



Research article

Probabilistic generalization of Spivey-type relation for degenerate Bell polynomials

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Abstract: Following Spivey’s pivotal discovery of a recurrence relation for Bell numbers, significant research has emerged concerning various generalizations of Bell numbers and polynomials. For example, Kim and Kim established a Spivey-type recurrence relation specifically for degenerate Bell and Dowling polynomials. In this paper, we extend this work by deriving a probabilistic generalization of Spivey-type recurrence relations for both degenerate Bell and degenerate *r*-Bell polynomials.

Keywords: probabilistic degenerate Bell polynomials; probabilistic degenerate *r*-Bell polynomials; Spivey-type relation

Mathematics Subject Classification: 11B73, 11B83

1. Introduction

The Stirling number of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ enumerates the number of partitions of a set with *n* objects into *k* nonempty disjoint subsets. The sum of the Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$, called the Bell number and denoted by ϕ_n , counts the total number of partitions of a set with *n* objects. That is, $\phi_n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. The Bell polynomial $\phi_n(x) = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x^k$ is a natural polynomial extension of the Bell number ϕ_n (see [1, 2, 4, 9, 13]).

Spivey found the following recurrence relation on the Bell numbers (see [13]):

$$\phi_{l+n} = \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} k^{l-m} \phi_m.$$

His relation was extended by Gould-Quaintance to the Bell polynomials (see [5]):

$$\phi_{l+n}(y) = \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} k^{l-m} y^k \phi_m(y).$$

Recently, in [8], the authors discovered the following relation for the degenerate Bell polynomials (see (1.2), (1.7), (1.8)):

$$\phi_{l+n,\lambda}(y) = \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda} (k - n\lambda)_{l-m,\lambda} y^k \phi_{m,\lambda}(y), \quad (1.1)$$

where the degenerate Stirling numbers of the second kind are defined as

$$(x)_{n,\lambda} = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda} (x)_k, \quad (n \geq 0), \quad (\text{see [9]}), \quad (1.2)$$

with the falling factorial and degenerate falling factorial sequences respectively given by

$$(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1),$$

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1).$$

Assume that Y is a random variable whose moment generating function exists in a neighborhood of the origin:

$$E[e^{tY}] = \sum_{n=0}^{\infty} E[Y^n] \frac{t^n}{n!}, \quad (|t| < r), \quad (1.3)$$

for some $r > 0$, (see [3]).

Let $(Y_j)_{j \geq 1}$ be a sequence of mutually independent copies of the random variable Y , and let

$$S_k = Y_1 + Y_2 + \cdots + Y_k, \quad (k \geq 1), \quad \text{with } S_0 = 0. \quad (1.4)$$

The aim of this paper is to generalize the Spivey-type relation for the degenerate Bell polynomials in (1.1) to that for the probabilistic degenerate Bell polynomials associated with Y (see (1.13), (1.14)). Indeed, we obtain (see Theorem 2.1):

$$\begin{aligned} \phi_{l+n,\lambda}^Y(y) &= \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \frac{y^k}{k!} \sum_{l_1+\cdots+l_k=n} \binom{n}{l_1, \dots, l_k} \\ &\times E\left[(S_k - n\lambda)_{l-m,\lambda} \prod_{i=1}^k (Y_i)_{l_i,\lambda}\right] \phi_{m,\lambda}^Y(y), \end{aligned} \quad (1.5)$$

where l_1, l_2, \dots, l_k are positive integers.

