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Research article

Semigroups of partial linear transformations whose restrictions belong to an injective partial linear transformation semigroup

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Abstract: Let V be a vector space over a field $\mathbb{F}$ and let W be a subspace of V. The semigroup of partial linear transformations on V whose restriction to W belongs to an injective partial linear transformation semigroup $I(W)$ is denoted by $P_{I(W)}(V)$. In this paper, we describe Green's relations for $P_{I(W)}(V)$, characterize its regular elements, and give necessary and sufficient conditions for $P_{I(W)}(V)$ to be regular, inverse, or completely regular. We also analyze the ideal structure of $P_{I(W)}(V)$, identifying its maximal and minimal ideals.

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1. Introduction

The *partial transformation semigroup* $P(X)$ on a set X consists of all partial maps $\alpha : A \to B$ with $A, B \subseteq X$ under composition, where we write $\operatorname{dom} \alpha$ and $X\alpha$ for the domain and range of α. The *full transformation semigroup* $T(X)$ comprises those $\alpha \in P(X)$ with $\operatorname{dom} \alpha = X$, while the *symmetric inverse semigroup* $I(X)$ consists of all injective elements of $P(X)$. It is well known that $P(X)$ and $T(X)$ are regular semigroups, and $I(X)$ is an inverse semigroup. Analogously, for a vector space V, the *partial linear transformation semigroup* $P(V)$ consists of all linear maps $\alpha : S \to T$, where S and T are subspaces of V, equipped with composition, so that both $\operatorname{dom} \alpha$ and $V\alpha$ are subspaces of V. The *full linear transformation semigroup* $L(V)$ comprises those $\alpha \in P(V)$ with $\operatorname{dom} \alpha = V$, while the *injective partial linear transformation semigroup* $I(V)$ consists of all injective elements of $P(V)$. Similarly, $P(V)$ and $L(V)$ are regular semigroups and $I(V)$ is an inverse semigroup, with Green's relations for these three semigroups completely characterized.

Jampachon, Saichalee and Sullivan [9] investigated factorizable and locally factorizable subsemigroups of the partial transformation semigroup $P(X)$ and extended these results to the full

linear transformation semigroup $L(V)$. In 2006, Ittharat and Sullivan [8] extended that work to the partial linear transformation semigroup $P(V)$ and considered the notion of local factorizability for such semigroups.

The semigroup $P(V)$ admits a partial order $\alpha \subseteq \beta$ defined by $\operatorname{dom} \alpha \subseteq \operatorname{dom} \beta$ and $\alpha = \beta|_{\operatorname{dom} \alpha}$, where $\beta|_{\operatorname{dom} \alpha}$ is the restriction of β to $\operatorname{dom} \alpha$. Sullivan [15] compared this subset order with the natural partial order $\leq$ on $P(V)$, determining their meets, joins and characterizing minimal, maximal and compatible elements under each. Similar results were obtained for the semigroup $L(V)$.

For a subspace $W \subseteq V$, define the subsemigroup $S(V, W)$ of $L(V)$ by

$$S(V, W) = \{\alpha \in L(V) : W\alpha \subseteq W\}.$$

Both the identity id_V and the zero map belong to $S(V, W)$. Nenthein and Kemprasit [10] characterized the regular elements of $S(V, W)$, proving that $\alpha \in S(V, W)$ is regular if and only if $V\alpha \cap W = W\alpha$ and that $S(V, W)$ is regular precisely when $W = V$ or $W = \{0\}$. Huisheng [7] described the $\mathcal{L}^*$ and $\mathcal{R}^*$ relations on $S(V, W)$, while Honyam and Sanwong [5] presented Green's relations and ideals and showed that $S(V, W)$ is never isomorphic to $L(U)$ for any vector space U if W is a nonzero proper subspace of V. Chinram and Baupradist [2] gave necessary and sufficient conditions for elements in $S(V, W)$ to be left [right] magnifying. Chaiya [1] studied the natural partial order, compatibility, maximal and minimal elements of $S(V, W)$, as well as conditions for it to be factorizable, unit-regular and directly finite. Recently, Sawatraksa and Tantong [14] examined left, right and complete regularity in this semigroup.

Let W be a subspace of a vector space V. For a subsemigroup $\mathbb{S}(W)$ of $L(W)$, Sarkar and Singh [13] introduced

$$L_{\mathbb{S}(W)}(V) = \{\alpha \in L(V) : \alpha|_W \in \mathbb{S}(W)\},$$

generalizing $S(V, W)$ and characterizing its regular, inverse and completely regular elements. They also described unit-regular elements and gave criteria for unit-regularity in $L_{\mathbb{S}(W)}(V)$.

The notation $GL(V)$ refers to the set of all automorphisms of V, that is all bijective linear transformations from V to itself; under composition, $GL(V)$ forms the well-known *general linear group*.

In [12], a special case of $L_{\mathbb{S}(W)}(V)$ was investigated where $\mathbb{S}(W)$ is the general linear group of W. So, for a subspace U of V over a field $\mathbb{F}$,

$$L_{GL(U)}(V) = \{\alpha \in L(V) : \alpha|_U \in GL(U)\}.$$

When $U = V$, $L_{GL(U)}(V)$ coincides with $GL(U)$. A recent work [12] presented a description of Green's relations and ideals for this semigroup, identified its minimal ideal and all minimal idempotents, proved an isomorphism theorem for the finite-dimensional case over a finite field, and determined a generating set.

Analogously to $L_{GL(U)}(V)$, define the semigroup of partial linear transformations on V whose restriction to a subspace W is in $\mathcal{I}(W)$:

$$P_{\mathcal{I}(W)}(V) = \{\alpha \in P(V) : \alpha|_W \in \mathcal{I}(W)\}.$$

This is a generalization of the injective partial linear transformation semigroup $\mathcal{I}(W)$. Notably, $P_{\mathcal{I}(W)}(V)$ contains the identity on V, making it a monoid.

In this paper, we explore the semigroup $P_{\mathcal{I}(W)}(V)$, providing a description of its Green's relations, characterizing its regular elements and giving necessary and sufficient conditions for $P_{\mathcal{I}(W)}(V)$ to be regular, inverse, or completely regular. Moreover, the ideal structure of $P_{\mathcal{I}(W)}(V)$ is analyzed, with its maximal and minimal ideals identified.

2. Preliminaries

We start by setting up notation and basic results from linear algebra and semigroup theory that will be used throughout. Readers are referred to [3, 4, 6, 11] for any terminology not explicitly defined here.

Throughout, W is a subspace of a vector space V and $\langle e_i \rangle = \langle \{e_i : i \in I\} \rangle$ denotes the subspace generated by a linearly independent set $\{e_i\} = \{e_i : i \in I\}$ in V. When $W = \langle e_i \rangle$, the set $\{e_i\}$ forms a basis of W, so $\dim W = |I|$. Standard results give $\dim W \leq \dim V$ and if $\dim W = \dim V$ with V finite-dimensional, then $W = V$.

For subspaces M, N of V, the *sum $M + N$* is defined by

$$M + N = \{m + n \mid m \in M, n \in N\},$$

which is the smallest subspace containing both M and N. We say V is the *internal direct sum* of S_1 and S_2, written $V = S_1 \oplus S_2$, if $V = S_1 + S_2$ and $S_1 \cap S_2 = \{0\}$. If a vector space V is the internal direct sum of a family $S_1, S_2, \ldots, S_n$ of subspaces of V, it is denoted as $V = S_1 \oplus S_2 \oplus \cdots \oplus S_n$. If $V = S \oplus T$, we call T a *complement* of S in V. While a subspace can have many complements, all complements are mutually isomorphic.

For a subspace W of V, the *quotient space V/W* consists of all cosets of W in V. All complements of W in V are isomorphic to V/W and hence to each other. The dimension of V/W, denoted by $\mathrm{codim}(W)$, is called the *codimension* of W in V; if V is finite-dimensional, then $\mathrm{codim}(W) = \dim V - \dim W$.

For a subset $\{u_i\}$ of V, the notation $\sum a_i u_i$ denotes a finite linear combination:

$$a_{i_1} u_{i_1} + a_{i_2} u_{i_2} + \cdots + a_{i_n} u_{i_n},$$

where n is a positive integer, $u_{i_1}, \ldots, u_{i_n}$ are elements from $\{u_i\}$ and $a_{i_1}, \ldots, a_{i_n}$ are scalars.

For $\alpha \in P(V)$ and a subspace $W \subseteq V$, we write $\ker \alpha$ for the kernel and $V\alpha$ for the range of α. The expression $W\alpha$ denotes the set of images of elements in $W \cap \mathrm{dom}\, \alpha$ under α; when $W\alpha = \langle w_j \alpha \rangle$, each w_j is an element of $W \cap \mathrm{dom}\, \alpha$ and the set $\{w_j\}$ is linearly independent. If $V\alpha = \langle v_i \alpha \rangle$, then $\mathrm{dom}\, \alpha = \ker \alpha \oplus \langle v_i \rangle$.

