$\begin{aligned}\sum_{k=m}^{n-1}\left(\frac{(q-1)\sum_{s=m}^{k-1} q^s f(q^s)}{q^k - q^m}\right)^p \frac{q^k}{q^{k+1} - q^m} &= \sum_{k=m}^{n-1}\left(\frac{q^{1/2}+1}{q^{k/2}+q^{m/2}}\right)^p \frac{q^k}{q^{k+1} - q^m} \\ &= \sum_{k=0}^{2}\left(\frac{2^{1/2}+1}{2^{k/2}+1}\right)^{-6} \frac{2^k}{2^{k+1} - 1} = 3.09385,\end{aligned}$$

while the right-hand side of (3.6) gives

$$\sum_{k=m}^{n-1}\left(1 - \frac{q^{k+1} - q^m}{q^n - q^m}\right) f^p(q^k) \frac{q^k}{q^{k+1} - q^m} = \sum_{k=0}^{2}\left(1 - \frac{2^{k+1} - 1}{7}\right) \frac{2^{4k}}{2^{k+1} - 1} = 3.9048,$$

which indicates that (3.6) holds for $p = -6 < 0$.

Example 3.6. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q = 2$, $m = 0$, $n = 3$, $p = 5$, and $f(x) = x$, then

$$\sum_{s=m}^{k-1} q^s f(q^s) = \sum_{s=m}^{k-1} q^{2s} = \frac{q^{2k} - q^{2m}}{q^2 - 1}$$

so that

$$\frac{(q-1)\sum_{s=m}^{k-1} q^s f(q^s)}{q^k - q^m} = \frac{q^k + q^m}{q + 1}.$$

Thus, the left-hand side of (3.6) becomes

$$\begin{aligned}\sum_{k=m}^{n-1}\left(\frac{(q-1)\sum_{s=m}^{k-1} q^s f(q^s)}{q^k - q^m}\right)^p \frac{q^k}{q^{k+1} - q^m} &= \sum_{k=m}^{n-1}\left(\frac{q^k + q^m}{q+1}\right)^p \frac{q^k}{q^{k+1} - q^m} \\ &= \sum_{k=0}^{2}\left(\frac{2^k + 1}{3}\right)^5 \frac{2^k}{2^{k+1} - 1} = 8.14697,\end{aligned}$$

and the right-hand side of (3.6) gives

$$\sum_{k=m}^{n-1}\left(1 - \frac{q^{k+1} - q^m}{q^n - q^m}\right) f^p(q^k) \frac{q^k}{q^{k+1} - q^m} = \sum_{k=0}^{2}\left(1 - \frac{2^{k+1} - 1}{7}\right) \frac{2^{6k}}{2^{k+1} - 1} = 13.0476,$$

which indicates that (3.6) holds.

By applying a similar technique as used previously, we can readily derive the following time scales version of (1.6).

Theorem 3.2. *Let* $\mathbb{T}$ *be a time scale with* $a, \ell \in \mathbb{T}$, $f \in C_{\mathrm{rd}}([a,\ell]_{\mathbb{T}}, \mathbb{R}^+)$, *and* $0 < p < 1$. *Then,*

$$\int_a^\ell \left(\frac{\int_a^x f(y)\Delta y}{x-a} \right)^p \frac{\Delta x}{\sigma(x)-a} \geq \int_a^\ell \left(1 - \frac{\sigma(x)-a}{\ell - a}\right) f^p(x) \frac{\Delta x}{\sigma(x)-a}. \tag{3.7}$$

Corollary 3.4. *Let* $\mathbb{T} = \mathbb{R}$ *and* $a = 0$. *Then,* (3.7) *reduces to* (1.6), *proved by Persson et al. [4].*

Remark 3.1. *Inequality* (3.7) *represents the reversed version of* (3.1) *when* $0 < p < 1$.

Corollary 3.5. *If* $\mathbb{T} = \mathbb{N}_0$, $a, \ell \in \mathbb{N}_0$, $0 < p < 1$, *and* f *is a positive sequence, then*

$$\sum_{x=a}^{\ell-1} \frac{1}{x+1-a} \left(\frac{\sum_{y=a}^{x-1} f(y)}{x-a} \right)^p \geq \sum_{x=a}^{\ell-1} \frac{1}{x+1-a} \left(1 - \frac{x+1-a}{\ell-a}\right) f^p(x). \tag{3.8}$$

Corollary 3.6. *If* $\mathbb{T} = q^{\mathbb{N}_0}$ *for* $q > 1$, $a = q^m$, $\ell = q^n$, $m, n \in \mathbb{N}_0$, $0 < p < 1$, *and* f *is a positive sequence, then*

$$\sum_{k=m}^{n-1} \left(\frac{\sum_{s=m}^{k-1} (q-1) q^s f(q^s)}{q^k - q^m} \right)^p \frac{q^k}{q^{k+1} - q^m} \geq \sum_{k=m}^{n-1} \left(1 - \frac{q^{k+1} - q^m}{q^n - q^m}\right) f^p(q^k) \frac{q^k}{q^{k+1} - q^m}. \tag{3.9}$$

Example 3.7. If $\mathbb{T} = \mathbb{R}$, $a = 0$, $\ell = 1$, $p = \frac{1}{2}$, and $f(x) = x^2$, then

$$\int_a^\ell \left(\frac{\int_a^x f(y)\Delta y}{x-a} \right)^p \frac{\Delta x}{\sigma(x)-a} = \int_0^1 \left(\frac{\int_0^x y^2 \mathrm{d}y}{x} \right)^{\frac{1}{2}} \frac{\mathrm{d}x}{x} = 0.57735026$$

and

$$\int_a^\ell \left(1 - \frac{\sigma(x)-a}{\ell-a}\right) f^p(x) \frac{\Delta x}{\sigma(x)-a} = \int_0^1 (1-x)\mathrm{d}x = 0.50.$$

So, (3.7) is fulfilled.