Furthermore, the relation in (1.5) is extended to the probabilistic degenerate r -Bell polynomials associated with Y (see (2.8)) as follows (see Theorem 2.3):

$$\begin{aligned} \phi_{j+n,\lambda}^{(r,Y)}(y) &= \sum_{l=0}^n \binom{n}{l} (r)_{n-l,\lambda} \sum_{k=0}^l \sum_{m=0}^j \frac{y^k}{k!} \binom{j}{m} \sum_{l_1+\cdots+l_k=l} \binom{l}{l_1, \dots, l_k} \\ &\times E\left[(S_k - n\lambda)_{j-m,\lambda} \prod_{i=1}^k (Y_i)_{l_i,\lambda}\right] \phi_{m,\lambda}^{(r,Y)}(y). \end{aligned}$$

This mathematical framework, which integrates probability theory (through random variables Y and their sums S_k) with combinatorics (via degenerate Bell polynomials and their Spivey-type recurrence relations), is useful for modeling complex systems involving both combinatorial structures and stochastic processes, suggesting potential applications in:

- **Stochastic Combinatorics:** Modeling the expected number of random partitions or clusters in a system where the size or weight of the elements (Y_i) is random.
- **Risk Modeling:** Calculating compound sums of random variables (S_k) in fields like insurance or finance, especially when the number of contributing events (k) is itself linked to a combinatorial structure.
- **Theoretical Physics:** Introducing stochastic fluctuations into combinatorial identities used in areas like statistical mechanics and quantum optics.

The degenerate exponentials are defined as

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} = (1 + \lambda t)^\frac{x}{\lambda}, \quad e_\lambda(t) = e_\lambda^1(t), \quad (\text{see [8, 9, 14]}). \quad (1.6)$$

The degenerate Bell polynomials are given by

$$\phi_{n,\lambda}(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_\lambda x^k, \quad (n \geq 0), \quad (\text{see [9]}). \quad (1.7)$$

When $x = 1$, $\phi_{n,\lambda} = \phi_{n,\lambda}(1)$ are called the degenerate Bell numbers. Note that

$$\lim_{\lambda \rightarrow 0} \phi_{n,\lambda}(x) = \phi_n(x), \quad \lim_{\lambda \rightarrow 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_\lambda = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$$

From (1.7), we have

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see [9]}). \quad (1.8)$$

For $r \in \mathbb{N}$ and $n \geq 0$, the degenerate r -Stirling numbers of the second kind are defined by

$$(x+r)_{n,\lambda} = \sum_{k=0}^n \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} (x)_k, \quad (n \geq 0), \quad (\text{see [10, 11, 15]}). \quad (1.9)$$

In view of (1.7), the degenerate r -Bell polynomials are given by

$$\phi_{n,\lambda}^{(r)}(x) = \sum_{k=0}^n \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} x^k, \quad (n \geq 0), \quad (\text{see [10, 11]}). \quad (1.10)$$

When $x = 1$, $\phi_{n,\lambda}^{(r)} = \phi_{n,\lambda}^{(r)}(1)$ are called the degenerate r -Bell numbers.

We note that

$$g(x+t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{t^n D_x^n}{n!} g(x) = e^{tD_x} g(x), \quad (\text{see [7]}), \quad (1.11)$$

where $D_x = \frac{d}{dx}$.

In [9], the probabilistic degenerate Stirling numbers of the second kind associated with Y are defined by

$$\frac{1}{k!} (E[e_\lambda^Y(t)] - 1)^k = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} \frac{t^n}{n!}. \quad (1.12)$$

Thus, by (1.12), we get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} = \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} E[(S_l)_{n,\lambda}], \quad (0 \leq k \leq n).$$

In view of (1.8), the probabilistic degenerate Bell polynomials associated with Y are defined by

$$e^{x(E[e_\lambda^Y(t)]-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}^Y(x) \frac{t^n}{n!}, \quad (\text{see [9]}). \quad (1.13)$$

By (1.10) and (1.12), we get

$$\phi_{n,\lambda}^Y(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Y,\lambda} x^k, \quad (n \geq 0), \quad (1.14)$$

and

$$\phi_{n+1,\lambda}^Y(x) = x \sum_{k=0}^n \binom{n}{k} E[(Y)_{k+1,\lambda}] \phi_{n-k,\lambda}^Y(x), \quad (\text{see [9]}).$$