We adopt the notation from [4, p. 241]: for any $\alpha \in P(X)$,

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix},$$

where i runs over some index set I, $\{a_i\} = \{a_i : i \in I\}$, $X\alpha = \{a_i\}$ and $a_i \alpha^{-1} = A_i$. This notation naturally extends to $P(V)$. To define $\alpha \in P(V)$, select a basis $\{e_i\}$ for $\mathrm{dom}\, \alpha$ and a set $\{a_i\} \subseteq V$, then define $e_i \alpha = a_i$ for each $i \in I$ and extend linearly. Thus, given $\{e_i\}$ and $\{a_i\}$, any $\alpha \in P(V)$ can be represented as

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

Specifically, if $\alpha \in P_{I(W)}(V)$, then $W\alpha \subseteq W$. In this case, the domain of α can be partitioned into three parts so that

$$\alpha = \begin{pmatrix} u_s & v_i & w_j \\ 0 & v_i\alpha & w_j\alpha \end{pmatrix},$$

where $\ker \alpha = \langle u_s \rangle$, $W\alpha = \langle w_j\alpha \rangle$ and $V\alpha = \langle v_i\alpha \rangle \oplus \langle w_j\alpha \rangle$. We also note by definition that $\ker \alpha \cap W = \{0\}$ for all $\alpha \in P_{I(W)}(V)$.

Our analysis relies on the following established results from linear algebra:

Theorem 2.1. *Let V and W be vector spaces over a field $\mathbb{F}$. Then $V \cong W$ if and only if $\dim V = \dim W$.*

Theorem 2.2. *Let S and T be subspaces of a vector space V such that $S \cap T = \{0\}$. Then $\dim(S \oplus T) = \dim S + \dim T$.*

3. Green's relations

Let S be a semigroup and let S^1 denote the semigroup obtained by adjoining an identity to S if it does not already have one; if S is a monoid, then $S^1 = S$. The *Green's relations* $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$, introduced by Green, are fundamental for studying semigroup structure. For any $a, b \in S$, we have $a \mathcal{L} b$ if and only if there exist $x, y \in S^1$ such that $a = xb$ and $b = ya$. Dually, $a \mathcal{R} b$ if and only if $a = bx$ and $b = ay$ for some $x, y \in S^1$. The relation $a \mathcal{J} b$ holds if and only if there exist $x, y, u, v \in S^1$ with $a = xby$ and $b = uav$. The other Green's relations are given by

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \quad \text{and} \quad \mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

It follows from their definitions that $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{J}$ are equivalence relations on S. Because the composition of $\mathcal{L}$ and $\mathcal{R}$ is commutative, $\mathcal{D}$ is the join $\mathcal{L} \vee \mathcal{R}$, that is the smallest equivalence relation containing $\mathcal{L} \cup \mathcal{R}$. Furthermore, $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence on S. The following inclusions always hold: $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. In the case where S is commutative, all these relations coincide, that is, $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$.

Green's relations for several semigroups of linear transformations have been characterized in the literature. Specifically, [8, Lemma 2.2] described these relations for the semigroup $P(V)$ of partial linear transformations, [3, Exercise 2.2.6] characterized them for the semigroup $L(V)$ of all linear transformations and [12, Theorem 3.5] established the following characterization for $L_{GL(U)}(V)$.

Theorem 3.1 ([12, Theorem 3.5]). *Let $\alpha, \beta \in L_{GL(U)}(V)$. Then the following statements hold.*

(i) *$(\alpha, \beta) \in \mathcal{L}$ if and only if $V\alpha = V\beta$.*

(ii) *$(\alpha, \beta) \in \mathcal{R}$ if and only if $\ker \alpha = \ker \beta$.*

(iii) *$(\alpha, \beta) \in \mathcal{H}$ if and only if $V\alpha = V\beta$ and $\ker \alpha = \ker \beta$.*

(iv) *$(\alpha, \beta) \in \mathcal{D}$ if and only if $\dim(V\alpha/U) = \dim(V\beta/U)$.*

(v) *$\mathcal{D} = \mathcal{J}$.*

In this section, we describe Green's relations for the semigroup $P_{I(W)}(V)$.

Theorem 3.2. *Let* $\alpha, \beta \in P_{\mathcal{I}(W)}(V)$. *Then* $\alpha = \gamma\beta$ *for some* $\gamma \in P_{\mathcal{I}(W)}(V)$ *if and only if* $V\alpha \subseteq V\beta$ *and* $W\alpha \subseteq W\beta$. *Consequently,* $\alpha \mathcal{L} \beta$ *if and only if* $V\alpha = V\beta$ *and* $W\alpha = W\beta$.

Proof. Assume that $\alpha = \gamma\beta$ for some $\gamma \in P_{\mathcal{I}(W)}(V)$. Then $V\alpha = V(\gamma\beta) \subseteq V\beta$ and $W\alpha = W(\gamma\beta) \subseteq W\beta$.

Conversely, assume that $V\alpha \subseteq V\beta$ and $W\alpha \subseteq W\beta$. Let $W\alpha = \langle w_j\alpha \rangle$, $V\alpha = \langle v_i\alpha \rangle \oplus \langle w_j\alpha \rangle$. For each $i \in I$ and $j \in J$, we have $v_i\alpha = v_i'\beta$ and $w_j\alpha = w_j'\beta$ for some $v_i' \in V$ and $w_j' \in W$. Thus $W\beta = \langle w_j'\beta \rangle \oplus \langle w_k\beta \rangle$ and $V\beta = \langle v_i'\beta \rangle \oplus \langle v_p\beta \rangle \oplus \langle w_j'\beta \rangle \oplus \langle w_k\beta \rangle$. We can write

$$\alpha = \begin{pmatrix} u_s & v_i & w_j \\ 0 & v_i\alpha & w_j\alpha \end{pmatrix} \text{ and } \beta = \begin{pmatrix} u_t & v_i' & v_p & w_j' & w_k \\ 0 & v_i'\beta & v_p\beta & w_j'\beta & w_k\beta \end{pmatrix}$$

where $\ker\alpha = \langle u_s \rangle$ and $\ker\beta = \langle u_t \rangle$. Define $\gamma \in P(V)$ by

$$\gamma = \begin{pmatrix} u_s & v_i & w_j \\ 0 & v_i' & w_j' \end{pmatrix}.$$

It is straightforward to check that $\gamma \in P_{\mathcal{I}(W)}(V)$ and $\alpha = \gamma\beta$. $\qquad\square$

Theorem 3.3. *Let* $\alpha, \beta \in P_{\mathcal{I}(W)}(V)$. *Then* $\alpha = \beta\gamma$ *for some* $\gamma \in P_{\mathcal{I}(W)}(V)$ *if and only if the following conditions hold:*

(i) $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$.

(ii) $\ker\beta \subseteq \ker\alpha$.

(iii) $W\beta^{-1} \cap \operatorname{dom}\alpha \subseteq W\alpha^{-1}$.

(iv) $\ker\alpha \cap W\beta^{-1} \subseteq \ker\beta$.

Consequently, $\alpha \mathcal{R} \beta$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$, $\ker\alpha = \ker\beta$ *and* $W\alpha^{-1} = W\beta^{-1}$.

Proof. Assume that $\alpha = \beta\gamma$ for some $\gamma \in P_{\mathcal{I}(W)}(V)$. It is clear that $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$. Let $u \in \ker\beta$. Then $u\beta = 0$ which implies that $0 = u\beta\gamma = u\alpha$. Hence $u \in \ker\alpha$. Thus $\ker\beta \subseteq \ker\alpha$. Let $v \in W\beta^{-1} \cap \operatorname{dom}\alpha$. Then $v\beta = w$ for some $w \in W$. We have $v\alpha = v\beta\gamma = w\gamma \in W$ since $\gamma|_W \in \mathcal{I}(W)$. Hence $v \in W\alpha^{-1}$ and so $W\beta^{-1} \cap \operatorname{dom}\alpha \subseteq W\alpha^{-1}$. Let $u \in \ker\alpha \cap W\beta^{-1}$. Then $u\alpha = 0$ and $u\beta = w$ for some $w \in W$. We have $0 = u\alpha = u\beta\gamma = w\gamma$ which implies that $w \in \ker\gamma|_W = \{0\}$ since $\gamma|_W \in \mathcal{I}(W)$. Thus $u\beta = w = 0$ from which it follows that $u \in \ker\beta$ and so $\ker\alpha \cap W\beta^{-1} \subseteq \ker\beta$.

Conversely, assume that the conditions hold. We consider the following two cases.

Case 1: $V\alpha = \{0\}$. Let $\ker\beta = \langle u_i \rangle$. By (ii), we can write $\ker\alpha = \langle u_i \rangle \oplus \langle u_j \rangle$ and

$$\alpha = \begin{pmatrix} u_i & u_j \\ 0 & 0 \end{pmatrix}.$$

We assert that $\{u_j\beta\}$ is linearly independent. Let $\sum a_j u_j\beta = 0$ for some $a_j \in \mathbb{F}$. Then $\sum a_j u_j \in \ker\beta = \langle u_i \rangle$ which implies that $a_j = 0$ for all j since $\{u_i, u_j\}$ is linearly independent. Thus $\{u_j\beta\}$ is linearly independent. Let $V\beta = \langle u_j\beta \rangle \oplus \langle v_k\beta \rangle$. We can write

$$\beta = \begin{pmatrix} u_i & u_j & v_k \\ 0 & u_j\beta & v_k\beta \end{pmatrix}.$$

Define $\gamma \in P(V)$ by

$$\gamma = \begin{pmatrix} u_j\beta \\ 0 \end{pmatrix}.$$

To show that $\gamma \in P_{\mathcal{I}(W)}(V)$, let $w \in W \cap \operatorname{dom}\gamma$. Then $w = \sum a_j u_j\beta$ for some $a_j \in \mathbb{F}$. Then $\sum a_j u_j \in \ker\alpha \cap W\beta^{-1} \subseteq \ker\beta = \langle u_i \rangle$ which implies that $a_j = 0$ for all j since $\{u_i, u_j\}$ is linearly independent. Thus $w = 0$ and so $W \cap \operatorname{dom}\gamma = \{0\}$. Hence $\gamma \in P_{\mathcal{I}(W)}(V)$. Moreover, it is straightforward to check that $\alpha = \beta\gamma$.

