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Research article

Novel oscillation criteria for general third-order nonlinear neutral differential equations

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Abstract: This research aims to introduce new criteria that guarantee the oscillation of all solutions to third-order nonlinear neutral differential equations in their canonical form. The proposed methodology integrates the comparison principle with first-order differential equations and employs the Riccati substitution technique, which simplifies the complex structure of the equations and transforms them into more analyzable forms. This approach contributes to establishing general and precise oscillation conditions, representing an extension and improvement over previously published work in this field. It is important to note that this study is purely analytical, focusing on the derivation of oscillatory properties and theoretical criteria. To validate the applicability of the results, three numerical examples are provided, demonstrating the capability of the proposed criteria to verify the oscillation of solutions and highlighting both the theoretical and practical significance of the methodology.

Keywords: neutral differential equations; oscillation; third order; nonlinear equations; canonical case
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

This study investigates the oscillatory nature of all solutions to a general class of third-order nonlinear neutral differential equations, defined as

$$\left(\mathfrak{a}_2(s)\left[\left(\mathfrak{a}_1(s)\left(\mathcal{Z}'(s)\right)^{\alpha_1}\right)'\right]^{\alpha_2}\right)' + q(s)x^{\beta}(\kappa(s)) = 0,\ s \geq s_0, \tag{1.1}$$

where $\mathcal{Z}(s) := x(s) + p(s)x(\tau(s))$. Throughout the paper, we will always assume that

(H1) α_1, α_2, and β are quotients of positive odd integers;
(H2) $\tau, \kappa \in C^1([s_0,\infty),\mathbb{R})$, with $\tau(s) \geq s$, $\kappa(s) \leq s$, $\kappa'(s) \geq 0$, and $\lim_{s\to\infty}\tau(s) = \lim_{s\to\infty}\kappa(s) = \infty$;
(H3) $\mathfrak{a}_1, \mathfrak{a}_2, q \in C^1([s_0,\infty),\mathbb{R}^+)$, with $q(s) \neq 0$ and $p \in C([s_0,\infty),\mathbb{R}^+)$, $0 < p(s) \leq p_0 < 1$. Moreover,

$$\int_{s_0}^{\infty} \frac{1}{\mathfrak{a}_1^{1/\alpha_1}(\omega)}d\omega = \infty,\ \text{and}\ \int_{s_0}^{\infty} \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)}d\omega = \infty. \tag{1.2}$$

By a solution of Eq (1.1), we mean a function $x(s)$ which is continuous on $[s_x,\infty)$ and satisfies (1.1) on $[s_x,\infty)$ for $s_x \geqslant s_0$. We consider only those solution $x(s)$ of (1.1) which satisfy

$$\sup\{|x(s)| : s \geqslant S\} > 0,\ \text{for all } S \geq s_x.$$

In addition, we implicitly assume that Eq (1.1) has solutions. A solution $x(s)$ to Eq (1.1) is said to be oscillatory if it possesses a sufficiently large number of zeros on the interval $[s_x,\infty)$; that is, for any $s_1 \in [s_x,\infty)$ there exists $s_2 \geq s_1$ such that $x(s_2) = 0$. Otherwise, the solution is considered nonoscillatory, meaning that it eventually remains either positive or negative. Extending this concept to the equation itself, Eq (1.1) is said to be oscillatory if all of its solutions are oscillatory; otherwise, it is considered non-oscillatory.

In dynamical models, delay and oscillation scenarios are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems. In recent years, the theory of oscillation in functional differential equations has received increasing attention due to its wide applications in engineering and natural sciences. For foundational and recent contributions addressing oscillatory behavior in various classes of such equations, we refer the reader to all references cited throughout this paper, in particular to [1–3].

Neutral differential equations are prominent mathematical models characterized by their special structure, in which the higher-order derivative of the unknown function depends not only on the current variable s, but also on delayed or advanced arguments. This property provides great flexibility in modeling complex phenomena that are difficult to represent with conventional differential equations. They have been applied to networks with lossless transmission lines [4, 5], mechanical oscillations of connected masses, and other areas such as electrical circuits, population dynamics, and stability analysis, reflecting their broad importance in applied mathematics, natural sciences, and engineering [6].

The problem of oscillation of solutions to differential equations has been extensively studied since Sturm's pioneering work on second-order linear differential equations. Over the past three decades, oscillation theory has advanced significantly for second-order neutral delay differential equations; see Li et al. [7], Tamilvanan et al. [8], Džurina et al. [9], Alrashdi et al. [10], and

Aljawi et al. [11]. Third-order neutral delay differential equations have also received attention, with several new results; see [12–14]. Compared to second-order equations, research on third-order cases remains limited, highlighting a clear research gap.

Below, we review key studies on the oscillatory behavior and Kneser-type solutions of third-order differential equations, which have contributed to the development of precise criteria and modern analytical methods. For instance, Saker [15] examined the oscillatory behavior of a nonlinear delay differential equation of the form

$$\left(\mathfrak{a}_2(\mathrm{s})\left(\mathfrak{a}_1(\mathrm{s})\,x'(\mathrm{s})\right)'\right)' + q(\mathrm{s})f\left(x(\mathrm{s}-\tau)\right) = 0,\ \mathrm{s} \geq \mathrm{s}_0,$$

in the typical case, and established criteria ensuring that every solution of this equation is oscillatory by employing Riccati transformation techniques.

Later, Jadlovská et al. [16] and Jadlovská and Li [17] formulated effective oscillation criteria for third-order delay differential equations of the form

$$\left(\mathfrak{a}_2(\mathrm{s})\left(\mathfrak{a}_1(\mathrm{s})\,x'(\mathrm{s})\right)'\right)' + q(\mathrm{s})x\left(\tau(\mathrm{s})\right) = 0 \tag{1.3}$$

in the canonical case characterized by the two conditions

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\mathfrak{a}_1(\omega)}\mathrm{d}\omega = \infty \text{ and } \int_{\mathrm{s}_0}^{\infty} \frac{1}{\mathfrak{a}_2(\omega)}\mathrm{d}\omega = \infty. \tag{1.4}$$

In contrast, Grace et al. [18] developed new oscillation criteria for the same class of equations in the noncanonical framework, characterized by

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\mathfrak{a}_1(\omega)}\mathrm{d}\omega < \infty \text{ and } \int_{\mathrm{s}_0}^{\infty} \frac{1}{\mathfrak{a}_2(\omega)}\mathrm{d}\omega < \infty. \tag{1.5}$$

Extending the results of [16], Masood et al. [19] considered quasilinear third-order delay differential equations of the type

$$\left(\mathfrak{a}_2(\mathrm{s})\left(\left(\mathfrak{a}_1(\mathrm{s})\,x'(\mathrm{s})\right)'\right)^{\alpha_2}\right)' + q(\mathrm{s})x^{\alpha_2}\left(\tau(\mathrm{s})\right) = 0,$$

and established new oscillation results under the assumptions

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\mathfrak{a}_1(\omega)}\mathrm{d}\omega = \infty \text{ and } \int_{\mathrm{s}_0}^{\infty} \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)}\mathrm{d}\omega = \infty,$$

where the results were obtained using an efficient iterative approach.

Similarly, Chatzarakis et al. [20] introduced criteria aimed at studying the oscillatory behavior of third-order neutral delay equations, such as

$$\mathcal{Z}'''(\mathrm{s}) + q(\mathrm{s})\,x^{\beta}\left(\kappa(\mathrm{s})\right) = 0,$$

and provided sharp conditions ensuring the nonexistence of Kneser-type solutions, thereby pushing the boundaries of oscillation theory for neutral differential equations.

Furthermore, Džurina et al. [21] established sufficient conditions for the absence of Kneser solutions in third-order neutral delay differential equations of the form

$$\left(\mathfrak{a}_2(\mathrm{s})\left(\mathfrak{a}_1(\mathrm{s})\,\mathcal{Z}'(\mathrm{s})\right)'\right)' + q(\mathrm{s})\,x\left(\kappa(\mathrm{s})\right) = 0. \tag{1.6}$$

By combining their recently acquired findings with pre-existing research, they achieved oscillation for all solutions of equations.

On the other hand, Nithyakala et al. [22] studied the nonexistence of Kneser-type solutions in noncanonical settings using Myshkis-type criteria. More recently, Purushothaman et al. [23] examined the existence and bounds of Kneser-type solutions for the noncanonical case of Eq (1.6) with $\kappa(\mathrm{s}) = \mathrm{s}$. In related studies, Almarri et al. [24] and Masood et al. [25] analyzed the oscillatory properties of Eq (1.6) using a substitution procedure that transformed it into an oscillatory form, followed by an iterative method to extract the results.

Finally, Graef et al. [26] established oscillation criteria for Eq (1.1) in the special case $\mathrm{p}(\mathrm{s}) = 0$, using comparison techniques with first-order delay differential equations. Subsequently, the same equation was studied by Grace et al. [27] within the framework of delay dynamic equations, where they obtained a class of oscillation results that are regarded as foundational contributions in the literature.