2. Probabilistic generalization of Spivey-type relation for degenerate Bell polynomials

Let $Y, (Y_j)_{j \geq 1}$, and $S_k, (k \geq 0)$, be as in (1.3) and (1.4). By (1.6), we get

$$\begin{aligned} e_\lambda^Y(x+t) &= (1 + \lambda(x+t))^{\frac{Y}{\lambda}} = (1 + \lambda x)^{\frac{Y}{\lambda}} \left(1 + \frac{\lambda t}{1 + \lambda x}\right)^{\frac{Y}{\lambda}} \\ &= e_\lambda^Y(x) e_\lambda^Y\left(\frac{t}{1 + \lambda x}\right). \end{aligned} \quad (2.1)$$

From (1.11) and (2.1), we note that

$$\begin{aligned} e^{tD_x} e^{y(E[e_\lambda^Y(x)]-1)} &= e^{y(E[e_\lambda^Y(x+t)]-1)} = e^{y(E[e_\lambda^Y(x)(e_\lambda^Y(\frac{t}{1+\lambda x})-1)+e_\lambda^Y(x)]-1)} \\ &= e^{yE[e_\lambda^Y(x)(e_\lambda^Y(\frac{t}{1+\lambda x})-1)]} e^{y(E[e_\lambda^Y(x)]-1)}. \end{aligned} \quad (2.2)$$

Now, we observe that

$$\begin{aligned} e^{yE[e_\lambda^Y(x)(e_\lambda^Y(\frac{t}{1+\lambda x})-1)]} &= \sum_{k=0}^{\infty} \frac{y^k}{k!} \left(E \left[e_\lambda^Y(x) \left(e_\lambda^Y\left(\frac{t}{1+\lambda x}\right) - 1 \right) \right] \right)^k \\ &= \sum_{k=0}^{\infty} \frac{y^k}{k!} E \left[e^{Y_1 + \dots + Y_k} (x) \left(e_\lambda^{Y_1} \left(\frac{t}{1+\lambda x} \right) - 1 \right) \cdots \left(e_\lambda^{Y_k} \left(\frac{t}{1+\lambda x} \right) - 1 \right) \right] \end{aligned} \quad (2.3)$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{n=k}^{\infty} \left(\sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} \right) E \left[e_{\lambda}^{S_k}(x)(Y_1)_{l_1, \lambda} (Y_2)_{l_2, \lambda} \cdots (Y_k)_{l_k, \lambda} \right] \frac{\left(\frac{t}{1+\lambda x}\right)^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{n=k}^{\infty} \left(\sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} \right) E \left[e_{\lambda}^{S_k-n\lambda}(x)(Y_1)_{l_1, \lambda} (Y_2)_{l_2, \lambda} \cdots (Y_k)_{l_k, \lambda} \right] \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{y^k}{k!} \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} \right) E \left[e_{\lambda}^{S_k-n\lambda}(x) \prod_{i=1}^k (Y_i)_{l_i, \lambda} \right] \frac{t^n}{n!},
\end{aligned}$$

where l_1, l_2, \dots, l_k are positive integers.

By (1.13), (2.2), and (2.3), we get

$$\begin{aligned}
e^{tD_x} e^{y(E[e_{\lambda}^y(x)]-1)} &= e^{yE[e_{\lambda}^y(x)(e_{\lambda}^y(\frac{t}{1+\lambda x})-1)]} e^{y(E[e_{\lambda}^y(x)]-1)} \tag{2.4} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{y^k}{k!} \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} \frac{t^n}{n!} \\
&\quad \times \sum_{j=0}^{\infty} E \left[(S_k - n\lambda)_{j, \lambda} \prod_{i=1}^k (Y_i)_{l_i, \lambda} \right] \frac{x^j}{j!} \sum_{m=0}^{\infty} \phi_{m, \lambda}^y(y) \frac{x^m}{m!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{y^k}{k!} \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} \frac{t^n}{n!} \\
&\quad \times \sum_{l=0}^{\infty} \sum_{m=0}^l \binom{l}{m} E \left[(S_k - n\lambda)_{l-m, \lambda} \prod_{i=1}^k (Y_i)_{l_i, \lambda} \right] \phi_{m, \lambda}^y(y) \frac{x^l}{l!} \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \frac{y^k}{k!} \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} \\
&\quad \times E \left[(S_k - n\lambda)_{l-m, \lambda} \prod_{i=1}^k (Y_i)_{l_i, \lambda} \right] \phi_{m, \lambda}^y(y) \frac{t^n}{n!} \frac{x^l}{l!}.
\end{aligned}$$