Case 2: $V\alpha \neq \{0\}$. Let $W\alpha = \langle w_i\alpha \rangle$ and $V\alpha = \langle v_i\alpha \rangle \oplus \langle w_i\alpha \rangle$. Let $\ker\beta = \langle u_r \rangle$. Then $\ker\alpha = \langle u_r \rangle \oplus \langle u_s \rangle$ since $\ker\beta \subseteq \ker\alpha$. We can write

$$\alpha = \begin{pmatrix} u_r & u_s & v_i & w_j \\ 0 & 0 & v_i\alpha & w_j\alpha \end{pmatrix}.$$

We claim that $\{u_s\beta, v_i\beta, w_j\beta\}$ is linearly independent. Consider

$$\sum a_s u_s\beta + \sum b_i v_i\beta + \sum c_j w_j\beta = 0$$

for some $a_s, b_i, c_j \in \mathbb{F}$. Then

$$\left(\sum a_s u_s + \sum b_i v_i + \sum c_j w_j\right)\beta = 0$$

which implies that

$$\sum a_s u_s + \sum b_i v_i + \sum c_j w_j \in \ker\beta = \langle u_r \rangle.$$

We obtain $a_s = b_i = c_j = 0$ for all s, i, j since $\{u_r, u_s, v_i, w_j\}$ is linearly independent. Thus $\{u_s\beta, v_i\beta, w_j\beta\}$ is linearly independent. Define $\gamma \in P(V)$ by

$$\gamma = \begin{pmatrix} u_s\beta & v_i\beta & w_j\beta \\ 0 & v_i\alpha & w_j\alpha \end{pmatrix}.$$

We assert that $\gamma \in P_{\mathcal{I}(W)}(V)$. Let $w \in W \cap \operatorname{dom}\gamma$. Then

$$w = \sum a_s u_s\beta + \sum b_i v_i\beta + \sum c_j w_j\beta = \left(\sum a_s u_s + \sum b_i v_i + \sum c_j w_j\right)\beta$$

for some $a_s, b_i, c_j \in \mathbb{F}$. We obtain

$$\sum a_s u_s + \sum b_i v_i + \sum c_j w_j \in W\beta^{-1} \cap \operatorname{dom}\alpha \subseteq W\alpha^{-1}$$

which implies that

$$w\gamma = \sum b_i v_i\alpha + \sum c_j w_j\alpha = \sum a_s u_s\alpha + \sum b_i v_i\alpha + \sum c_j w_j\alpha \in W.$$

Hence $W\gamma \subseteq W$. To show that $\ker\gamma \cap W = \{0\}$, let $u \in \ker\gamma \cap W$. Then $u = \sum a_s u_s\beta$ for some $a_s \in \mathbb{F}$. We have $\sum a_s u_s \in \ker\alpha \cap W\beta^{-1} \subseteq \ker\beta$ which implies that $u = 0$. Thus $\ker\gamma \cap W = \{0\}$ and so $\gamma \in P_{\mathcal{I}(W)}(V)$.

Finally, we prove that $\alpha = \beta\gamma$. To show that $\operatorname{dom}\alpha = \operatorname{dom}(\beta\gamma)$, let $v \in \operatorname{dom}\alpha$. Then $v = \sum a_r u_r + \sum b_s u_s + \sum c_i v_i + \sum d_j w_j$ for some $a_r, b_s, c_i, d_j \in \mathbb{F}$. Then $v\beta = \sum b_s u_s\beta + \sum c_i v_i\beta + \sum d_j w_j\beta \in \operatorname{dom}\gamma$

from which it follows that $v \in \text{dom}(\beta\gamma)$. Thus $\text{dom }\alpha \subseteq \text{dom}(\beta\gamma)$. On the other hand, let $v \in \text{dom}(\beta\gamma)$. Then $v\beta \in \text{dom }\gamma$ which implies that $v\beta = \sum a_s u_s \beta + \sum b_i v_i \beta + \sum c_j w_j \beta$ for some $a_s, b_i, c_j \in \mathbb{F}$. Hence

$$v - \sum a_s u_s - \sum b_i v_i - \sum c_j w_j \in \ker \beta = \langle u_r \rangle,$$

and so $v = \sum d_r u_r + \sum a_s u_s + \sum b_i v_i + \sum c_j w_j$ for some $d_r \in \mathbb{F}$. Thus $v \in \text{dom }\alpha$ which implies that $\text{dom}(\beta\gamma) \subseteq \text{dom }\alpha$. It is concluded that $\text{dom }\alpha = \text{dom}(\beta\gamma)$. Let $v \in \text{dom }\alpha$. Then $v = \sum a_r u_r + \sum b_s u_s + \sum c_i v_i + \sum d_j w_j$ for some $a_r, b_s, c_i, d_j \in \mathbb{F}$. We have

$$v\beta\gamma = \sum a_r u_r \beta\gamma + \sum b_s u_s \beta\gamma + \sum c_i v_i \beta\gamma + \sum d_j w_j \beta\gamma = \sum c_i v_i \alpha + \sum d_j w_j \alpha$$

and

$$v\alpha = \sum a_r u_r \alpha + \sum b_s u_s \alpha + \sum c_i v_i \alpha + \sum d_j w_j \alpha = \sum c_i v_i \alpha + \sum d_j w_j \alpha,$$

from which it follows that $v\beta\gamma = v\alpha$. Therefore, $\alpha = \beta\gamma$. $\qquad\square$

Combining Theorems 3.2 and 3.3 yields the following result.

Theorem 3.4. *Let $\alpha, \beta \in P_{I(W)}(V)$. Then $\alpha \mathcal{H} \beta$ if and only if $V\alpha = V\beta$, $W\alpha = W\beta$, $\text{dom }\alpha = \text{dom }\beta$, $\ker \alpha = \ker \beta$ and $W\alpha^{-1} = W\beta^{-1}$.*

Next, we consider the relations $\mathcal{J}$ and $\mathcal{D}$ for the semigroup $P_{I(W)}(V)$. To do this, we need to analyze the dimensions of the vector spaces involved.

Lemma 3.5. *Let $\alpha, \beta \in P_{I(W)}(V)$. Then*

$$\dim(V\alpha\beta/(V\alpha\beta \cap W)) \leq \dim(V\alpha/(V\alpha \cap W))$$

and

$$\dim(V\alpha\beta/(V\alpha\beta \cap W)) \leq \dim(V\beta/(V\beta \cap W)).$$

Proof. Let $V\alpha\beta \cap W = \langle v_i \alpha\beta \rangle$ and $V\alpha\beta = \langle v_i \alpha\beta \rangle \oplus \langle v_j \alpha\beta \rangle$. Then $\dim(V\alpha\beta/(V\alpha\beta \cap W)) = |J|$. We can write

$$\alpha\beta = \begin{pmatrix} u_s & v_i & v_j \\ 0 & v_i \alpha\beta & v_j \alpha\beta \end{pmatrix},$$

where $\ker(\alpha\beta) = \langle u_s \rangle$. We assert that $\{v_i \alpha\} \cup \{v_j \alpha\}$ is linearly independent. Consider $\sum a_i v_i \alpha + \sum b_j v_j \alpha = 0$ for some $a_i, b_j \in \mathbb{F}$. Then $\sum a_i v_i \alpha\beta + \sum b_j v_j \alpha\beta = 0$ which implies that $\sum a_i v_i + \sum b_j v_j \in \ker(\alpha\beta) = \langle u_s \rangle$. Since $\{u_s\} \dot\cup \{v_i\} \dot\cup \{v_j\}$ is linearly independent, we obtain $a_i = b_j = 0$ for all i, j. Thus $\{v_i \alpha\} \cup \{v_j \alpha\}$ is linearly independent.

We claim that $\langle v_j \alpha \rangle \cap W = \{0\}$. Let $v \in \langle v_j \alpha \rangle \cap W$. Then $v = \sum a_j v_j \alpha$ for some $a_j \in \mathbb{F}$. We have $v\beta = \sum a_j v_j \alpha\beta \subseteq V\alpha\beta \cap W$ and $v\beta \in \langle v_j \alpha\beta \rangle$ from which it follows that $v\beta = 0$ since $\langle v_j \alpha\beta \rangle$ is a complement of $V\alpha\beta \cap W$ in $V\alpha\beta$. Hence $0 = v\beta = \sum a_j v_j \alpha\beta$ which implies that $a_j = 0$ for all j. Thus $v = 0$ and so $\langle v_j \alpha \rangle \cap W = \{0\}$ as claimed.