Example 3.8. If $\mathbb{T} = \mathbb{N}$, $a = 1$, $\ell = 7$, $p = \frac{1}{3}$, and $f(x) = 1$, then

$$\sum_{x=a}^{\ell-1} \frac{1}{x+1-a} \left(\frac{\sum_{y=a}^{x-1} f(y)}{x-a} \right)^p = \sum_{x=1}^{6} \frac{1}{x} = 2.45.$$

Moreover,

$$\sum_{x=a}^{\ell-1} \frac{1}{x+1-a} \left(1 - \frac{x+1-a}{\ell-a}\right) f^p(x) = \sum_{x=1}^{6} \frac{1}{x} \left(1 - \frac{x}{6}\right) = 1.45.$$

Therefore, (3.8) is fulfilled.

Example 3.9. If $\mathbb{T} = 2^{\mathbb{N}_0}$ where $q = 2$, $m = 0$, $n = 3$, $p = \frac{1}{2}$, and $f(x) = 4x$, then

$$\sum_{s=m}^{k-1} q^s f(q^s) = 4 \sum_{s=m}^{k-1} q^{2s} = \frac{4(q^{2k} - q^{2m})}{q^2 - 1}$$

so that

$$\frac{(q-1)\sum_{s=m}^{k-1} q^s f(q^s)}{q^k - q^m} = \frac{4(q^k + q^m)}{q+1}.$$

Thus, the left-hand side of (3.9) gives

$$\begin{aligned}\sum_{k=m}^{n-1} \left(\frac{(q-1)\sum_{s=m}^{k-1} q^s f(q^s)}{q^k - q^m} \right)^p \frac{q^k}{q^{k+1} - q^m} &= \sum_{k=m}^{n-1} \frac{q^k}{q^{k+1} - q^m} \sqrt{\frac{4(q^k + q^m)}{q+1}} \\ &= \sum_{k=0}^{2} \frac{2^k}{2^{k+1} - 1} \sqrt{\frac{4(2^k + 1)}{3}} = 4.44175,\end{aligned}$$

and the right-hand side of (3.9) becomes

$$\sum_{k=m}^{n-1} \left(1 - \frac{q^{k+1} - q^m}{q^n - q^m}\right) f^p(q^k) \frac{q^k}{q^{k+1} - q^m} = \sum_{k=0}^{2} \left(1 - \frac{2^{k+1} - 1}{7}\right) \frac{2^{1+\frac{3k}{2}}}{2^{k+1} - 1} = 2.7918,$$

which implies that (3.9) is satisfied.

In the following, we establish the time scales version of (1.7).

Theorem 3.3. *Assume* $a, \ell \in \mathbb{T}$*,* $p \geq 1$*,* $\alpha > 0$*, and* $f \in C_{\rm rd}([a, \ell]_{\mathbb{T}}, \mathbb{R}^+)$ *is nonincreasing. Then,*

$$\int_a^\ell \left(\int_a^x f(y)\Delta y \right)^p (x-a)^{-\alpha-1} \Delta x \geq \frac{p}{\alpha} \int_a^\ell \frac{(x-a)^{p-1}}{(\sigma(x)-a)^\alpha} f^p(x) \left[1 - \left(\frac{\sigma(x)-a}{\ell - a} \right)^\alpha \right] \Delta x. \tag{3.10}$$

Proof. Applying (2.8) by replacing b with x, we see that

$$\left(\int_a^x f(y)\Delta y \right)^p \geq p \int_a^x (y-a)^{p-1} f^p(y) \Delta y.$$

Therefore,

$$\int_a^\ell \left(\int_a^x f(y)\Delta y \right)^p (x-a)^{-\alpha-1} \Delta x \geq p \int_a^\ell \left(\int_a^x (y-a)^{p-1} f^p(y) \Delta y \right) (x-a)^{-\alpha-1} \Delta x. \tag{3.11}$$

Employing the integration by parts formula (2.2) on the right-hand side of (3.11) with

$$\varphi(x) = \int_a^x (y-a)^{p-1} f^p(y)\Delta y \quad \text{and} \quad \eta(x) = -\int_x^\ell (t-a)^{-\alpha-1} \Delta t,$$

we get

$$\begin{aligned}&\int_a^\ell \left(\int_a^x (y-a)^{p-1} f^p(y)\Delta y \right) (x-a)^{-\alpha-1} \Delta x \\ &= \int_a^\ell \varphi(x) \eta^\Delta(x) \Delta x \\ &= \varphi(\ell)\eta(\ell) - \varphi(a)\eta(a) - \int_a^\ell \varphi^\Delta(x) \eta(\sigma(x)) \Delta x\end{aligned}$$

$$= -\int_a^\ell \varphi^\Delta(x)\eta(\sigma(x))\Delta x$$
$$= \int_a^\ell (x-a)^{p-1} f^p(x)\left(\int_{\sigma(x)}^\ell (t-a)^{-\alpha-1}\Delta t\right)\Delta x,$$

and substituting this into (3.11), we see that

$$\int_a^\ell \left(\int_a^x f(y)\Delta y\right)^p (x-a)^{-\alpha-1}\Delta x \geq p\int_a^\ell (x-a)^{p-1} f^p(x)\left(\int_{\sigma(x)}^\ell (t-a)^{-\alpha-1}\Delta t\right)\Delta x. \tag{3.12}$$

Employing the chain rule (2.1) with $\eta(t) = t - a$ and $\varphi(x) = x^{-\alpha}$, we get

$$(\varphi\circ\eta)^\Delta(t) = \varphi'(\eta(c))\eta^\Delta(t) = -\alpha(c-a)^{-\alpha-1} \quad \text{for some} \quad c \in [t, \sigma(t)].$$

For $c \geq t$ and $\alpha > 0$, we observe that

$$(\varphi\circ\eta)^\Delta(t) \geq -\alpha(t-a)^{-\alpha-1},$$

and then

$$\int_{\sigma(x)}^\ell (t-a)^{-\alpha-1}\Delta t \geq \frac{-1}{\alpha}\int_{\sigma(x)}^\ell (\varphi\circ\eta)^\Delta(t)\Delta t = \frac{1}{\alpha}\left[(\sigma(x)-a)^{-\alpha} - (\ell-a)^{-\alpha}\right]. \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain

$$\int_a^\ell \left(\int_a^x f(y)\Delta y\right)^p (x-a)^{-\alpha-1}\Delta x$$
$$\geq \frac{p}{\alpha}\int_a^\ell (x-a)^{p-1} f^p(x)\left[(\sigma(x)-a)^{-\alpha} - (\ell-a)^{-\alpha}\right]\Delta x$$
$$= \frac{p}{\alpha}\int_a^\ell \frac{(x-a)^{p-1}}{(\sigma(x)-a)^\alpha} f^p(x)\left[1 - \left(\frac{\sigma(x)-a}{\ell-a}\right)^\alpha\right]\Delta x,$$

which is (3.10). The proof is complete.

