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Research article

Convergence, stability, and error analysis of the method of lines for solving loaded parabolic equations

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Abstract: This paper studies the method of lines for solving loaded parabolic equations and provides a theoretical analysis of its convergence, stability, and error estimates. The spatial discretization reduces the original equation to a system of loaded ordinary differential equations solved by the Dzhumabaev parameterization method. Sufficient conditions for the existence and uniqueness of the solution are established, and it is proved that the method achieves second-order accuracy in space and stable convergence. A numerical example confirms the efficiency and reliability of the proposed approach.

Keywords: loaded parabolic equations; nonlocal problem; method of lines; system of loaded ordinary differential equations; Dzhumabaev's parameterization method; convergence analysis
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1. Introduction and problem statement

Loaded parabolic equations arise in various biological and ecological processes, including population dynamics in ecology, mathematical biology, reaction-diffusion systems, epidemiology (such as the spread of infectious diseases), and pharmacokinetics [1–3]. These equations offer a more realistic and flexible framework for modeling systems with memory, feedback mechanisms, or spatial averaging.

Over the past decades, mathematical models based on loaded parabolic equations have been successfully applied to a variety of life science problems. These include modeling oxygen transport in tissues [1], predicting tumor growth patterns [4], simulating disease outbreaks [2], analyzing the propagation of neural activity [5], and investigating the mechanisms of wound healing [6]. In all these applications, loaded parabolic equations effectively capture the essential dynamics of complex biological systems, which highlights their broad relevance in life sciences.

In parallel with applied research, the theoretical study of loaded parabolic equations has also attracted considerable attention. Numerous classes of such problems have been extensively studied in works [7–9]. The solution of loaded parabolic equations and boundary value problems for loaded pseudoparabolic equations were investigated in [10, 11], while broader theoretical issues related to fractional, impulsive, and semilinear parabolic equations were addressed in [12–14]. These results have established the analytical foundations necessary for developing accurate and stable numerical methods.

To obtain numerical solutions, various well-established approaches have been proposed, including finite difference schemes, finite element methods, spectral methods, and the method of lines. Each method has its own advantages and range of applicability depending on the problem structure and boundary conditions. Among these, the method of lines has proven particularly effective for parabolic-type problems [15–17], as it allows the transformation of a partial differential equation into a system of ordinary differential equations that can be treated with well-developed ODE techniques [18–20].

Motivated by these considerations, this paper focuses on the analysis of the following nonlocal problem for a loaded parabolic equation:

$$\frac{\partial u}{\partial t} = a(t,x)\frac{\partial^2 u}{\partial x^2} + b(t,x)u(t,x) + \sum_{j=0}^{m} k_j(t,x)u(t_j,x) + f(t,x), \quad (t,x) \in \Omega = (0,T)\times(0,\omega), \tag{1.1}$$

$$B(x)u(0,x) + C(x)\frac{\partial u(0,x)}{\partial x} + D(x)u(T,x) + E(x)\frac{\partial u(T,x)}{\partial x} = \varphi(x), \quad x \in [0,\omega], \tag{1.2}$$

$$u(t,0) = \psi_0(t), \quad u(t,\omega) = \psi_1(t), \quad t \in [0,T]. \tag{1.3}$$

Let $a(t,x)$, $b(t,x)$, $k_j(t,x)$ $(j = 0,\dots,m)$, and $f(t,x)$ be functions that are continuous in t and satisfy Holder continuity in x; $T > 0$ and $\omega > 0$; $t_j \in [0,T]$, $j = \overline{0,m}$ are loading points. The functions $B(x)$, $C(x)$, $D(x)$, and $E(x)$ are continuous on $[0,\omega]$. It is assumed that the functions $\varphi(x)$, $\psi_0(t)$, and $\psi_1(t)$ are fully smooth and fulfill the consistency conditions.

Let $u(t,x)$ be a function that provides a solution to the problem defined by Eqs. (1.1)–(1.3). This function remains continuous throughout the closed domain $\Omega_c = [0,T]\times[0,\omega]$. Moreover, $u(t,x)$ possesses continuous partial derivatives of the first order with respect to t and second order with respect to x. It satisfies the corresponding loaded parabolic Eq. (1.1) and adheres to the conditions specified in (1.2) and (1.3).

The method of lines transforms loaded parabolic equations into systems of loaded ordinary differential equations by discretizing the spatial variables. To solve the resulting system, we utilize the parameterization method introduced by D. Dzhumabaev [21, 22], which has demonstrated effectiveness in addressing boundary value problems associated with such systems. Additionally, many researchers have investigated a wide range of problems involving loaded ordinary differential equations, contributing valuable theoretical and numerical insights [23–25].

Building on these developments, several studies have explored different formulations and extensions of the method of lines for loaded and nonlocal parabolic equations.

In particular, study [26] examines a nonlocal boundary value problem for a loaded linear parabolic equation under dynamic boundary conditions. The authors provide results concerning existence and uniqueness in Sobolev spaces under certain assumptions, establish a maximum principle, and apply the method of lines to obtain numerical approximations, including convergence analysis.

A complementary approach is presented in [27], where a numerical solution to a nonlocal problem for a loaded parabolic equation is obtained via the method of lines. The authors develop explicit computational formulas and propose an implementable algorithm for practical calculations.

In [28], a two-point boundary value problem for a loaded parabolic equation is reduced, via spatial discretization, to a boundary value problem for a system of loaded ODEs. The resulting system is then solved using the Dzhumabaev parameterization method. In that work, an auxiliary theorem is proved to establish the theoretical foundation of the approach, and solvability conditions for the discretized problem are rigorously derived.

Further, [29] provides solvability conditions and a priori estimates for the initial-boundary value problems involving loaded parabolic equations. A numerical experiment is provided to illustrate that the discrepancy between numerical and exact results remains within an error of 10^{-7}, showcasing the high accuracy of the method.

The novelty of the present study lies in the integration of the method of lines with the Dzhumabaev parameterization method to address nonlocal boundary conditions in loaded parabolic equations. This combination enables a rigorous analysis of convergence, stability, and error estimates while maintaining computational simplicity and second-order spatial accuracy. These advantages distinguish the proposed approach from classical methods.

The present paper further develops this line of research and extends the analysis to a broader class of nonlocal problems. The structure of the paper is as follows. Section 2 presents the application of the method of lines to the nonlocal problem for loaded parabolic equations (1.1)–(1.3). In Section 3, the Dzhumabaev parameterization method is applied to solve the two-point boundary value problem for a loaded ordinary differential equation, and a theorem on the existence and uniqueness of its solution is presented. Section 4 is devoted to the proof of convergence of the solution of the discretized problem to the solution of the original nonlocal parabolic problem. In Section 5, the stable convergence of the solution of the discretized problem to the solution of problems (1.1)–(1.3) is established. Finally, Section 6 includes a numerical example that demonstrates the effectiveness and accuracy of the proposed method.

