AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 29582–29594.
DOI: 10.3934/math.20251299
Received: 05 November 2025
Revised: 03 December 2025
Accepted: 08 December 2025
Published: 15 December 2025

Research article

Norm of the point evaluation functionals on the Zygmund-type spaces

Stevo Stević[1,2,3,*]

[1] Mathematical Institute of the Serbian Academy of Sciences and Arts, Knez Mihailova 36/III, 11000 Beograd, Serbia

[2] Serbian Academy of Sciences and Arts, Knez Mihailova 35, 11000 Beograd, Serbia

[3] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: sscite1@gmail.com.

Abstract: Norms of the point evaluation functionals $\delta_z(f) = f(z)$ on the Zygmund-type spaces $\mathcal{Z}^\alpha(\mathbb{D})$, where $\alpha > 0$, with suitably chosen standard weights on the open unit disk $\mathbb{D}$ in the complex plane, are calculated. More precisely, when $\alpha = 1$ we calculate the norm of the functional δ_z on the quotient space $\mathcal{Z}/\mathbb{P}_1$ for the weight $(1 - |z|^2)^\alpha$, whereas when $\alpha \in (0, +\infty)$ we calculate the norm of the functional on the quotient space $\mathcal{Z}^\alpha/\mathbb{P}_1$ for the weight $(1 - |z|)^\alpha$, where $\mathbb{P}_1$ is the space of linear holomorphic functions (a two-dimensional linear space). The choice of the weight functions is caused by the fact that for such chosen weight functions the norms of the point evaluation functionals can be calculated due to the possibility of calculating some integrals that appear during the calculation of the norms of the functionals.

Keywords: point evaluation functional; norm of a functional; Zygmund-type spaces; holomorphic functions; open unit disc
Mathematics Subject Classification: 47A30, 47B38

1. Introduction and preliminaries

Throughout the paper the set of positive natural numbers (i.e., the set $\{1, 2, 3, \ldots\}$) is denoted by $\mathbb{N}$, by $\mathbb{N}_0$ we denote the set $\mathbb{N} \cup \{0\}$, the complex plane is denoted by $\mathbb{C}$, by $\mathbb{D}$ we denote the open unit disc in $\mathbb{C}$, that is, the set $\{z \in \mathbb{C} : |z| < 1\}$, by $\partial\mathbb{D}$ we denote the boundary of $\mathbb{D}$, by $H(\mathbb{D})$ the linear space of holomorphic functions on $\mathbb{D}$, whereas by $H^\infty(\mathbb{D})$ we denote the space of bounded holomorphic functions on $\mathbb{D}$ with the supremum norm

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|$$

(for some basics on this and other spaces of holomorphic functions, see, e.g., [19]).

If X is a vector space with the norm $\|\cdot\|_X$, then by B_X we denote the closed unit ball in X, that is, the set

$$\{x \in X : \|x\|_X \leq 1\}.$$

Let $L : X \to Y$ be a linear operator between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. If there is $C \geq 0$ such that

$$\|Lx\|_Y \leq C\|x\|_X \quad \text{for} \quad x \in X,$$

then it is said that the operator is bounded.

The quantity

$$\|L\|_{X\to Y} := \sup_{x\in B_X} \|Lx\|_Y,$$

is called the norm of the linear operator L. If $Y = \mathbb{C}$, then the linear operator is a linear functional. If f is a linear functional on X, then by $\|f\|_{X^*}$ we denote the norm of the functional f. For some basic facts on bounded linear operators, see, for example, [8, 19, 20, 32].

Investigation of various spaces of holomorphic functions with some weights [1,22,31] and operators from or to them has attracted a lot of attention from many authors (see, for example, [10, 13, 24, 30] and the references cited therein).

The space consisting of all $f \in H(\mathbb{D})$ such that

$$b'_\alpha(f) := \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f''(z)| < +\infty, \tag{1.1}$$

where $\alpha > 0$, is called the Zygmund-type space with the standard weight and is denoted by $\mathcal{Z}^\alpha(\mathbb{D}) = \mathcal{Z}^\alpha$. For $\alpha = 1$, it is the Zygmund space $\mathcal{Z}(\mathbb{D}) = \mathcal{Z}$ (the notation was introduced by the present author in [15], whereas the notations for the standard and other generalizations of Zygmund spaces were introduced in later papers of ours on these spaces; see, e.g., [16, 29]; see also [3, 6, 11, 34].

