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Research article

Novel optical solitons for the cubic–quintic nonlinear Schrödinger equation with an additional anti-cubic nonlinear term using the symmetry group method

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Abstract: This paper examines the cubic-quintic nonlinear Schrödinger equation (CQNLSE) with an additional anti-cubic nonlinear term, by using Stainberg's symmetry technique. The CQNLSE with the additional anti-cubic nonlinear term is a generalized model of higher-order nonlinear effects, offering a more accurate description of optical pulse propagation in nonlinear media with complex nonlinear responses, which makes the CQNLSE have a wide range of applications in several fields like optics, communications, spectroscopy, and computing. In our study, we used symmetry group analysis to derive a finite Lie group of transformations, and as a result, a novel similarity transformation, not previously reported in the literature, was obtained from this group. By using this transformation, the CQNLSE with the anti-cubic term was reduced to a nonlinear ordinary differential equation, which can be solved using the Jacobi elliptic expansion method, and a variety of wave solutions were obtained. These solutions include periodic waves, kink solitons, and bright solitons, contain other solutions, shown in the previous literature. We have also introduced a new solution, which has not been achived before in studies. The 3D and 2D plots of the periodic sn wave and its limit as a kink solitary wave were given to declare the dynamical behavior of the wave propagation by controlling the parameters contained in the solution.

Keywords: symmetry group method; cubic-quintic nonlinear Schrödinger equation with an additional anti-cubic nonlinear term; optical solitons
Mathematics Subject Classification: 35-XX, 35C08

1. Introduction

The nonlinear Schrödinger equation (NLSE) and its extensions play a vital role in understanding and modeling the propagation of optical waves in various nonlinear media. This equation, which incorporates the effects of nonlinearity, dispersion, and attenuation, provides a fundamental framework for studying phenomena such as solitons, supercontinuum generation, and optical switching [1–3].

Solitons, which are very important for their self-localized optical pulses that maintain their shape and intensity over long distances, are described by specific solutions of the NLSE. These solitons have a wide range of applications in optical communication systems, where they can be used to transmit information efficiently and reliably [4–6].

Supercontinuum generation, the process of generating a broad spectrum of optical frequencies from a single input pulse, is another area where the NLSE and its extensions are crucial. This phenomenon provides applications in spectroscopy, optical coherence tomography, and metrology [7].

Optical switching, the ability to control the propagation of light using nonlinear effects, is also described by the NLSE. This technology has potential applications in optical computing and all-optical networks [8].

Therefore, the nonlinear Schrödinger equation and its extensions provide a powerful tool for understanding and modeling the complex behavior of optical waves in nonlinear media. These equations have broad applications in various fields of optics, including communications, spectroscopy, and computing.

Regarding the previously highlighted importance for NLSEs extensions, we have chosen to study the cubic-quintic nonlinear Schrödinger equation (CQNLSE) with an additional anti-cubic nonlinear term which is consider the generalization form of the CQNLSE that incorporates higher-order nonlinear effects. It is used to describe the propagation of optical pulses in nonlinear media where the nonlinear response is more complex than the simple cubic-quintic model and can be written as [9, 10]:

$$i\vartheta_{\tau} + \alpha\vartheta_{\xi\xi} + \left(\beta\,|\vartheta|^{-4} + \lambda\,|\vartheta|^{2} + \delta\,|\vartheta|^{4}\right)\vartheta = 0, \tag{1.1}$$

where τ and ξ are the time and spatial variables, $\vartheta = \vartheta(\tau, \xi)$ is a complex wave function denoting the soliton profile, where the first term depicts the temporal evolution, $i^2 = -1$, α represents the group velocity dispersion, and β, λ, and δ are anti-cubic, cubic, and quintic nonlinearities, respectively. When $\beta = 0$, it denotes the soliton profile parabolic law nonlinearity [11, 12]. The anti-cubic term can be introduced to stabilize solutions that would otherwise collapse or blow up in finite time under the influence of other terms [13]. The CQNLSE with an additional anti-cubic nonlinear term has been solved by using many traveling wave methodologies like the conservation laws [9], mapping method [10], tanh-coth scheme, modified simple equation method [11], polynomial method [12], and Jacobi elliptic function method [14], in addition to classical Lie group analysis [15].

This paper employs symmetry group analysis to derive a novel transformation that reduces Eq (1.1) to a nonlinear ordinary differential equation (ODE). The traveling wave transformations used previously were without any references reported in literature, this is the reason why it takes that form. So in our study, we have chosen the symmetry group due to the evidence in [9–12, 14]. This approach is distinct from the previous classical Lie group study [15] because it yields a general transformation that encompasses all the special cases derived in [15], where the authors only solved for specific vector fields. The reduced ODE is then solved using the Jacobi expansion method to obtain new optical solitary wave solutions in various forms, which cover the two cases given in [14] and provide many other solutions. Finally, we analyze the dynamic behavior of periodic waves, kink solitons, and bright solitons to highlight their intriguing properties.