2. Application of the method of lines

We apply the method of lines to problems (1.1)–(1.3) by discretizing it with respect to the spatial variable x, where $x_i = i\tau$, $i = \overline{0, N}$, and $N\tau = \omega$. To facilitate the formulation of the discretized problem, we define the following notations: $u_i(t) = u(t, i\tau)$, $a_i(t) = a(t, i\tau)$, $b_i(t) = b(t, i\tau)$, $k_i^j(t) = k_j(t, i\tau)$ $(j = 0, \dots, m)$, $f_i(t) = f(t, i\tau)$, $B_i = B(i\tau)$, $C_i = C(i\tau)$, $D_i = D(i\tau)$, $E_i = E(i\tau)$, $\varphi_i = \varphi(i\tau)$, $i = 0, \dots, N$. The reformulated form of problem (1.1)-(1.3) in terms of the introduced notations is given as follows:

$$\frac{du_i}{dt} = a_i(t)\frac{u_{i+1} - 2u_i + u_{i-1}}{\tau^2} + b_i(t)u_i + \sum_{j=0}^{m} k_i^j(t)u_i(t_j) + f_i(t), \quad i = 1, \dots, N-1, \tag{2.1}$$

$$B_i u_i(0) + C_i \frac{u_{i+1}(0) - u_{i-1}(0)}{2\tau} + D_i u_i(T) + E_i \frac{u_{i+1}(T) - u_{i-1}(T)}{2\tau} = \varphi_i, \tag{2.2}$$

$$u_0(t) = \psi_0(t), \quad u_N(t) = \psi_1(t), \quad t \in [0, T]. \tag{2.3}$$

A solution to the discretized problems (2.1)–(2.3) is the $\{u_1(t), u_2(t), \dots, u_{N-1}(t)\}$ system, where $u_i(t)$ represents an approximation of the solution $u(t, x)$ at the spatial grid points x_i. This system satisfies the discretized Eq. (2.1) and the conditions (2.2) and (2.3).

This article aims to establish the interrelation between the nonlocal problem for the loaded parabolic equation and its corresponding discretized problem. To achieve this goal, we analyze the convergence, stability, and error estimates of the method of lines when applied to such problems. The analysis builds upon and extends the results presented in our previous work [28], where the Dzhumabaev parameterization method was employed for solving the resulting system of loaded ODEs.

Through this investigation, we aim to provide both a theoretical foundation and practical validation for the proposed scheme, thereby reinforcing its applicability to various types of nonlocal parabolic problems arising in life sciences and engineering.

The discretized problems (2.1)–(2.3) can be written in matrix-vector notation as

$$\frac{dU}{dt} = A(t)U(t) + \sum_{j=0}^{m} M_j(t)U(t_j) + F(t), \quad U \in R^{N-1}, \tag{2.4}$$

$$\widetilde{B}U(0) + \widetilde{C}U(T) = \Phi, \quad \Phi \in R^{N-1}, \quad t \in [0, T]. \tag{2.5}$$

In this formulation, $A(t)$ and $M_j(t)$ for $j = 0, \dots, m$, are $(N-1)\times(N-1)$ matrices, while $F(t)$ is a vector function of size $(N-1)$, each defined and continuous on the interval $[0, T]$. The constant matrices $\widetilde{B}$ and $\widetilde{C}$ also belongs to $\mathbb{R}^{(N-1)\times(N-1)}$.

$$A(t) = \begin{pmatrix} \frac{-2a_1(t)}{\tau^2} + b_1(t) & \frac{a_1(t)}{\tau^2} & \dots & 0 & 0 \\ \frac{a_2(t)}{\tau^2} & \frac{-2a_2(t)}{\tau^2} + b_2(t) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{-2a_{N-2}(t)}{\tau^2} + b_{N-2}(t) & \frac{a_{N-2}(t)}{\tau^2} \\ 0 & 0 & \dots & \frac{a_{N-1}(t)}{\tau^2} & \frac{-2a_{N-1}(t)}{\tau^2} + b_{N-1}(t) \end{pmatrix},$$

$$F(t) = \begin{pmatrix} \frac{a_1(t)\psi_0(t)}{\tau^2} + f_1(t) \\ f_2(t) \\ \vdots \\ f_{N-2}(t) \\ \frac{a_{N-1}(t)\psi_1(t)}{\tau^2} + f_{N-1}(t) \end{pmatrix}, \quad M_j(t) = \begin{pmatrix} k_1^j(t) & 0 & \dots & 0 & 0 \\ 0 & k_2^j(t) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & k_{N-2}^j(t) & 0 \\ 0 & 0 & \dots & 0 & k_{N-1}^j(t) \end{pmatrix}, \quad j = 0, \dots, m,$$

$$\widetilde{B}=\begin{pmatrix} B_1 & \frac{C_1}{2\tau} & \dots & 0 & 0 \\ -\frac{C_2(t)}{2\tau} & B_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_{N-2} & -\frac{C_{N-2}(t)}{2\tau} \\ 0 & 0 & \dots & -\frac{C_{N-1}(t)}{2\tau} & B_{N-1} \end{pmatrix}, \quad \Phi=\begin{pmatrix} \varphi_1+\frac{C_1\psi_0(0)+E_1\psi_0(T)}{2\tau} \\ \varphi_2 \\ \vdots \\ \varphi_{N-2} \\ \varphi_{N-1}-\frac{C_{N-1}\psi_1(0)+E_{N-1}\psi_1(T)}{2\tau} \end{pmatrix},$$

$$\widetilde{C}=\begin{pmatrix} D_1 & \frac{E_1}{2\tau} & \dots & 0 & 0 \\ -\frac{E_2(t)}{2\tau} & D_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & D_{N-2} & -\frac{E_{N-2}(t)}{2\tau} \\ 0 & 0 & \dots & -\frac{E_{N-1}(t)}{2\tau} & D_{N-1} \end{pmatrix}, \quad U(t)=\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N-2}(t) \\ u_{N-1}(t) \end{pmatrix}.$$

The solution $U(t)$ to the problem defined by Eq. (2.4), (2.5) is assumed to be continuously differentiable on the interval $[0,T]$, satisfies the system of loaded differential equations (2.4), and fulfills the condition (2.5).

We introduce the space $C([0,T],\mathbb{R}^{N-1})$, consisting of continuous vector-valued functions $U:[0,T]\to R^{N-1}$, equipped with the norm $\|U\|_1=\max\limits_{t\in[0,T]}\|U(t)\|$.

Additionally, we define the space $C([0,T],t_r,\mathbb{R}^{(N-1)(m+1)})$ for system functions of the form $U[t]=(U_1(t),U_2(t),\dots,U_{m+1}(t))$ where each component function U_r: $[t_{r-1},t_r)\to\mathbb{R}^{N-1}$ is continuous on its corresponding subinterval and satisfies the condition $\lim\limits_{t\to t_r-0}U_r(t)$ for all $r=1,\dots,m+1$. The associated norm in this space is given by $\|U[\cdot]\|_2=\max\limits_{r=\overline{1,m+1}}\sup\limits_{t\in[t_{r-1},t_r)}\|U_r(t)\|$.

3. The Dzhumabaev parameterization method and conditions for unique solvability of problems (2.4) and (2.5).

To solve the problem defined by Eqs. (2.4) and (2.5), we apply the Dzhumabaev parameterization method. The interval $[0,T]$ is partitioned into $m+1$ subintervals using a set of loading points as follows:

$$[0,T)=\bigcup_{r=1}^{m+1}[t_{r-1},t_r),\quad \text{where}\quad 0=t_0<t_1<t_2<\dots<t_{m+1}=T.$$

For each subinterval $t\in[t_{r-1},t_r)$, with $r=1,\dots,m+1$, we define the function $U_r(t)=U(t)$, which denotes the restriction of $U(t)$ to the r-th subinterval.