Corollary 3.7. *Let $\mathbb{T} = \mathbb{R}$ and $a = 0$. Then,* (3.10) *reduces to* (1.7)*, proved by Persson et al. [4].*

Corollary 3.8. *Let $\mathbb{T} = \mathbb{N}_0$, $a, \ell \in \mathbb{N}_0$, $p \geq 1$, $\alpha > 0$, and f be a nonnegative nonincreasing sequence. Then,*

$$\sum_{x=a}^{\ell-1}\left(\sum_{y=a}^{x-1} f(y)\right)^p (x-a)^{-\alpha-1} \geq \frac{p}{\alpha}\sum_{x=a}^{\ell-1}\frac{(x-a)^{p-1}}{(x+1-a)^\alpha} f^p(x)\left[1 - \left(\frac{x+1-a}{\ell-a}\right)^\alpha\right]. \tag{3.14}$$

Corollary 3.9. *Let $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, $a = q^m$, $\ell = q^n$, $m, n \in \mathbb{N}_0$, $p \geq 1$, $\alpha > 0$, and f be a nonnegative nonincreasing sequence. Then,*

$$\sum_{k=m}^{n-1}\left(\sum_{s=m}^{k-1}(q-1)q^s f(q^s)\right)^p q^k(q^k - q^m)^{-\alpha-1} \geq \frac{p}{\alpha}\sum_{k=m}^{n-1}\frac{(q^k-q^m)^{p-1}}{(q^{k+1}-q^m)^\alpha}\left(1 - \left[\frac{q^{k+1}-q^m}{q^n-q^m}\right]^\alpha\right) q^k f^p(q^k). \tag{3.15}$$

Example 3.10. If $\mathbb{T} = \mathbb{R}$, $a = 0$, $\ell = 1$, $p = 2$, $\alpha = 1$, and $f(x) = 2 - x$, then

$$\int_a^\ell \left(\int_a^x f(y)\Delta y\right)^p (x-a)^{-\alpha-1}\Delta x = \int_0^1 \left(\int_0^x (2-y)\mathrm{d}y\right)^2 x^{-2}\mathrm{d}x = \frac{37}{12}$$

and

$$\frac{p}{\alpha}\int_a^\ell \frac{(x-a)^{p-1}}{(\sigma(x)-a)^\alpha} f^p(x)\left[1-\left(\frac{\sigma(x)-a}{\ell-a}\right)^\alpha\right]\Delta x = 2\int_0^1 (2-x)^2(1-x)\mathrm{d}x = \frac{17}{6}.$$

Thus, (3.10) is achieved.

Example 3.11. If $\mathbb{T} = \mathbb{N}_0$, $a = 0$, $\ell = 6$, $p = 2$, $\alpha = 1$, and $f(x) = 1$, then

$$\sum_{x=a}^{\ell-1}\left(\sum_{y=a}^{x-1} f(y)\right)^p (x-a)^{-\alpha-1} = \sum_{x=0}^{5}\left(\sum_{y=0}^{x-1} 1\right)^2 x^{-2} = 6$$

and

$$\frac{p}{\alpha}\sum_{x=a}^{\ell-1}\frac{(x-a)^{p-1}}{(x+1-a)^\alpha} f^p(x)\left[1-\left(\frac{x+1-a}{\ell-a}\right)^\alpha\right] = 2\sum_{x=0}^{5}\frac{x}{x+1}\left(1-\frac{x+1}{6}\right) = 2.1.$$

So, (3.14) holds.

Example 3.12. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q = 2$, $m = 0$, $n = 3$, $p = 2$, $\alpha = 1$, and $f(x) = 1$, then

$$\sum_{s=m}^{k-1} q^s f(q^s) = \sum_{s=m}^{k-1} q^s = \frac{q^k - q^m}{q-1}.$$

Thus, the left-hand side of (3.15) gives

$$\sum_{k=m}^{n-1}\left((q-1)\sum_{s=m}^{k-1} q^s f(q^s)\right)^p q^k (q^k - q^m)^{-\alpha-1} = \sum_{k=0}^{2} 2^k = 7,$$

and the right-hand side of (3.15) gives

$$\frac{p}{\alpha}\sum_{k=m}^{n-1}\frac{(q^k-q^m)^{p-1}}{(q^{k+1}-q^m)^\alpha}\left(1-\left[\frac{q^{k+1}-q^m}{q^n-q^m}\right]^\alpha\right) q^k f^p(q^k) = 2\sum_{k=0}^{2}\frac{2^k(2^k-1)}{2^{k+1}-1}\left(1-\frac{2^{k+1}-1}{7}\right) = 0.7619.$$

Therefore, (3.15) holds.

The following theorem is established for a nondecreasing function f.

Theorem 3.4. *Let $a, \ell \in \mathbb{T}$, $p \geq 1$, $\alpha > 0$, and $f \in C_{\mathrm{rd}}([a, \ell]_{\mathbb{T}}, \mathbb{R}^+)$ be a nondecreasing function. Then,*

$$\begin{aligned} &\int_a^\ell (\sigma(x)-a)^{-\alpha}\left(\int_a^x f(s)\Delta s\right)^p \frac{\Delta x}{\sigma(x)-a} \\ &\leq \frac{p}{\alpha}\int_a^\ell (\sigma(x)-a)^{p-\alpha} f^p(x)\left[1-\left(\frac{\sigma(x)-a}{\ell-a}\right)^\alpha\right]\frac{\Delta x}{\sigma(x)-a}. \end{aligned} \tag{3.16}$$

Proof. Applying (2.14), since f is nondecreasing, we have

$$\begin{aligned}
&\int_a^\ell (\sigma(x)-a)^{-\alpha}\left(\int_a^x f(s)\Delta s\right)^p \frac{\Delta x}{\sigma(x)-a} \\
&= \int_a^\ell \left(\int_a^x f(s)\Delta s\right)^p (\sigma(x)-a)^{-\alpha-1}\Delta x \qquad (3.17)\\
&\le p\int_a^\ell (\sigma(x)-a)^{-\alpha-1}\left(\int_a^x (\sigma(t)-a)^{p-1} f^p(t)\Delta t\right)\Delta x.
\end{aligned}$$