Based on the previous studies and the results presented in Graef et al. [26] and Grace et al. [27], Eq (1.1) introduced in this paper is a novel extension of the equations discussed in the literature. This work aims to establish new sufficient conditions that guarantee the oscillation of all solutions to Eq (1.1). The results further contribute to expanding the scope of prior research by addressing some special cases not considered in earlier studies. Given the complexity of the studied equation, these findings can be generalized to broader classes of differential equations, including sublinear or higher-order nonlinear differential equations, thereby highlighting the novelty and potential applicability of the proposed approach.

Remark 1.1. *The results established in this study provide a unified framework that significantly extends several earlier contributions, each of which emerges as a special case of Eq (1.1) under more restrictive assumptions on the parameters and coefficient functions. For clarity, these particular cases from the literature are summarized in Table 1.*

Table 1. Special cases of Eq (1.1) considered in previous studies.

Case	References	Conditions
1	[15, 16, 18]	$\alpha_1 = \alpha_2 = \beta = 1, \quad p(s) = 0$
2	[19]	$\alpha_1 = 1, \quad \alpha_2 = \beta, \quad p(s) = 0$
3	[20]	$\alpha_1 = \alpha_2 = 1, \quad \mathfrak{a}_1(s) = \mathfrak{a}_2(s) = 1$
4	[21, 22, 25]	$\alpha_1 = \alpha_2 = \beta = 1$
5	[26]	$p(s) = 0$

Remark 1.2. *In what follows, all functional inequalities are assumed to hold for sufficiently large* s*. Without loss of generality, we restrict our attention to positive solutions of (1.1), because the existence of* $x(\mathrm{s})$ *automatically implies that* $-x(\mathrm{s})$ *is also a solution.*

Remark 1.3. *In the sequel, we assume* $\mathrm{s} \geq \mathrm{s}_2 \geq \mathrm{s}_1$*, where* s_2 *is sufficiently large. Consequently,* $\mathrm{s} \in [\mathrm{s}_2, \infty) \subseteq [\mathrm{s}_1, \infty)$.

The rest of this paper is organized as follows: Section 2 provides the basic notations and introductory lemmas necessary for the main results. In Section 3, we present theorems containing necessary and sufficient criteria for the oscillation of all solutions (1.1). Section 4 provides illustrative numerical examples of the application of the theoretical results, while Section 5 concludes with a summary of the main results, concluding remarks, and suggestions for future research.

2. Preliminary results

In this section, we introduce the notations that will be employed in the subsequent analysis. We begin with the following notation:

$$\pi(\mathrm{s},\mathrm{s}_0) := \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} \mathrm{d}\omega, \ \pi^*(\mathrm{s},\mathrm{s}_0) := \int_{\mathrm{s}_0}^{\mathrm{s}} \left(\frac{\pi(\omega,\mathrm{s}_0)}{\mathfrak{a}_1(\omega)} \right)^{1/\alpha_1} \mathrm{d}\omega$$

and

$$\widehat{q}(\mathrm{s}) = q(\mathrm{s}) \left(1 - \mathrm{p}(\kappa(\mathrm{s})) \frac{\pi^*(\tau(\kappa(\mathrm{s})),\mathrm{s}_1)}{\pi^*(\kappa(\mathrm{s}),\mathrm{s}_1)} \right)^{\beta}.$$

Remark 2.1. *For any solution x of (1.1), we denote by*

$$L_1(\mathcal{Z}(\mathrm{s})) = (\mathcal{Z}'(\mathrm{s}))^{\alpha_1} \ \text{ and } L_2(\mathcal{Z}(\mathrm{s})) = (\mathfrak{a}_1(\mathrm{s})(L_1(\mathcal{Z}(\mathrm{s})))')^{\alpha_2}$$

on $[\mathrm{s}_0,\infty)$. *Then, Eq (1.1) can be rewritten as*

$$(\mathfrak{a}_2(\mathrm{s}) L_2(\mathcal{Z}(\mathrm{s})))' + q(\mathrm{s}) x^{\beta}(\kappa(\mathrm{s})) = 0. \tag{2.1}$$

To establish our main results, we make use of the following lemmas, the first of which is well known.

Lemma 2.1. *[28] Let $q \in ([\mathrm{s}_0,\infty), \mathbb{R}^+)$, $\kappa \in ([\mathrm{s}_0,\infty), \mathbb{R})$ be continuous functions, and $f \in (\mathbb{R},\mathbb{R})$ be a nondecreasing continuous function. Moreover, $xf(x) > 0$, $\forall x \neq 0$ and $\kappa(\mathrm{s}) \leq \mathrm{s}$ with $\lim_{\mathrm{s}\to\infty} \kappa(\mathrm{s}) = \infty$. If the first-order delay differential inequality*

$$\mathcal{Z}'(\mathrm{s}) + q(\mathrm{s}) f(\mathcal{Z}(\kappa(\mathrm{s}))) \leq 0, \ \mathrm{s} \geq \mathrm{s}_0 \geq 0$$

has an eventually positive solution, then the first-order delay differential equation

$$\mathcal{Z}'(\mathrm{s}) + q(\mathrm{s}) f(\mathcal{Z}(\kappa(\mathrm{s}))) = 0, \ \mathrm{s} \geq \mathrm{s}_0 \geq 0$$

has an eventually positive solution.

The following lemma presents fundamental properties of positive (nonoscillatory) solutions of Eq (1.1), and represents a generalization of a classical result originally due to Kiguradze (see, e.g., [29]).

Lemma 2.2. *Assume that $x(\mathrm{s})$ is a positive solution of (1.1) and its corresponding function $\mathcal{Z}(\mathrm{s})$. Then either*

$$\mathcal{Z}(\mathrm{s}) \in \mathrm{N}_0 \Leftrightarrow \ L_1(\mathcal{Z}(\mathrm{s})) < 0, \ L_2(\mathcal{Z}(\mathrm{s})) > 0, \ (\mathfrak{a}_2(\mathrm{s}) L_2(\mathcal{Z}(\mathrm{s})))' < 0,$$

or

$$\mathcal{Z}(\mathrm{s}) \in \mathrm{N}_2 \Leftrightarrow L_1(\mathcal{Z}(\mathrm{s})) > 0, \ L_2(\mathcal{Z}(\mathrm{s})) > 0, \ (\mathfrak{a}_2(\mathrm{s}) L_2(\mathcal{Z}(\mathrm{s})))' < 0,$$

eventually.

We now establish sufficient conditions guaranteeing the nonexistence of N_0-type solutions. In the subsequent analysis, we assume the existence of a nondecreasing function $\varrho(s) \in C([s_0,\infty),\mathbb{R})$ satisfying

$$\varrho(s) \geq s,\ \varrho(\kappa(s)) \geq \kappa(s),\ \text{and}\ \varrho(\varrho(\kappa(s))) \leq s. \tag{2.2}$$

Moreover, we define

$$\widetilde{\pi}^*(\varrho(s),s) := \int_s^{\varrho(s)} \left(\frac{\pi(\varrho(\omega),\omega)}{\mathfrak{a}_1(\omega)}\right)^{1/\alpha_1} d\omega.$$

The following lemma provides some fundamental monotonicity properties of positive solutions of (1.1).

Lemma 2.3. *Let $x(s)$ be a positive solution of (1.1) with the corresponding function $\mathcal{Z} \in N_2$. Then*
(i) $\frac{\mathfrak{a}_1(s)L_1(\mathcal{Z}(s))}{\pi(s,s_1)}$ *is decreasing;*
(ii) $\frac{\mathcal{Z}(s)}{\pi^*(s,s_1)}$ *is decreasing;*
(iii) $(\mathfrak{a}_2(s)(L_2\mathcal{Z}(s)))' + \widehat{q}(s)\mathcal{Z}^\beta(\kappa(s)) \leq 0.$

Proof. Let $x(s)$ be a positive solution of (1.1) with the corresponding function $\mathcal{Z} \in N_2$ for $s \geq s_1$.
(i) Because $\mathfrak{a}_2(s)\left[(\mathfrak{a}_1(s)(\mathcal{Z}'(s))^{\alpha_1})'\right]^{\alpha_2}$ is decreasing, we have

$$\begin{aligned}
\mathfrak{a}_1(s)(\mathcal{Z}'(s))^{\alpha_1} &\geq \int_{s_1}^{s} \mathfrak{a}_2^{1/\alpha_2}(\omega)(\mathfrak{a}_1(\omega)(\mathcal{Z}'(\omega))^{\alpha_1})' \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} d\omega \\
&= \int_{s_1}^{s} \mathfrak{a}_2^{1/\alpha_2}(\omega) L_2^{1/\alpha_2}(\mathcal{Z}(\omega)) \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} d\omega \\
&\geq \mathfrak{a}_2^{1/\alpha_2}(s) L_2^{1/\alpha_2}(\mathcal{Z}(s)) \int_{s_1}^{s} \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} d\omega \\
&= \mathfrak{a}_2^{1/\alpha_2}(s) L_2^{1/\alpha_2}(\mathcal{Z}(s)) \pi(s,s_1).
\end{aligned}$$