The functional $b'_\alpha(\cdot)$ is a semi-norm on $\mathcal{Z}^\alpha$. A norm on this space can be introduced by

$$\|f\|_{\mathcal{Z}^{\alpha,1}} := |f(0)| + |f'(0)| + b'_\alpha(f),$$

as it was the case in [15].

Another semi-norm, which is obviously equivalent to (1.1), can be introduced by

$$b_\alpha(f) := \sup_{z\in\mathbb{D}} (1 - |z|)^\alpha |f''(z)|, \tag{1.2}$$

whereas the corresponding norm on $\mathcal{Z}^\alpha$ is introduced by

$$\|f\|_{\mathcal{Z}^\alpha} := |f(0)| + |f'(0)| + b_\alpha(f).$$

Remark 1. In some cases, it is more suitable to choose the weight function $(1 - |z|^2)^\alpha$, whereas in some other cases it is more suitable to choose the weight function $(1 - |z|)^\alpha$. Such situations have already appeared in some related investigations. For example, in [25] we managed to calculate the norm of the weighted composition operators from the standard Bloch space $\mathcal{B}(\mathbb{D}) = \mathcal{B}$ to the weighted-type space, where the norm on the Bloch space was given by

$$|f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)|,$$

whereas in [27] we managed to calculate the norm of the weighted composition operators from the α-Bloch space $\mathcal{B}^\alpha(\mathbb{D}) = \mathcal{B}^\alpha$ with $\alpha \in (0, +\infty) \setminus \{1\}$, to the weighted-type space, where the norm on the α-Bloch space was given by

$$|f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|)^\alpha |f'(z)|.$$

The different choice of the weight functions in these two cases is caused by the fact that, for such chosen weight functions, the norms of the weighted composition operators can be calculated due to the ability to calculate some integrals that appear during the calculation of the norms. This idea will also be used in the present investigation.

Note that the semi-norms in (1.1) and (1.2) are norms on the quotient space $\mathcal{Z}^\alpha/\mathbb{P}_1$, where $\mathbb{P}_1$ is the space of linear holomorphic functions (a two-dimensional linear space).

Zygmund-type spaces and various linear operators from or to them have been studied a lot (beside the already mentioned papers, see, for instance, [7, 9, 33]; see also [5, 12, 13] where the essential norms of some operators from or to the spaces have been estimated). For some generalizations of the spaces, see [4, 35].

In some of our papers, we have given some estimates for the point evaluation functionals

$$\delta_z(f) = f(z) \tag{1.3}$$

and

$$\delta'_z(f) = f'(z), \tag{1.4}$$

where z is a fixed point in $\mathbb{D}$, on some spaces of holomorphic functions on $\mathbb{D}$.

For example, in [15], we showed that

$$|f(z)| \le \|f\|_{\mathcal{Z}} \sup_{z\in\mathbb{D}}(|z| + (1 - |z|)\ln(1 - |z|)), \tag{1.5}$$

for every $f \in \mathcal{Z}$.

Let

$$g(t) := t + (1 - t)\ln(1 - t), \quad t \in [0, 1).$$

Then, we have

$$g(0) = 0 \tag{1.6}$$

and

$$g'(t) = -\ln(1 - t) > 0, \tag{1.7}$$

for $t \in (0, 1)$.

Now note that from (1.7) it follows that the function g is increasing on the interval $(0, 1)$, from which, along with (1.6), it follows that $g(t) \ge 0$ for $t \in [0, 1)$.

From this and since

$$\lim_{t\to 1-0} g(t) = 1,$$

we have

$$0 < g(t) < 1, \tag{1.8}$$

for $t \in (0,1)$.

From (1.5) and (1.8), we obtain

$$|f(z)| \le \|f\|_{\mathcal{Z}}, \tag{1.9}$$

for each $z \in \mathbb{D}$.