Applying the integration by parts formula (2.2) on the right-hand side of (3.17) with

$$\varphi(x) = \int_a^x (\sigma(t)-a)^{p-1} f^p(t)\Delta t \quad \text{and} \quad \eta(x) = -\int_x^\ell (\sigma(t)-a)^{-\alpha-1}\Delta t,$$

we get

$$\begin{aligned}
&\int_a^\ell (\sigma(x)-a)^{-\alpha-1}\left(\int_a^x (\sigma(t)-a)^{p-1} f^p(t)\Delta t\right)\Delta x \\
&= \int_a^\ell \varphi(x)\eta^\Delta(x)\Delta x = \varphi(\ell)\eta(\ell) - \varphi(a)\eta(a) - \int_a^\ell \varphi^\Delta(x)\eta^\sigma(x)\Delta x \\
&= -\int_a^\ell \varphi^\Delta(x)\eta^\sigma(x)\Delta x = -\int_a^\ell (\sigma(x)-a)^{p-1} f^p(x)\eta^\sigma(x)\Delta x \\
&= \int_a^\ell (\sigma(x)-a)^{p-1} f^p(x)\left(\int_{\sigma(x)}^\ell (\sigma(t)-a)^{-\alpha-1}\Delta t\right)\Delta x,
\end{aligned}$$

and substituting this into (3.17), we observe that

$$\begin{aligned}
&\int_a^\ell \left(\int_a^x f(s)\Delta s\right)^p (\sigma(x)-a)^{-\alpha-1}\Delta x \\
&\le p\int_a^\ell (\sigma(x)-a)^{p-1} f^p(x)\left(\int_{\sigma(x)}^\ell (\sigma(t)-a)^{-\alpha-1}\Delta t\right)\Delta x.
\end{aligned} \qquad (3.18)$$

Applying the chain rule formula (2.1) with $\eta(t) = t-a$ and $\varphi(x) = x^{-\alpha}$, we get

$$(\varphi\circ\eta)^\Delta(t) = \varphi'(\eta(c))\eta^\Delta(t) = -\alpha(c-a)^{-\alpha-1} \quad \text{for some} \quad c\in[t,\sigma(t)].$$

Since $c \le \sigma(t)$ and $\alpha > 0$, we obtain

$$\frac{-1}{\alpha}(\varphi\circ\eta)^\Delta(t) \ge (\sigma(t)-a)^{-\alpha-1},$$

and so

$$\begin{aligned}
\int_{\sigma(x)}^\ell (\sigma(t)-a)^{-\alpha-1}\Delta t &\le \frac{-1}{\alpha}\int_{\sigma(x)}^\ell (\varphi\circ\eta)^\Delta(t)\Delta t \\
&= \frac{-1}{\alpha}[\varphi(\eta(\ell)) - \varphi(\eta(\sigma(x)))] = \frac{1}{\alpha}\left[(\sigma(x)-a)^{-\alpha} - (\ell-a)^{-\alpha}\right].
\end{aligned} \qquad (3.19)$$

Substituting (3.19) into (3.18), we obtain

$$\begin{aligned}
&\int_a^\ell \left(\int_a^x f(s)\Delta s\right)^p (\sigma(x)-a)^{-\alpha-1}\Delta x\\
&\leq \frac{p}{\alpha}\int_a^\ell (\sigma(x)-a)^{p-1} f^p(x)\left[(\sigma(x)-a)^{-\alpha}-(\ell-a)^{-\alpha}\right]\Delta x\\
&= \frac{p}{\alpha}\int_a^\ell (\sigma(x)-a)^{p-\alpha} f^p(x)\left[1-\left(\frac{\sigma(x)-a}{\ell-a}\right)^{\alpha}\right]\frac{\Delta x}{\sigma(x)-a},
\end{aligned}$$

which is (3.16). The proof is complete.

Remark 3.2. *Using the nondecreasing function f along with $\sigma(x)$ in* (3.16) *shows that* (3.16) *is a reversed version of* (3.10).

In the following, we establish a time scales version of (1.9).

Theorem 3.5. *Let $\mathbb{T}$ be a time scale with $\ell \in \mathbb{T}$, $\ell \geq 0$, $p \geq 1$, $\alpha > 0$, and let $f \in C_{\mathrm{rd}}((\ell,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ be nonincreasing. If $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is also a time scale with $\nu(x) = \frac{x}{y}$, $x, y \in \mathbb{T}$, and $y > \ell$, then*

$$\int_\ell^\infty \left(\int_x^\infty f(y)\Delta y\right)^p x^{\alpha-1}\Delta x \geq \frac{p}{\alpha}\int_\ell^\infty y^{p+\alpha-1} f^p(y) T(y)\Delta y, \tag{3.20}$$

where

$$T(y) = \alpha\beta_{\frac{\ell}{y}}(\alpha,p) \quad \textit{with} \quad \beta_\lambda(u,v) := \int_\lambda^1 s^{u-1}(1-s)^{v-1}\tilde{\Delta}s \quad \textit{for} \quad 0 \leq \lambda < 1.$$

Proof. Applying (2.9) by replacing a with x, we get

$$\left(\int_x^\infty f(y)\Delta y\right)^p \geq p\int_x^\infty (y-x)^{p-1} f^p(y)\Delta y.$$

Then,

$$\begin{aligned}
\int_\ell^\infty \left(\int_x^\infty f(y)\Delta y\right)^p x^{\alpha-1}\Delta x &\geq p\int_\ell^\infty \left(\int_x^\infty (y-x)^{p-1} f^p(y)\Delta y\right) x^{\alpha-1}\Delta x\\
&= p\int_\ell^\infty \int_x^\infty x^{\alpha-1}(y-x)^{p-1} f^p(y)\Delta y\Delta x.
\end{aligned} \tag{3.21}$$