On the other hand, by (1.13), we get

$$\begin{aligned}
e^{tD_x} e^{y(E[e_{\lambda}^y(x)]-1)} &= e^{tD_x} \sum_{l=0}^{\infty} \phi_{l, \lambda}^y(y) \frac{x^l}{l!} \tag{2.5} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} D_x^n \sum_{l=0}^{\infty} \phi_{l, \lambda}^y(y) \frac{x^l}{l!} \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \phi_{l+n, \lambda}^y(y) \frac{t^n}{n!} \frac{x^l}{l!}.
\end{aligned}$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

Theorem 2.1. For $n, l \geq 0$, we have

$$\begin{aligned} \phi_{l+n,\lambda}^Y(y) &= \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \frac{y^k}{k!} \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} \\ &\times E\left[(S_k - n\lambda)_{l-m,\lambda} \prod_{i=1}^k (Y_i)_{l_i,\lambda}\right] \phi_{m,\lambda}^Y(y), \end{aligned} \quad (2.6)$$

where l_1, l_2, \dots, l_k are positive integers.

When $Y = 1$, from (2.6) we have the following relation in (1.1):

$$\begin{aligned} \phi_{l+n,\lambda}(y) &= \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} (k - n\lambda)_{l-m,\lambda} y^k \phi_{m,\lambda}(y) \\ &\times \frac{1}{k!} \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, l_2, \dots, l_k} \prod_{i=1}^k (1)_{l_i,\lambda} \\ &= \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda} (k - n\lambda)_{l-m,\lambda} y^k \phi_{m,\lambda}(y). \end{aligned}$$

Here we note that

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda} = \frac{1}{k!} \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, l_2, \dots, l_k} \prod_{i=1}^k (1)_{l_i,\lambda},$$

where l_1, l_2, \dots, l_k are positive integers.

Further, letting $\lambda \rightarrow 0$ yields Gould-Quaintance extension for Bell polynomials of Spivey's recurrence relation for Bell numbers.

$$\phi_{l+n}(y) = \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k^{l-m} y^k \phi_m(y), \quad (\text{see [5, 7]}).$$

Letting $y = 1$ in (2.6), we get the following Spivey-like relation for the probabilistic degenerate Bell numbers associated with Y .

Corollary 2.2. For $n, l \geq 0$, we have

$$\phi_{l+n,\lambda}^Y = \sum_{k=0}^n \sum_{m=0}^l \binom{l}{m} \frac{1}{k!} \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} E\left[(S_k - n\lambda)_{l-m,\lambda} \prod_{i=1}^k (Y_i)_{l_i,\lambda}\right] \phi_{m,\lambda}^Y,$$

where l_1, l_2, \dots, l_k are positive integers.

For $r \in \mathbb{N}$, we define the probabilistic degenerate r -Stirling numbers of the second kind associated with Y as

$$\frac{1}{k!} (E[e_\lambda^Y(t)] - 1)^k e_\lambda^r(t) = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda}^Y \frac{t^n}{n!}, \quad (k \geq 0). \quad (2.7)$$

When $Y = 1$, we have (see (1.9))

$$\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda}^Y = \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda}, \quad (n \geq k \geq 0).$$

Also, we define the probabilistic degenerate r -Bell polynomials associated with Y by

$$e^{x(E[e_\lambda^Y(t)]-1)} e_\lambda^r(t) = \sum_{n=0}^{\infty} \phi_{n,\lambda}^{(r,Y)}(x) \frac{t^n}{n!}. \quad (2.8)$$

When $x = 1$, $\phi_{n,\lambda}^{(r,Y)} = \phi_{n,\lambda}^{(r,Y)}(1)$ are called the probabilistic degenerate r -Bell numbers associated with Y .