Relations (1.3) and (1.9) imply

$$|\delta_z(f)| \le \|f\|_{\mathcal{Z}}. \tag{1.10}$$

From (1.10), we obtain

$$\|\delta_z\|_{\mathcal{Z}^*} \le 1. \tag{1.11}$$

For the case of the unit ball, see [29]. Note also that from (1.9), it follows that

$$\|f\|_\infty \le \|f\|_{\mathcal{Z}}.$$

Calculating norms of linear operators between normed vector spaces is one of the basic problems in operator theory, which is frequently very difficult. Namely, it is a rare situation that the norm of a linear operator can be calculated in terms of including symbols and parameters. A more frequent situation is that the norm of a linear operator is estimated by some quantities that include the symbols and parameters [2, 21]. Beside [2] and [21], the following references [8, 18, 32] contain some of the classical formulas for the norms of some operators, including some functionals. Beside the already mentioned papers [25] and [27], some recent formulas for norms of some concrete linear operators can be found, for example, in [14, 17, 26, 28]. It seems that, besides the spaces with weights, the spaces of Cauchy transforms are suitable for dealing with the problem [22, 23, 31]. In some of the papers Hilbert–Schmidt norms of some operators were calculated (see, e.g., [2, 30]).

Motivated by the above-mentioned investigations, in this note, we calculate the norm of some point evaluation functionals on the Zygmund-type spaces $\mathcal{Z}^\alpha$. It should be mentioned that the formulas for the functionals that we obtained here depend on the choice of the norm that is chosen on the Zygmund-type space. Namely, when $\alpha = 1$ we calculate the norm of the point evaluation functional δ_z on the quotient space $\mathcal{Z}/\mathbb{P}_1$ for the weight function $(1-|z|^2)^\alpha$ (i.e., for the weight function $1-|z|^2$), whereas when $\alpha \in (0,+\infty)$, we calculate the norm of the functional on the quotient space $\mathcal{Z}^\alpha/\mathbb{P}_1$ for the weight function $(1-|z|)^\alpha$. The different choice of the weight functions is caused by the fact that for such chosen weight functions, the norms of the point evaluation functionals can be calculated thanks to the ability to calculate some integrals that appear during the calculation of the norms.

2. Main results

Here, we formulate and prove our main results. Before this we give some formulas that are used in the proofs of the main results.

Note that

$$\int_0^z \int_0^\zeta f''(u)dud\zeta = f(z) - f(0) - zf'(0), \tag{2.1}$$

for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

We also have

$$\begin{aligned}\left|\int_0^z \int_0^\zeta f''(u)dud\zeta\right| =& \left|\int_0^z \int_0^1 f''(\zeta t)\zeta dtd\zeta\right| \\ =& \left|z^2 \int_0^1 \int_0^1 f''(zst)dtsds\right| \\ \le& |z|^2 \int_0^1 \int_0^1 |f''(zst)|dtsds,\end{aligned} \tag{2.2}$$

for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

Theorem 1. *Let $\alpha > 0$. Then, the following statements hold:*
(a) if $\alpha = 1$, then

$$\|\delta_z\|_{\mathcal{Z}^{1,1}/\mathbb{P}_1 \to \mathbb{C}} = \frac{1}{2}\int_0^{|z|} \ln \frac{1+s}{1-s} ds; \tag{2.3}$$

(b) if $1 \neq \alpha \neq 2$, then

$$\|\delta_z\|_{\mathcal{Z}^{\alpha}/\mathbb{P}_1 \to \mathbb{C}} = \frac{(1-|z|)^{2-\alpha} + (2-\alpha)|z| - 1}{(1-\alpha)(2-\alpha)}; \tag{2.4}$$

(c) if $\alpha = 2$, then

$$\|\delta_z\|_{\mathcal{Z}^{2}/\mathbb{P}_1 \to \mathbb{C}} = \ln \frac{1}{1-|z|} - |z|; \tag{2.5}$$

(d) if $\alpha = 1$, then

$$\|\delta_z\|_{\mathcal{Z}/\mathbb{P}_1 \to \mathbb{C}} = |z| + (1-|z|)\ln(1-|z|), \tag{2.6}$$

for $z \in \mathbb{D}$.