Applying Fubini's theorem (2.5) on the right-hand side of (3.21), we obtain

$$\begin{aligned}
\int_\ell^\infty \left(\int_x^\infty f(y)\Delta y\right)^p x^{\alpha-1}\Delta x &\geq p\int_\ell^\infty \int_\ell^{\sigma(y)} x^{\alpha-1}(y-x)^{p-1} f^p(y)\Delta x\Delta y\\
&= p\int_\ell^\infty f^p(y)\left(\int_\ell^{\sigma(y)} x^{\alpha-1}(y-x)^{p-1}\Delta x\right)\Delta y\\
&\geq p\int_\ell^\infty f^p(y)\left(\int_\ell^{y} x^{\alpha-1}(y-x)^{p-1}\Delta x\right)\Delta y.
\end{aligned} \tag{3.22}$$

Applying (2.17) with $\nu(x) = \frac{x}{y}$, $\nu^\Delta(x) > 0$, and $g(x) = x^{\alpha-1}(y-x)^{p-1}$, we observe that

$$\int_\ell^y x^{\alpha-1}(y-x)^{p-1}\Delta x = \int_{\frac{\ell}{y}}^1 y^{\alpha+p-1} s^{\alpha-1}(1-s)^{p-1}\tilde{\Delta}s. \tag{3.23}$$

Substituting (3.23) into (3.22), we get

$$\int_{\ell}^{\infty}\left(\int_{x}^{\infty} f(y)\Delta y\right)^{p} x^{\alpha-1}\Delta x \geq \frac{p}{\alpha}\int_{\ell}^{\infty} y^{p+\alpha-1} f^{p}(y)\left(\alpha\int_{\frac{\ell}{y}}^{1} s^{\alpha-1}(1-s)^{p-1}\tilde{\Delta} s\right)\Delta y,$$

which is (3.20). The proof is complete.

Corollary 3.10. *If $\mathbb{T}=\mathbb{R}$, then* (3.20) *reduces to* (1.9)*, proved by Persson et al. [4].*

Corollary 3.11. *If $\ell=0$, then $\beta_{\frac{\ell}{y}}(\alpha,p)=\beta_0(\alpha,p)=\beta(\alpha,p)$, where $\beta(u,v)=\int_0^1 s^{u-1}(1-s)^{v-1}\tilde{\Delta}s$, and*

$$\int_{0}^{\infty}\left(\int_{x}^{\infty} f(y)\Delta y\right)^{p} x^{\alpha-1}\Delta x \geq p\beta(\alpha,p)\int_{0}^{\infty} y^{p+\alpha-1} f^{p}(y)\Delta y. \tag{3.24}$$

Corollary 3.12. *Let $\mathbb{T}=\mathbb{N}$ with $\ell\in\mathbb{N}$, $p\geq 1$, $\alpha>0$, and f be a nonnegative nonincreasing sequence. If $\tilde{\mathbb{T}}=\nu(\mathbb{T})$ such that $\nu(x)=\frac{x}{y}$, $x,y\in\mathbb{T}$, and $y>\ell$, then*

$$\sum_{x=\ell}^{\infty}\left(\sum_{y=x}^{\infty} f(y)\right)^{p} x^{\alpha-1} \geq \frac{p}{\alpha}\sum_{y=\ell}^{\infty} y^{p+\alpha-1} f^{p}(y) T(y), \tag{3.25}$$

where

$$T(y)=\alpha\beta_{\frac{\ell}{y}}(\alpha,p)=\frac{\alpha}{y}\sum_{x=\ell}^{y-1}\left(\frac{x}{y}\right)^{\alpha-1}\left(1-\frac{x}{y}\right)^{p-1}.$$

Corollary 3.13. *Let $\mathbb{T}=q^{\mathbb{N}_0}$ for $q>1$ with $\ell=q^m$, $p\geq 1$, $\alpha>0$, and f be a nonnegative nonincreasing sequence. If $\tilde{\mathbb{T}}=\nu(\mathbb{T})$ is also a time scale with $\nu(x)=\frac{x}{y}$, $x,y\in\mathbb{T}$, and $y>\ell$, then*

$$\sum_{k=m}^{\infty} q^{k\alpha}\left(\sum_{s=k}^{\infty}(q-1)q^{s} f(q^{s})\right)^{p} \geq p\sum_{k=m}^{\infty} q^{k(p+\alpha)} f^{p}(q^{k})\left(\sum_{s=m}^{k-1}(q-1)q^{\alpha(s-k)}[1-q^{s-k}]^{p-1}\right). \tag{3.26}$$

Example 3.13. If $\mathbb{T}=\mathbb{R}$ with $\ell=0$, $p=2$, $\alpha=1$, and $f(y)=\frac{1}{(y+1)^2}$, then the left-hand side of (3.24) becomes

$$\int_{0}^{\infty}\left(\int_{x}^{\infty} f(y)\mathrm{d}y\right)^{p} x^{\alpha-1}\mathrm{d}x=\int_{0}^{\infty}\left(\int_{x}^{\infty}\frac{1}{(y+1)^2}\mathrm{d}y\right)^{2}\mathrm{d}x=1,$$

and the right-hand side of (3.24) becomes

$$p\left(\int_{0}^{1} s^{\alpha-1}(1-s)^{p-1}\mathrm{d}s\right)\int_{0}^{\infty} y^{p+\alpha-1} f^{p}(y)\mathrm{d}y=2\left(\int_{0}^{1}(1-s)\mathrm{d}s\right)\int_{0}^{\infty}\frac{y^2}{(y+1)^4}\mathrm{d}y=\frac{1}{3}.$$

So, (3.24) is satisfied.

Example 3.14. Let $\mathbb{T}=\mathbb{N}$ with $\ell=1$, $p=1$, $\alpha=1$, and $f(y)=\frac{1}{y^3}$. If $\tilde{\mathbb{T}}=\nu(\mathbb{T})$ such that $\nu(x)=\frac{x}{y}$, $x,y\in\mathbb{T}$, and $y>\ell$, then the left-hand side of (3.25) becomes

$$\sum_{x=\ell}^{\infty}\left(\sum_{y=x}^{\infty} f(y)\right)^{p} x^{\alpha-1}=\sum_{x=1}^{\infty}\left(\sum_{y=x}^{\infty}\frac{1}{y^3}\right)=\frac{\pi^2}{6}=1.644934,$$

and the right-hand side of (3.25) gives

$$\frac{p}{\alpha}\sum_{y=\ell}^{\infty} y^{p+\alpha-1} f^p(y)\left(\frac{\alpha}{y}\sum_{x=\ell}^{y-1}\left(\frac{x}{y}\right)^{\alpha-1}\left(1-\frac{x}{y}\right)^{p-1}\right)=\sum_{y=1}^{\infty}\frac{1}{y^2}\left(1-\frac{1}{y}\right)=0.442877.$$

Thus, (3.25) holds.