Then

$$\mathfrak{a}_2^{1/\alpha_2}(s) L_2^{1/\alpha_2}(\mathcal{Z}(s)) \pi(s,s_1) - \mathfrak{a}_1(s) L_1(\mathcal{Z}(s)) \leq 0.$$

Therefore,

$$\left(\frac{\mathfrak{a}_1(s) L_1(\mathcal{Z}(s))}{\pi(s,s_1)}\right)' = \frac{L_2^{1/\alpha_2}(\mathcal{Z}(s))\pi(s,s_1) - \mathfrak{a}_1(s) L_1(\mathcal{Z}(s)) \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(s)}}{\pi^2(s,s_1)} \leq 0.$$

Thus, $\frac{\mathfrak{a}_1(s)L_1(\mathcal{Z}(s))}{\pi(s,s_1)}$ is decreasing.
(ii) Because $\frac{\mathfrak{a}_1(s)L_1(\mathcal{Z}(s))}{\pi(s,s_1)}$ is decreasing, this fact yields

$$\begin{aligned}
\mathcal{Z}(s) &\geq \int_{s_1}^{s} \frac{\mathfrak{a}_1^{1/\alpha_1}(\omega)\pi^{1/\alpha_1}(\omega,s_1)\mathcal{Z}'(\omega)}{\mathfrak{a}_1^{1/\alpha_1}(\omega)\pi^{1/\alpha_1}(\omega,s_1)} d\omega = \int_{s_1}^{s} \frac{\mathfrak{a}_1^{1/\alpha_1}(\omega)\pi^{1/\alpha_1}(\omega,s_1)L_1^{1/\alpha_1}(\mathcal{Z}(\omega))}{\mathfrak{a}_1^{1/\alpha_1}(\omega)\pi^{1/\alpha_1}(\omega,s_1)} d\omega \\
&\geq \frac{\mathfrak{a}_1^{1/\alpha_1}(s) L_1^{1/\alpha_1}(\mathcal{Z}(s))}{\pi^{1/\alpha_1}(s,s_1)} \int_{s_1}^{s} \left(\frac{\pi(\omega,s_1)}{\mathfrak{a}_1(\omega)}\right)^{1/\alpha_1} d\omega \\
&= \frac{\mathfrak{a}_1^{1/\alpha_1}(s) L_1^{1/\alpha_1}(\mathcal{Z}(s))}{\pi^{1/\alpha_1}(s,s_1)} \pi^*(s,s_1).
\end{aligned}$$

Hence,

$$\left(\frac{\mathcal{Z}(\mathrm{s})}{\pi^{*}(\mathrm{s},\mathrm{s}_1)}\right)' = \frac{\pi^{*}(\mathrm{s},\mathrm{s}_1)\,L_1^{1/\alpha_1}(\mathcal{Z}(\mathrm{s})) - \mathcal{Z}(\mathrm{s})\left(\frac{\pi(\mathrm{s},\mathrm{s}_1)}{\mathfrak{a}_1(\mathrm{s})}\right)^{1/\alpha_1}}{[\pi^{*}(\mathrm{s},\mathrm{s}_1)]^2} \le 0,$$

which implies that $\frac{\mathcal{Z}(\mathrm{s})}{\pi^{*}(\mathrm{s},\mathrm{s}_1)}$ is decreasing.
(iii) By using the definition of $\mathcal{Z}$, we have $\mathcal{Z}(\mathrm{s}) \ge x(\mathrm{s})$ and

$$x(\mathrm{s}) = \mathcal{Z}(\mathrm{s}) - \mathrm{p}(\mathrm{s})\,x(\tau(\mathrm{s})) \ge \mathcal{Z}(\mathrm{s}) - \mathrm{p}(\mathrm{s})\,\mathcal{Z}(\tau(\mathrm{s})). \tag{2.3}$$

Because $\frac{\mathcal{Z}(\mathrm{s})}{\pi^{*}(\mathrm{s},\mathrm{s}_1)}$ is decreasing and $\tau(\mathrm{s}) \ge \mathrm{s}$, we obtain

$$\mathcal{Z}(\tau(\mathrm{s})) \le \frac{\pi^{*}(\tau(\mathrm{s}),\mathrm{s}_1)}{\pi^{*}(\mathrm{s},\mathrm{s}_1)}\mathcal{Z}(\mathrm{s}).$$

By using the above inequality in (2.3), we deduce that

$$x(\mathrm{s}) = \mathcal{Z}(\mathrm{s}) - \mathrm{p}(\mathrm{s})\,x(\tau(\mathrm{s})) \ge \left(1 - \mathrm{p}(\mathrm{s})\,\frac{\pi^{*}(\tau(\mathrm{s}),\mathrm{s}_1)}{\pi^{*}(\mathrm{s},\mathrm{s}_1)}\right)\mathcal{Z}(\mathrm{s}).$$

Now, from (2.1), we get

$$\begin{aligned}
(\mathfrak{a}_2(\mathrm{s})(L_2\mathcal{Z}(\mathrm{s})))' &\le -q(\mathrm{s})x^{\beta}(\kappa(\mathrm{s})) \\
&\le -q(\mathrm{s})\left(1 - \mathrm{p}(\kappa(\mathrm{s}))\,\frac{\pi^{*}(\tau(\kappa(\mathrm{s})),\mathrm{s}_1)}{\pi^{*}(\kappa(\mathrm{s}),\mathrm{s}_1)}\right)^{\beta}\mathcal{Z}^{\beta}(\kappa(\mathrm{s})) \\
&= -\widehat{q}(\mathrm{s})\mathcal{Z}^{\beta}(\kappa(\mathrm{s})).
\end{aligned}$$

3. Main results

In this section, we review a number of theorems that include sufficient criteria to ensure that all solutions of (1.1) are oscillatory.

Theorem 3.1. *Assume that the following first-order delay differential equations*

$$\omega'(\mathrm{s}) + (1-\mathrm{p}_0)^{\beta}\,q(\mathrm{s})\,[\widetilde{\pi}^{*}(\varrho(\kappa(\mathrm{s})),\kappa(\mathrm{s}))]^{\beta}\,\omega^{\frac{\beta}{\alpha_1\alpha_2}}(\varrho(\varrho(\kappa(\mathrm{s})))) = 0 \tag{3.1}$$

and

$$\omega_1'(\mathrm{s}) + \widehat{q}(\mathrm{s})\,[\pi^{*}(\kappa(\mathrm{s}),\mathrm{s}_1)]^{\beta}\,\omega_1^{\frac{\beta}{\alpha_1\alpha_2}}(\kappa(\mathrm{s})) = 0 \tag{3.2}$$

are both oscillatory. Then Eq (1.1) is oscillatory.

Proof. We proceed by contradiction. Assume that Eq (1.1) has an eventually positive solution, that is, there exists a solution $x(\mathrm{s})$ such that $x(\mathrm{s}) > 0$ for all $\mathrm{s} \ge \mathrm{s}_0$, where s_0 is a sufficiently large value. By Lemma 2.2, it follows that the behavior of $\mathcal{Z}$ and its derivatives must fall into one of the two possible cases, N_0 or N_2. We will consider each case and demonstrate that both lead to a contradiction.
Case 1: Assume that $\mathcal{Z}(\mathrm{s}) \in \mathrm{N}_0$ for all $\mathrm{s} \ge \mathrm{s}_1$, where $\mathrm{s}_1 \ge \mathrm{s}_0$ is sufficiently large. Because $\mathcal{Z}(\mathrm{s}) \ge x(\mathrm{s})$, we have

$$x(\mathrm{s}) = \mathcal{Z}(\mathrm{s}) - \mathrm{p}(\mathrm{s})\,x(\tau(\mathrm{s})) \ge \mathcal{Z}(\mathrm{s}) - \mathrm{p}(\mathrm{s})\,\mathcal{Z}(\tau(\mathrm{s})) \ge (1-\mathrm{p}_0)\,\mathcal{Z}(\mathrm{s}).$$

Substituting this inequality into Eq (2.1) gives

$$\left(\mathfrak{a}_2(s)\left(L_2\mathcal{Z}(s)\right)\right)' = -q(s)x^{\beta}(\kappa(s)) \leq -\left(1-p_0\right)^{\beta} q(s)\mathcal{Z}^{\beta}(\kappa(s)). \tag{3.3}$$

Indeed, we note that

$$\begin{aligned} \mathfrak{a}_1(\varrho(s))L_1(\mathcal{Z}(\varrho(s))) - \mathfrak{a}_1(s)L_1(\mathcal{Z}(s)) &= \int_s^{\varrho(s)} \left(\mathfrak{a}_1(\omega)L_1(\mathcal{Z}(\omega))\right)' d\omega \\ &= \int_s^{\varrho(s)} \frac{\mathfrak{a}_2^{\frac{1}{\alpha_2}}(\omega)L_2^{\frac{1}{\alpha_2}}(\mathcal{Z}(\omega))}{\mathfrak{a}_2^{\frac{1}{\alpha_2}}(\omega)} d\omega. \end{aligned}$$