2. Symmetry group analysis

Despite the emergence of various methods for solving nonlinear partial differential equations (PDEs), Lie group analysis and its associated symmetry methods remain a cornerstone [16–18]. Many of these methods rely on transformations to reduce the complexity of the PDE before proceeding with further steps. These transformations often originate from Lie group theory, making it a fundamental tool in the field of PDEs. The following shows the main steps of symmetry group analysis [19–21]:

For any two differential operators Γ_1 and Γ_2 given by

$$\Gamma_1(u) = \frac{\partial u}{\partial \tau} - L_1(u, v), \tag{2.1}$$

$$\Gamma_2(v) = \frac{\partial v}{\partial \tau} - L_2(u, v), \tag{2.2}$$

where $u = u(\tau, \xi)$, $v = v(\tau, \xi)$, and L_1, L_2 are functions on τ, ξ, u, and v, then the symmetry operators can be defined as:

$$S_1(u) = C(\tau, \xi, u, v)\frac{\partial u}{\partial \tau} + D(\tau, \xi, u, v)\frac{\partial u}{\partial \xi} + E_1(\tau, \xi, u, v), \tag{2.3}$$

$$S_2(v) = C(\tau, \xi, u, v)\frac{\partial v}{\partial \tau} + D(\tau, \xi, u, v)\frac{\partial v}{\partial \xi} + E_2(\tau, \xi, u, v). \tag{2.4}$$

The Frèchet derivatives f_1 and f_2 can be written as

$$f_1(\Gamma_1, u, v) = \frac{d}{d\epsilon}\Gamma_1(u + \epsilon S_1)|_{\epsilon=0}, \tag{2.5}$$

$$f_2(\Gamma_2, u, v) = \frac{d}{d\epsilon}\Gamma_2(v + \epsilon S_2)|_{\epsilon=0}. \tag{2.6}$$

Once we obtain the Fréchet derivatives, we substitute the expressions for $\frac{\partial u}{\partial \tau}$ and $\frac{\partial v}{\partial \tau}$ (and their derivatives) from Eqs (2.1) and (2.2). By setting all coefficients for the derivatives of u and v to be zero, we achive a partial differential system. Solving this system yields the infinitesimal generators C, D, E_1, E_2. Finally, we solve the characteristic equations to determine the transformation and reduce the original system to a nonlinear ordinary differential system.

3. Lie group and novel wave solutions for Eq (1.1)

At first, we assumed that the CQNLSE with an additional anti-cubic nonlinear term has a solution in the form [19–21]:

$$\vartheta(\tau, \xi) = u(\tau, \xi)\, e^{iv(\tau,\xi)}. \tag{3.1}$$

By separating the real and imaginary parts, the following partial differential system is obtained:

$$\delta u^8 + \lambda u^6 - \alpha u^4 v_\xi^2 - u^4 v_\tau + \alpha u^3 u_{\xi\xi} + \beta = 0, \tag{3.2}$$

$$\alpha u^4 v_{\xi\xi} + 2\alpha u^3 u_\xi v_\xi + u^3 u_\tau \ = \ 0. \tag{3.3}$$

From the symmetry technique steps, Eqs (3.2) and (3.3) can be rewritten in the folllowing form:

$$\Gamma_1(u) \ = \ \frac{\partial u}{\partial \tau} - \left(-\alpha u v_{\xi\xi} - 2\alpha u_\xi v_\xi\right), \tag{3.4}$$

$$\Gamma_2(v) \ = \ \frac{\partial v}{\partial \tau} - \left(\delta u^4 + \lambda u^2 - \alpha v_\xi^2 + \frac{\alpha}{u} u_{\xi\xi} + \frac{\beta}{u^4}\right). \tag{3.5}$$

Then the Frèchet derivatives f_1 and f_2 are given by

$$\begin{aligned} f_1 \ &= \ 8\delta u^7 S_1 + 6\lambda u^5 S_1 - 4\alpha u^3 v_\xi^2 S_1 - 2\alpha u^4 v_\xi S_{2\xi} \\ &\quad -4u^3 v_\tau S_1 - u^4 S_{2\tau} + 3\alpha u^2 u_{\xi\xi} S_1 + \alpha u^3 S_{1\xi\xi}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} f_2 \ &= \ 4\alpha u^3 v_{\xi\xi} S_1 + \alpha u^4 S_{2\xi\xi} + 6\alpha u^2 u_\xi v_\xi S_1 + 2\alpha u^3 S_{1\xi} v_\xi \\ &\quad +2\alpha u^3 u_\xi S_{2\xi} + u^3 S_{1\tau} + 3u^2 u_\tau S_1. \end{aligned} \tag{3.7}$$