Next, we introduce the parameters $\lambda_r=U_r(t_{r-1})$, for $r=1,\dots,m+1$, and set $\lambda_{m+2}=\lim\limits_{t\to T-0}U_{m+1}(t)$. By expressing $U_r(t)$ as $U_r(t)=\widetilde{U}_r(t)+\lambda_r$, in each interval $[t_{r-1},t_r)$, we reformulate the problem in terms of the functions $\widetilde{U}_r(t)$ and the parameters λ_r

$$\frac{d\widetilde{U}_r}{dt}=A(t)(\widetilde{U}_r+\lambda_r)+\sum_{j=0}^{m}M_j(t)\lambda_{j+1}+F(t),\quad t\in[t_{r-1},t_r), \tag{3.1}$$

$$\widetilde{U}(t_{r-1})=0,\qquad r=1,\dots,m+1, \tag{3.2}$$

$$\widetilde{B}\lambda_1+\widetilde{C}\lambda_{m+2}=\Phi, \tag{3.3}$$

$$\lambda_s + \lim_{t\to t_s-0}\widetilde{U}_s(t) = \lambda_{s+1}, \quad s = 1,\dots,m+1. \tag{3.4}$$

A pair $(\lambda^*, \widetilde{U}^*[t])$, where the functions $\widetilde{U}_r^*(t)$ are continuously differentiable on each subinterval $[t_{r-1}, t_r)$ for $r = 1,\dots,m+1$, and satisfy the system (3.1) along with the conditions (3.2)–(3.4) at $\lambda_r = \lambda_r^*$, represents a solution to the problems (3.1)–(3.4).

Hence, in case the function $U^*(t)$ is the solution to problems (2.4) and (2.5), then the pair $(\lambda^*, \widetilde{U}^*[t])$, where $\lambda^* = (U^*(t_0), U^*(t_1),\dots,U^*(t_{m+1}))$, $\widetilde{U}^*[t] = (U^*(t)-U^*(t_0), U^*(t)-U^*(t_1),\dots,U^*(t)-U^*(t_m))$ will be the solution to the problems (3.1)–(3.4). Reversely, in case $(\lambda^{**}, \widetilde{U}^{**}[t])$, where $\lambda^{**} = (\lambda_1^{**},\dots,\lambda_{m+2}^{**})$, $\widetilde{U}^{**}[t] = (\widetilde{U}_1^{**}(t),\dots,\widetilde{U}_{m+1}^{**}(t))'$ is a solution to the problems (3.1)–(3.4), then the function $U_r^{**}(t) = \widetilde{U}_r^{**}(t)+\lambda_r^{**}$, $t\in[t_{r-1},t_r)$ $(r = 1,\dots,m+1)$, $U_{m+1}^{**}(T) = \lambda_{m+2}^{**}$, is the solution to the problems (2.4), (2.5).

The use of the initial conditions $\widetilde{U}_r(t_{r-1}) = 0$, for $r = 1,\dots,m+1$, under the assumption $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_{m+2})$, leads to the following system of Volterra integral equations of the second kind:

$$\widetilde{U}_r(t) = \int_{t_{r-1}}^t A(\xi)(\widetilde{U}_r(\xi)+\lambda_r)d\xi + \int_{t_{r-1}}^t \sum_{j=0}^m M_j(\xi)d\xi\lambda_{j+1} + \int_{t_{r-1}}^t F(\xi)d\xi, \quad r = 1,\dots,m+1. \tag{3.5}$$

By iteratively substituting $\widetilde{U}_r(\xi)$, into the right-hand side of (3.5), we obtain a representation of the solution $\widetilde{U}_r(t)$, after ν steps of iteration $(\nu = 1,2,\dots)$, in the form:

$$\widetilde{U}_r(t) = D_{\nu,r}(t)\lambda_r + \sum_{j=0}^m H_{\nu,r}(t,M_j)\lambda_{j+1} + G_{\nu,r}(t,\widetilde{U}_r) + \widetilde{F}_{\nu,r}(t), \quad r = 1,\dots,m+1, \tag{3.6}$$

where

$$D_{\nu,r}(t) = \int_{t_{r-1}}^t A(\xi_1)d\xi_1 + \int_{t_{r-1}}^t A(\xi_1)\int_{t_{r-1}}^{\xi_1} A(\xi_2)d\xi_2 d\xi_1 + ... + \int_{t_{r-1}}^t A(\xi_1)\dots \int_{t_{r-1}}^{\xi_{\nu-2}} A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}} A(\xi_\nu)d\xi_\nu\dots d\xi_1,$$

$$H_{\nu,r}(t,M_j) = \int_{t_{r-1}}^t M_j(\xi_1)d\xi_1 + \int_{t_{r-1}}^t A(\xi_1)\int_{t_{r-1}}^{\xi_1} M_j(\xi_2)d\xi_2 d\xi_1 + ... + \int_{t_{r-1}}^t A(\xi_1)\dots \int_{t_{r-1}}^{\xi_{\nu-2}} A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}} M_j(\xi_\nu)d\xi_\nu\dots d\xi_1, \quad j = 0,\dots,m,$$

$$G_{\nu,r}(t,\widetilde{U}_r) = \int_{t_{r-1}}^t A(\xi_1)\dots\int_{t_{r-1}}^{\xi_{\nu-2}} A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}} A(\xi_\nu)\widetilde{U}_r(\xi_\nu)d\xi_\nu\dots d\xi_1,$$

$$\widetilde{F}_{\nu,r}(t) = \int_{t_{r-1}}^t F(\xi_1)d\xi_1 + \int_{t_{r-1}}^t A(\xi_1)\int_{t_{r-1}}^{\xi_1} F(\xi_2)d\xi_2 d\xi_1 + ... + \int_{t_{r-1}}^t A(\xi_1)\dots \int_{t_{r-1}}^{\xi_{\nu-2}} A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}} F(\xi_\nu)d\xi_\nu\dots d\xi_1.$$

From (3.6), we determine the left-hand side limits:

$$\lim_{t\to t_r-0} \widetilde{U}_r(t) = D_{\nu,r}(t_r)\lambda_r + \sum_{j=0}^{m} H_{\nu,r}(t_r, M_j)\lambda_{j+1} + G_{\nu,r}(t_r, \widetilde{U}_r) + \widetilde{F}_{\nu,r}(t_r), \quad r = 1,\ldots,m+1. \tag{3.7}$$

Substituting the right-hand side of Eq. (3.7) into conditions (3.3) and (3.4), and multiplying Eq. (3.3) by $l = \max_s(t_s - t_{s-1}), \quad s = 1,\ldots,m+1$, we derive the following set of expressions:

$$\widetilde{B}\lambda_1 \cdot l + \widetilde{C}\lambda_{m+2} \cdot l = \Phi \cdot l, \tag{3.8}$$

$$[I + D_{\nu,s}(t_s)]\lambda_s + \sum_{j=0}^{m} H_{\nu,s}(t_s, M_j)\lambda_{j+1} - \lambda_{s+1} = -G_{\nu,s}(t_s, \widetilde{U}_s) - \widetilde{F}_{\nu,s}(t_s), \quad s = 1,\ldots,m+1. \tag{3.9}$$

Here, I denotes the identity matrix of size $(N-1)\times(N-1)$. Let $Q_\nu(l)$ represent the matrix that collects the left-hand sides of Eqs. (3.8) and (3.9). Then the system can be compactly written as

$$Q_\nu(l)\lambda = -\widetilde{F}_\nu(l) - G_\nu(\widetilde{U}, l), \tag{3.10}$$

where

$$\widetilde{F}_\nu(l) = (-\Phi l, \widetilde{F}_{\nu,1}(t_1),\ldots,\widetilde{F}_{\nu,m+1}(T)), \quad G_\nu(\widetilde{U}, l) = (0, G_{\nu,1}(\widetilde{U}_1, t_1),\ldots,G_{\nu,m+1}(\widetilde{U}_{m+1}, T)).$$

To determine the unknown pair $(\lambda, \widetilde{U}(t))$ that satisfies the systems (3.1)–(3.4), we construct a self-consistent iterative scheme based on Eqs. (3.5) and (3.10). The solution to the original problems (2.4) and (2.5) is obtained via a sequence of functions $U^{(k)}(t)$, for $k = 0, 1, 2, \ldots$, computed according to the following iterative algorithm:

Step 0.