It is concluded that $\langle v_j \alpha \rangle \cap (V\alpha \cap W) = \{0\}$. Let $V\alpha \cap W = \langle v_k \alpha \rangle$. We can write $V\alpha = \langle v_r \alpha \rangle \oplus \langle v_j \alpha \rangle \oplus \langle v_k \alpha \rangle$. Therefore,

$$\dim(V\alpha/(V\alpha \cap W)) = |R| + |J| \geq |J| = \dim(V\alpha\beta/(V\alpha\beta \cap W)).$$

Since $\langle v_j \alpha\beta \rangle \cap W = \{0\}$, it is straightforward to check that

$$\dim(V\beta/(V\beta \cap W)) \geq |J| = \dim(V\alpha\beta/(V\alpha\beta \cap W)).$$

$\qquad\square$

Theorem 3.6. *Let $\alpha, \beta \in P_{\mathcal{I}(W)}(V)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in P_{\mathcal{I}(W)}(V)$ if and only if* $\dim(V\alpha) \leq \dim(V\beta)$, $\dim(W\alpha) \leq \dim(W\beta)$ *and* $\dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\beta/(V\beta \cap W))$. *Consequently, $\alpha \mathcal{J} \beta$ if and only if* $\dim(V\alpha) = \dim(V\beta)$, $\dim(W\alpha) = \dim(W\beta)$ *and* $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$.

Proof. Assume that $\dim(V\alpha) \leq \dim(V\beta)$, $\dim(W\alpha) \leq \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\beta/(V\beta \cap W))$. Let $W\alpha = \langle w_k\alpha \rangle$, $V\alpha \cap W = \langle v_j \rangle \oplus \langle w_k\alpha \rangle$ and $V\alpha = \langle v_i\alpha \rangle \oplus \langle v_j\alpha \rangle \oplus \langle w_k\alpha \rangle$. Let $W\beta = \langle w'_k\beta \rangle \oplus \langle w_t\beta \rangle$, $V\beta \cap W = \langle v_s\beta \rangle \oplus \langle w'_k\beta \rangle \oplus \langle w_t\beta \rangle$. Since $|I| = \dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\beta/(V\beta \cap W))$, we can write $V\beta = \langle v'_i\beta \rangle \oplus \langle v_r\beta \rangle \oplus V\beta \cap W$. Then $V\beta = \langle v'_i\beta \rangle \oplus \langle v_r\beta \rangle \oplus \langle v_s\beta \rangle \oplus \langle w'_k\beta \rangle \oplus \langle w_t\beta \rangle$. Hence, we can write

$$\alpha = \begin{pmatrix} u_p & v_i & v_j & w_k \\ 0 & v_i\alpha & v_j\alpha & w_k\alpha \end{pmatrix} \text{ and } \beta = \begin{pmatrix} u_q & v'_i & v_r & v_s & w'_k & w_t \\ 0 & v'_i\beta & v_r\beta & v_s\beta & w'_k\beta & w_t\beta \end{pmatrix},$$

where $\ker\alpha = \langle u_p \rangle$ and $\ker\beta = \langle u_q \rangle$. Since $\dim(V\alpha) \leq \dim(V\beta)$, we have $|I| + |J| + |K| \leq |I| + |R| + |S| + |K| + |T|$. Consider the following two cases:

Case 1: $|J| \leq |R| + |S| + |T|$. In this case, we can write $\{v_r\beta\} \dot{\cup} \{v_s\beta\} \dot{\cup} \{w_t\beta\} = \{v'_j\beta\} \dot{\cup} \{v_n\beta\}$ which implies that

$$\beta = \begin{pmatrix} u_q & v'_i & v'_j & v_n & w'_k \\ 0 & v'_i\beta & v'_j\beta & v_n\beta & w'_k\beta \end{pmatrix}.$$

Define $\lambda, \mu \in P(V)$ by

$$\lambda = \begin{pmatrix} u_p & v_i & v_j & w_k \\ 0 & v'_i & v'_j & w'_k \end{pmatrix} \text{ and } \mu = \begin{pmatrix} v'_i\beta & v'_j\beta & w'_k\beta \\ v_i\alpha & v_j\alpha & w_k\alpha \end{pmatrix}.$$

One can check that $\lambda, \mu \in P_{\mathcal{I}(W)}(V)$ and $\alpha = \lambda\beta\mu$.

Case 2: $|J| > |R| + |S| + |T|$. Suppose to the contrary that $|I| + |K|$ is finite. Then $|I| + |J| + |K| > |I| + |R| + |S| + |K| + |T|$, which contradicts our assumption. Thus $|I| + |K|$ must be infinite. We claim that $|I| + |K| \geq |J|$. If not, then $|I| + |K| < |J|$, which implies that $|J|$ is infinite and

$$|I| + |J| + |K| = |J| = |J| + |R| + |S| + |T| > |I| + |R| + |S| + |K| + |T|.$$

However, this contradicts the assumption that $|I| + |J| + |K| \leq |I| + |R| + |S| + |K| + |T|$. Thus $|I| + |K| < |J|$ is impossible. Hence $|I| + |K| \geq |J|$. Consider the following two subcases:

Subcase 2.1: $|I| \geq |K|$. Then $|I|$ must be infinite and $|I| = |I| + |K| \geq |J|$. We can write $\{v'_i\beta\} = \{v''_i\beta\} \dot{\cup} \{v''_j\beta\}$, which implies that

$$\beta = \begin{pmatrix} u_q & v''_i & v_r & v_s & v''_j & w'_k & w_t \\ 0 & v''_i\beta & v_r\beta & v_s\beta & v''_j\beta & w'_k\beta & w_t\beta \end{pmatrix}.$$

Define $\lambda, \mu \in P(V)$ by

$$\lambda = \begin{pmatrix} u_p & v_i & v_j & w_k \\ 0 & v''_i & v''_j & w'_k \end{pmatrix} \text{ and } \mu = \begin{pmatrix} v''_i\beta & v''_j\beta & w'_k\beta \\ v_i\alpha & v_j\alpha & w_k\alpha \end{pmatrix}.$$

It is straightforward to verify that $\lambda, \mu \in P_{\mathcal{I}(W)}(V)$ and $\alpha = \lambda\beta\mu$.

Subcase 2.2: $|K| > |I|$. Then $|K|$ must be infinite and $|K| = |I| + |K| \geq |J|$. We can write $\{w'_k\beta\} = \{w''_j\beta\} \dot\cup \{w''_k\beta\}$, which implies that

$$\beta = \begin{pmatrix} u_q & v'_i & v_r & v_s & w''_j & w''_k & w_t \\ 0 & v'_i\beta & v_r\beta & v_s\beta & w''_j\beta & w''_k\beta & w_t\beta \end{pmatrix}.$$

Define $\lambda, \mu \in P(V)$ by

$$\lambda = \begin{pmatrix} u_p & v_i & v_j & w_k \\ 0 & v'_i & w''_j & w''_k \end{pmatrix} \text{ and } \mu = \begin{pmatrix} v'_i\beta & w''_j\beta & w''_k\beta \\ v_i\alpha & v_j\alpha & w_k\alpha \end{pmatrix}.$$

Similar to the previous subcase, it is straightforward to check that $\lambda, \mu \in P_{I(W)}(V)$ and $\alpha = \lambda\beta\mu$.

Conversely, assume that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in P_{I(W)}(V)$. By Lemma 3.5, we have $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\lambda\beta\mu/(V\lambda\beta\mu \cap W)) \leq \dim(V\beta/(V\beta \cap W))$. Moreover, since $V\alpha = V\lambda\beta\mu \subseteq V\beta\mu$, we have $\dim(V\alpha) \leq \dim(V\beta\mu) \leq \dim(V\beta)$. Similarly, we obtain $\dim(W\alpha) \leq \dim(W\beta)$. □

Let $1 \leq r \leq \dim V$, $1 \leq s \leq \dim W$ and $1 \leq t \leq \dim V/W$. Define a subset $J(r,s,t)$ of $P_{I(W)}(V)$ by

$$J(r,s,t) = \{\alpha \in P_{I(W)}(V) : \dim(V\alpha) = r, \ \dim(W\alpha) = s, \ \dim(V\alpha/(V\alpha \cap W)) = t\}.$$

Then $J(r,s,t)$ is a $\mathcal{J}$-class of $P_{I(W)}(V)$.