By using the symmetry operators defined in Eqs (2.3) and (2.4) in Eqs (3.5) and (3.6), together with substituting the drivatives of $\frac{\partial u}{\partial \tau}$ and $\frac{\partial v}{\partial \tau}$ to make the differentaition of u, v with respect to τ disappear, then equating all partial drivatives of u, v with respect to ξ with zero, the following partial differential system is obtained:

$$\begin{gathered} C_\xi = 0, E_{1uu} = 0, E_{1uv} = 0, E_{1\xi u} = 0, 2E_{1\xi u} + D_{\xi\xi} = 0, \\ u^2 E_{2u} + E_{1v} = 0, 2\alpha E_{1\xi v} - 2u\alpha E_{2\xi} - uD_\tau = 0, \\ E_{1vv} - 2uD_\xi - uE_{2v} + uC_\tau = 0, \\ 2uD_\xi + uE_{1u} - E_1 - uC_\tau - uE_{2v} = 0, \\ 2u^5\lambda E_1 + 4u^7\delta E_1 - u^8\delta C_\tau - u^6\lambda E_{2v} - u^6\lambda C_\tau - u^8\delta E_{2v} \\ -\frac{4\beta}{u} E_1 + \alpha u^3 E_{1\xi\xi} - u^4 E_{2\tau} - \beta C_\tau - \beta E_{2v} = 0. \end{gathered} \tag{3.8}$$

C, D, E_1, and E_2 can be determined by solving system (3.8) using Maple:

$$C = a_1, \ D = -2a_2\alpha\tau + a_3, \ \ E_1 = 0, \ \ E_2 = a_2\xi + a_4, \tag{3.9}$$

where a_1, a_2, a_3, and a_4 are constants, whereas in a physical context, they would typically relate to the velocity of a soliton or measurable quantities like gain/loss, background potential, or initial conditions. Therefore, we can construct the following four-vector Lie group:

$$\chi_1 = \frac{\partial}{\partial \tau}, \chi_2 = -2\alpha\tau\frac{\partial}{\partial \xi} + \xi\frac{\partial}{\partial v}, \chi_3 = \frac{\partial}{\partial \xi}, \chi_4 = \frac{\partial}{\partial v}, \tag{3.10}$$

with the commutator relations $[\chi_i, \chi_i] = 0$, $[\chi_1, \chi_2] = -2\alpha\chi_3$, $[\chi_1, \chi_3] = 0$, $[\chi_1, \chi_4] = 0, [\chi_2, \chi_1] = 2\alpha\chi_3, [\chi_2, \chi_3] = -\chi_4, [\chi_2, \chi_4] = 0, [\chi_3, \chi_1] = 0, [\chi_3, \chi_2] = \chi_4, [\chi_3, \chi_4] = 0, [\chi_4, \chi_i] = 0, \forall i = 1, 2, 3, 4$.

We have taken the general case of all vector fields as a linear combination, which is defined by:

$$a_1\chi_1 + a_2\chi_2 + a_3\chi_3 + a_4\chi_4. \tag{3.11}$$

Since the similarity variables can be determined by solving the characteristic equation

$$\frac{d\tau}{a_1} = \frac{d\xi}{-2a_2\alpha\tau + c_3} = \frac{-du}{0} = \frac{-dv}{a_2\xi + a_4}, \tag{3.12}$$

we get

$$\begin{aligned} \eta &= a_1\xi + a_2\alpha\tau^2 - a_3\tau,\ u = U(\eta), \\ v &= V(\eta) + \frac{a_4}{a_1}\tau - \frac{a_3a_2}{2a_1^2}\tau^2 + \frac{a_2^2}{3a_1^2}\alpha\tau^3 - \frac{a_2}{a_1^2}\left(a_1\xi + a_2\alpha\tau^2 - a_3\tau\right)\tau. \end{aligned} \tag{3.13}$$