Proof. Since $f \in \mathcal{Z}^{1,1}/\mathbb{P}_1$, we may assume that

$$f(0) = f'(0) = 0. \tag{2.7}$$

Using (2.7) in (2.1) and combining such obtained relation with the inequality in (2.2), we have

$$|f(z)| \le |z|^2 \int_0^1 \int_0^1 |f''(zst)|dtsds, \tag{2.8}$$

for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

(a) From (2.8) and the definition of the semi-norm (1.1), we have

$$\begin{aligned}|f(z)| \leq & b_1'(f)\int_0^1\int_0^1 \frac{|z|^2}{1-|z\hat{s}t|^2}dt\hat{s}d\hat{s}\\ =& b_1'(f)|z|\int_0^1\int_0^{|z|\hat{s}}\frac{du}{1-u^2}d\hat{s}\\ =& b_1'(f)\frac{|z|}{2}\int_0^1 \ln\frac{1+|z|\hat{s}}{1-|z|\hat{s}}d\hat{s}\\ =& b_1'(f)\frac{1}{2}\int_0^{|z|}\ln\frac{1+s}{1-s}ds. \end{aligned} \tag{2.9}$$

Now note that the semi-norm (1.1) is a norm on the space $\mathcal{Z}^{1,1}/\mathbb{P}_1$, so from (2.9), we get

$$\|\delta_z\|_{\mathcal{Z}^{1,1}/\mathbb{P}_1\to\mathbb{C}} \leq \frac{1}{2}\int_0^{|z|}\ln\frac{1+s}{1-s}ds. \tag{2.10}$$

Let

$$f_t(z) = \frac{1}{2}\int_0^z \ln\frac{1+t\zeta}{1-t\zeta}d\zeta, \tag{2.11}$$

where $t\in\mathbb{D}$ and $z\in\mathbb{D}$.

Then, we have $f_t(0)=0$ and

$$f_t'(z) = \frac{1}{2}\ln\frac{1+tz}{1-tz}. \tag{2.12}$$

From (2.12) it follows that

$$f_t'(0)=0 \quad \text{and} \quad f_t''(z) = \frac{t}{1-t^2z^2}.$$

Hence,

$$b_1'(f_t) = \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{|t|}{|1-t^2z^2|} \leq \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{|t|}{(1-|t^2z^2|)} \leq |t|. \tag{2.13}$$

Let

$$h_{|t|}(z) = (1-|z|^2)\frac{|t|}{(1-|t|^2|z|^2)}.$$

Then,

$$h_{|t|}(0) = |t|. \tag{2.14}$$

From (2.14), together with (2.13), we obtain

$$b_\alpha'(f_t) = |t|. \tag{2.15}$$

Further, we have

$$|\delta_z(f_t)| = \left|\frac{1}{2}\int_0^z \ln\frac{1+t\zeta}{1-t\zeta}d\zeta\right| = \left|\frac{z}{2}\int_0^1 \ln\frac{1+t\hat{s}z}{1-t\hat{s}z}d\hat{s}\right|. \tag{2.16}$$

For $t = |t|e^{-i\arg z}$, from (2.16), we get

$$|\delta_z(f_t)| = \frac{|z|}{2}\int_0^1 \ln\frac{1+|t|\hat{s}|z|}{1-|t|\hat{s}|z|}d\hat{s}. \tag{2.17}$$

Now note that the integrand in (2.17) is an increasing function in variable $|t|$. By letting $|t| \to 1-0$ in (2.17), using Bepo-Levi's theorem and a change of variables, we get

$$\begin{aligned}\lim_{|t|\to 1-0}|\delta_z(f_t)| &= \lim_{|t|\to 1-0}\frac{|z|}{2}\int_0^1 \ln\frac{1+|t|\hat{s}|z|}{1-|t|\hat{s}|z|}d\hat{s}\\ &=\frac{|z|}{2}\int_0^1 \ln\frac{1+\hat{s}|z|}{1-\hat{s}|z|}d\hat{s}\\ &=\frac{1}{2}\int_0^{|z|}\ln\frac{1+s}{1-s}ds. \end{aligned}\tag{2.18}$$

From (2.15) and (2.18), and since $|t| < 1$, we get

$$\|\delta_z\|_{\mathcal{Z}^{1,1}/\mathbb{P}_1\to\mathbb{C}} \ge \frac{1}{2}\int_0^{|z|}\ln\frac{1+s}{1-s}ds. \tag{2.19}$$

From (2.10) and (2.19), formula (2.3) follows.