By (1.8), (2.1)–(2.3), and (2.8), we get

$$\begin{aligned} e^{tD_x} \left(e^{y(E[e_\lambda^Y(x)]-1)} e_\lambda^r(x) \right) &= e^{y(E[e_\lambda^Y(x+t)]-1)} e_\lambda^r(x+t) \\ &= e^{yE[e_\lambda^Y(x)(e_\lambda^Y(\frac{t}{1+\lambda x}-1)]} e_\lambda^r\left(\frac{t}{1+\lambda x}\right) e^{y(E[e_\lambda^Y(x)]-1)} e_\lambda^r(x) \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^l \frac{y^k}{k!} \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} \frac{t^l}{l!} E \left[e_\lambda^{S_k-l\lambda}(x) \prod_{i=1}^k (Y_i)_{l_i,\lambda} \right] \\ &\quad \times \sum_{s=0}^{\infty} (r)_{s,\lambda} \frac{t^s}{s!} e_\lambda^{-s\lambda}(x) \sum_{m=0}^{\infty} \phi_{m,\lambda}^{(r,Y)}(y) \frac{x^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} (r)_{n-l,\lambda} \sum_{k=0}^l \frac{y^k}{k!} \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} \frac{t^n}{n!} \\ &\quad \times E \left[e_\lambda^{S_k-l\lambda-(n-l)\lambda}(x) \prod_{i=1}^k (Y_i)_{l_i,\lambda} \right] \sum_{m=0}^{\infty} \phi_{m,\lambda}^{(r,Y)}(y) \frac{x^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} (r)_{n-l,\lambda} \sum_{k=0}^l \frac{y^k}{k!} \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} \frac{t^n}{n!} \\ &\quad \times \sum_{a=0}^{\infty} E \left[(S_k - n\lambda)_{a,\lambda} \prod_{i=1}^k (Y_i)_{l_i,\lambda} \right] \frac{x^a}{a!} \sum_{m=0}^{\infty} \phi_{m,\lambda}^{(r,Y)}(y) \frac{x^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} (r)_{n-l,\lambda} \sum_{k=0}^l \frac{y^k}{k!} \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} \frac{t^n}{n!} \\ &\quad \times \sum_{j=0}^{\infty} \sum_{m=0}^j \binom{j}{m} E \left[(S_k - n\lambda)_{j-m,\lambda} \prod_{i=1}^k (Y_i)_{l_i,\lambda} \right] \phi_{m,\lambda}^{(r,Y)}(y) \frac{x^j}{j!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^j \sum_{l=0}^n \binom{n}{l} (r)_{n-l,\lambda} \binom{j}{m} \sum_{k=0}^l \frac{y^k}{k!} \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} \\ &\quad \times E \left[(S_k - n\lambda)_{j-m,\lambda} \prod_{i=1}^k (Y_i)_{l_i,\lambda} \right] \phi_{m,\lambda}^{(r,Y)}(y) \frac{t^n}{n!} \frac{x^j}{j!}, \end{aligned} \quad (2.9)$$

where l_1, l_2, \dots, l_k are positive integers.

On the other hand, by (2.8), we get

$$\begin{aligned} & e^{tD_x} \left(e^{y(E[e_\lambda^y(x)]-1)} e_\lambda^r(x) \right) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} D_x^n \sum_{j=0}^{\infty} \phi_{j,\lambda}^{(r,Y)}(y) \frac{x^j}{j!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \phi_{j+n,\lambda}^{(r,Y)}(y) \frac{t^n}{n!} \frac{x^j}{j!}. \end{aligned} \quad (2.10)$$

Therefore, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.3. For $n, j \geq 0$, we have

$$\begin{aligned} \phi_{j+n,\lambda}^{(r,Y)}(y) &= \sum_{l=0}^n \binom{n}{l} (r)_{n-l,\lambda} \sum_{k=0}^l \sum_{m=0}^j \frac{y^k}{k!} \binom{j}{m} \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} \\ &\quad \times E \left[(S_k - n\lambda)_{j-m,\lambda} \prod_{i=1}^k (Y_i)_{l_i,\lambda} \right] \phi_{m,\lambda}^{(r,Y)}(y), \end{aligned}$$

where l_1, l_2, \dots, l_k are positive integers.