By using Eq (3.13) in systems (3.2) and (3.3), this can be reduced to the folllowing nonlinear ordinary system:

$$\delta U^8 + \lambda U^6 - a_1^2\alpha U^4 V'^2 + a_3 U^4 V' - \frac{a_4}{a_1}U^4 + \frac{a_2}{a_1^2}\eta U^4 + \alpha a_1^2 U^3 U'' + \beta = 0, \tag{3.14}$$

$$\alpha a_1^2 U^2 V'' + 2\alpha a_1^2 U U' V' - a_3 U U' = 0. \tag{3.15}$$

Integrate Eq (3.15) with respect to η,

$$V' = \frac{a_3}{2\alpha a_1^2} + \frac{b_1}{U^2}, \tag{3.16}$$

where b_1 is the integration constant. Then inserting (3.16) into (3.14), we get

$$\delta U^8 + \lambda U^6 + \left(\frac{a_2}{a_1^2}\eta - \frac{a_4}{a_1} - \frac{a_3^2}{4\alpha a_1^2}\right)U^4 + \alpha a_1^2 U^3 U'' + \beta - \alpha a_1^2 b_1^2 = 0. \tag{3.17}$$

Assume that

$$U(\eta) = F^{\frac{1}{2}}(\eta). \tag{3.18}$$

Therefore, Eq (3.17) becomes

$$\delta F^4 + \lambda F^3 - \left(\frac{a_4}{a_1} - \frac{a_2}{a_1^2}\eta + \frac{a_3^2}{4\alpha a_1^2}\right)F^2 + \frac{\alpha a_1^2}{2}\left(F F'' - \frac{F'^2}{2}\right) + \beta - \alpha a_1^2 b_1^2 = 0. \tag{3.19}$$

To solve Eq (3.19), assume $a_2 = 0$, and by using the Jacobi expansion technique [25, 26],

$$F(\eta) = \sum_{i=0}^{M} \Lambda_i \varphi^i(\eta), \tag{3.20}$$

where Λ_i are constants to be found later. M is an integer determined from the balance between the terms F^4 and FF'', so $M = 1$. Therefore,

$$F(\eta) = \Lambda_0 + \Lambda_1\varphi(\eta), \tag{3.21}$$

where $\varphi(\eta)$ satisfies the following elliptic equation:

$$\left(\frac{d\varphi(\eta)}{d\varsigma}\right)^2 = r_0 + r_2\varphi(\eta)^2 + r_4\varphi(\eta)^4, \tag{3.22}$$

where r_0, r_2, r_4 are constants with known values. By substituting from (3.21) and (3.22) in Eq (3.19), sorting the powers of $\varphi(\eta)$, and equating it to zero, the following system of algebraic equations is given:

$$\begin{aligned}
3\alpha^2 a_1^4 r_4 \Lambda_1^2 + 4\alpha a_1^2 \delta \Lambda_1^4 &= 0,\\
\alpha^2 a_1^4 r_4 \Lambda_0 \Lambda_1 + 4\alpha a_1^2 \delta \Lambda_0 \Lambda_1^3 + \alpha a_1^2 \lambda \Lambda_1^3 &= 0,\\
\alpha^2 a_1^4 \Lambda_1^2 r_2 + 24\alpha a_1^2 \delta \Lambda_0^2 \Lambda_1^2 + 12\alpha a_1^2 \lambda \Lambda_0 \Lambda_1^2 + 4\alpha a_1 a_4 \Lambda_1^2 - a_3^2 \Lambda_1^2 &= 0,\\
\alpha^2 a_1^4 \Lambda_0 \Lambda_1 r_2 + 8\alpha a_1^2 \delta \Lambda_0^3 \Lambda_1 + 6\alpha a_1^2 \lambda \Lambda_0^2 \Lambda_1 - 4\alpha a_1 a_4 \Lambda_0 \Lambda_1 + a_3^2 \Lambda_0 \Lambda_1 &= 0,\\
\alpha^2 a_1^4 \Lambda_1^2 r_0 + 4b_1^2 \alpha^2 a_1^4 - 4\alpha a_1^2 \delta \Lambda_0^4 - 4\alpha a_1^2 \lambda + \Lambda_0^3 + 4\alpha a_1 a_4 \Lambda_0^2 - 4\alpha\beta a_1^2 - a_3^2 \Lambda_0^2 &= 0. \qquad (3.23)
\end{aligned}$$

By solving the above system, the following solutions are considered:

$$\begin{aligned}
\Lambda_0 &= -\frac{S}{3\alpha\lambda a_1^2}, \Lambda_1 = \frac{1}{\lambda}\sqrt{\frac{2}{3} r_4 S}, \text{where } \delta = \frac{9\alpha\lambda^2 a_1^2}{8S}, S = \left(\alpha^2 a_1^4 r_2 - 4\alpha a_1 a_4 + a_3^2\right),\\
\beta &= \frac{1}{216\alpha^3 a_1^6 \lambda^2}\left(\alpha^6 a_1^{12}\left(5r_2^3 - 36 r_0 r_2 r_4\right) + 36\alpha^5 a_1^9 a_4 \left(4 r_0 r_4 - r_2^2\right) + 216 b_1^2 \alpha^4 a_1^8 \lambda^2 - 9\alpha^4 a_1^8 a_3^2 \left(4 r_0 r_4 - r_2^2\right)\right.\\
&\left. +48\alpha^4 a_1^6 a_4^2 r_2 - 24\alpha^3 a_1^5 a_3^2 a_4 r_2 + 3\alpha^2 a_1^4 a_3^4 r_2 + 64\alpha^3 a_1^3 a_4^3 - 48\alpha^2 a_1^2 a_3^2 a_4^2 + 12\alpha a_1 a_3^4 a_4 - a_3^6\right). \qquad (3.24)
\end{aligned}$$