(b) From (2.8) and the definition of the semi-norm (1.2), we have

$$\begin{aligned}|f(z)| \le& b_\alpha(f)\int_0^1\int_0^1 \frac{|z|^2}{(1-|z\hat{s}t|)^\alpha}dt\hat{s}d\hat{s}\\ =&b_\alpha(f)|z|\int_0^1\int_0^{|z|\hat{s}}\frac{du}{(1-u)^\alpha}d\hat{s} \qquad (2.20)\\ =&b_\alpha(f)\frac{|z|}{1-\alpha}\int_0^1(1-(1-|z|\hat{s})^{1-\alpha})d\hat{s}\\ =&\frac{b_\alpha(f)}{1-\alpha}\int_0^{|z|}(1-(1-s)^{1-\alpha})ds \qquad (2.21)\\ =&b_\alpha(f)\frac{(1-|z|)^{2-\alpha}+(2-\alpha)|z|-1}{(1-\alpha)(2-\alpha)}. \qquad (2.22)\end{aligned}$$

Since the semi-norm (1.2) is a norm on the space $\mathcal{Z}^\alpha/\mathbb{P}_1$, from (2.22), we get

$$\|\delta_z\|_{\mathcal{Z}^\alpha/\mathbb{P}_1\to\mathbb{C}} \le \frac{(1-|z|)^{2-\alpha}+(2-\alpha)|z|-1}{(1-\alpha)(2-\alpha)}. \tag{2.23}$$

Let

$$\widehat{f_t}(z) = \frac{1}{1-\alpha}\int_0^z(1-(1-t\zeta)^{1-\alpha})d\zeta, \tag{2.24}$$

where $t \in \mathbb{D}$ and $z \in \mathbb{D}$.

Then, we have $\widehat{f_t}(0) = 0$ and

$$\widehat{f_t}'(z) = \frac{1-(1-tz)^{1-\alpha}}{1-\alpha}. \tag{2.25}$$

From (2.25), it follows that

$$\widehat{f_t}'(0) = 0 \quad \text{and} \quad \widehat{f_t}''(z) = \frac{t}{(1-tz)^{\alpha}}.$$

Hence,

$$b_{\alpha}(\widehat{f_t}) = \sup_{z\in\mathbb{D}}(1-|z|)^{\alpha}\frac{|t|}{|1-tz|^{\alpha}} \le \sup_{z\in\mathbb{D}}(1-|z|)^{\alpha}\frac{|t|}{(1-|tz|)^{\alpha}} \le |t|. \tag{2.26}$$

Let

$$\widehat{h}_{|t|}(z) = (1-|z|)^{\alpha}\frac{|t|}{(1-|t||z|)^{\alpha}}. \tag{2.27}$$

Then,

$$\widehat{h}_{|t|}(0) = |t|. \tag{2.28}$$

From (2.28), together with (2.26), we obtain

$$b_{\alpha}(\widehat{f_t}) = |t|. \tag{2.29}$$

Further, we have

$$|\delta_z(\widehat{f_t})| = \left|\int_0^z \frac{(1-(1-t\zeta)^{1-\alpha})}{1-\alpha}d\zeta\right| = \left|z\int_0^1 \frac{(1-(1-t\hat{s}z)^{1-\alpha})}{1-\alpha}d\hat{s}\right|. \tag{2.30}$$

For $t = |t|e^{-i\arg z}$, from (2.30), we get

$$|\delta_z(\widehat{f_t})| = |z|\int_0^1 \frac{1-(1-|t||z|\hat{s})^{1-\alpha}}{1-\alpha}d\hat{s}. \tag{2.31}$$

The integrand in (2.31) is an increasing function in variable $|t|$ for $\alpha \neq 1$. So, by letting $|t| \to 1-0$ in (2.31), using Bepo-Levi's theorem and a change of variables, we get

$$\begin{aligned}\lim_{|t|\to 1-0}|\delta_z(\widehat{f_t})| &= \lim_{|t|\to 1-0}|z|\int_0^1 \frac{1-(1-|t||z|\hat{s})^{1-\alpha}}{1-\alpha}d\hat{s}\\ &=|z|\int_0^1 \frac{1-(1-|z|\hat{s})^{1-\alpha}}{1-\alpha}d\hat{s}\\ &=\int_0^{|z|} \frac{1-(1-s)^{1-\alpha}}{1-\alpha}ds. \end{aligned} \tag{2.32}$$

From (2.22), (2.29) and (2.32), and since $|t| < 1$, we get

$$\|\delta_z\|_{\mathcal{Z}^{\alpha}/\mathbb{P}_1\to\mathbb{C}} \ge \frac{(1-|z|)^{2-\alpha}+(2-\alpha)|z|-1}{(1-\alpha)(2-\alpha)}, \tag{2.33}$$

when $1 \neq \alpha \neq 2$.