Using the fact that $L_2(\mathcal{Z}(s))$ is decreasing, we find

$$-\mathfrak{a}_1(s)\left(\mathcal{Z}'(s)\right)^{\alpha_1} \geq \pi(\varrho(s), s)\,\mathfrak{a}_2^{\frac{1}{\alpha_2}}(\varrho(s))\,L_2^{\frac{1}{\alpha_2}}(\mathcal{Z}(\varrho(s))).$$

This inequality implies

$$-\mathcal{Z}'(s) \geq \left(\frac{\pi(\varrho(s), s)}{\mathfrak{a}_1(s)}\right)^{\frac{1}{\alpha_1}} \left(\mathfrak{a}_2(\varrho(s))L_2(\mathcal{Z}(\varrho(s)))\right)^{\frac{1}{\alpha_1\alpha_2}}.$$

Integrating from s to $\varrho(s)$ yields

$$-\mathcal{Z}(\varrho(s)) + \mathcal{Z}(s) \geq \left(\mathfrak{a}_2(\varrho(\varrho(s)))L_2(\mathcal{Z}(\varrho(\varrho(s))))\right)^{\frac{1}{\alpha_1\alpha_2}} \int_s^{\varrho(s)} \left(\frac{\pi(\varrho(\omega), \omega)}{\mathfrak{a}_1(\omega)}\right)^{\frac{1}{\alpha_1}} d\omega.$$

Consequently,

$$\mathcal{Z}(s) \geq \widetilde{\pi}^*(\varrho(s), s)\left(\mathfrak{a}_2(\varrho(\varrho(s)))L_2(\mathcal{Z}(\varrho(\varrho(s))))\right)^{\frac{1}{\alpha_1\alpha_2}}. \tag{3.4}$$

Substituting (3.4) into (3.3) gives

$$-\left(\mathfrak{a}_2(s)L_2(\mathcal{Z}(s))\right)' \geq (1-p_0)^{\beta} q(s)\left[\widetilde{\pi}^*(\varrho(\kappa(s)), \kappa(s))\left(\mathfrak{a}_2(\varrho(\varrho(\kappa(s))))L_2(\mathcal{Z}(\varrho(\varrho(\kappa(s)))))\right)^{\frac{1}{\alpha_1\alpha_2}}\right]^{\beta}.$$

Defining $\omega(s) := \mathfrak{a}_2(s)L_2(\mathcal{Z}(s))$, we get

$$\omega'(s) + (1-p_0)^{\beta} q(s)\left[\widetilde{\pi}^*(\varrho(\kappa(s)), \kappa(s))\right]^{\beta}\left[\omega(\varrho(\varrho(\kappa(s))))\right]^{\frac{\beta}{\alpha_1\alpha_2}} \leq 0. \tag{3.5}$$

By Lemma 2.1, the corresponding equation (3.1) has a positive solution with $\omega(s) \to \infty$ as $s \to \infty$, which contradicts our assumption.

Case 2: Assume that $\mathcal{Z}(s) \in N_2$ for all $s \geq s_1$, where $s_1 \geq s_0$ is sufficiently large. Because $L_2(\mathcal{Z}(s))$ is decreasing, we have

$$\begin{aligned} \mathfrak{a}_1(s)L_1(\mathcal{Z}(s)) &\geq \int_{s_1}^{s} \left(\mathfrak{a}_1(\omega)L_1(\mathcal{Z}(\omega))\right)' d\omega \\ &= \int_{s_1}^{s} \frac{\mathfrak{a}_2^{1/\alpha_2}(\omega)L_2^{1/\alpha_2}(\mathcal{Z}(\omega))}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} d\omega \end{aligned}$$

$$\begin{aligned} &\geq \mathfrak{a}_2^{1/\alpha_2}(\mathrm{s})\, L_2^{1/\alpha_2}(\mathcal{Z}(\mathrm{s})) \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} \mathrm{d}\omega \\ &= \pi(\mathrm{s}, \mathrm{s}_1)\, (\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))^{1/\alpha_2}. \end{aligned}$$

Equivalently,

$$\mathcal{Z}'(\mathrm{s}) \geq \left(\frac{\pi(\mathrm{s}, \mathrm{s}_1)}{\mathfrak{a}_1(\mathrm{s})}\right)^{1/\alpha_1} (\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))^{\frac{1}{\alpha_1\alpha_2}}. \tag{3.6}$$

Integrating from s_1 to s yields

$$\mathcal{Z}(\mathrm{s}) \geq (\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))^{\frac{1}{\alpha_1\alpha_2}} \int_{\mathrm{s}_1}^{\mathrm{s}} \left(\frac{\pi(\omega, \mathrm{s}_1)}{\mathfrak{a}_1(\omega)}\right)^{1/\alpha_1} \mathrm{d}\omega = \pi^*(\mathrm{s}, \mathrm{s}_1)\, (\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))^{\frac{1}{\alpha_1\alpha_2}}. \tag{3.7}$$

Using (3.7) in Lemma 2.3(iii), we obtain

$$-(\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))' \geq \widehat{q}(\mathrm{s})\mathcal{Z}^{\beta}(\kappa(\mathrm{s})) \geq \widehat{q}(\mathrm{s})\, [\pi^*(\kappa(\mathrm{s}), \mathrm{s}_1)]^{\beta}\, [\mathfrak{a}_2(\kappa(\mathrm{s}))\, L_2(\mathcal{Z}(\kappa(\mathrm{s})))]^{\frac{\beta}{\alpha_1\alpha_2}}.$$

Defining $\omega_1(\mathrm{s}) = \mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s}))$, we get

$$\omega_1'(\mathrm{s}) + \widehat{q}(\mathrm{s})\, [\pi^*(\kappa(\mathrm{s}), \mathrm{s}_1)]^{\beta}\, \omega_1^{\frac{\beta}{\alpha_1\alpha_2}}(\kappa(\mathrm{s})) \leq 0. \tag{3.8}$$

By Lemma 2.1, the corresponding equation (3.2) has a positive solution with $\omega_1(\mathrm{s}) \to \infty$ as $\mathrm{s} \to \infty$, which is impossible.

Because both cases lead to contradictions, the initial assumption is false, completing the proof.

Theorem 3.2. *Let $\beta = \alpha_1\alpha_2$. If there exists a nondecreasing function $\rho \in C^1([\mathrm{s}_0, \infty), \mathbb{R}^+)$ such that*

$$\limsup_{\mathrm{s}\to\infty} \int_{\kappa(\mathrm{s})}^{\mathrm{s}} q(\omega)\, (\widetilde{\pi}^*(\kappa(\mathrm{s}), \kappa(\omega)))^{\beta}\, \mathrm{d}\omega \geq \frac{1}{(1-\mathrm{p}_0)^{\beta}} \tag{3.9}$$

and

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_1}^{\mathrm{s}} \left[\rho(\omega)\widehat{q}(\omega) - \frac{1}{(1+\alpha_1\alpha_2)^{1+\alpha_1\alpha_2}} \frac{[\rho'(\omega)]^{1+\alpha_1\alpha_2}}{[\rho(\omega)\,\kappa'(\omega)]^{\alpha_1\alpha_2}} \left(\frac{\pi(\kappa(\omega), \mathrm{s}_1)}{\mathfrak{a}_1(\kappa(\omega))}\right)^{-\alpha_2}\right] \mathrm{d}\omega = \infty \tag{3.10}$$

hold, then Eq (1.1) is oscillatory.

Proof. We proceed by contradiction. Assume that Eq (1.1) has an eventually positive solution, that is, there exists a solution $x(\mathrm{s})$ such that $x(\mathrm{s}) > 0$ for all $\mathrm{s} \geq \mathrm{s}_0$, where s_0 is a sufficiently large value. By Lemma 2.2, it follows that the behavior of $\mathcal{Z}$ and its derivatives must fall into one of the two possible cases, N_0 or N_2. We will consider each case and demonstrate that both lead to a contradiction.
Case 1: Assume that $\mathcal{Z}(\mathrm{s}) \in \mathrm{N}_0$ for all $\mathrm{s} \geq \mathrm{s}_1$, where $\mathrm{s}_1 \geq \mathrm{s}_0$ is sufficiently large. Clearly, for $\mathrm{s} \geq l \geq \mathrm{s}_1$,

$$\begin{aligned} \mathfrak{a}_1(\mathrm{s})\, L_1(\mathcal{Z}(\mathrm{s})) - \mathfrak{a}_1(l)\, L_1(\mathcal{Z}(l)) &= \int_l^{\mathrm{s}} (\mathfrak{a}_1(\omega)\, L_1(\mathcal{Z}(\omega)))'\, \mathrm{d}\omega \\ &= \int_l^{\mathrm{s}} \frac{\mathfrak{a}_2^{1/\alpha_2}(\omega)\, L_2^{1/\alpha_2}(\mathcal{Z}(\omega))}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} \mathrm{d}\omega \end{aligned}$$

$$
\begin{aligned}
&\geq \mathfrak{a}_2^{1/\alpha_2}(\mathrm{s})\, L_2^{1/\alpha_2}(\mathcal{Z}(\mathrm{s})) \int_l^{\mathrm{s}} \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} \mathrm{d}\omega \\
&= \pi(\mathrm{s}, l)\, \mathfrak{a}_2^{1/\alpha_2}(\mathrm{s})\, L_2^{1/\alpha_2}(\mathcal{Z}(\mathrm{s})),
\end{aligned}
$$

that is,

$$
-\mathfrak{a}_1(l)\, L_1(\mathcal{Z}(l)) \geq \pi(\mathrm{s}, l)\, \mathfrak{a}_2^{1/\alpha_2}(\mathrm{s})\, L_2^{1/\alpha_2}(\mathcal{Z}(\mathrm{s})).
$$