When $Y = 1$, we have

$$\begin{aligned} \phi_{j+n,\lambda}^{(r)}(y) &= \sum_{l=0}^n \sum_{m=0}^j \binom{n}{l} \binom{j}{m} (r)_{n-l,\lambda} \sum_{k=0}^l \frac{y^k}{k!} \\ &\quad \times \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} \prod_{i=1}^k (1)_{l_i,\lambda} (k - n\lambda)_{j-m,\lambda} \phi_{m,\lambda}^{(r)}(y) \\ &= \sum_{l=0}^n \sum_{m=0}^j \binom{n}{l} \binom{j}{m} (r)_{n-l,\lambda} \sum_{k=0}^l y^k \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_\lambda (k - n\lambda)_{j-m,\lambda} \phi_{m,\lambda}^{(r)}(y) \\ &= \sum_{k=0}^n \sum_{m=0}^j \sum_{l=k}^n \binom{n}{l} (r)_{n-l,\lambda} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_\lambda \binom{j}{m} y^k (k - n\lambda)_{j-m,\lambda} \phi_{m,\lambda}^{(r)}(y). \end{aligned} \quad (2.11)$$

From (2.7), we note that

$$\begin{aligned} \sum_{n=k}^{\infty} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} \frac{t^n}{n!} &= \frac{1}{k!} (e_\lambda(t) - 1)^k e_\lambda^r(t) \\ &= \sum_{l=k}^{\infty} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_\lambda \frac{t^l}{l!} \sum_{j=0}^{\infty} (r)_{j,\lambda} \frac{t^j}{j!} \\ &= \sum_{n=k}^{\infty} \sum_{l=k}^n \binom{n}{l} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_\lambda (r)_{n-l,\lambda} \frac{t^n}{n!}. \end{aligned} \quad (2.12)$$

Thus, by comparing the coefficients on both sides of (2.12), we get

$$\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} = \sum_{l=k}^n \binom{n}{l} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_\lambda (r)_{n-l,\lambda}, \quad (n \geq k \geq 0). \quad (2.13)$$

Therefore, by (2.11) and (2.13), we obtain the following corollary.

Corollary 2.4. For $j, n \geq 0$, we have

$$\phi_{j+n,\lambda}^{(r)}(y) = \sum_{k=0}^n \sum_{m=0}^j \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} \binom{j}{m} (k-n\lambda)_{j-m,\lambda} y^k \phi_{m,\lambda}^{(r)}(y).$$

3. Conclusions

In 2008, Spivey's discovery of a recurrence relation for the Bell numbers ϕ_n opened a new avenue of research. Since then, significant work has been dedicated to exploring various Spivey-type recurrence relations (see [5–8, 12]). This includes generalizing Spivey's relation to probabilistic Bell and r -Bell polynomials associated with Y . Here Y is a random variable whose moment generating function exists in a neighborhood of the origin.

In this paper, we further expanded this area by proving Spivey-type recurrence relations for probabilistic degenerate Bell and r -Bell polynomials associated with Y . These polynomials are degenerate versions of the aforementioned probabilistic Bell and r -Bell polynomials associated with Y .

Moving forward, our future research will focus on continuing to investigate degenerate versions and probabilistic extensions of various other special polynomials and numbers.

Author contributions

In this paper, the authors consistently discussed, reviewed, and revised the paper equally during the writing, conception, and review of the paper. In other words, the authors declare that all contributions and work were made equally..

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the reviewers for their helpful comments and suggestions that helped improve the original manuscript in its present form.

Conflict of interest

Prof. T. Kim and Prof. D. S. Kim are the Guest Editors of special issue "Probabilistic Number theory, Computation and their applications Theory and Computation" for AIMS Mathematics. They were not involved in the editorial review and the decision to publish this article. The authors declare no conflict of interest.

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