From (2.23) and (2.33), formula (2.4) follows.

(c) From (2.21), in this case, we have

$$|f(z)| \le b_2(f)\int_0^{|z|}\Big(\frac{1}{1-s}-1\Big)ds = b_2(f)\Big(\ln\frac{1}{1-|z|}-|z|\Big). \tag{2.34}$$

Since the semi-norm (1.2) with $\alpha = 2$, is a norm on the space $\mathcal{Z}^2/\mathbb{P}_1$, from (2.34), we get

$$\|\delta_z\|_{\mathcal{Z}^2/\mathbb{P}_1\to\mathbb{C}} \le \ln\frac{1}{1-|z|} - |z|. \tag{2.35}$$

Consider the family of functions in (2.24) with $\alpha = 2$. Then, as in the previous case, we have that the relations (2.29)–(2.32) also hold for $\alpha = 2$. Hence, we have

$$\lim_{|t|\to 1-0}|\delta_z(\widehat{f_t})| = \int_0^{|z|}((1-s)^{-1}-1)ds. \tag{2.36}$$

From (2.29) and (2.36), and since $|t| < 1$, we get

$$\|\delta_z\|_{\mathcal{Z}^2/\mathbb{P}_1\to\mathbb{C}} \ge \ln\frac{1}{1-|z|} - |z|. \tag{2.37}$$

From (2.35) and (2.37), formula (2.5) follows.

(d) From (2.20) and some calculations, we have

$$\begin{aligned}|f(z)| \le& b_1(f)|z|\int_0^1\int_0^{|z|\hat{s}}\frac{du}{1-u}d\hat{s}\\ =& b_1(f)|z|\int_0^1\ln\frac{1}{1-\hat{s}|z|}d\hat{s}\\ =& b_1(f)\int_0^{|z|}\ln\frac{1}{1-s}ds\\ =& b_1(f)(|z|+(1-|z|)\ln(1-|z|)).\end{aligned} \tag{2.38}$$

Since the semi-norm (1.2) with $\alpha = 1$, is a norm on the space $\mathcal{Z}/\mathbb{P}_1$, from (2.38), we get

$$\|\delta_z\|_{\mathcal{Z}/\mathbb{P}_1\to\mathbb{C}} \le |z| + (1-|z|)\ln(1-|z|). \tag{2.39}$$

Let

$$\widetilde{f_t}(z) = \int_0^z \ln\frac{1}{1-t\zeta}d\zeta, \tag{2.40}$$

where $t \in \mathbb{D}$ and $z \in \mathbb{D}$.

Then, we have $\widetilde{f_t}(0) = 0$,

$$\widetilde{f_t}'(z) = \ln\frac{1}{1-tz}. \tag{2.41}$$

From (2.41), it follows that

$$\widetilde{f_t}'(0) = 0 \quad \text{and} \quad \widetilde{f_t}''(z) = \frac{t}{1-tz}.$$

Hence,

$$b_1(\widetilde{f_t}) = \sup_{z\in\mathbb{D}}(1-|z|)\frac{|t|}{|1-tz|} \le \sup_{z\in\mathbb{D}}(1-|z|)\frac{|t|}{(1-|t||z|)} \le |t|. \tag{2.42}$$

By considering the function (2.27) with $\alpha = 1$, we have that (2.28) also holds in this case, from which, together with (2.42), we obtain

$$b_1(\widetilde{f_t}) = |t|. \tag{2.43}$$

Further, we have

$$|\delta_z(\widetilde{f_t})| = \left|\int_0^z \ln\frac{1}{1-t\zeta}d\zeta\right| = \left|z\int_0^1 \ln\frac{1}{1-t\hat{s}z}d\hat{s}\right|. \tag{2.44}$$

For $t = |t|e^{-i\arg z}$, from (2.44), we get

$$|\delta_z(\widetilde{f_t})| = |z|\int_0^1 \ln\frac{1}{1-|t|\hat{s}|z|}d\hat{s}. \tag{2.45}$$