Then, clearly,

$$
-\mathfrak{a}_1(l)\,(\mathcal{Z}'(l))^{\alpha_1} \geq \pi(\mathrm{s}, l)\,(\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))^{1/\alpha_2},
$$

or

$$
-\mathcal{Z}'(l) \geq \left(\frac{\pi(\mathrm{s}, l)}{\mathfrak{a}_1(l)}\right)^{1/\alpha_1} (\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))^{\frac{1}{\alpha_1\alpha_2}},
$$

which by integrating from l to s gives

$$
\mathcal{Z}(l) - \mathcal{Z}(\mathrm{s}) \geq (\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))^{\frac{1}{\alpha_1\alpha_2}} \int_l^{\mathrm{s}} \left(\frac{\pi(\mathrm{s}, \omega)}{\mathfrak{a}_1(\omega)}\right)^{1/\alpha_1} \mathrm{d}\omega,
$$

that is,

$$
\mathcal{Z}(l) \geq (\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))^{\frac{1}{\alpha_1\alpha_2}}\, \widetilde{\pi}^*(\mathrm{s}, l).
$$

Now, for $\mathrm{s} \geq \omega > \mathrm{s}_2$ for some $\mathrm{s}_2 > \mathrm{s}_1$, setting $l = \kappa(\omega)$ and $\mathrm{s} = \kappa(\mathrm{s})$ in the preceding inequality, we get

$$
\mathcal{Z}(\kappa(\omega)) \geq (\mathfrak{a}_2(\kappa(\mathrm{s}))\, L_2(\mathcal{Z}(\kappa(\mathrm{s}))))^{\frac{1}{\alpha_1\alpha_2}}\, \widetilde{\pi}^*(\kappa(\mathrm{s}), \kappa(\omega)). \tag{3.11}
$$

Integrating inequality (3.3) from $\kappa(\mathrm{s})$ to s and then applying (3.11), we get

$$
\begin{aligned}
\mathfrak{a}_2(\kappa(\mathrm{s}))\, L_2(\mathcal{Z}(\kappa(\mathrm{s}))) &\geq \int_{\kappa(\mathrm{s})}^{\mathrm{s}} (1-\mathrm{p}_0)^{\beta} q(\omega) \mathcal{Z}^{\beta}(\kappa(\omega))\, \mathrm{d}\omega \\
&\geq \int_{\kappa(\mathrm{s})}^{\mathrm{s}} (1-\mathrm{p}_0)^{\beta} q(\omega)\, (\widetilde{\pi}^*(\kappa(\mathrm{s}), \kappa(\omega)))^{\beta} (\mathfrak{a}_2(\kappa(\mathrm{s}))\, L_2(\mathcal{Z}(\kappa(\mathrm{s}))))^{\frac{\beta}{\alpha_1\alpha_2}}\, \mathrm{d}\omega \\
&\geq (\mathfrak{a}_2(\kappa(\mathrm{s}))\, L_2(\mathcal{Z}(\kappa(\mathrm{s}))))^{\frac{\beta}{\alpha_1\alpha_2}} \int_{\kappa(\mathrm{s})}^{\mathrm{s}} (1-\mathrm{p}_0)^{\beta} q(\omega)\, (\widetilde{\pi}^*(\kappa(\mathrm{s}), \kappa(\omega)))^{\beta}\, \mathrm{d}\omega.
\end{aligned}
$$

Because $\beta = \alpha_1\alpha_2$, we have $\frac{\beta}{\alpha_1\alpha_2} = 1$, and thus it follows that

$$
\int_{\kappa(\mathrm{s})}^{\mathrm{s}} q(\omega)\, (\widetilde{\pi}^*(\kappa(\mathrm{s}), \kappa(\omega)))^{\beta}\, \mathrm{d}\omega \leq \frac{1}{(1-\mathrm{p}_0)^{\beta}},
$$

a contradiction to (3.9).

Case 2: Assume that $\mathcal{Z}(\mathrm{s}) \in \mathrm{N}_2$ for all $\mathrm{s} \geq \mathrm{s}_1$, where $\mathrm{s}_1 \geq \mathrm{s}_0$ is sufficiently large. Define

$$
\psi(\mathrm{s}) := \rho(\mathrm{s}) \frac{\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s}))}{\mathcal{Z}^{\beta}(\kappa(\mathrm{s}))} > 0. \tag{3.12}
$$

Now,

$$
\psi'(\mathrm{s}) = \rho'(\mathrm{s}) \frac{\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s}))}{\mathcal{Z}^{\beta}(\kappa(\mathrm{s}))} + \rho(\mathrm{s}) \frac{(\mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})))'}{\mathcal{Z}^{\beta}(\kappa(\mathrm{s}))} - \beta \frac{\rho(\mathrm{s})\, \mathfrak{a}_2(\mathrm{s})\, L_2(\mathcal{Z}(\mathrm{s})) \cdot \mathcal{Z}'(\kappa(\mathrm{s})) \cdot \kappa'(\mathrm{s})}{\mathcal{Z}^{1+\beta}(\kappa(\mathrm{s}))}
$$

$$\leq \frac{\rho'(s)}{\rho(s)}\psi(s) - \rho(s)\widehat{q}(s) - \beta\kappa'(s)\frac{\mathcal{Z}'(\kappa(s))}{\mathcal{Z}(\kappa(s))}\psi(s).$$

By using (3.6), we have

$$\psi'(s) \leq -\rho(s)\widehat{q}(s) + \frac{\rho'(s)}{\rho(s)}\psi(s) - \beta\kappa'(s)\frac{\left(\frac{\pi(\kappa(s),s_1)}{\mathfrak{a}_1(\kappa(s))}\right)^{1/\alpha_1}(\mathfrak{a}_2(\kappa(s))L_2(\mathcal{Z}(\kappa(s))))^{\frac{1}{\alpha_1\alpha_2}}}{\mathcal{Z}(\kappa(s))}\psi(s).$$

By using the decreasing nature of $\mathfrak{a}_2(s)L_2(\mathcal{Z}(s))$ and the increasing nature of $\mathcal{Z}(s)$, we get

$$\begin{aligned}\psi'(s) &\leq -\rho(s)\widehat{q}(s) + \frac{\rho'(s)}{\rho(s)}\psi(s) - \beta\kappa'(s)\left(\frac{\pi(\kappa(s),s_1)}{\mathfrak{a}_1(\kappa(s))}\right)^{\frac{1}{\alpha_1}}\frac{[\mathfrak{a}_2(s)L_2(\mathcal{Z}(s))]^{\frac{1}{\alpha_1\alpha_2}}}{\mathcal{Z}(\kappa(s))}\psi(s)\\ &= -\rho(s)\widehat{q}(s) + \frac{\rho'(s)}{\rho(s)}\psi(s) - \frac{\beta\kappa'(s)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(s)}\left(\frac{\pi(\kappa(s),s_1)}{\mathfrak{a}_1(\kappa(s))}\right)^{\frac{1}{\alpha_1}}\psi^{1+\frac{1}{\alpha_1\alpha_2}}(s).\end{aligned}$$

By using the inequality [30]

$$By - Ay^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha},\ A > 0,$$

where

$$\alpha = \alpha_1\alpha_2,\ B = \frac{\rho'(s)}{\rho(s)},\ A = \frac{\beta\kappa'(s)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(s)}\left(\frac{\pi(\kappa(s),s_1)}{\mathfrak{a}_1(\kappa(s))}\right)^{\frac{1}{\alpha_1}},\ \text{and } y(s) = \psi(s),$$

we have

$$\psi'(s) \leq -\rho(s)\widehat{q}(s) + \frac{1}{(\alpha_1\alpha_2+1)^{\alpha_1\alpha_2+1}}\frac{[\rho'(s)]^{1+\alpha_1\alpha_2}}{[\rho(s)\kappa'(s)]^{\alpha_1\alpha_2}}\left(\frac{\pi(\kappa(s),s_1)}{\mathfrak{a}_1(\kappa(s))}\right)^{-\alpha_2}.$$

Integrating the preceding inequality from s_1 to s, we get

$$\int_{s_1}^{s}\left[\rho(\omega)\widehat{q}(\omega) - \frac{1}{(\alpha_1\alpha_2+1)^{\alpha_1\alpha_2+1}}\frac{[\rho'(\omega)]^{1+\alpha_1\alpha_2}}{[\rho(\omega)\kappa'(\omega)]^{\alpha_1\alpha_2}}\left(\frac{\pi(\kappa(\omega),s_1)}{\mathfrak{a}_1(\kappa(\omega))}\right)^{-\alpha_2}\right]d\omega \leq \psi(s_1).$$

This leads to a contradiction with condition (3.10) as $s \to \infty$, thus concluding the proof.