(a) Assuming the matrix $Q_\nu(l)$ is invertible for a given $l \in \mathbb{R}^+$, $\nu \in \mathbb{N}$, we define the initial approximation of the parameter vector $\lambda^{(0)} = (\lambda_1^{(0)},\ldots,\lambda_{m+2}^{(0)}) \in \mathbb{R}^{(N-1)(m+2)}$ by solving $Q_\nu(l)\lambda^{(0)} = -\widetilde{F}_\nu(l)$, which yields $\lambda^{(0)} = -[Q_\nu(l)]^{-1}\widetilde{F}_\nu(l)$.

(b) Using the components of $\lambda^{(0)} \in \mathbb{R}^{(N-1)(m+2)}$, we solve the Cauchy problems (3.1) and (3.2) on each interval $[t_{r-1}, t_r]$, $r = 1,\ldots,m+1$, and obtain the functions $\widetilde{U}_r^{(0)}(t)$.

(c) Then, the solution on each interval is given by $U^{(0)}(t) = \lambda_r^{(0)} + \widetilde{U}_r^{(0)}(t), \quad t \in [t_{r-1}, t_r], \quad r = 1,\ldots,m+1$, and $U^{(0)}(T) = \lambda_{m+2}^{(0)}$.

Step 1.

(a) Substituting the functions $\widetilde{U}_r^{(0)}(t)$ into the right-hand side of Eq. (3.10), we compute the updated parameter vector $\lambda^{(1)} = (\lambda_1^{(1)},\ldots,\lambda_{m+2}^{(1)}) \in \mathbb{R}^{(N-1)(m+2)}$, by solving $Q_\nu(l)\lambda^{(1)} = -\widetilde{F}_\nu(l) - G_\nu(\widetilde{U}^{(0)}, l)$.

(b) With the new parameters $\lambda_r^{(1)}$, we solve the Cauchy problems (3.1) and (3.2) for each $r = 1,\ldots,m+1$, and determine the functions $\widetilde{U}_r^{(1)}(t)$.

(c) The solution on each subinterval is then updated as $U^{(1)}(t) = \lambda_r^{(1)} + \widetilde{U}_r^{(1)}(t), \quad t \in [t_{r-1}, t_r], \quad r = 1,\ldots,m+1$, and $U^{(1)}(T) = \lambda_{m+2}^{(1)}$.

This procedure is repeated iteratively. At each step k, we obtain the approximations $U^{(k)}[t]$. It is important to note that, for fixed parameter values λ_r, the Cauchy problems are solved independently on each subinterval $t \in [t_{r-1}, t_r]$, $r = 1, \dots, m+1$.

The estimates used in the convergence analysis are provided below:

$$\|\Phi\| \leq \max\Big(\Big|\varphi_1 + \frac{C_1\psi_0(0) + E_1\psi_0(T)}{2\tau}\Big|, |\varphi_2|, |\varphi_3|, \dots, |\varphi_{N-2}|, \Big|\varphi_{N-1} - \frac{C_{N-1}\psi_1(0) + E_{N-1}\psi_1(T)}{2\tau}\Big|\Big)$$
$$\leq \max\Big(\|\varphi_1\| + \frac{|C_1| \cdot |\psi_0(0)| + |E_1| \cdot |\psi_0(T)|}{2\tau}, |\varphi_2|, |\varphi_3|, \dots, |\varphi_{N-1}| - \frac{|C_{N-1}| \cdot |\psi_1(0)| + |E_{N-1}| \cdot |\psi_1(T)|}{2\tau}\Big)$$
$$\leq \max_{i=\overline{1,N-1}} |\varphi_i| + \frac{|C_1| + |E_1|}{2\tau} \max_{t\in[0,T]} |\psi_0(t)| + \frac{|C_{N-1}| + |E_{N-1}|}{2\tau} \max_{t\in[0,T]} |\psi_1(t)|,$$

$$\|F\|_1 = \max_{t\in[0,T]} |F(t)| \leq \max_{t\in[0,T]} \Big(\frac{|a_1(t)| \cdot |\psi_0(t)|}{\tau^2} + |f_1(t)|, \max_{i=\overline{2,N-2}} |f_i(t)|, \frac{|a_{N-1}(t)| \cdot |\psi_1(t)|}{\tau^2} + |f_{N-1}(t)|\Big)$$
$$\leq \max_{t\in[0,T]} \frac{1}{\tau^2}\Big(|a_1(t)|, |a_{N-1}(t)|\Big) \cdot \max_{t\in[0,T]}(|\psi_0(t)|, |\psi_1(t)|) + \max_{t\in[0,T]} \max_{i=\overline{1,N-1}} |f_i(t)|$$
$$\leq \Big(\max_{t\in[0,T]} \max_{i=\overline{1,N-1}} \frac{1}{\tau^2}|a_i(t)| + 1\Big) \cdot \max\Big(\max_{t\in[0,T]}(|\psi_0(t)|, |\psi_1(t)|), \max_{t\in[0,T]} \max_{i=\overline{1,N-1}} |f_i(t)|\Big).$$

The convergence of the proposed iterative scheme, along with the existence and uniqueness of the solution to problems (2.4) and (2.5), is ensured by the following result.

Theorem 1. *Let the matrix $Q_\nu(l) : \mathbb{R}^{(N-1)(m+2)} \to \mathbb{R}^{(N-1)(m+2)}$ be invertible for all $l \in \mathbb{R}^+, \nu \in N$. Suppose the following conditions are satisfied:*

$$\|[Q_\nu(l)]^{-1}\| \leq p_\nu(l), \tag{3.11}$$

$$g_\nu(l) = p_\nu(l) \cdot \Big[e^{\alpha l} - \sum_{j=0}^{\nu} \frac{(\alpha l)^j}{j!} + \sum_{i=0}^{m} \beta_i l \cdot \Big(e^{\alpha l} - \sum_{k=0}^{\nu-1} \frac{(\alpha l)^k}{k!}\Big)\Big] < 1. \tag{3.12}$$

Under these assumptions, the two-point boundary value problem associated with the loaded differential equations (2.4) and (2.5) admit a unique solution $U^*(t)$, and the following estimate holds:

$$\|U^*\|_1 \leq \widetilde{K}_\nu(l) \max(\|F\|_1, \|\Phi\|), \tag{3.13}$$

$$\widetilde{K}_\nu(l) = \Big\{\Big(e^{\alpha l} - 1 + e^{\alpha l} \sum_{i=0}^{m} \beta_i l\Big) \cdot \frac{p_\nu(l)}{1 - g_\nu(l)} \cdot \frac{(\alpha l)^\nu}{\nu!} + \frac{p_\nu(l)}{1 - g_\nu(l)} \cdot \frac{(\alpha l)^\nu}{\nu!} + 1\Big\}$$
$$\times \Big\{\Big(e^{\alpha l} - 1 + e^{\alpha l} \cdot \sum_{i=0}^{m} \beta_i l\Big) p_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\Big) + e^{\alpha l}\Big\} l + p_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\Big) l,$$

where $\|A(t)\| \leq \alpha = const$, $\|M_i(t)\| \leq \beta_i = const$, $i = 0, \dots, m$.