Example 3.15. Assume $\mathbb{T}=2^{\mathbb{N}_0}$ with $q=2$, $m=0$, $p=2$, $\alpha=1$, and $f(x)=\frac{1}{x^2}$. If $\tilde{\mathbb{T}}=\nu(\mathbb{T})$ is also a time scale with $\nu(x)=\frac{x}{y}$, $x,y\in\mathbb{T}$, and $y>\ell$, then

$$\sum_{s=k}^{\infty}(q-1)q^s f(q^s)=(q-1)\sum_{s=k}^{\infty}\left(\frac{1}{q}\right)^s=q^{1-k},$$

the left-hand side of (3.26) becomes

$$\sum_{k=m}^{\infty} q^{k\alpha}\left(\sum_{s=k}^{\infty}(q-1)q^s f(q^s)\right)^p=\sum_{k=0}^{\infty} q^{k\alpha+p(1-k)}=\sum_{k=0}^{\infty}2^{-k+2}=8,$$

and the right-hand side of (3.26) becomes

$$p\sum_{k=m}^{\infty} q^{k(p+\alpha)} f^p(q^k)\left(\sum_{s=m}^{k-1}(q-1)q^{\alpha(s-k)}[1-q^{s-k}]^{p-1}\right)=2\sum_{k=0}^{\infty}2^{-k}\left(\sum_{s=0}^{k-1}2^{s-k}[1-2^{s-k}]\right)=0.7619.$$

This shows that (3.26) holds.

In the following, we establish the time scales version of (1.8) by applying Fubini's theorem and the integration by substitution from a time scale $\mathbb{T}$ to another one $\tilde{\mathbb{T}}=\nu(\mathbb{T})$.

Theorem 3.6. *Let $\mathbb{T}$ be a time scale with $\ell\in\mathbb{T}$, $\ell>0$, $p\geq 1$, and $\alpha>0$, and let $f\in C_{\mathrm{rd}}((0,\ell]_{\mathbb{T}},\mathbb{R}^+)$ be nondecreasing. If $\tilde{\mathbb{T}}=\nu(\mathbb{T})$ is also a time scale with $\nu(x)=\frac{\sigma(y)}{x}$ for $x,y\in\mathbb{T}$, then*

$$\int_0^{\ell}\left(\int_0^x f(y)\Delta y\right)^p x^{-\alpha-1}\Delta x \geq \frac{p}{\alpha}\int_0^{\ell}(\sigma(y))^{p-\alpha-1} f^p(y)T(y)\Delta y, \tag{3.27}$$

where

$$T(y)=\alpha\int_{\frac{\sigma(y)}{\ell}}^{1}(1-s)^{p-1}\frac{s^{1-p+\alpha}}{\tilde{\sigma}(s)}\tilde{\Delta}s.$$

Proof. Applying (2.10) with $a=0$ and $b=x$, we get

$$\left(\int_0^x f(y)\Delta y\right)^p \geq p\int_0^x (x-\sigma(y))^{p-1} f^p(y)\Delta y.$$

Consequently,

$$\int_0^{\ell}\left(\int_0^x f(y)\Delta y\right)^p x^{-\alpha-1}\Delta x \geq p\int_0^{\ell}\left(\int_0^x (x-\sigma(y))^{p-1} f^p(y)\Delta y\right)x^{-\alpha-1}\Delta x. \tag{3.28}$$

Applying Fubini's theorem (2.5) on the right-hand side of (3.28), we obtain

$$\begin{aligned}\int_0^\ell \left(\int_0^x (x-\sigma(y))^{p-1} f^p(y)\Delta y\right) x^{-\alpha-1}\Delta x &= \int_0^\ell \int_0^x (x-\sigma(y))^{p-1} f^p(y) x^{-\alpha-1}\Delta y\Delta x \\ &= \int_0^\ell \int_{\sigma(y)}^\ell (x-\sigma(y))^{p-1} f^p(y) x^{-\alpha-1}\Delta x\Delta y \\ &= \int_0^\ell f^p(y)\left(\int_{\sigma(y)}^\ell (x-\sigma(y))^{p-1} x^{-\alpha-1}\Delta x\right)\Delta y.\end{aligned}$$

Hence, (3.28) becomes

$$\int_0^\ell \left(\int_0^x f(y)\Delta y\right)^p x^{-\alpha-1}\Delta x \geq p\int_0^\ell f^p(y)\left(\int_{\sigma(y)}^\ell (x-\sigma(y))^{p-1} x^{-\alpha-1}\Delta x\right)\Delta y. \tag{3.29}$$

Applying (2.17) on the term

$$\int_{\sigma(y)}^\ell (x-\sigma(y))^{p-1} x^{-\alpha-1}\Delta x$$

with $\nu(x) = \frac{\sigma(y)}{x}$, $\nu^\Delta(x) < 0$, and $g(x) = (x-\sigma(y))^{p-1}x^{-\alpha-1}$, we see that

$$\begin{aligned}\int_{\sigma(y)}^\ell (x-\sigma(y))^{p-1} x^{-\alpha-1}\Delta x &= \int_{\nu(\ell)}^{\nu(\sigma(y))} \left(\frac{\sigma(y)}{s}-\sigma(y)\right)^{p-1}\left(\frac{\sigma(y)}{s}\right)^{-\alpha-1}\left(\frac{\sigma(y)}{s\tilde{\sigma}(s)}\right)\tilde{\Delta}s \\ &= \int_{\nu(\ell)}^{\nu(\sigma(y))} (\sigma(y))^{p-\alpha-1}\left(\frac{1}{s}-1\right)^{p-1}\left(\frac{1}{s}\right)^{-\alpha-1}\left(\frac{1}{s\tilde{\sigma}(s)}\right)\tilde{\Delta}s \\ &= \int_{\frac{\sigma(y)}{\ell}}^{1} (\sigma(y))^{p-\alpha-1}\left(\frac{1}{s}-1\right)^{p-1}\left(\frac{1}{s}\right)^{-\alpha-1}\left(\frac{1}{s\tilde{\sigma}(s)}\right)\tilde{\Delta}s \\ &= \int_{\frac{\sigma(y)}{\ell}}^{1} (\sigma(y))^{p-\alpha-1}(1-s)^{p-1}\frac{s^{1-p+\alpha}}{\tilde{\sigma}(s)}\tilde{\Delta}s.\end{aligned}$$