Theorem 3.7. *Let $\alpha, \beta \in P_{I(W)}(V)$. Then $\alpha\mathcal{D}\beta$ if and only if $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$, $\dim((V\alpha \cap W)/W\alpha) = \dim((V\beta \cap W)/W\beta)$ and $\dim(W\alpha) = \dim(W\beta)$.*

Proof. Assume that $\alpha\mathcal{D}\beta$. Then $\alpha\mathcal{J}\beta$ since $\mathcal{D} \subseteq \mathcal{J}$. Thus $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$, $\dim(V\alpha) = \dim(V\beta)$ and $\dim(W\alpha) = \dim(W\beta)$. Let $W\alpha = \langle w_k\alpha \rangle$, $W\beta = \langle w'_k\beta \rangle$, $V\alpha \cap W = \langle v_s\alpha \rangle \oplus \langle w_k\alpha \rangle$ and $V\beta \cap W = \langle v_t\beta \rangle \oplus \langle w'_k\beta \rangle$. Since $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$, we can write $V\alpha = \langle v_i\alpha \rangle \oplus (V\alpha \cap W)$ and $V\beta = \langle v'_i\beta \rangle \oplus (V\beta \cap W)$, which implies that

$$\alpha = \begin{pmatrix} u_p & v_i & v_s & w_k \\ 0 & v_i\alpha & v_s\alpha & w_k\alpha \end{pmatrix} \text{ and } \beta = \begin{pmatrix} u_q & v'_i & v_t & w'_k \\ 0 & v'_i\beta & v_t\beta & w'_k\beta \end{pmatrix},$$

where $\ker\alpha = \langle u_p \rangle$ and $\ker\beta = \langle u_q \rangle$. Since $\alpha\mathcal{D}\beta$, there exists $\gamma \in P_{I(W)}(V)$ such that $\alpha\mathcal{L}\gamma\mathcal{R}\beta$. By Theorem 3.3, we have $\text{dom}\,\gamma = \text{dom}\,\beta = \langle u_q \rangle \oplus \langle v'_i \rangle \oplus \langle v_t \rangle \oplus \langle w'_k \rangle$, $\ker\gamma = \ker\beta = \langle u_q \rangle$ and $W\gamma^{-1} = W\beta^{-1} = \langle v_t \rangle \oplus \langle w'_k \rangle$. We assert that $\{v'_i\gamma, v_t\gamma, w'_k\gamma\}$ is linearly independent. Consider

$$\sum a_i v'_i\gamma + \sum b_t v_t\gamma + \sum c_k w'_k\gamma = 0$$

for some $a_i, b_t, c_k \in \mathbb{F}$. Then

$$\sum a_i v'_i + \sum b_t v_t + \sum c_k w'_k \in \ker\gamma = \langle u_q \rangle$$

which implies that $a_i = b_t = c_k = 0$ for all i, t, k since $\{u_q, v'_i, v_t, w'_k\}$ is linearly independent. Thus $\{v'_i\gamma, v_t\gamma, w'_k\gamma\}$ is linearly independent. We can write

$$\gamma = \begin{pmatrix} u_q & v'_i & v_t & w'_k \\ 0 & v'_i\gamma & v_t\gamma & w'_k\gamma \end{pmatrix}.$$

Since $W\gamma^{-1} = \langle v_t \rangle \oplus \langle w'_k \rangle$, we obtain $V\gamma \cap W = \langle v_t\gamma \rangle \oplus \langle w'_k\gamma \rangle$ from which it follows that $\dim((V\gamma \cap W)/W\gamma) = |T|$. By Theorem 3.2, we have

$$\dim((V\alpha \cap W)/W\alpha) = \dim((V\gamma \cap W)/W\gamma) = |T| = \dim((V\beta \cap W)/W\beta).$$

Conversely, assume that $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$, $\dim((V\alpha \cap W)/W\alpha) = \dim((V\beta \cap W)/W\beta)$ and $\dim(W\alpha) = \dim(W\beta)$. Let $W\alpha = \langle w_k\alpha \rangle$, $V\alpha \cap W = \langle v_j\alpha \rangle \oplus \langle w_k\alpha \rangle$ and $V\alpha = \langle v_i\alpha \rangle \oplus (V\alpha \cap W)$. By the assumptions, we can write $W\beta = \langle w'_k\beta \rangle$, $V\beta \cap W = \langle v'_j\beta \rangle \oplus \langle w'_k\beta \rangle$ and $V\beta = \langle v'_i\beta \rangle \oplus (V\beta \cap W)$. Hence

$$\alpha = \begin{pmatrix} u_p & v_i & v_j & w_k \\ 0 & v_i\alpha & v_j\alpha & w_k\alpha \end{pmatrix} \text{ and } \beta = \begin{pmatrix} u_q & v'_i & v'_j & w'_k \\ 0 & v'_i\beta & v'_j\beta & w'_k\beta \end{pmatrix},$$

where $\ker\alpha = \langle u_p \rangle$ and $\ker\beta = \langle u_q \rangle$. Define $\gamma \in P(V)$ by

$$\gamma = \begin{pmatrix} u_q & v'_i & v'_j & w'_k \\ 0 & v_i\alpha & v_j\alpha & w_k\alpha \end{pmatrix}.$$

It is straightforward to verify that $\gamma \in P_{I(W)}(V)$ and $\alpha \mathcal{L} \gamma \mathcal{R} \beta$. Therefore, $\alpha \mathcal{D} \beta$. $\qquad\square$

The following example shows that, in general, the relations $\mathcal{D}$ and $\mathcal{J}$ are not equal on $P_{I(W)}(V)$.

Example 3.8. *Let $V = \langle v_1, v_2, v_3, \ldots \rangle$ and $W = \langle v_2, v_3, v_4, \ldots \rangle$ be vector spaces over a field $\mathbb{F}$. Define $\alpha, \beta \in P_{I(W)}(V)$ by*

$$\alpha = \begin{pmatrix} v_2 & v_3 & v_4 & v_5 & \cdots \\ v_2 & v_3 & v_4 & v_5 & \cdots \end{pmatrix} \text{ and } \beta = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & \cdots \\ v_2 & v_3 & v_4 & v_5 & \cdots \end{pmatrix}.$$

It is routinely verified that $\alpha \mathcal{J} \beta$, but α does not $\mathcal{D}$-relate to β since $\dim((V\alpha \cap W)/W\alpha) = 0 \neq 1 = \dim((V\beta \cap W)/W\beta)$.

We now present a necessary and sufficient condition for when the relations $\mathcal{D}$ and $\mathcal{J}$ coincide on $P_{I(W)}(V)$.

Theorem 3.9. $\mathcal{D} = \mathcal{J}$ *on $P_{I(W)}(V)$ if and only if $V = W$ or $W = \{0\}$ or $\dim V$ is finite.*

Proof. Suppose that $V \neq W$, $W \neq \{0\}$ and $\dim V$ is infinite. We consider the following two cases.

Case 1: $\dim W$ is infinite. Then there exists a countable subset $\{w_1, w_2, \ldots\}$ of a basis of W. Moreover, since $V \neq W$, there exists a vector $v \in V$ such that $v \notin W$. Define $\alpha, \beta \in P_{I(W)}(V)$ by

$$\alpha = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & \cdots \\ w_1 & w_2 & w_3 & w_4 & \cdots \end{pmatrix} \text{ and } \beta = \begin{pmatrix} v & w_1 & w_2 & w_3 & \cdots \\ w_1 & w_2 & w_3 & w_4 & \cdots \end{pmatrix}.$$

We can check that $\alpha \mathcal{J} \beta$ but α does not $\mathcal{D}$-relate to β since $\dim((V\alpha \cap W)/W\alpha) = 0 \neq 1 = \dim((V\beta \cap W)/W\beta)$.

Case 2: $\dim W$ is finite. Let w be a nonzero vector in W. Since $\dim V$ is infinite, there exists a countable linearly independent subset $\{v_1, v_2, \ldots\}$ of V such that $\langle v_1, v_2, \ldots \rangle \cap W = \{0\}$. Define $\alpha, \beta \in P_{I(W)}(V)$ by

$$\alpha = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & \cdots \\ w & v_2 & v_3 & v_4 & \cdots \end{pmatrix} \text{ and } \beta = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & \cdots \\ v_1 & v_2 & v_3 & v_4 & \cdots \end{pmatrix}.$$

It is straightforward to check that $\alpha \mathcal{J} \beta$, but α does not $\mathcal{D}$-relate to β since $\dim((V\alpha \cap W)/W\alpha) = 1 \neq 0 = \dim((V\beta \cap W)/W\beta)$.

Conversely, assume that the assumptions hold. If $V = W$ or $W = \{0\}$, then $P_{\mathcal{I}(W)}(V) = \mathcal{I}(V)$ or $P_{\mathcal{I}(W)}(V) = P(V)$, respectively. Hence $\mathcal{D} = \mathcal{J}$. Now, assume that $\dim V$ is finite. Let $\alpha, \beta \in P_{\mathcal{I}(W)}(V)$ be such that $\alpha \mathcal{J} \beta$. By Theorem 3.6, we have $\dim(V\alpha) = \dim(V\beta)$, $\dim(W\alpha) = \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$. We obtain

$$\dim(V\alpha) - \dim(V\alpha \cap W) = \dim(V\alpha/(V\alpha \cap W))$$
$$= \dim(V\beta/(V\beta \cap W))$$
$$= \dim(V\beta) - \dim(V\beta \cap W)$$
$$= \dim(V\alpha) - \dim(V\beta \cap W).$$

Thus $\dim(V\alpha \cap W) = \dim(V\beta \cap W)$. It follows that

$$\dim((V\alpha \cap W)/W\alpha) = \dim(V\alpha \cap W) - \dim(W\alpha)$$
$$= \dim(V\beta \cap W) - \dim(W\beta)$$
$$= \dim((V\beta \cap W)/W\beta).$$

Therefore, $\alpha \mathcal{D} \beta$. $\qquad \square$

4. Regularity

Recall that an element a in a semigroup S is *regular* if there exists $x \in S$ such that $axa = a$. The semigroup S is called *regular* if every element is regular. An element b is an *inverse* of a if

$$aba = a \quad \text{and} \quad bab = b.$$

A semigroup S is called an *inverse semigroup* if each element possesses a unique inverse.