Theorem 3.3. *Let $\beta = \alpha_1\alpha_2$. Suppose that condition (3.9) is satisfied. Then, the oscillatory nature of the first-order delay differential equation (3.2) ensures the oscillation of Eq (1.1).*

Proof. The proof follows immediately by combining condition (3.2) of Theorem 3.1 with condition (3.9) in Theorem 3.2.

Theorem 3.4. *Let $\beta = \alpha_1\alpha_2$. Suppose that there exists a nondecreasing function $\rho \in C^1([s_0,\infty),\mathbb{R}^+)$ such that (3.10) is satisfied. Then, the oscillatory nature of the first-order delay differential equation (3.1) ensures the oscillation of Eq (1.1).*

Proof. The proof follows immediately by combining condition (3.1) of Theorem 3.1 with condition (3.10) of Theorem 3.2.

Theorem 3.5. *Let $\beta \leq \alpha_1\alpha_2$. Assume there exists a nondecreasing function $\varrho(\mathrm{s}) \in C([\mathrm{s}_0,\infty),\mathbb{R})$ such that (2.2) holds. If*

$$\lim_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}} q(\omega)\left[\widetilde{\pi}^*(\varrho(\kappa(\omega)),\kappa(\omega))\right]^{\beta}\mathrm{d}\omega=\infty \tag{3.13}$$

and

$$\lim_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}} \widehat{q}(\omega)\left[\pi^*(\kappa(\omega),\mathrm{s}_1)\right]^{\beta}\mathrm{d}\omega=\infty, \tag{3.14}$$

then Eq (1.1) is oscillatory.

Proof. We proceed by contradiction. Assume that Eq (1.1) has an eventually positive solution, that is, there exists a solution $x(\mathrm{s})$ such that $x(\mathrm{s})>0$ for all $\mathrm{s}\geq \mathrm{s}_0$, where s_0 is a sufficiently large value. By Lemma 2.2, it follows that the behavior of $\mathcal{Z}$ and its derivatives must fall into one of the two possible cases, N_0 or N_2. We will consider each case and demonstrate that both lead to a contradiction.
Case 1: Assume that $\mathcal{Z}(\mathrm{s})\in \mathrm{N}_0$ for all $\mathrm{s}\geq\mathrm{s}_1$, where $\mathrm{s}_1\geq\mathrm{s}_0$ is sufficiently large. Proceeding as in the proof of Theorem 3.1, we get the inequality (3.5) for $\mathrm{s}\geq\mathrm{s}_2$. Upon using the fact that $\varrho(\varrho(\kappa(\mathrm{s})))\leq \mathrm{s}$ and $\omega(\mathrm{s}):=\mathfrak{a}_2(\mathrm{s})(L_2\mathcal{Z}(\mathrm{s}))$ is positive and nonincreasing, we have $\omega(\varrho(\varrho(\kappa(\mathrm{s}))))\geq\omega(\mathrm{s})$. Thus, inequality (3.5) takes the form

$$\omega'(\mathrm{s})+(1-\mathrm{p}_0)^{\beta}q(\mathrm{s})\left[\widetilde{\pi}^*(\varrho(\kappa(\mathrm{s})),\kappa(\mathrm{s}))\right]^{\beta}\left[\omega(\mathrm{s})\right]^{\frac{\beta}{\alpha_1\alpha_2}}\leq 0,$$

that is,

$$(1-\mathrm{p}_0)^{\beta}q(\mathrm{s})\left[\widetilde{\pi}^*(\varrho(\kappa(\mathrm{s})),\kappa(\mathrm{s}))\right]^{\beta}\leq-\frac{\omega'(\mathrm{s})}{\omega^{\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s})}=-\frac{\left(\omega^{1-\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s})\right)'}{1-\frac{\beta}{\alpha_1\alpha_2}},\ \beta\leq\alpha_1\alpha_2.$$

Integrating the preceding inequality from s_2 to s, we get

$$\int_{\mathrm{s}_2}^{\mathrm{s}}(1-\mathrm{p}_0)^{\beta}q(\omega)\left[\widetilde{\pi}^*(\varrho(\kappa(\omega)),\kappa(\omega))\right]^{\beta}\mathrm{d}\omega\leq\frac{\omega^{1-\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s}_2)}{1-\frac{\beta}{\alpha_1\alpha_2}}-\frac{\omega^{1-\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s})}{1-\frac{\beta}{\alpha_1\alpha_2}}\leq\frac{1}{1-\frac{\beta}{\alpha_1\alpha_2}}\omega^{1-\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s}_2),$$

a contradiction to (3.13) as $\mathrm{s}\to\infty$.
Case 2: Assume that $\mathcal{Z}(\mathrm{s})\in \mathrm{N}_2$ for all $\mathrm{s}\geq\mathrm{s}_1$, where $\mathrm{s}_1\geq\mathrm{s}_0$ is sufficiently large. Proceeding as in the proof of condition (3.2) in Theorem 3.1, we get the inequality (3.8) for $\mathrm{s}\geq\mathrm{s}_2$. Upon using the fact that $\kappa(\mathrm{s})\leq\mathrm{s}$ and $\omega_1(\mathrm{s})=\mathfrak{a}_2(\mathrm{s})L_2(\mathcal{Z}(\mathrm{s}))$ is positive and nonincreasing, we have $\omega_1(\kappa(\mathrm{s}))\geq\omega_1(\mathrm{s})$. Thus, inequality (3.8) takes the form

$$\omega_1'(\mathrm{s})+\widehat{q}(\mathrm{s})\left[\pi^*(\kappa(\mathrm{s}),\mathrm{s}_1)\right]^{\beta}\omega_1^{\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s})\leq 0,$$

that is,

$$\widehat{q}(\mathrm{s})\left[\pi^*(\kappa(\mathrm{s}),\mathrm{s}_1)\right]^{\beta}\leq-\frac{\omega_1'(\mathrm{s})}{\omega_1^{\frac{\beta}{\alpha_1\alpha_2}}}=-\frac{\left(\omega_1^{1-\frac{\beta}{\alpha_1\alpha_2}}\right)'}{1-\frac{\beta}{\alpha_1\alpha_2}},\ \beta\leq\alpha_1\alpha_2.$$

Integrating the preceding inequality from s_2 to s, we get

$$\int_{\mathrm{s}_2}^{\mathrm{s}}\widehat{q}(\omega)\left[\pi^*(\kappa(\omega),\mathrm{s}_1)\right]^{\beta}\mathrm{d}\omega\leq\frac{\omega_1^{1-\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s}_2)}{1-\frac{\beta}{\alpha_1\alpha_2}}-\frac{\omega_1^{1-\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s})}{1-\frac{\beta}{\alpha_1\alpha_2}}\leq\frac{1}{1-\frac{\beta}{\alpha_1\alpha_2}}\omega^{1-\frac{\beta}{\alpha_1\alpha_2}}(\mathrm{s}_2),$$

a contradiction to (3.14) as $\mathrm{s}\to\infty$. The proof is complete.

Theorem 3.6. *Let $\beta < \alpha_1\alpha_2$. Suppose that (3.14) is satisfied. Then, the oscillatory nature of the first-order delay differential equation (3.1) ensures the oscillation of Eq (1.1).*

Proof. The proof follows immediately by combining condition (3.1) of Theorem 3.1 with condition (3.14) of Theorem 3.5.

Theorem 3.7. *Let $\beta < \alpha_1\alpha_2$. Suppose that (3.13) is satisfied. Then, the oscillatory nature of the first-order delay differential equation (3.2) ensures the oscillation of Eq (1.1).*

Proof. The proof follows immediately by combining condition (3.2) of Theorem 3.1 with condition (3.13) of Theorem 3.5.

4. Examples

In this section, we present some examples that support and illustrate our results.