By using the known values of r_1, r_2, and r_4 as given in [22–24], the following new Jacobi periodic wave solutions are obtained for Eq (3.19):

$$F_1 = \frac{\alpha^2 a_1^4 \left(m^2+1\right) + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{m}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 (m^2+1) + 4\alpha a_1 a_4 - a_3^2\right)} sn(\eta, m), \qquad (3.25)$$

$$F_2 = \frac{\alpha^2 a_1^4 \left(m^2+1\right) + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{m}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 (m^2+1) + 4\alpha a_1 a_4 - a_3^2\right)} cd(\eta, m), \qquad (3.26)$$

$$F_3 = -\frac{\alpha^2 a_1^4 \left(2m^2-1\right) - 4\alpha a_1 a_4 + a_3^2}{3\lambda\alpha a_1^2} + \frac{m}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 (2m^2-1) - 4\alpha a_1 a_4 + a_3^2\right)} cn(\eta, m), \qquad (3.27)$$

$$F_4 = -\frac{\alpha^2 a_1^4 \left(2-m^2\right) - 4\alpha a_1 a_4 + a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 (2-m^2) - 4\alpha a_1 a_4 + a_3^2\right)} dn(\eta, m), \qquad (3.28)$$

$$F_5 = \frac{\alpha^2 a_1^4 \left(1+m^2\right) + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 (1+m^2) + 4\alpha a_1 a_4 - a_3^2\right)} ns(\eta, m), \qquad (3.29)$$

$$F_6 = \frac{\alpha^2 a_1^4 \left(1+m^2\right) + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 (1+m^2) + 4\alpha a_1 a_4 - a_3^2\right)} dc(\eta, m), \qquad (3.30)$$

$$F_7 = \frac{\alpha^2 a_1^4 \left(1-2m^2\right) + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}(m^2-1)\left(\alpha^2 a_1^4 (2m^2-1) - 4\alpha a_1 a_4 + a_3^2\right)} nc(\eta, m), \qquad (3.31)$$

$$F_8 = \frac{\alpha^2 a_1^4 \left(m^2-2\right) + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}(1-m^2)\left(\alpha^2 a_1^4 (2-m^2) - 4\alpha a_1 a_4 + a_3^2\right)} nd(\eta, m), \qquad (3.32)$$

$$F_9 = \frac{\alpha^2 a_1^4 \left(m^2-2\right) + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}(m^2-1)\left(\alpha^2 a_1^4 (2-m^2) - 4\alpha a_1 a_4 + a_3^2\right)} sc(\eta, m), \qquad (3.33)$$

$$F_{10} = \frac{\alpha^2 a_1^4 \left(1-2m^2\right) + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}m^2(1-m^2)\left(\alpha^2 a_1^4 (2m^2-1) - 4\alpha a_1 a_4 + a_3^2\right)} sd(\eta, m), \qquad (3.34)$$

$$F_{11} = \frac{\alpha^2 a_1^4\left(m^2-2\right)+4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4\left(m^2-2\right)+4\alpha a_1 a_4 - a_3^2\right)}cs\left(\eta, m\right), \tag{3.35}$$

$$F_{12} = \frac{\alpha^2 a_1^4\left(m^2-\frac{1}{2}\right)+4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{2\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4\left(m^2-\frac{1}{2}\right)+4\alpha a_1 a_4 - a_3^2\right)}\left(ns\left(\eta, m\right) \pm cs\left(\eta, m\right)\right), \tag{3.36}$$

$$F_{13} = \frac{4\alpha a_1 a_4 - \frac{\alpha^2 a_1^4}{2}\left(m^2+1\right) - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{2\lambda}\sqrt{\frac{2}{3}\left(m^2-1\right)\left(\frac{\alpha^2 a_1^4}{2}\left(m^2+1\right)-4\alpha a_1 a_4 + a_3^2\right)}\left(nc\left(\eta, m\right) \pm sc\left(\eta, m\right)\right).$$