An element a in a semigroup S is *left regular* if there exists $x \in S$ such that $a = xa^2$ and *right regular* if $a = a^2x$ for some $x \in S$. The element a is *completely regular* if there exists $x \in S$ such that $a = axa$ and $ax = xa$. Every completely regular element is both left and right regular, in fact, an element is completely regular if and only if it is both left and right regular. If every element in S is left (right, completely) regular, then S is called a left (right, completely) regular semigroup.

While $P(V)$ and $L(V)$ are regular semigroups, [12, Theorem 3.2] established that $L_{GL(U)}(V)$ is regular as well.

In this section, we provide necessary and sufficient criteria for an element of $P_{\mathcal{I}(W)}(V)$ to be regular, left regular, or right regular. We also characterize when $P_{\mathcal{I}(W)}(V)$ is regular, left regular, right regular, or completely regular as a semigroup. Additionally, we identify the conditions under which $P_{\mathcal{I}(W)}(V)$ forms an inverse semigroup.

Theorem 4.1. *Let $\alpha \in P_{\mathcal{I}(W)}(V)$. Then α is regular if and only if $V\alpha \cap W = W\alpha$.*

Proof. Assume that α is regular. Then there is $\beta \in P_{\mathcal{I}(W)}(V)$ such that $\alpha = \alpha\beta\alpha$. Let $w \in V\alpha \cap W$. Then $w = v\alpha$ for some $v \in V$. We have $w = v\alpha = v\alpha\beta\alpha = w\beta\alpha$. Since $w \in W$, we obtain $w\beta\alpha \in W\alpha$. Thus $w \in W\alpha$, which implies that $V\alpha \cap W \subseteq W\alpha \subseteq V\alpha \cap W$. Therefore, $V\alpha \cap W = W\alpha$.

Conversely, assume that $V\alpha \cap W = W\alpha$. We can write $V\alpha \cap W = \langle w_k\alpha \rangle$. Let $V\alpha = \langle v_j\alpha \rangle \oplus \langle w_k\alpha \rangle$. Then

$$\alpha = \begin{pmatrix} u_i & v_j & w_k \\ 0 & v_j\alpha & w_k\alpha \end{pmatrix},$$

where $\ker \alpha = \langle u_i \rangle$. Define $\beta \in P(V)$ by

$$\beta = \begin{pmatrix} v_j\alpha & w_k\alpha \\ v_j & w_k \end{pmatrix}.$$

It is straightforward to check that $\beta \in P_{\mathcal{I}(W)}(V)$ and $\alpha = \alpha\beta\alpha$. Thus α is regular. $\square$

Theorem 4.2. *The semigroup $P_{\mathcal{I}(W)}(V)$ is regular if and only if $V = W$ or $W = \{0\}$.*

Proof. It is clear that if $V = W$ or $W = \{0\}$, then $P_{\mathcal{I}(W)}(V)$ is regular since $P_{\mathcal{I}(W)}(V) = \mathcal{I}(V)$ or $P_{\mathcal{I}(W)}(V) = P(V)$.

Conversely, suppose that $V \neq W$ and $W \neq \{0\}$. Then there exist nonzero vectors $w \in W$ and $v \in V \setminus W$. Define $\alpha \in P_{\mathcal{I}(W)}(V)$ by $\alpha = \begin{pmatrix} v \\ w \end{pmatrix}$. We obtain by Theorem 4.1 that α is not regular since $w \in V\alpha \cap W$ but $w \notin W\alpha$. Therefore, $P_{\mathcal{I}(W)}(V)$ is not regular. $\square$

Theorem 4.3. *$P_{\mathcal{I}(W)}(V)$ is an inverse semigroup if and only if $V = W$.*

Proof. Suppose that $V \neq W$. Then there exists a nonzero vector $v \in V \setminus W$. Define $\alpha, \beta \in P_{\mathcal{I}(W)}(V)$ by $\alpha = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} v \\ 0 \end{pmatrix}$. It is clear that $\alpha = \alpha\alpha\alpha$, $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$. Hence both α and β are distinct inverse elements of α. Thus $P_{\mathcal{I}(W)}(V)$ is not an inverse semigroup. The converse is clear since if $V = W$, then $P_{\mathcal{I}(W)}(V) = \mathcal{I}(V)$, which is an inverse semigroup. $\square$

Theorem 4.4. *Let $\alpha \in P_{\mathcal{I}(W)}(V)$. Then α is left regular if and only if $V\alpha = V\alpha^2$ and $W\alpha = W\alpha^2$.*

Proof. Assume that α is left regular. Then there exists $\beta \in P_{\mathcal{I}(W)}(V)$ such that $\alpha = \beta\alpha^2$. By Theorem 3.2, we have $V\alpha \subseteq V\alpha^2$ and $W\alpha \subseteq W\alpha^2$. On the other hand, it is clear that $V\alpha^2 \subseteq V\alpha$ and $W\alpha^2 \subseteq W\alpha$. Thus $V\alpha = V\alpha^2$ and $W\alpha = W\alpha^2$.

Conversely, assume that $V\alpha = V\alpha^2$ and $W\alpha = W\alpha^2$. Again by Theorem 3.2, there exists $\beta \in P_{\mathcal{I}(W)}(V)$ such that $\alpha = \beta\alpha^2$. Therefore, α is left regular. $\square$

Theorem 4.5. *$P_{\mathcal{I}(W)}(V)$ is left regular if and only if $\dim V \leq 1$.*

Proof. Assume that $\dim V > 1$. Then there exist nonzero vectors u and v in V such that $\{u, v\}$ is linearly independent. Define $\alpha \in P_{\mathcal{I}(W)}(V)$ by

$$\alpha = \begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} & \text{if } v \in W, \\ \begin{pmatrix} v \\ u \end{pmatrix} & \text{otherwise.} \end{cases}$$

It follows from Theorem 4.4 that α is not left regular since $V\alpha = \langle u \rangle$ or $V\alpha = \langle v \rangle$, but $V\alpha^2 = \{0\}$.

Conversely, assume that $\dim V \leq 1$. If $\dim V = 0$, then $P_{\mathcal{I}(W)}(V) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$, which is trivially left regular. Now, suppose that $\dim V = 1$. If $\dim W = 0$, then $V = \langle v \rangle$ and $W = \{0\}$, which implies that

$$P_{\mathcal{I}(W)}(V) = P(V) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} v \\ av \end{pmatrix} : a \in \mathbb{F} \right\}.$$

By direct computation, we can verify that $V\alpha = V\alpha^2$ and $W\alpha = W\alpha^2$ for all $\alpha \in P_{\mathcal{I}(W)}(V)$. If $\dim W = 1$, then $W = \langle w \rangle = V$ and so

$$P_{\mathcal{I}(W)}(V) = \mathcal{I}(V) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} w \\ aw \end{pmatrix} : a \in \mathbb{F} \setminus \{0\} \right\}.$$

Again, it is straightforward to check that $V\alpha = V\alpha^2$ and $W\alpha = W\alpha^2$ for all $\alpha \in P_{\mathcal{I}(W)}(V)$. Therefore, $P_{\mathcal{I}(W)}(V)$ is left regular. $\qquad \square$

Theorem 4.6. *Let $\alpha \in P_{\mathcal{I}(W)}(V)$. Then α is right regular if and only if $\mathrm{dom}\, \alpha = \mathrm{dom}\, \alpha^2$, $\ker \alpha = \ker \alpha^2$ and $(V\alpha \setminus W)\alpha \subseteq V \setminus W$.*

Proof. Assume that α is right regular. Then there exists $\beta \in P_{\mathcal{I}(W)}(V)$ such that $\alpha = \alpha^2\beta$. By Theorem 3.3 (i), we have $\mathrm{dom}\, \alpha \subseteq \mathrm{dom}\, \alpha^2 \subseteq \mathrm{dom}\, \alpha$ which implies that $\mathrm{dom}\, \alpha = \mathrm{dom}\, \alpha^2$. In addition, we have by Theorem 3.3 (ii) that $\ker \alpha^2 \subseteq \ker \alpha \subseteq \ker \alpha^2$ and so $\ker \alpha = \ker \alpha^2$. Let $v\alpha \in V\alpha \setminus W$. If $v\alpha^2 \in W$, then $v\alpha = v\alpha^2\beta \in W$, which is a contradiction. Thus $v\alpha^2 \notin W$ and so $(V\alpha \setminus W)\alpha \subseteq V \setminus W$.