Example 4.1. Consider the third-order neutral differential equation

$$\left(\mathrm{s}\left[\left(\left[\left(x(\mathrm{s})+\frac{1}{2}x(2\mathrm{s})\right)'\right]^{1/3}\right)'\right]^{3}\right)'+\frac{9}{\mathrm{s}^3}x^3\left(\frac{1}{3}\mathrm{s}\right)=0,\ \mathrm{s}\geq 1, \tag{4.1}$$

where $\alpha_1 = \frac{1}{3}, \alpha_2 = 3, \beta = 3, \mathfrak{a}_1(\mathrm{s}) = 1, \mathfrak{a}_2(\mathrm{s}) = \mathrm{s}, \mathrm{p}(\mathrm{s}) = \frac{1}{16}, q(\mathrm{s}) = \frac{9}{\mathrm{s}^3}, \tau(\mathrm{s}) = 2\mathrm{s}$, and $\kappa(\mathrm{s}) = \frac{1}{3}\mathrm{s}$. Then assumptions (H_1)–(H_3) hold, and

$$\int_{\mathrm{s}_0}^{\infty}\frac{1}{\mathfrak{a}_1^{1/\alpha_1}(\omega)}\mathrm{d}\omega=\int_1^{\infty}1\mathrm{d}\omega=\infty,$$

$$\int_{\mathrm{s}_0}^{\infty}\frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)}\mathrm{d}\omega=\int_1^{\infty}\omega^{-1/3}\mathrm{d}\omega=\infty.$$

Moreover,

$$\pi(\mathrm{s},1)=\int_1^{\mathrm{s}}\omega^{-1/3}\mathrm{d}\omega=\frac{3}{2}\left(\mathrm{s}^{2/3}-1\right),$$

and

$$\pi^*(\mathrm{s},1)=\frac{27}{8}\int_1^{\mathrm{s}}\left(\omega^{2/3}-1\right)^3\mathrm{d}\omega=\frac{9}{8}\mathrm{s}^3-\frac{243}{56}\mathrm{s}^{7/3}+\frac{243}{40}\mathrm{s}^{5/3}-\frac{27}{8}\mathrm{s}+\frac{18}{35}.$$

Hence, $\pi^*\left(\frac{2\mathrm{s}}{3},1\right)$ and $\pi^*\left(\frac{\mathrm{s}}{3},1\right)$ are obtained by substituting $\mathrm{s}\mapsto\frac{\mathrm{s}}{6}$ and $\mathrm{s}\mapsto\frac{\mathrm{s}}{3}$. For large s, one has $\pi^*(\mathrm{s},1)\sim\frac{9}{8}\mathrm{s}^3$, which yields

$$\frac{\pi^*\left(\frac{2\mathrm{s}}{3},1\right)}{\pi^*\left(\frac{\mathrm{s}}{3},1\right)}\sim 8,$$

and therefore,

$$\widehat{q}(\mathrm{s})=\frac{9}{\mathrm{s}^3}\left(1-\frac{1}{16}\frac{\pi^*\left(\frac{2}{3}\mathrm{s},1\right)}{\pi^*\left(\frac{1}{3}\mathrm{s},1\right)}\right)^3\sim\frac{1.125}{\mathrm{s}^3}.$$

With $\varrho(\mathrm{s}) = \frac{3}{2}\mathrm{s}$ $\left(\mathrm{so}, \varrho(\kappa(\mathrm{s})) = \frac{\mathrm{s}}{2}, \ \varrho(\varrho(\kappa(\mathrm{s}))) = \frac{3}{4}\mathrm{s} < \mathrm{s}\right)$, one obtains

$$\pi(\varrho(\mathrm{s}), \mathrm{s}) = \int_{\mathrm{s}}^{\frac{3}{2}\mathrm{s}} \omega^{-1/3} \mathrm{d}\omega = \frac{3}{2}\mathrm{s}^{2/3}\left(\left(\frac{3}{2}\right)^{2/3} - 1\right) = 0.46556\mathrm{s}^{2/3}.$$

Consequently,

$$\widetilde{\pi}^*(\varrho(\mathrm{s}), \mathrm{s}) := \int_{\mathrm{s}}^{\frac{3}{2}\mathrm{s}} (\pi(\varrho(\omega), \omega))^3 \mathrm{d}\omega = \int_{\mathrm{s}}^{\frac{3}{2}\mathrm{s}} \left(0.46556\omega^{2/3}\right)^3 \mathrm{d}\omega \simeq 0.0799\mathrm{s}^3.$$

Similarly, one obtains

$$\widetilde{\pi}^*(\varrho(\kappa(\mathrm{s})), \kappa(\mathrm{s})) \simeq 0.00296\mathrm{s}^3.$$

Therefore, (3.1) becomes

$$\omega'(\mathrm{s}) + 1.92 \times 10^{-7}\mathrm{s}^6\omega^3\left(\frac{3}{4}\mathrm{s}\right) = 0. \tag{4.2}$$

It is not difficult to see that (3.2) becomes

$$\omega_1'(\mathrm{s}) + 8.138 \times 10^{-5}\mathrm{s}^6\omega_1^3\left(\frac{1}{3}\mathrm{s}\right) = 0. \tag{4.3}$$

Clearly, Lemma 2.1 guarantees that all solutions of Eqs (3.1) and (3.2) are oscillatory. Thus, by Theorem 3.1, every solution of Eq (4.1) is oscillatory.

Note that the results in [19, 24, 25] are not applicable to Eq (4.1); thus, the findings of the present study extend and improve upon these earlier works.

Example 4.2. Consider the third-order neutral differential equation

$$\left(\frac{1}{\mathrm{s}}\left(\frac{1}{\mathrm{s}}\left(x(\mathrm{s}) + \frac{1}{2}x(2\mathrm{s})\right)'\right)'\right)' + \frac{q_0}{\mathrm{s}^5}x\left(\frac{1}{3}\mathrm{s}\right) = 0, \ \mathrm{s} \geq 1, \tag{4.4}$$

where $q_0 > 0$. Clearly:

$$\alpha_1 = \alpha_2 = \beta = 1, \ \mathfrak{a}_1(\mathrm{s}) = \frac{1}{\mathrm{s}}, \ \mathfrak{a}_2(\mathrm{s}) = \frac{1}{\mathrm{s}}, \ \mathrm{p}(\mathrm{s}) = \frac{1}{25}, \ q(\mathrm{s}) = \frac{q_0}{\mathrm{s}^5}, \ \tau(\mathrm{s}) = 2\mathrm{s}, \text{ and } \kappa(\mathrm{s}) = \frac{1}{3}\mathrm{s}.$$

Hence, assumptions (H_1)–(H_3) and (1.2) are satisfied. Moreover, we obtain

$$\pi(\mathrm{s}, l) = \int_l^{\mathrm{s}} \frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)} \mathrm{d}\omega = \int_l^{\mathrm{s}} \omega \mathrm{d}\omega = \frac{1}{2}\left(\mathrm{s}^2 - l^2\right),$$

$$\pi^*(\mathrm{s}, 1) = \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{2}\left(\mathrm{s}^3 - \mathrm{s}l^2\right) \mathrm{d}\omega = \frac{\mathrm{s}^4}{8} - \frac{\mathrm{s}^2}{4} + \frac{1}{8},$$

and

$$\widetilde{\pi}^*(\kappa(\mathrm{s}), \kappa(l)) = \int_{\frac{l}{3}}^{\frac{\mathrm{s}}{3}} \frac{1}{2}\left(\frac{\mathrm{s}^2}{9}\omega - \omega^3\right) \mathrm{d}\omega = \frac{\left(\mathrm{s}^2 - l^2\right)^2}{648}.$$

Condition (3.9) then yields

$$\begin{aligned}\limsup_{s\to\infty}\int_{\kappa(s)}^{s} q(\omega)\left(\widetilde{\pi}^{*}\left(\kappa(s),\kappa(\omega)\right)\right)^{\beta}\mathrm{d}\omega &= \limsup_{s\to\infty}\int_{s/2}^{s}\frac{q_0}{\omega^5}\frac{\left(s^2-\omega^2\right)^2}{648}\mathrm{d}\omega = \frac{q_0}{648}\int_{s/2}^{s}\frac{\left(s^2-\omega^2\right)^2}{\omega^5}\mathrm{d}\omega\\ &\simeq 0.00223\cdot q_0,\end{aligned}$$

which holds provided that

$$0.00223\cdot q_0 > \frac{1}{\left(1-\frac{1}{25}\right)} \Longrightarrow q_0 > 467.1.$$

For sufficiently large s, we have

$$\pi^{*}(s,1)\sim\frac{s^4}{8},\qquad \frac{\pi^{*}\left(\frac{2s}{3},1\right)}{\pi^{*}\left(\frac{s}{3},1\right)}\sim 16.$$

Thus,

$$\widehat{q}(s)=\frac{q_0}{s^5}\left(1-\frac{1}{25}\frac{\pi^{*}\left(\frac{2s}{3},1\right)}{\pi^{*}\left(\frac{s}{3},1\right)}\right)\simeq\frac{q_0}{s^5}\left(1-\frac{16}{25}\right)\simeq\frac{9q_0}{25s^5}.$$

For condition (3.10) with $\rho(s)=s^4$, we obtain for sufficiently large s:

$$\limsup_{s\to\infty}\int_1^s\left[\omega^4\frac{9q_0}{25\omega^5}-\frac{3}{2^2}\frac{\left(4\omega^3\right)^2}{\omega^4}\left(\frac{\omega}{3}\cdot\frac{1}{2}\cdot\frac{\omega^2}{9}\right)^{-1}\right]\mathrm{d}\omega=\limsup_{s\to\infty}\int_1^s\left[\frac{9q_0}{25}-648\right]\frac{1}{\omega}\mathrm{d}\omega=\infty,$$

which holds when

$$\frac{9q_0}{25}-648>0\Longrightarrow q_0>1800.$$

Therefore, by Theorems 3.1–3.3, every solution of (4.5) is oscillatory whenever $q_0 > 1800$.