When the module of the previous Jacobi functions becomes 1, the following new hyperbolic functions are obtained:

$$F_{14} = \frac{2\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(2\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2\right)}\tanh(\eta), \tag{3.37}$$

$$F_{15} = \frac{4\alpha a_1 a_4 - \alpha^2 a_1^4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 - 4\alpha a_1 a_4 + a_3^2\right)}sech(\eta), \tag{3.38}$$

$$F_{16} = \frac{2\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(2\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2\right)}\coth(\eta), \tag{3.39}$$

$$F_{17} = \frac{4\alpha a_1 a_4 - \alpha^2 a_1^4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(4\alpha a_1 a_4 - \alpha^2 a_1^4 - a_3^2\right)}\ \mathrm{csch}(\eta), \tag{3.40}$$

$$F_{18} = \frac{\frac{1}{2}\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{1}{6}\left(\frac{1}{2}\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2\right)}\left(\coth(\eta) \pm \mathrm{csch}(\eta)\right). \tag{3.41}$$

As the module of the Jacobi functions becomes zero, the following periodic functions achieved:

$$F_{19} = \frac{4\alpha a_1 a_4 - 2\alpha^2 a_1^4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(4\alpha a_1 a_4 - 2\alpha^2 a_1^4 - a_3^2\right)}\tan(\eta), \tag{3.42}$$

$$F_{20} = \frac{\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2\right)}\sec(\eta), \tag{3.43}$$

$$F_{21} = \frac{4\alpha a_1 a_4 - 2\alpha^2 a_1^4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(4\alpha a_1 a_4 - 2\alpha^2 a_1^4 - a_3^2\right)}\cot(\eta), \tag{3.44}$$

$$F_{22} = \frac{\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{2}{3}\left(\alpha^2 a_1^4 + 4\alpha a_1 a_4 - a_3^2\right)}\csc(\eta), \tag{3.45}$$

$$F_{23} = \frac{4\alpha a_1 a_4 - \frac{1}{2}\alpha^2 a_1^4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{\lambda}\sqrt{\frac{1}{6}\left(4\alpha a_1 a_4 - \frac{1}{2}\alpha^2 a_1^4 - a_3^2\right)}\left(\csc(\eta) \pm \cot(\eta)\right), \tag{3.46}$$

$$F_{24} = \frac{4\alpha a_1 a_4 - \frac{1}{2}\alpha^2 a_1^4 - a_3^2}{3\lambda\alpha a_1^2} + \frac{1}{2\lambda}\sqrt{\frac{2}{3}\left(4\alpha a_1 a_4 - \frac{1}{2}\alpha^2 a_1^4 - a_3^2\right)}\left(\sec(\eta) \pm \tan(\eta)\right). \tag{3.47}$$

Novel solutions are obtained for the CQNLSE by back substitution from (3.25)–(3.47) into (3.18), (3.13), and (3.16) under conditions given in (3.24) for δ and β; the following solution are attained:

$$\vartheta(\tau,\xi) = F_j^{\frac{1}{2}}(a_1\xi - a_3\tau)e^{i(\frac{a_3}{2\alpha a_1^2}(a_1\xi - a_3\tau) + \frac{b_1}{F_j(\varsigma)} + \frac{a_4}{a_1}\tau)}, j = 1, ..., 24. \tag{3.48}$$

To physically interpret the obtained solutions given by Eq (3.48), we have found that the traveling wave transformation $\eta = a_1\xi - a_3\tau, a_1$is the wave number component and a_3 is the frequency. F_j can describe the amplitude envelope of the soliton, where $\frac{1}{a_1}$ is proportional to the characteristic width of the soliton. Moreover, η can be rewritten as $\eta = a_1(\xi - \frac{a_3}{a_3}\tau)$, and therefore the group velocity of the soliton's envelope is given by $V_g = \frac{a_3}{a_3}$. According to that, a_1, along with a_3, both determine the speed at which the soliton travels, making it an essential parameter defining the soliton's motion [9–11].