Conversely, assume that $\mathrm{dom}\, \alpha = \mathrm{dom}\, \alpha^2$, $\ker \alpha = \ker \alpha^2$ and $(V\alpha \setminus W)\alpha \subseteq V \setminus W$. Let $v \in V\alpha^2$. Then there exists $v' \in V\alpha$ such that $v = v'\alpha$. Define a function $\beta : V\alpha^2 \to V$ by $v\beta = v'$ for all $v \in V\alpha^2$. First, we assert that β is well-defined. Let $v \in V\alpha^2$ be such that $v = v'\alpha$ and $v = v''\alpha$ for some $v', v'' \in V\alpha$. Then $v' = u'\alpha$ and $v'' = u''\alpha$ for some $u', u'' \in V$. We obtain

$$(u' - u'')\alpha^2 = (u'\alpha - u''\alpha)\alpha = (v' - v'')\alpha = 0,$$

which implies that $u' - u'' \in \ker \alpha^2 = \ker \alpha$. Then $v' = u'\alpha = u''\alpha = v''$. Thus β is well-defined. Next, we show that β is a linear function. Let $u, v \in V\alpha^2$ and $a, b \in \mathbb{F}$. Then $u = u'\alpha$ and $v = v'\alpha$ for some $u', v' \in V\alpha$. We have $au + bv = (au' + bv')\alpha$, which implies that $(au + bv)\beta = au' + bv' = au\beta + bv\beta$. Therefore, β is linear. To show that $\ker \beta = \{0\}$, let $v \in \ker \beta$. Then $v = v'\alpha$ for some $v' \in V\alpha$. We have $v' = v\beta = 0$, which implies that $v = 0$. Thus $\ker \beta = \{0\}$. Finally, we show that $W\beta \subseteq W$. Let $w \in W \cap V\alpha^2$. Then $w = w'\alpha$ for some $w' \in V\alpha$. If $w' \in V\alpha \setminus W$, then $w = w'\alpha \in (V\alpha \setminus W)\alpha \subseteq V \setminus W$, which is a contradiction. Thus $w' \in W$, which implies that $w\beta = w' \in W$ and so $W\beta \subseteq W$. Therefore, $\beta \in P_{\mathcal{I}(W)}(V)$.

It is straightforward to verify that $\alpha = \alpha^2\beta$. Thus α is right regular. $\qquad \square$

The proof of the following theorem is similar to that of Theorem 4.5, so we omit it.

Theorem 4.7. $P_{\mathcal{I}(W)}(V)$ *is right regular if and only if* $\dim V \leq 1$.

By combining Theorem 4.5 and Theorem 4.7, we obtain the following result.

Theorem 4.8. $P_{\mathcal{I}(W)}(V)$ *is completely regular if and only if* $\dim V \leq 1$.

5. Ideals

Let I be a nonempty subset of a semigroup S. Recall that I is called an *ideal* of S if $SI \subseteq I$ and $IS \subseteq I$. In particular, the entire semigroup S and, if S has a zero element 0, the set $\{0\}$ are both examples of ideals. An ideal I is said to be *proper* if $\{0\} \subsetneq I \subsetneq S$. For any $a \in S$, the smallest ideal of S containing a is the set S^1aS^1, called the *principal ideal* generated by a. An ideal I is *minimal* if

there does not exist an ideal J with $J \subsetneq I$. Conversely, I is *maximal* if there is no proper ideal J with $I \subsetneq J$.

Ideals of $L(V)$ are described in [3, Exercise 2.2.6], while [8] showed that proper ideals of $P(V)$ are precisely

$$P_r = \{\alpha \in P(V) : \operatorname{rank} \alpha < r\},$$

where $2 \le r \le \dim V$. For $L_{GL(U)}(V)$, [12] provided the following characterization.

Theorem 5.1 ([12, Theorem 3.6]). *The proper ideals of $L_{GL(U)}(V)$ are precisely the sets*

$$Q(k) = \{\alpha \in L_{GL(U)}(V) : \dim(V\alpha/U) < k\},$$

where $1 \le k \le \dim(V/U)$.

In this section, we present a detailed description of all the ideals of $P_{I(W)}(V)$ and discuss their properties. Additionally, we examine the principal, minimal and maximal ideals associated with these semigroups.

Let p be an arbitrary cardinal number and define

$$p' = \min\{q : q > p\}.$$

This value p' always exists because cardinal numbers are well-ordered; specifically, if p is finite, then $p' = p + 1$, which is simply the next cardinal after p.

To characterize the ideals of $P_{I(W)}(V)$, set

$$\dim V = a, \quad \dim W = b, \quad \text{and} \quad \dim V/W = c.$$

For any cardinals r, s, t such that $1 \le r \le a'$, $1 \le s \le b'$ and $1 \le t \le c'$, define

$$Q(r, s, t) = \{\alpha \in P_{I(W)}(V) : \dim(V\alpha) < r, \ \dim(W\alpha) < s, \ \dim(V\alpha/(V\alpha \cap W)) < t\}.$$

When $r = a'$, $s = b'$ and $t = c'$, this gives $Q(r, s, t) = Q(a', b', c') = P_{I(W)}(V)$. Observe that if $r \le u$, $s \le v$ and $t \le w$, then $Q(r, s, t) \subseteq Q(u, v, w)$.

Theorem 5.2. *The set $Q(r, s, t)$ is an ideal of $P_{I(W)}(V)$ for any cardinals r, s, t such that $1 \le r \le a'$, $1 \le s \le b'$ and $1 \le t \le c'$.*

Proof. Let $\alpha \in Q(r, s, t)$ and $\beta \in P_{I(W)}(V)$. Then $\dim(V\alpha) < r$, $\dim(W\alpha) < s$ and $\dim(V\alpha/(V\alpha \cap W)) < t$. We obtain $\dim(V\alpha\beta) \le \dim(V\alpha) < r$ and $\dim(W\alpha\beta) \le \dim(W\alpha) < s$. By Lemma 3.5, we have $\dim(V\alpha\beta/(V\alpha\beta \cap W)) \le \dim(V\alpha/(V\alpha \cap W)) < t$. Therefore, $\alpha\beta \in Q(r, s, t)$. Similarly, we can show that $\beta\alpha \in Q(r, s, t)$. Thus $Q(r, s, t)$ is an ideal of $P_{I(W)}(V)$. $\square$

Nevertheless, the following example illustrates that there exists an ideal in $P_{I(W)}(V)$ which does not have the form $Q(r, s, t)$ for any choice of r, s, t.

Example 5.3. *Let $V = \langle v_1, v_2, w_1, w_2 \rangle$ and $W = \langle w_1, w_2 \rangle$. One can easily verify that $Q = Q(3, 3, 1) \cup Q(4, 2, 3)$ is an ideal of $P_{I(W)}(V)$. We will show that Q does not have the form $Q(r, s, t)$ where $1 \le r \le 5$, $1 \le s \le 3$ and $1 \le t \le 3$. Suppose to the contrary that $Q = Q(r, s, t)$. Define $\alpha, \beta \in P_{I(W)}(V)$ by*

$$\alpha = \begin{pmatrix} w_1 & w_2 \\ w_1 & w_2 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} v_1 & v_2 & w_1 \\ v_1 & v_2 & w_1 \end{pmatrix}.$$

Obviously, $\alpha \in Q(3,3,1) \subseteq Q = Q(r,s,t)$ and $\beta \in Q(4,2,3) \subseteq Q = Q(r,s,t)$. If $r < 4$ or $t < 3$, then $\beta \notin Q(r,s,t)$. If $s < 3$, then $\alpha \notin Q(r,s,t)$. Thus we must have $r \geq 4$, $s \geq 3$ and $t \geq 3$. We can see that $\gamma \in P_{\mathcal{I}(W)}(V)$ defined by

$$\gamma = \begin{pmatrix} v_1 & w_1 & w_2 \\ v_1 & w_1 & w_2 \end{pmatrix} \in Q(4,3,3) \subseteq Q(r,s,t)$$

but $\gamma \notin Q$. This is a contradiction.

For each nonempty subset Z of $P_{\mathcal{I}(W)}(V)$, we define the set

$$Q(Z) = \{\alpha \in P_{\mathcal{I}(W)}(V) : \dim(V\alpha) \leq \dim(V\beta), \dim(W\alpha) \leq \dim(W\beta)$$
$$\text{and } \dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\beta/(V\beta \cap W)) \text{ for some } \beta \in Z\}.$$

It is straightforward to verify that $Z \subseteq Q(Z)$ and $Q(Z_1) \subseteq Q(Z_2)$ for any two nonempty subsets $Z_1 \subseteq Z_2$ of $P_{\mathcal{I}(W)}(V)$.

Theorem 5.4. *The ideals of $P_{\mathcal{I}(W)}(V)$ are precisely the sets of the form $Q(Z)$ for some nonempty subset Z of $P_{\mathcal{I}(W)}(V)$.*

Proof. First, we assert that $Q(Z)$ is an ideal of $P_{\mathcal{I}(W)}(V)$ for any nonempty subset Z of $P_{\mathcal{I}(W)}(V)$. Let $\alpha \in Q(Z)$ and $\beta \in P_{\mathcal{I}(W)}(V)$. Then there exists $\gamma \in Z$ such that $\dim(V\alpha) \leq \dim(V\gamma)$, $\dim(W\alpha) \leq \dim(W\gamma)$ and $\dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\gamma/(V\gamma \cap W))$. We obtain $\dim(V\alpha\beta) \leq \dim(V\alpha) \leq \dim(V\gamma)$, $\dim(W\alpha\beta) \leq \dim(W\alpha) \leq \dim(W\gamma)$ and $\dim(V\alpha\beta/(V\alpha\beta \cap W)) \leq \dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\gamma/(V\gamma \cap W))$. Thus $\alpha\beta \in Q(Z)$. Similarly, we can show that $\beta\alpha \in Q(Z)$.