The integrand in (2.45) is an increasing function in variable $|t|$. So, by letting $|t| \to 1-0$ in (2.45), using Bepo-Levi's theorem and a change of variables, we get

$$\begin{aligned}\lim_{|t|\to 1-0}|\delta_z(\widetilde{f_t})| &= \lim_{|t|\to 1-0}|z|\int_0^1 \ln\frac{1}{1-|t|\hat{s}|z|}d\hat{s}\\ &=|z|\int_0^1 \ln\frac{1}{1-\hat{s}|z|}d\hat{s}\\ &=\int_0^{|z|}\ln\frac{1}{1-s}ds\\ &=|z| + (1-|z|)\ln(1-|z|).\end{aligned} \tag{2.46}$$

From (2.43) and (2.46), and since $|t| < 1$, we get

$$\|\delta_z\|_{\mathcal{Z}/\mathbb{P}_1\to\mathbb{C}} \ge |z| + (1-|z|)\ln(1-|z|). \tag{2.47}$$

From (2.39) and (2.47), formula (2.6) follows. □

3. Conclusions

Here, we calculate norms of the point evaluation functionals on the Zygmund-type spaces on the open unit disc in the complex plane. The norms of the functionals can be calculated thanks to the ability to calculate some integrals that appear during the calculation of the norms. Such obtained formulas can be used in some further investigations on the spaces of functions and operators on them, and can also serve as a motivation for some related investigations.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest.

References

1. K. D. Bierstedt, W. H. Summers, Biduals of weighted Banach spaces of analytic functions, *J. Austral. Math. Soc. (Series A)*, **54** (1993), 70–79. https://doi.org/10.1017/S1446788700036983
2. C. C. Cowen, B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.
3. J. Du, S. Li, Weighted composition operators from Zygmund type spaces into Bloch type spaces, *Math. Ineq. Appl.*, **20** (2017), 247–262. https://doi.org/10.7153/mia-20-19
4. J. Du, S. Li, Z. Liu, Stević-Sharma-type operators between Bergman spaces induced by doubling weights, *Math. Methods Appl. Sci.*, **47** (2024), 10006–10023. https://doi.org/10.1002/mma.10107
5. J. Du, S. Li, Y. Zhang, Essential norm of generalized composition operators on Zygmund type spaces and Bloch type spaces, *Annal. Polon. Math.*, **119** (2017), 107–119. https://doi.org/10.4064/ap170405-12-7
6. J. Du, S. Li, Y. Zhang, Essential norm of weighted composition operators on Zygmund-type spaces with normal weight, *Math. Inequal. Appl.*, **21** (2018), 701–714. https://doi.org/10.7153/mia-2018-21-49
7. J. Du, X. Zhu, Essential norm of an integral-type operator from ω-Bloch spaces to μ-Zygmund spaces on the unit ball, *Opusc. Math.*, **38** (2018), 829–839. https://doi.org/10.7494/OpMath.2018.38.6.829
8. N. Dunford, J. T. Schwartz, *Linear operators I,* Interscience Publishers, Jon Willey and Sons, New York, 1958.
9. X. Fu, S. Li, Composition operators from Zygmund spaces into Q_K spaces, *J. Inequal. Appl.*, **2013** (2013), 175. https://doi.org/10.1186/1029-242X-2013-175
10. L. Hu, S. Li, R. Yang, Generalized weighted composition operators on weighted Hardy spaces, *Oper. Matrices*, **17** (2023), 1109–1124. https://doi.org/10.7153/oam-2023-17-73
11. Q. Hu, S. Li, Essential norm of weighted composition operators from the Bloch space and the Zygmund space to the Bloch space, *Filomat*, **32** (2018), 681–691. https://doi.org/10.2298/FIL1802681H
12. Q. Hu, S. Li, Y. Zhang, Essential norm of weighted composition operators from analytic Besov spaces into Zygmund type spaces, *J. Contemp. Math. Anal.*, **54** (2019), 129–142. https://doi.org/10.3103/S1068362319030014
13. Q. Hu, X. Zhu, Essential norm of weighted composition operators from the Lipschtiz space to the Zygmund space, *Bull. Malays. Math. Sci. Soc.*, **41** (2018), 1293–1307.
14. H. Li, S. Li, Norm of an integral operator on some analytic function spaces on the unit disk, *J. Inequal. Appl.*, **2013** (2013), 342. https://doi.org/10.1186/1029-242X-2013-342