The proof of this Theorem 1 is omitted as it is analogous to that of Theorem 2 in [28].

Theorem 1 provides sufficient conditions ensuring the existence and uniqueness of the solution to problems (2.4) and (2.5). The matrix $Q_\nu(l)$ is a constant matrix constructed through repeated definite integrals of the matrices $A(t)$, $M_j(t)$, B, and C. Therefore, by using computational software packages, the invertibility of $Q_\nu(l)$ and the fulfillment of conditions (3.11) and (3.12) can be constructively verified for specific problems, since all matrices $A(t)$, $M_j(t)$, B, and C are explicitly defined by the given coefficients and boundary data of the original problems (1.1)–(1.3).

4. Convergence of the method of lines

We now establish the convergence of the solution to the discretized problems (2.1)–(2.3) toward the solution of the original nonlocal problems (1.1)–(1.3).

Theorem 2. *Let $a(t,x)$, $b(t,x)$, $k_j(t,x)$ for $j = 0,\dots,m$, and $f(t,x)$ be continuous in the domain Ω. Suppose the functions $B(x), C(x), D(x), E(x)$ are continuous on the interval $[0,\omega]$, and that $\varphi(x), \psi_0(t), \psi_1(t)$ are sufficiently smooth and satisfy the necessary consistency conditions. Then the solution of the discretized problems (2.1)–(2.3) converges to the solution of the nonlocal problem for the loaded parabolic equations (1.1)–(1.3) with an accuracy of order $O(\tau^2)$ as $\tau \to 0$.*

Proof. Let $u_i(t)$ be the solution of the discretized problems (2.1)–(2.3), and $u(t,x_i)$ be the solution at the grid points of problems (1.1)–(1.3).

The function $u(t,x)$ at x_i satisfies the following relation:

$$\frac{\partial u(t,x_i)}{\partial t} = a(t,x_i)\frac{\partial^2 u}{\partial x^2}\Big|_{x=x_i} + b(t,x_i)u(t,x_i) + \sum_{j=0}^{m} k_j(t,x_i)u(t_j,x_i) + f(t,x_i),$$

$$B(x_i)u(0,x_i) + C(x_i)\frac{\partial u(0,x)}{\partial x}\Big|_{x=x_i} + D(x_i)u(T,x_i) + E(x_i)\frac{\partial u(T,x)}{\partial x}\Big|_{x=x_i} = \varphi(x_i), \quad i = 1,\dots,N-1.$$

The second-order accurate central difference formula for approximating the second derivative of a function $u(t,x)$ at a point x_i is given by [30]: $\frac{\partial^2 u}{\partial x^2}\Big|_{x=x_i} = \frac{u(t,x_{i+1})-2u(t,x_i)+u(t,x_{i-1})}{\tau^2} + O(\tau^2)$, while the central difference approximation for the first derivative at x_i is expressed as $\frac{\partial u}{\partial x}\Big|_{x=x_i} = \frac{u(t,x_{i+1})-u(t,x_{i-1})}{2\tau} + O(\tau^2)$, we get

$$\begin{aligned}\frac{\partial u(t,x_i)}{\partial t} &= a(t,x_i)\frac{u(t,x_{i+1}) - 2u(t,x_i) + u(t,x_{i-1})}{\tau^2} + b(t,x_i)u(t,x_i)\\ &\quad + \sum_{j=0}^{m} k_j(t,x_i)u(t_j,x_i) + f(t,x_i) + O(\tau^2),\end{aligned} \tag{4.1}$$

$$\begin{aligned}&B(x_i)u(0,x_i) + C(x_i)\frac{u_{i+1}(0) - u_{i-1}(0)}{2\tau} + D(x_i)u(T,x_i)\\ &+ E(x_i)\frac{u_{i+1}(T) - u_{i-1}(T)}{2\tau} = \varphi(x_i) + O(\tau^2), \quad i = 1,\dots,N-1,\end{aligned} \tag{4.2}$$

$$u_0(t) = \psi_0(t), \quad u_N(t) = \psi_1(t), \quad t \in [0,T]. \tag{4.3}$$

Subtracting the discretized problems (2.1)–(2.3) from problems (4.1)–(4.3) by subtracting the corresponding equations and boundary conditions gives the error evolution equation $\delta_i(t) = u(t,x_i) - u_i(t)$:

$$\frac{\partial \delta_i}{\partial t} = a(t,x_i)\frac{\delta_{i+1} - 2\delta_i + \delta_{i-1}}{\tau^2} + b(t,x_i)\delta_i + \sum_{j=0}^{m} k_j(t,x_i)\delta(t_j) + O(\tau^2), \tag{4.4}$$

$$B(x_i)\delta(0) + C(x_i)\frac{\delta_{i+1}(0) - \delta_{i-1}(0)}{2\tau} + D(x_i)\delta(T) + E(x_i)\frac{\delta_{i+1}(T) - \delta_{i-1}(T)}{2\tau} = O(\tau^2), \quad i = 1,\dots,N-1, \tag{4.5}$$

$$\delta_0(t) = 0, \quad \delta_N(t) = 0, \quad t \in [0, T]. \tag{4.6}$$

Here, the term $O(\tau^2)$ represents the truncation error introduced by the finite difference approximation. The corresponding error equation, derived from (4.4)–(4.6), retains the structural form of the original discretized problems (2.1)–(2.3).

Therefore, the estimate provided in Theorem 1 can be applied directly, leading to

$$\max_{t\in[0,T]} \max_{i=1,\ldots,N-1} |\delta_i(t)| \leq \widetilde{K}_\nu(l) \cdot O(\tau^2), \quad i = 1, \ldots, N-1,$$

which completes the proof of Theorem 2. □

It should be noted that the overall accuracy of the method of lines depends both on the spatial discretization and on the numerical accuracy of solving the resulting system of ordinary differential equations.

In this work, the ODE system is solved using the Dzhumabaev parameterization method, whose convergence properties are given in Theorem 1. Since this method provides an approximation of the ODE solution with an error of order $O(\tau^2)$, the total convergence order of the method of lines remains $O(\tau^2)$.

Thus, the accuracy of the ODE solution does not reduce the overall precision of the proposed approach.

5. Investigation of stability and error analysis

Suppose that the discretized systems (2.1)–(2.3) is solved using perturbed input data rather than exact values. Let the approximated coefficients and functions be denoted by

$$\tilde{a}_i(t),\ \tilde{b}_i(t),\ \tilde{f}_i(t),\ \tilde{k}_i^j(t),\ \tilde{\varphi}_i,\ \tilde{\psi}_0(t),\ \tilde{\psi}_1(t),\ \widetilde{B}_i,\ \widetilde{C}_i,\ \widetilde{D}_i,\ \widetilde{E}_i,$$

where $j = 0, \ldots, m$ and i denotes the spatial grid index.