Then, (3.29) becomes

$$\begin{aligned}\int_0^\ell \left(\int_0^x f(y)\Delta y\right)^p x^{-\alpha-1}\Delta x &\geq p\int_0^\ell f^p(y)\left(\int_{\frac{\sigma(y)}{\ell}}^{1} (\sigma(y))^{p-\alpha-1}(1-s)^{p-1}\frac{s^{1-p+\alpha}}{\tilde{\sigma}(s)}\tilde{\Delta}s\right)\Delta y \\ &= p\int_0^\ell (\sigma(y))^{p-\alpha-1} f^p(y)\left(\int_{\frac{\sigma(y)}{\ell}}^{1} (1-s)^{p-1}\frac{s^{1-p+\alpha}}{\tilde{\sigma}(s)}\tilde{\Delta}s\right)\Delta y \\ &= \frac{p}{\alpha}\int_0^\ell (\sigma(y))^{p-\alpha-1} f^p(y) T(y)\Delta y,\end{aligned}$$

which is (3.27). The proof is complete.

Remark 3.3. *If $\mathbb{T} = \mathbb{R}$, then* (3.27) *reduces to* (1.8)*, proved by Persson et al. [4].*

Corollary 3.14. *Let $\mathbb{T} = \mathbb{N}$ with $\ell \in \mathbb{N}$, $p \geq 1$, and $\alpha > 0$, and f be a nonnegative, nondecreasing sequence on $(0, \ell]_{\mathbb{T}}$. If $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is also a time scale with $\nu(x) = \frac{\sigma(y)}{x}$ for $x, y \in \mathbb{T}$, then*

$$\sum_{x=1}^{\ell-1}\left(\sum_{y=1}^{x-1} f(y)\right)^p x^{-\alpha-1} \geq p\sum_{y=1}^{\ell-1}(y+1)^{p-\alpha-1} f^p(y)\left(\sum_{x=y+1}^{\ell}\frac{x-1}{x(x+1)}\left(1-\frac{y+1}{x}\right)^{p-1}\left(\frac{y+1}{x}\right)^{1-p+\alpha}\right). \tag{3.30}$$

Corollary 3.15. *Let $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, $m, n \in \mathbb{N}_0$, $p \geq 1$, $\alpha > 0$, and let $f \in C_{\rm rd}((0, \ell]_{\mathbb{T}}, \mathbb{R}^+)$ be nondecreasing. If $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is also a time scale with $\nu(x) = \frac{\sigma(y)}{x}$ for $x, y \in \mathbb{T}$, then*

$$\begin{aligned} &\sum_{k=m}^{n-1} q^{-\alpha k}\left(\sum_{s=m}^{k-1}(q-1)q^s f(q^s)\right)^p \\ &\geq p\sum_{k=m}^{n-1} q^k q^{(k+1)(p-\alpha-1)} f^p(q^k)\left(\sum_{s=k+1}^{n-1}\frac{(q-1)(q^s-1)}{q^{s+1}}(1-q^{k+1-s})^{p-1} q^{(k+1-s)(1-p+\alpha)}\right). \end{aligned} \tag{3.31}$$

Example 3.16. Let $\mathbb{T} = \mathbb{R}$ with $\ell = 1$, $p = 2$, $\alpha = 1$, and $f(x) = x$. In this case, $\tilde{\mathbb{T}} = \mathbb{R}$, the left-hand side of (3.27) gives

$$\int_0^\ell\left(\int_0^x f(y)\mathrm{d}y\right)^p x^{-\alpha-1}\mathrm{d}x = \int_0^1\left(\int_0^x y\mathrm{d}y\right)^2 x^{-2}\mathrm{d}x = \frac{1}{12},$$

and the right-hand side of (3.27) becomes

$$p\int_0^\ell y^{p-\alpha-1} f^p(y)\left(\int_{\frac{y}{\ell}}^1 (1-s)^{p-1}\frac{s^{1-p+\alpha}}{s}\mathrm{d}s\right)\mathrm{d}y = 2\int_0^1 y^2\left(\int_y^1 \frac{1-s}{s}\mathrm{d}s\right)\mathrm{d}y = \frac{1}{18}.$$

This shows that (3.27) holds.

Example 3.17. Let $\mathbb{T} = \mathbb{N}$ with $\ell = 4$, $p = 2$, $\alpha = 1$, and $f(x) = x$. If $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ with $\nu(x) = \frac{\sigma(y)}{x}$ for $x, y \in \mathbb{T}$, then the left-hand side of (3.30) becomes

$$\sum_{x=1}^{\ell-1}\left(\sum_{y=1}^{x-1} f(y)\right)^p x^{-\alpha-1} = \sum_{x=1}^{3}\left(\sum_{y=1}^{x-1} y\right)^2 x^{-2} = 1.25,$$

and the right-hand side of (3.30) gives

$$\begin{aligned} &p\sum_{y=1}^{\ell-1}(y+1)^{p-\alpha-1} f^p(y)\left(\sum_{x=y+1}^{\ell}\frac{x-1}{x(x+1)}\left(1-\frac{y+1}{x}\right)^{p-1}\left(\frac{y+1}{x}\right)^{1-p+\alpha}\right) \\ &= 2\sum_{y=1}^{3} y^2\left(\sum_{x=y+1}^{4}\frac{x-1}{x(x+1)}\left(1-\frac{y+1}{x}\right)\right) = 0.5611. \end{aligned}$$

Thus, (3.30) is satisfied.