Conversely, let I be an ideal of $P_{\mathcal{I}(W)}(V)$. We will show that $I = Q(I)$. As we mentioned earlier, $I \subseteq Q(I)$. Now, let $\alpha \in Q(I)$. Then there exists $\beta \in I$ such that $\dim(V\alpha) \leq \dim(V\beta)$, $\dim(W\alpha) \leq \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\beta/(V\beta \cap W))$. By Theorem 3.6, there exist $\lambda, \mu \in P_{\mathcal{I}(W)}(V)$ such that $\alpha = \lambda\beta\mu$. Since I is an ideal, we have $\alpha = \lambda\beta\mu \in I$. Therefore, $Q(I) \subseteq I$ which implies that $I = Q(I)$. □

Let A be any nonempty subset of a semigroup S. The intersection of all ideals of S that contain A (including S itself) yields an ideal of S that contains A and is contained in every ideal that includes A. This is known as the *ideal of S generated by A*. It is well known that this ideal is exactly $S^1 A S^1$. By using this result together with Theorem 3.6, the following corollary follows immediately.

Corollary 5.5. *The set $Q(Z)$ is the ideal of $P_{\mathcal{I}(W)}(V)$ generated by Z for any nonempty subset Z of $P_{\mathcal{I}(W)}(V)$. In particular, the principal ideals of $P_{\mathcal{I}(W)}(V)$ are precisely the sets of the form $Q(Z)$ for some singleton subset Z of $P_{\mathcal{I}(W)}(V)$.*

We end this section with a discussion of the minimal and maximal ideals of $P_{\mathcal{I}(W)}(V)$.

Theorem 5.6. *Let W be a nonzero proper subspace of V. Then the set $J(0,0,0)$ is the unique minimal ideal of $P_{\mathcal{I}(W)}(V)$.*

Proof. It is straightforward to verify that $J(0,0,0) = Q\left(\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}\right)$ is an ideal of $P_{\mathcal{I}(W)}(V)$.

To show that $J(0,0,0)$ is minimal, let I be an ideal of $P_{\mathcal{I}(W)}(V)$ such that $I \subseteq J(0,0,0)$. Let $\alpha \in J(0,0,0)$. Since I is an ideal, I is nonempty, which implies that there exists $\beta \in I \subseteq J(0,0,0)$. Then $\alpha \mathcal{J} \beta$. By Theorem 3.6, there exist $\lambda, \mu \in P_{\mathcal{I}(W)}(V)$ such that $\alpha = \lambda\beta\mu$. Since $\beta \in I$ and I is an ideal, we have $\alpha \in I$. Thus $J(0,0,0) \subseteq I$, which implies that $I = J(0,0,0)$.

Finally, we show that $J(0, 0, 0)$ is the unique minimal ideal of $P_{\mathcal{I}(W)}(V)$. Let I be a minimal ideal of $P_{\mathcal{I}(W)}(V)$. Let $\alpha \in J(0, 0, 0)$. Then $\dim(V\alpha) = 0$, $\dim(W\alpha) = 0$ and $\dim(V\alpha/(V\alpha \cap W)) = 0$. Since I is an ideal, I is nonempty. There is $\beta \in I$. Hence $\dim(V\alpha) = 0 \leq \dim(V\beta)$, $\dim(W\alpha) = 0 \leq \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) = 0 \leq \dim(V\beta/(V\beta \cap W))$. By Theorem 3.6, we have $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in P_{\mathcal{I}(W)}(V)$. Since I is an ideal, we obtain $\alpha \in I$. Thus $J(0, 0, 0) \subseteq I$, which implies by minimality of I that $I = J(0, 0, 0)$. Therefore, $J(0, 0, 0)$ is the unique minimal ideal of $P_{\mathcal{I}(W)}(V)$. $\qquad\square$

Observe that $J(0, 0, 0)$ consists exactly of those elements α in $P(V)$ for which $\operatorname{dom}\alpha \cap W = \{0\}$ and every vector in the domain of α is mapped to the zero vector.

Theorem 5.7. $P_{\mathcal{I}(W)}(V) \setminus J(a, b, c)$ *is the unique maximal ideal of* $P_{\mathcal{I}(W)}(V)$.

Proof. It is routine to verify that $P_{\mathcal{I}(W)}(V) \setminus J(a, b, c) = Q(P_{\mathcal{I}(W)}(V) \setminus J(a, b, c))$ is an ideal of $P_{\mathcal{I}(W)}(V)$. To prove maximality, let I be a proper ideal of $P_{\mathcal{I}(W)}(V)$ such that $P_{\mathcal{I}(W)}(V) \setminus J(a, b, c) \subseteq I$. Suppose to the contrary that $I \neq P_{\mathcal{I}(W)}(V) \setminus J(a, b, c)$. Then there exists $\alpha \in I$ such that $\alpha \notin P_{\mathcal{I}(W)}(V) \setminus J(a, b, c)$. We have $\alpha \in J(a, b, c)$. Hence $\dim(V\alpha) = \dim V$, $\dim(W\alpha) = \dim W$ and $\dim(V\alpha/(V\alpha \cap W)) = \dim(V/W)$. Let $\beta \in P_{\mathcal{I}(W)}(V)$. Then $\dim(V\beta) \leq \dim(V\alpha)$, $\dim(W\beta) \leq \dim(W\alpha)$ and $\dim(V\beta/(V\beta \cap W)) \leq \dim(V\alpha/(V\alpha \cap W))$. By Theorem 3.6, there exist $\lambda, \mu \in P_{\mathcal{I}(W)}(V)$ such that $\beta = \lambda\alpha\mu$. Since I is an ideal, we have $\beta \in I$. Thus $I = P_{\mathcal{I}(W)}(V)$, which contradicts the assumption that I is proper.

Finally, we show that $P_{\mathcal{I}(W)}(V) \setminus J(a, b, c)$ is the unique maximal ideal of $P_{\mathcal{I}(W)}(V)$. Let I be a maximal ideal of $P_{\mathcal{I}(W)}(V)$. Note that I does not contain the identity map id_V otherwise $I = P_{\mathcal{I}(W)}(V)$. We obtain $I \cup (P_{\mathcal{I}(W)}(V) \setminus J(a, b, c))$ is an ideal of $P_{\mathcal{I}(W)}(V)$ such that $\mathrm{id}_V \notin I \cup (P_{\mathcal{I}(W)}(V) \setminus J(a, b, c))$. Thus $I \cup (P_{\mathcal{I}(W)}(V) \setminus J(a, b, c))$ is a proper ideal of $P_{\mathcal{I}(W)}(V)$. Since $I \subseteq I \cup (P_{\mathcal{I}(W)}(V) \setminus J(a, b, c))$ and I is maximal, we must have $I = I \cup (P_{\mathcal{I}(W)}(V) \setminus J(a, b, c))$. In a similar manner, we can prove that $P_{\mathcal{I}(W)}(V) \setminus J(a, b, c) = I \cup (P_{\mathcal{I}(W)}(V) \setminus J(a, b, c))$. Hence $I = P_{\mathcal{I}(W)}(V) \setminus J(a, b, c)$. Therefore, $P_{\mathcal{I}(W)}(V) \setminus J(a, b, c)$ is the unique maximal ideal of $P_{\mathcal{I}(W)}(V)$. $\qquad\square$

6. Conclusions

In this paper, we have conducted a comprehensive study of the semigroup $P_{\mathcal{I}(W)}(V)$ of partial linear transformations on V whose restrictions to W belong to the injective partial linear transformation semigroup $\mathcal{I}(W)$, establishing complete characterizations of Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ in terms of dimensional and structural properties of the associated vector spaces. Our analysis revealed that an element $\alpha \in P_{\mathcal{I}(W)}(V)$ is regular if and only if $V\alpha \cap W = W\alpha$, that $P_{\mathcal{I}(W)}(V)$ is regular as a semigroup precisely when $V = W$ or $W = \{0\}$ and that $P_{\mathcal{I}(W)}(V)$ forms an inverse semigroup if and only if $V = W$, while both left and right regularity occur if and only if $\dim V \leq 1$. We completely described the ideal structure of $P_{\mathcal{I}(W)}(V)$, showing that all ideals have the form $Q(Z)$ for some nonempty subset Z of $P_{\mathcal{I}(W)}(V)$ and identified the unique minimal ideal as $J(0, 0, 0)$ when W is a nonzero proper subspace of V and the unique maximal ideal as $P_{\mathcal{I}(W)}(V) \setminus J(a, b, c)$ where $J(a, b, c)$ denotes the $\mathcal{J}$-class of the identity transformation. These results contribute significantly to transformation semigroup theory, providing a well-understood example that demonstrates the interplay between vector space structure and algebraic semigroup properties and establishing a foundation for future investigations into the natural partial order on $P_{\mathcal{I}(W)}(V)$, including characterizations of when elements are comparable, identification of maximal and minimal elements and analysis of compatibility relations within the ordered structure.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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