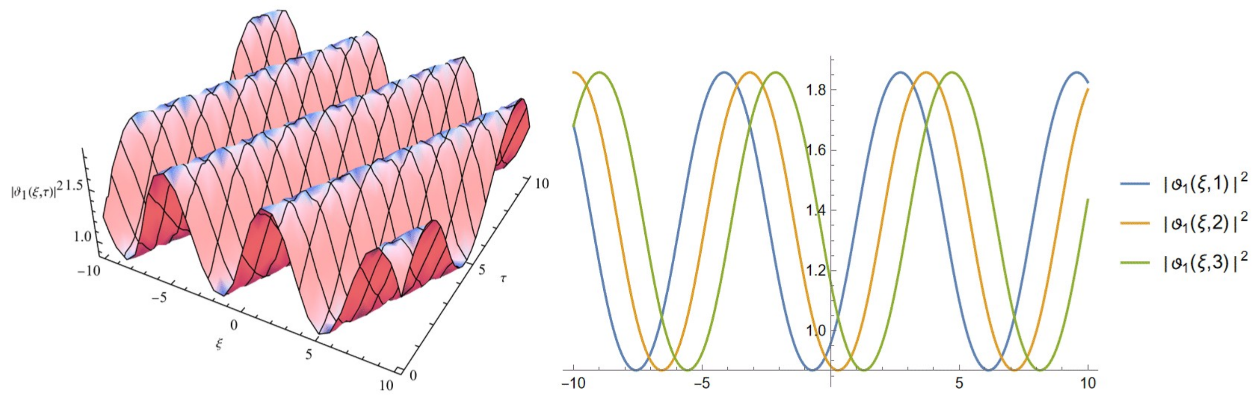
4. Results and discussion

The ability of solitary waves or solitons to propagate in a non-dispersive manner allows them to be useful in a range of scientific and engineering application fields. They are used in geophysics and fluid dynamics to study non-dispersive internal ocean waves and the Morning Glory cloud. They are also used in condensed matter physics to explain the flow of energy in crystal lattices and the formation of skyrmions. Furthermore, in the study of acoustics and the propagation of pressured pulses in two-phase systems, especially bubbly liquids, solitons are of very useful in such areas [25–27].

Optical solitary waves are fascinating phenomena in nonlinear optics that have significant implications for various applications, including telecommunications, optical computing, and materials science. These self-sustaining pulses of light can propagate long distances without dispersing or changing their shape, which can be considered as the most critical commercial application [28–30].

To visualize this unique behavior, we will plot the the intensity of the wave solution $|\vartheta_1(\tau,\xi)|^2$ obtained in the previous section. This will allow us to investigate the wave's profile and dynamics.

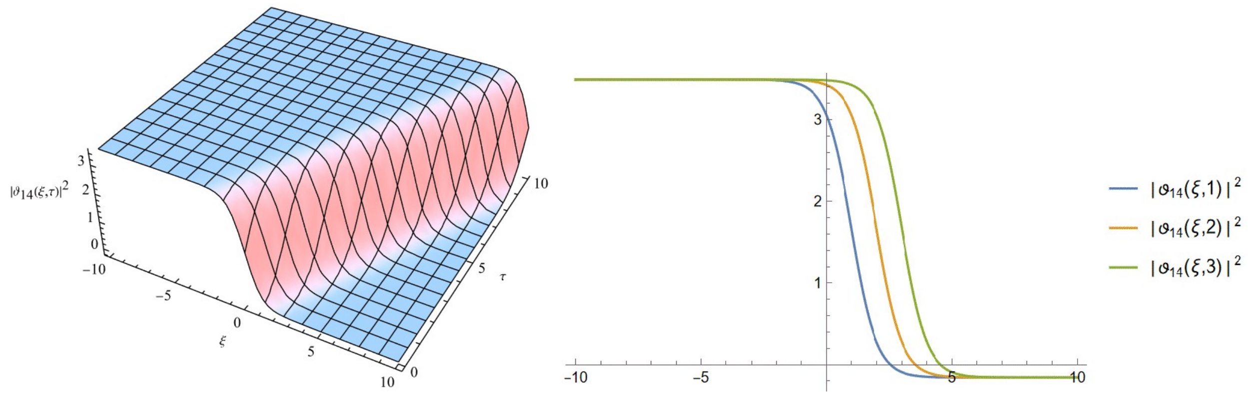
Figure 1 shows the 3D and 2D plots of the periodic wave amplitude $|\vartheta_1(\tau,\xi)|^2$according to the Jacobi sn function solution given by Eq (3.25), where the parameters are chosen to be $a_1 = a_3 = a_4 = b_1 = 1, \alpha = \lambda = 1, m = 0.3$, and according to the consideration of condition (3.24), $\beta = 0.872$ and $\delta = -0.275$.



(a) (b)

Figure 1. (a) The intensity of the periodic wave ϑ_1. (b) The 2D plot of the intensity of the periodic wave ϑ_1.

Figure 2 shows the 3D and 2D plots of the kink soliton amplitude $|\vartheta_{14}(\tau, \xi)|^2$, where the module m et al. tends to 1 in the Jacobi sn function solution given by Eq (3.25), and the parameters are given as $a_1 = a_3 = a_4 = b_1 = 1, \alpha = \lambda = 1$. Therefore, $\beta = 1.023$ and $\delta = 0.225$.