Example 3.18. Assume $\mathbb{T} = 2^{\mathbb{N}_0}$, $q = 2$, $m = 0$, $n = 3$, $p = 2$, $\alpha = 1$, and let $f(x) = x$. If $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is also a time scale with $\nu(x) = \frac{\sigma(y)}{x}$ for $x, y \in \mathbb{T}$, then the left-hand side of (3.31) becomes

$$\sum_{k=m}^{n-1} q^{-\alpha k}\left(\sum_{s=m}^{k-1}(q-1)q^s f(q^s)\right)^p = \sum_{k=0}^{2} 2^{-k}\left(\sum_{s=0}^{k-1} 2^{2s}\right)^2 = 6.75,$$

and the right-hand side of (3.31) gives

$$\begin{aligned} &p\sum_{k=m}^{n-1} q^k q^{(k+1)(p-\alpha-1)} f^p(q^k)\left(\sum_{s=k+1}^{n-1}\frac{(q-1)(q^s-1)}{q^{s+1}}(1-q^{k+1-s})^{p-1} q^{(k+1-s)(1-p+\alpha)}\right) \\ &= 2\sum_{k=0}^{2} 2^{3k}\left(\sum_{s=k+1}^{2}\frac{(2^s-1)}{2^{s+1}}(1-2^{k+1-s})\right) = 0.375. \end{aligned}$$

Thus, (3.31) is satisfied.

4. Conclusions

In this paper, we presented a novel formulation of Hardy-type inequalities in the Haar measure $\mathrm{d}x/x$ on time scales, where the classical constant $C = \left(\frac{p}{p-1}\right)^p$ known from Lebesgue setting is replaced by a sharp one $C = 1$. We developed a technique that applies the chain rule on a time scale $\mathbb{T}$ and on a second, associated time scale $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ (for strictly increasing ν), together with a chain rule and the derivative of inverse functions. This was crucial to our proofs. In this way, we extended finite-interval Hardy-type inequalities of Persson et al. [4] to time scales: the convexity form with sharp constant $C = 1$ (time-scale analogue of (1.5)), the reversed case for $0 < p < 1$ (analogue of (1.6)), and the parameterized reversed forms with $\alpha > 0$ and weights $T(\cdot), T_0(\cdot)$ (analogues of (1.7), (1.8), (1.9)), preserving the sharp constant $C = p/\alpha$. Hence, our approach recovers classical integral inequalities in the continuous case as special cases while yielding fundamentally new inequalities in the discrete case. Finally, we provided illustrative examples in both continuous and discrete calculi, showcasing the broad applicability of our approach. Moreover, we extended our findings to the quantum setting, deriving fundamentally new inequalities.

As further research directions, it would be interesting to establish a dynamic Hardy-type formulation with parameters p, q and two weight functions, together with a characterization of the admissible weights. Another interesting task is to investigate a Hardy-type generalization including a positive kernel $k(x, y)$.

Author contributions

Martin Bohner, Iren Jadlovská and Ahmed Saied: Conceptualization, Writing, Methodology, Editing. All authors read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. G. H. Hardy, Note on a theorem of Hilbert, *Math. Z.*, **6** (1920), 314–317. https://doi.org/10.1007/BF01199965

2. G. H. Hardy, Notes on some points in the integral calculus, LX. An inequality between integrals, *Messenger Math.*, **54** (1925), 150–156.
3. G. H. Hardy, Notes on some points in the integral calculus, LXIV. Futher inequalities between integrals, *Messenger Math.*, **57** (1928), 12–16.
4. L. E. Persson, N. Samko, G. Tephnadze, Sharpness of some Hardy-type inequalities, *J. Inequal. Appl.*, **2023** (2023), 155. https://doi.org/10.1186/s13660-023-03066-1
5. S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18–56. https://doi.org/10.1007/BF03323153
6. B. Benaissa, Some inequalities on time scales similar to reverse Hardy's inequality, *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.*, **26** (2022), 113–126. https://doi.org/10.21857/mnlqgcr4ry
7. M. Bohner, A. Nosheen, J. Pečarić, A. Younus, Some dynamic Hardy-type inequalities with general kernel, *J. Math. Inequal.*, **8** (2014), 185–199. https://doi.org/10.7153/jmi-08-12
8. E. Awwad, A. I. Saied, Some new multidimensional Hardy-type inequalities with general kernels on time scales, *J. Math. Inequal.*, **16** (2022), 393–412. https://doi.org/10.7153/jmi-2022-16-29
9. J. A. Oguntuase, L. E. Persson, Time scales Hardy-type inequalities via superquadracity, *Ann. Funct. Anal.*, **5** (2014), 61–73. https://doi.org/10.15352/afa/1396833503
10. P. Řehák, Hardy inequality on time scales and its application to half-linear dynamic equations, *J. Inequal. Appl.*, **2005** (2005), 942973. https://doi.org/10.1155/JIA.2005.495
11. S. H. Saker, J. Alzabut, A. I. Saied, D. O'Regan, New characterizations of weights on dynamic inequalities involving a Hardy operator, *J. Inequal. Appl.*, **2021** (2021), 73. https://doi.org/10.1186/s13660-021-02606-x
12. S. H. Saker, E. Awwad, A. Saied, Some new dynamic inequalities involving monotonic functions on time scales, *J. Funct. Spaces*, **2019** (2019), 7584836. https://doi.org/10.1155/2019/7584836
13. S. H. Saker, A. I. Saied, M. Krnić, Some new weighted dynamic inequalities for monotone functions involving kernels, *Mediterr. J. Math.*, **17** (2020), 39. https://doi.org/10.1007/s00009-020-1473-0
14. R. Agarwal, M. Bohner, S. Saker, Dynamic Littlewood-type inequalities, *Proc. Amer. Math. Soc.*, **143** (2015), 667–677. https://doi.org/10.1090/S0002-9939-2014-12492-0
15. R. P. Agarwal, D. O'Regan, S. H. Saker, *Hardy type inequalities on time scales*, Cham: Springer, 2016. https://doi.org/10.1007/978-3-319-44299-0
16. S. H. Saker, D. O'Regan, Hardy and Littlewood inequalities on time scales, *Bull. Malays. Math. Sci. Soc.*, **39** (2016), 527–543. https://doi.org/10.1007/s40840-015-0300-4
17. M. Bohner, A. Peterson, *Dynamic equations on time scales: an introduction with applications*, Boston, MA: Birkhäuser, 2001. https://doi.org/10.1007/978-1-4612-0201-1
18. R. Bibi, M. Bohner, J. Pečarić, S. Varošanec, Minkowski and Beckenbach-Dresher inequalities and functionals on time scales, *J. Math. Inequal.*, **7** (2013), 299–312. https://doi.org/10.7153/jmi-07-28