15. S. Li, S. Stević, Volterra-type operators on Zygmund spaces, *J. Inequal. Appl.*, **2007** (2007), 32124. https://doi.org/10.1155/2007/32124

16. S. Li, S. Stević, Integral-type operators from Bloch-type spaces to Zygmund-type spaces, *Appl. Math. Comput.*, **215** (2009), 464–473. https://doi.org/10.1016/j.amc.2009.05.011

17. X. Liu, S. Li, Norm and essential norm of a weighted composition operator on the Bloch space, *Integr. Equat. Oper. Th.*, **87** (2017), 309–325. https://doi.org/10.1007/s00020-017-2349-y

18. A. Pelczynski, Norms of classical operators in function spaces, *Astérisque*, **131** (1985), 137–162.

19. W. Rudin, *Real and complex analysis,* Higher Mathematics Series, 3Eds., McGraw-Hill Education, London, New York, Sidney, 1976.

20. W. Rudin, *Functional analysis*, 2Eds., McGraw-Hill, Inc., New York, 1991.

21. H. J. Schwartz, *Composition operators on* H^p, Thesis, University of Toledo, 1969.

22. A. K. Sharma, Weighted composition operators from Cauchy integral transforms to logarithmic weighted-type spaces, *Ann. Funct. Anal.*, **4** (2013), 163–174. https://doi.org/10.15352/afa/1399899844

23. A. K. Sharma, Norm of a composition operator from the space of Cauchy transforms into Zygmund-type spaces, *Ukr. Math. J.*, **71** (2020), 1945–1958. https://doi.org/10.1007/s11253-020-01757-2

24. A. K. Sharma, R. Krishan, Difference of composition operators from the space of Cauchy integral transforms to the Dirichlet space, *Complex Anal. Oper. Theory*, **10** (2016), 141–152. https://doi.org/10.1007/s11785-015-0487-2

25. S. Stević, Norm of weighted composition operators from Bloch space to H_μ^∞ on the unit ball, *Ars. Combin.*, **88** (2008), 125–127.

26. S. Stević, Norms of some operators from Bergman spaces to weighted and Bloch-type space, *Util. Math.*, **76** (2008), 59–64.

27. S. Stević, Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces, *Appl. Math. Comput.*, **215** (2009), 818–820. https://doi.org/10.1016/j.amc.2009.06.005

28. S. Stević, Norm and essential norm of an integral-type operator from the Dirichlet space to the Bloch-type space on the unit ball, *Abstr. Appl. Anal.*, **2010** (2010), 134969.

29. S. Stević, On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball, *Abstr. Appl. Anal.*, **2010** (2010), 198608. https://doi.org/10.1155/2010/198608

30. S. Stević, Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball, *Abstr. Appl. Anal.*, **2010** (2010), 801264. https://doi.org/10.1155/2010/801264

31. S. Stević, Norm of the general polynomial differentiation composition operator from the space of Cauchy transforms to the *m*th weighted-type space on the unit disk, *Math. Methods Appl. Sci.*, **47** (2024), 3893–3902. https://doi.org/10.1002/mma.9681

32. V. A. Trenogin, B. M. Pisarevskiy, T. S. Soboleva, Zadachi i Uprazhneniya po Funktsional'nomu Analizu, Nauka, Moskva, 1984. (In Russian)

33. R. Yang, X. Zhu, Weighted composition operators from weighted Bergman-Orlicz spaces with doubling weights to weighted Zygmund spaces, *Math. Inequal. Appl.*, **27** (2024), 627–645. https://doi.org/10.7153/mia-2024-27-43

34. X. Zhu, A new characterization of the generalized weighted composition operator from H^∞ into the Zygmund space, *Math. Inequal. Appl.*, **18** (2015), 1135–1142. https://doi.org/10.7153/mia-18-87

35. X. Zhu, Weighted composition operators from Dirichlet-Zygmund-type spaces into Stević-type spaces, *Georgian Math. J.*, **30** (2023), 629–637. https://doi.org/10.1515/gmj-2022-2209

AIMS Press