Let $\tilde{u}_i(t)$ denote the solution of the discretized problem with the perturbed data and define the total error as $\tilde{\delta}_i(t) = u(t, x_i) - \tilde{u}_i(t)$, where $u(t, x_i)$ is the exact solution evaluated at the grid point x_i.

Assume that the perturbed discretized problem admits a unique solution. Following [15], define the perturbations in the data as

$$\begin{aligned}
&\tilde{\delta}a = a(t, x_i) - \tilde{a}_i(t), \quad \tilde{\delta}b = b(t, x_i) - \tilde{b}_i(t), \quad \tilde{\delta}f = f(t, x_i) - \tilde{f}_i(t),\\
&\tilde{\delta}k_j = k_j(t, x_i) - \tilde{k}_i^j(t) \quad (j = 0, \ldots, m), \quad \tilde{\delta}\varphi = \varphi(x_i) - \tilde{\varphi}_i, \quad \tilde{\delta}\psi_0(t) = \psi_0(t) - \tilde{\psi}_0(t),\\
&\tilde{\delta}\psi_1(t) = \psi_1(t) - \tilde{\psi}_1(t), \quad \tilde{\delta}B = B(x_i) - \widetilde{B}_i, \quad \tilde{\delta}C = C(x_i) - \widetilde{C}_i,\\
&\tilde{\delta}D = D(x_i) - \widetilde{D}_i, \quad \tilde{\delta}E = E(x_i) - \widetilde{E}_i.
\end{aligned}$$

For example, the quantity $\tilde{\delta}a$ represents the deviation between the exact coefficient $a(t, x_i)$ and its perturbed value $\tilde{a}(t, x)$. The remaining perturbations are interpreted similarly.

Definition 1. *The discretized problems (2.1)–(2.3) are said to converge stably to the nonlocal problems (1.1)–(1.3) if the following conditions hold:*

$$\max_{1\leq i\leq N-1} |\tilde{\delta}_i(t)| \to 0 \quad \textit{as} \quad \tau \to 0,$$

and

$$|\tilde{\delta}a|, |\tilde{\delta}b|, |\tilde{\delta}f|, |\tilde{\delta}k_j|, |\tilde{\delta}\varphi|, |\tilde{\delta}\psi_0|, |\tilde{\delta}\psi_1|, |\tilde{\delta}B|, |\tilde{\delta}C|, |\tilde{\delta}D|, |\tilde{\delta}E| \to 0, \quad j = 0, \ldots, m.$$

Theorem 3. *The solution of the discretized problems (2.1)–(2.3) converges stably to the solution of the nonlocal problem for loaded parabolic equations (1.1)–(1.3).*

Proof. Let $\tilde{u}_i(t)$ denote the solution of the discretized systems (2.1)–(2.3), where the input coefficients and boundary data are subject to perturbations.

The governing equations for $\tilde{u}_i(t)$ take the form:

$$\frac{d\tilde{u}_i}{dt} = \tilde{a}_i(t)\frac{\tilde{u}_{i+1} - 2\tilde{u}_i + \tilde{u}_{i-1}}{\tau^2} + \tilde{b}_i(t)\tilde{u}_i + \sum_{j=0}^{m} \tilde{k}_i^j(t)\tilde{u}_i(t_j) + \tilde{f}_i(t), \qquad i = 1, \ldots, N-1, \tag{5.1}$$

$$\widetilde{B}_i\tilde{u}_i(0) + \widetilde{C}_i\frac{\tilde{u}_{i+1}(0) - \tilde{u}_{i-1}(0)}{2\tau} + \widetilde{D}_i\tilde{u}_i(T) + \widetilde{E}_i\frac{\tilde{u}_{i+1}(T) - \tilde{u}_{i-1}(T)}{2\tau} = \tilde{\varphi}_i, \tag{5.2}$$

$$\tilde{u}_0(t) = \tilde{\psi}_0(t), \quad \tilde{u}_N(t) = \tilde{\psi}_1(t), \quad t \in [0, T]. \tag{5.3}$$

Using the perturbed data along with the identities $\tilde{a}(t, x_i) = \tilde{a}_i(t)$, $\tilde{b}(t, x_i) = \tilde{b}_i(t)$, $\tilde{k}_j(t, x_i) = \tilde{k}_i^j(t)$, $\tilde{f}(t, x_i) = \tilde{f}_i(t)$, $\widetilde{B}(x_i) = \widetilde{B}_i$, $\widetilde{C}(x_i) = \widetilde{C}_i$, $\widetilde{D}(x_i) = \widetilde{D}_i$, $\widetilde{E}(x_i) = \widetilde{E}_i$, we reformulate the problem in the form of systems (4.1)–(4.3) with perturbed data:

$$\begin{aligned} \frac{\partial u(t, x_i)}{\partial t} &= \tilde{a}(t, x_i)\frac{u(t, x_{i+1}) - 2u(t, x_i) + u(t, x_{i-1})}{\tau^2} + \tilde{b}(t, x_i)u(t, x_i) + \sum_{j=0}^{m} \tilde{k}_j(t, x_i)u(t_j, x_i) + \tilde{f}(t, x_i) \\ &+ \tilde{\delta}a\frac{u(t, x_{i+1}) - 2u(t, x_i) + u(t, x_{i-1})}{\tau^2} + \tilde{\delta}bu(t, x_i) + \sum_{j=0}^{m} \tilde{\delta}k_j(t, x_i)u(t_j, x_i) + \tilde{\delta}f(t, x_i) + O(\tau^2), \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\widetilde{B}(x_i)u(0, x_i) + \widetilde{C}(x_i)\frac{u_{i+1}(0) - u_{i-1}(0)}{2\tau} + \widetilde{D}(x_i)u(T, x_i) + \widetilde{E}(x_i)\frac{u_{i+1}(T) - u_{i-1}(T)}{2\tau} \\ &+ \tilde{\delta}B(x_i)u(0, x_i) + \tilde{\delta}C(x_i)\frac{u_{i+1}(0) - u_{i-1}(0)}{2\tau} + \tilde{\delta}D(x_i)u(T, x_i) + \tilde{\delta}E(x_i)\frac{u_{i+1}(T) - u_{i-1}(T)}{2\tau} \\ &= \tilde{\varphi}(x_i) + \tilde{\delta}\varphi(x_i) + O(\tau^2), \quad i = 1, \ldots, N-1, \end{aligned} \tag{5.5}$$

$$u_0(t) = \tilde{\psi}_0(t) + \tilde{\delta}\psi_0(t), \quad u_N(t) = \tilde{\psi}_1(t) + \tilde{\delta}\psi_1(t), \quad t \in [0, T]. \tag{5.6}$$