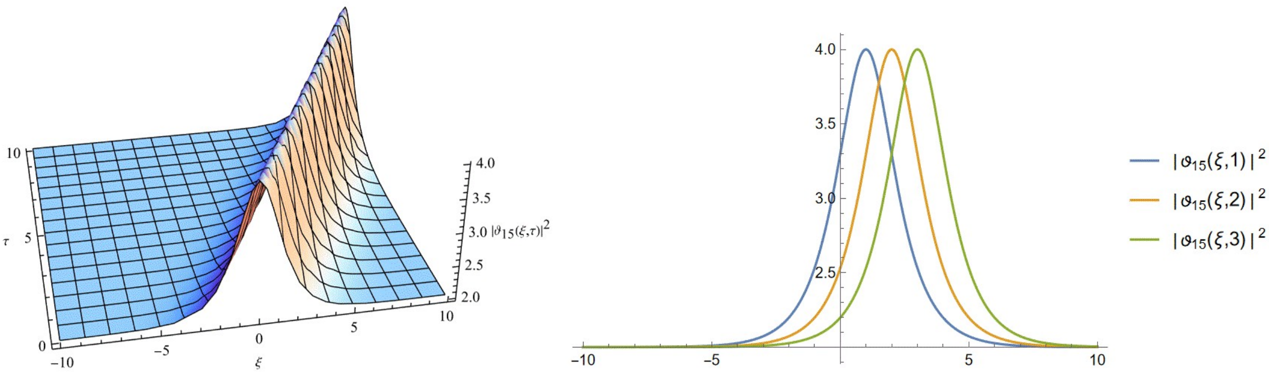


(a) (b)

Figure 2. (a) The intensity of the kink wave soliton ϑ_{14}. (b) The 2D plot of the intensity kink wave soliton ϑ_{14}.

Figure 3 shows the bright soliton wave solution $|\vartheta_{15}(\tau, \xi)|^2$ given by Eq (3.25), and the parameters are given as $a_1 = -1, a_3 = a_4 = b_1 = 1, \alpha = \lambda = 1$. Therefore, $\beta = 1$ and $\delta = 0.1875$.

In all Figures 1–3, the periodic, kink, and bright soliton waves were plotted with fixed values. $a_3 = a_4 = b_1 = 1, \alpha = \lambda = 1$, and $a_1 = 1$ in Figures 1(a),(b), and 2(a),(b) but $a_1 = -1$ in Figures 3(a),(b) and in every figure, β and δ change according to the condition given in Eq (3.24). The 2D plots were done corresponding to fixed values of τ as 1, 2, and 3 and fixed line colors as blue, orange, and green, respectively. As shown in the figures, the waves exhibit stable propagation. Additionally, our solutions extend beyond those previously documented in the literature [10–16], offering a more comprehensive understanding of the system's behavior. The soliton stability of the CQNLSE with an additional anti-cubic nonlinear term was discussed recently by Xiang in [30].



(a) (b)

Figure 3. (a) The intensity of the bright wave soliton ϑ_{15}. (b) The 2D plot of the intensity bright wave soliton ϑ_{15}.

5. Conclusions

This study investigates the cubic-quintic nonlinear Schrödinger equation with an additional anti-cubic nonlinear term using symmetry analysis. A finite Lie group of transformations is derived, yielding a general transformation that encompasses the traditional traveling wave transformation employed in previous studies [9–14]. Applying the general transformation obtained from the linear combination of the vector fields, the CQNLSE with an additional anti-cubic nonlinear term is reduced to a nonlinear ordinary differential equation. By solving the reduced equation using the Jacobi expansion method, various wave solutions are obtained. Comparing those solutions with the previous solutions in literature, we have found that the solutions cover the solitary wave solutions obtained before in [9–13] and the Jacobi wave solutions obtained in [14], where ϑ_1 is the only Jacobi wave solution obtained. Therefore, the other solutions here are new. Finally, we have plotted the intensity of the wave solution $|\vartheta_1(\tau,\xi)|^2$ in both 3D and 2D plots given by Figures 1(a),(b) to show the periodic behavior of the wave. Then when the modulus of the wave solution $\vartheta_1(\tau,\xi) \to 1$, it became the kink soliton solution ϑ_{14}, so we have plotted the intensity of $|\vartheta_1(\tau,\xi)|^2$ when $a_1 = 1$. It became like a kink soliton in the 3D and 2D plots given by Figures 2(a),(b) and when $a_1 = -1$, it changed to a bright soliton wave given in Figures 3(a),(b). So the sign of a_1 is very important and affects the soliton type as it is physically presents the velocity and shape width of the solitary wave.

Author contributions

Rehab M. Elshiekh wrote and applied different methodologies and Mahmoud Gaballah made physical applications and figures. All authors read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no conflict of interest.

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