Example 4.3. Consider

$$\left(\frac{1}{s^2}\left[\left(\frac{1}{15s^2}\left(x(s)+\frac{1}{180}x(3s)\right)'\right)'\right]^3\right)'+\frac{q_0}{s^{15}}x^3\left(\frac{1}{2}s\right)=0, \tag{4.5}$$

where $q_0 > 0$. We have

$$\begin{aligned}\alpha_1 &= 1,\ \alpha_2=\beta=3,\ \mathfrak{a}_1(s)=\frac{1}{15s^2},\ \mathfrak{a}_2(s)=\frac{1}{s^2},\\ p(s) &= \frac{1}{350},\ q(s)=\frac{q_0}{s^{15}},\ \tau(s)=3s,\ \text{and}\ \kappa(s)=\frac{1}{2}s.\end{aligned}$$

Then assumptions (H_1)–(H_3) and (1.2) are satisfied. Moreover, we have

$$\pi(s,l)=\int_l^s\frac{1}{\mathfrak{a}_2^{1/\alpha_2}(\omega)}\mathrm{d}\omega=\int_l^s\omega^{2/3}\mathrm{d}\omega=\frac{3}{5}\left(s^{5/3}-l^{5/3}\right).$$

Next,

$$\begin{aligned}\pi^*(\mathrm{s},1) &= \int_{\mathrm{s}_0}^{\mathrm{s}}\left(\frac{\pi(\omega,\mathrm{s}_0)}{\mathfrak{a}_1(\omega)}\right)^{1/\alpha_1}\mathrm{d}\omega = \int_1^{\mathrm{s}}\left(15\omega^2\cdot\frac{3}{5}\left(\omega^{5/3}-1\right)\right)\mathrm{d}\omega\\ &= 9\int_1^{\mathrm{s}}\left(\omega^{11/3}-\omega^2\right)\mathrm{d}\omega = \frac{27}{14}\mathrm{s}^{14/3}-3\mathrm{s}^3+\frac{15}{14}.\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{\pi}^*(\kappa(\mathrm{s}),\kappa(l)) &= \widetilde{\pi}^*\left(\frac{\mathrm{s}}{2},\frac{l}{2}\right) = \int_{\frac{l}{2}}^{\frac{\mathrm{s}}{2}}\frac{45}{5}\omega^2\left(\left(\frac{\mathrm{s}}{2}\right)^{5/3}-\omega^{5/3}\right)\mathrm{d}\omega\\ &= 9\int_{\frac{l}{2}}^{\frac{\mathrm{s}}{2}}\left(\left(\frac{\mathrm{s}}{2}\right)^{5/3}\omega^2-\omega^{11/3}\right)\mathrm{d}\omega\\ &= 9\left[\left(\left(\frac{\mathrm{s}}{2}\right)^{5/3}\cdot\frac{1}{3}\left(\left(\frac{\mathrm{s}}{2}\right)^3-\left(\frac{l}{2}\right)^3\right)\right)-\frac{3}{14}\left(\left(\frac{\mathrm{s}}{2}\right)^{14/3}-\left(\frac{l}{2}\right)^{14/3}\right)\right]\\ &= \frac{9}{2^{14/3}}\left[\frac{5}{42}\mathrm{s}^{14/3}-\frac{1}{3}\mathrm{s}^{5/3}l^3+\frac{3}{14}l^{14/3}\right].\end{aligned}$$

So for large s,

$$\widetilde{\pi}^*(\kappa(\mathrm{s}),\kappa(l)) \sim \frac{9}{2^{14/3}}\cdot\frac{5}{42}\mathrm{s}^{14/3}.$$

Then the condition (3.9) gives

$$\begin{aligned}&\limsup_{\mathrm{s}\to\infty}\int_{\kappa(\mathrm{s})}^{\mathrm{s}} q(\omega)\left(\widetilde{\pi}^*(\kappa(\mathrm{s}),\kappa(\omega))\right)^{\beta}\mathrm{d}\omega\\ =\ &\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}/2}^{\mathrm{s}}\frac{q_0}{\omega^{15}}\left(\frac{9}{2^{14/3}}\cdot\frac{5}{42}\mathrm{s}^{14/3}\right)^3\mathrm{d}\omega\\ =\ &q_0\frac{45^3}{2^{14}}\cdot\frac{1}{42^3}\limsup_{\mathrm{s}\to\infty}\mathrm{s}^{14}\int_{\mathrm{s}/2}^{\mathrm{s}}\frac{1}{\omega^{15}}\mathrm{d}\omega\\ =\ &q_0\frac{45^3}{2^{14}}\cdot\frac{1}{42^3}\limsup_{\mathrm{s}\to\infty}\mathrm{s}^{14}\left(\frac{2^{14}-1}{14\mathrm{s}^{14}}\right)\\ =\ &\frac{45^3}{2^{14}}\cdot\frac{1}{42^3}\left(\frac{2^{14}-1}{14}\right)q_0 = 0.08785q_0,\end{aligned}$$

which holds when

$$0.08785q_0 > \frac{1}{\left(1-\frac{1}{350}\right)^3} \Longrightarrow q_0 > 11.485.$$

For condition (3.10) with $\rho(\mathrm{s}) = \mathrm{s}^{14}$, we obtain for large s:

$$\frac{\pi^*\left(\frac{3\mathrm{s}}{2},1\right)}{\pi^*\left(\frac{\mathrm{s}}{2},1\right)} \sim 3^{14/3}.$$

Then

$$
\begin{aligned}
& \limsup_{s\to\infty}\int_1^s\left[\omega^{14}\frac{q_0}{\omega^{15}}\left(1-\frac{1}{180}\frac{\pi^*\left(\frac{3\omega}{2},1\right)}{\pi^*\left(\frac{\omega}{2},1\right)}\right)^3-\frac{1}{4^4}\frac{\left[14\omega^{13}\right]^4}{\left[\omega^{14}\cdot\frac{1}{2}\right]^3}\left(15\left(\frac{\omega}{2}\right)^2\cdot\frac{3}{5}\omega^{5/3}\right)^{-3}\right]d\omega \\
= & \limsup_{s\to\infty}\int_1^s\left[\frac{q_0}{\omega}\left(1-\frac{1}{350}\frac{\frac{27}{14}\left(\frac{3\omega}{2}\right)^{14/3}}{\frac{27}{14}\left(\frac{3\omega}{2}\right)^{14/3}}\right)^3-\frac{105.39}{\omega}\right]d\omega \\
= & \limsup_{s\to\infty}\int_1^s\left[q_0\left(1-\frac{3^{14/3}}{350}\right)^3-105.39\right]\frac{1}{\omega}d\omega \\
= & \left[0.1395q_0-105.39\right]\limsup_{s\to\infty}\ln s=\infty,
\end{aligned}
$$

which holds when

$$q_0 > 755.6.$$

Therefore, by Theorems 3.1–3.3, every solution of (4.5) is oscillatory for $q_0 > 755.6$. The results in [19, 24, 25] do not apply to Eq (4.5); the present study extends and improves them.

5. Conclusions

The study of oscillation in neutral differential equations is a fundamental direction in the theory of differential equations, given its pivotal role in exploring the qualitative structure of solutions. Despite the numerous results available in the literature, most are limited to linear or quasi-linear cases, or they assume strict conditions that limit their generality. In this paper, we developed new and sufficient criteria for the oscillation of all solutions of third-order nonlinear neutral differential equations in the standard case, by innovatively combining the comparison principle with first-order equations and the Riccati substitution technique. This methodology results in the formulation of more flexible and accurate conditions, contributing to the expansion of known results in this field. Remarkably, these results open promising horizons for future research aimed at studying oscillation in higher-order neutral equations or those involving sublinear terms such as

$$\left(\mathfrak{a}_2(s)\left[\left(\mathfrak{a}_1(s)\left(\left(x(s)+p(s)x^{\gamma}(\tau(s))\right)'\right)^{\alpha_1}\right)'\right]^{\alpha_2}\right)'+q(s)x^{\beta}(\kappa(s))=0,$$

thus promoting the construction of a more comprehensive and in-depth theory of oscillation.

Author contributions

Fahd Masood and Khalil S. Al-Ghafri: Methodology, Investigation, Writing–original draft preparation, Writing–review and editing; Alanoud Almutairi: Methodology, Investigation; Loredana Florentina Iambor: Methodology, Investigation, Writing–original draft preparation; Omar Bazighifan: Methodology, Investigation, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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