We subtract the equation of the problems (5.4)–(5.6) from that of the problems (5.1)–(5.3), and subtract the corresponding boundary conditions as well. As a result, we obtain

$$\frac{d\tilde{\delta}_i}{dt} = \tilde{a}_i(t)\frac{\tilde{\delta}_{i+1} - 2\tilde{\delta}_i + \tilde{\delta}_{i-1}}{\tau^2} + \tilde{b}_i(t)\tilde{\delta}_i + \sum_{j=0}^{m} \tilde{k}_i^j(t)\tilde{\delta}_i(t_j) + \widetilde{R}_i(t), \qquad i = 1, \ldots, N-1, \tag{5.7}$$

$$\widetilde{B}_i\tilde{\delta}_i(0) + \widetilde{C}_i\frac{\tilde{\delta}_{i+1}(0) - \tilde{\delta}_{i-1}(0)}{2\tau} + \widetilde{D}_i\tilde{\delta}_i(T) + \widetilde{E}_i\frac{\tilde{\delta}_{i+1}(T) - \tilde{\delta}_{i-1}(T)}{2\tau} = \widetilde{P}_i, \tag{5.8}$$

$$\tilde{\delta}_0(t) = \tilde{\delta}\psi_0(t), \quad \tilde{\delta}_N(t) = \tilde{\delta}\psi_1(t), \quad t \in [0, T]. \tag{5.9}$$

where

$$\widetilde{R}_i(t) = \tilde{\delta}a\frac{u(t, x_{i+1}) - 2u(t, x_i) + u(t, x_{i-1})}{\tau^2} + \tilde{\delta}bu(t, x_i) + \sum_{j=0}^{m} \tilde{\delta}k_j(t, x_i)u(t_j, x_i) + \tilde{\delta}f(t, x_i) + O(\tau^2),$$

$$\widetilde{P}_i = \tilde{\delta}B(x_i)u(0,x_i) + \tilde{\delta}C(x_i)\frac{u_{i+1}(0) - u_{i-1}(0)}{2\tau} + \tilde{\delta}D(x_i)u(T,x_i)$$
$$+ \tilde{\delta}E(x_i)\frac{u_{i+1}(T) - u_{i-1}(T)}{2\tau} + \tilde{\delta}\varphi(x_i) + O(\tau^2).$$

The subsequent analysis proceeds in a manner analogous to that used for the problems (2.4) and (2.5). By applying the estimate from Eq. (3.13) to the solution of the perturbed problems (5.7)–(5.9), we obtain:

$$\max_{t\in[0,T]} \max_{1\le i\le N-1} |\tilde{\delta}_i^*(t)| = \|\tilde{\delta}^*\|_1 \le \widetilde{K}_\nu(l)\max(\|\widetilde{R}\|_1, \|\widetilde{P}\|), \tag{5.10}$$

$$\widetilde{R}(t) = \begin{pmatrix} \frac{\tilde{a}_1(t)\tilde{\delta}\psi_0(t)}{\tau^2} + \widetilde{R}_1(t) \\ \widetilde{R}_2(t) \\ \vdots \\ \widetilde{R}_{N-2}(t) \\ \frac{\tilde{a}_{N-1}(t)\tilde{\delta}\psi_1(t)}{\tau^2} + \widetilde{R}_{N-1}(t) \end{pmatrix}, \quad \widetilde{P} = \begin{pmatrix} \widetilde{P}_1 + \frac{\widetilde{C}_1\tilde{\delta}\psi_0(0) + \widetilde{E}_1\tilde{\delta}\psi_0(T)}{2\tau} \\ \widetilde{P}_2 \\ \vdots \\ \widetilde{P}_{N-2} \\ \widetilde{P}_{N-1} - \frac{\widetilde{C}_{N-1}\tilde{\delta}\psi_1(0) + \widetilde{E}_{N-1}\tilde{\delta}\psi_1(T)}{2\tau} \end{pmatrix}.$$

$$\|\widetilde{R}\|_1 = \max_{t\in[0,T]} \|\widetilde{R}(t)\| \le \Big(\max_{t\in[0,T]} \max_{i=\overline{1,N-1}} \frac{1}{\tau^2}|\tilde{a}_i(t)| + 1\Big) \cdot \max\Big(\max_{t\in[0,T]}(|\tilde{\delta}\psi_0(t)|, |\tilde{\delta}\psi_1(t)|), \max_{t\in[0,T]} \max_{i=\overline{1,N-1}} |\widetilde{R}_i(t)|\Big),$$

$$\|\widetilde{P}\| \le \max_{i=\overline{1,N-1}} |\widetilde{P}_i| + \frac{|\widetilde{C}_1| + |\widetilde{E}_1|}{2\tau} \max_{t\in[0,T]} |\tilde{\delta}\psi_0(t)| + \frac{|\widetilde{C}_{N-1}| + |\widetilde{E}_{N-1}|}{2\tau} \max_{t\in[0,T]} |\tilde{\delta}\psi_1(t)|.$$

Using estimate (5.10), in combination with Definition 1, we complete the proof of Theorem 3. □

The method of lines is proven to be both convergent and stable, provided that the coefficients and boundary data meet the required regularity conditions. Under such assumptions, the solution obtained from the discretized system approaches the exact solution with an error of order $O(\tau^2)$ as $\tau \to 0$.

Moreover, stability guarantees that minor perturbations in the problem's input data produce only small variations in the resulting solution, thereby ensuring the robustness of the discretized scheme.

Lemma 1. *The total error Δ can be expressed as the sum of two components: $\Delta = \delta_i(t) + \tilde{\delta}_i(t)$. where $\delta_i(t)$ represents the discretization error, and $\tilde{\delta}_i(t)$ denotes the error introduced by perturbations in the input data. The former arises from the finite-difference approximation of spatial derivatives, while the latter is caused by inaccuracies in the coefficients, boundary values, or initial conditions.*

As the grid spacing $\tau \to 0$, the solution of the discretized problem converges to the exact solution with an error of order $O(\tau^2)$. This confirms that the method of lines possesses second-order accuracy in space.

The stability of the method ensures that small perturbations in the input data do not cause large deviations in the numerical solution. That is, the total error remains uniformly bounded over the time interval of interest. Consequently, the total error Δ satisfies the estimate $|\Delta| \le O(\tau^2)$, which shows that the method is both convergent and stable, and the overall error converges at a rate of $O(\tau^2)$ as $\tau \to 0$.

In the next section, we present example to demonstrate the efficiency, accuracy, and practical relevance of the proposed method for solving nonlocal problems involving loaded parabolic equations. The numerical experiment was carried out using Mathcad, which offers a convenient environment for both symbolic manipulation and numerical computation.

6. Example

Let us consider the following nonlocal problem for the loaded parabolic equation defined on the domain $(t, x) \in \Omega = (0; 0.5) \times (0; 0.25)$:

$$\frac{\partial u}{\partial t} = xt^5 \frac{\partial^2 u}{\partial x^2} - xtu(t, x) + u(0.1; x) + xu(0.2; x) + tu(0.3, x) - u(0.4, x) + \frac{3e^x}{20} + \frac{191tx^4}{100} + \frac{3x^4}{20} - \frac{x^5}{25} + t^3x^5 - 12t^3x + \frac{191te^x}{100} - \frac{xe^x}{25} - t^3x^5e^x + t^3xe^x, \quad (t, x) \in \Omega, \tag{6.1}$$

$$-30u(0, x) + 20\frac{\partial u(0; x)}{\partial x} - 20u(0.5; x) + 30x\frac{\partial u(0.5; x)}{\partial x} = 25x^4 - 5e^x + \frac{15xe^x}{2}, \qquad x \in [0; 0.25], \tag{6.2}$$

$$u(t; 0) = t^2, \quad u(t; 0.5) = t^2\left(e^{\frac{1}{4}} + \frac{1}{256}\right), \quad t \in [0; 0.5]. \tag{6.3}$$

The *exact solution* of given problems (6.1)–(6.3) is $u_{exact}(t, x) = (e^x + x^4)t^2$.

We discretize the spatial variable uniformly ($N = 4$) and use a time step $\tau = 0.0625$. This reduces the problems (6.1)–(6.3) to a system of loaded differential equations with nonlocal boundary conditions for a vector function $U(t)$. The exact vector solution corresponds to the samples of u_{exact} at

$$x \in \left\{\frac{1}{16}, \frac{1}{8}, \frac{3}{16}\right\}, i.e., U_{\text{exact}}(t) = \left(t^2\left(e^{\frac{1}{16}} + \frac{1}{65536}\right), t^2\left(e^{\frac{1}{8}} + \frac{1}{4096}\right), t^2\left(e^{\frac{3}{16}} + \frac{81}{65536}\right)\right)^\top.$$

Following the approach described in Section 3, we apply the Dzhumabaev parametrization method. The interval $[0, 0.5]$ is divided by the loading points. For the parameters $\alpha = 0.1827$, $\beta_1 = 1$, $\beta_2 = 0.1875$, $\beta_3 = 0.5$, $\beta_4 = 1$, and $l = 0.1$, the corresponding operator $Q_1(0.1) : \mathbb{R}^{18} \to \mathbb{R}^{18}$ satisfies $\|[Q_1(0.1)]^{-1}\| \leq 4.7144$ and $g_1(0.1) = 0.0241 < 1$. According to Theorem 1, these conditions guarantee the existence and uniqueness of the solution to the problem.

We then run the iterative algorithm introduced in Section 3 on the time nodes $\xi_j = 0.0125 \cdot j$, $j = 0, \ldots, 40$. The error decreases rapidly from the initial iteration to the next ones, and by the second iteration the desired accuracy is already achieved; thus, the computation is stopped at that point. The final approximation u_{approx} satisfies the uniform bound

$$\max_{j=0,\ldots,40} \max_{i=1,2,3} \left\| u_{\text{exact}}(\xi_j, x_i) - u_{\text{approx}}(\xi_j, x_i) \right\| < 6.47 \times 10^{-5},$$

which indicates that the obtained accuracy is satisfactory for practical computations and confirms the efficiency of the proposed method.

7. Conclusions

This paper is devoted to the study of a nonlocal problem for a loaded parabolic equation. We investigated the convergence, stability, and error behavior of the method of lines applied to such problems.

The method of lines has been shown to be a reliable and effective approach for solving parabolic partial differential equations. It provides second-order accuracy in space, stability, and robust error control, making it particularly suitable for problems with smooth solutions and well-posed boundary conditions. The method is both consistent and stable, which ensures its convergence to the exact solution. Under appropriate smoothness assumptions on the coefficients and boundary data, the total error decays at the rate of $O(\tau^2)$ as the grid is refined.

These theoretical results lay a solid foundation for the further application of the method to a wider class of nonlocal problems involving loaded parabolic equations. In future work, a more detailed numerical analysis will be conducted to further explore the method's practical performance and extend it to more complex problem settings.

Author contributions

Anar Assanova: Conceptualization, Methodology, Supervision, Funding acquisition; Liu Zhenhai: Validation, Formal analysis, Project administration; Saule Kuanysh: Resources, Writing original draft, Visualization; Zhazira Kadirbayeva: Investigation, Data Curation, Writing review and editing. All authors confirm that they have read and agreed to the final version of the manuscript prior to publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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References

1. J. Keener, J. Sneyd, *Mathematical Physiology*, New York: Springer-Verlag, 2009.
2. F. Brauer, P. Driessche, J. Wu, *Mathematical Epidemiology*, Berlin: Springer-Verlag, 2008. https://doi.org/10.1007/978-3-540-78911-6
3. A. M. Nakhushev, *Loaded Equations and Their Applications*, Moscow: Nauka, 2012.
4. A. D'Onofrio, A. Gandolfi, *Mathematical Oncology 2013*, Switzerland: Springer, 2014. https://doi.org/10.1007/978-1-4939-0458-7
5. B. G. Ermentrout, D. Terman, *Mathematical Foundations of Neuroscience*, New York: Springer, 2010. https://doi.org/10.1007/978-0-387-87708-2
6. A. Friedman, C. Y. Kao, *Mathematical Modeling of Biological Processes. Lecture Notes on Mathematical Modelling in the Life Sciences*, Switzerland: Springer International Publishing, 2014. https://doi.org/10.1007/978-3-319-08314-8

7. A. Kh. Attaev, M. I. Ramazanov, M. T. Omarov, On the correctness of boundary value problems for the two-dimensional loaded parabolic equation, *Bull. Karaganda Uni. Math. Ser.*, **108** (2022), 34–41. https://doi.org/10.31489/2022M/34-41
8. B. Islomov, U. I. Baltaeva, Boundary-value problems for a third-order loaded parabolic-hyperbolic equation with variable coefficients, *Elect. J. Diff. Equ.*, **221** (2015), 1–10.
9. T. K. Yuldashev, B. I. Islomov, E. K. Alikulov, Boundary-value problems for loaded third-order parabolic-hyperbolic equations in infinite three-dimensional domains, *Lobachevskii J. Math.*, **41** (2020), 926–944. https://doi.org/10.1134/S1995080220050145
10. V. M. Abdullaev, K. R. Aida-zade, Finite-difference methods for solving loaded parabolic equations, *Comput. Math. Math. Phys.,* **56** (2016), 93–105. https://doi.org/10.1134/S0965542516010036
11. S. Aitzhanov, K. Bekenayeva, Z. Abdikalikova, Boundary value problem for a loaded pseudoparabolic equation with a fractional Caputo operator, *Mathematics*, **11** (2023), 3987. https://doi.org/10.3390/math11183987
12. Y. Huang, Z. Liu, C. F. Wen, Approximate controllability for fractional semilinear parabolic equations, *Comput. Math. Appl.*, **77** (2019), 2971–2979. https://doi.org/10.1016/j.camwa.2018.08.003
13. O. Kapustyan, S. Temesheva, A. Tleulessova, Attracting sets in sup-norms for mild solutions of impulsive-perturbed parabolic semilinear problems, *AIMS Math.*, **10** (2025), 19173–19188. https://doi.org/10.3934/math.2025857
14. Z. Liu, On the identification of coefficients of semilinear parabolic equations, *Acta Math. Appl. Sin. Engl. Ser.*, **10** (1994), 356–367. https://doi.org/10.1007/BF02016326
15. B. M. Budak, The method of straight lines for certain boundary-value problems of the parabolic type, *Zh. Vychisl. Mat. Mat. Fiz.*, **6** (1961), 1105–1112.
16. D. S. Dzhumabaev, Substantiation of the method of lines for a single boundary value problem of a linear parabolic equation, *Izvestia Academy Sci. Kazakh SSR. Ser. Phys. Math.*, **1** (1983), 8–11.
17. J. S. Hicks, J. Wei, Numerical solution of parabolic partial differential equations with two-point boundary conditions by use of the method of lines, *J. Assoc. Comput. Mach.*, **14** (1967), 549–562. https://doi.org/10.1145/321406.321417
18. I. K. Karimov, I. Q. Khujaev, J. I. Khujaev, Application of the method of lines for solving one-dimensional equation of parabolic type under the boundary conditions of the second and first genera, *Vestnik KRAUNC. Fiz.-Mat. Nauki.*, **27** (2018), 78–92. https://doi.org/10.18454/2079-6641-2018-21-1-78-92
19. J. G. Verwer, J. M. Sanz-Serna, Convergence of method lines approximations to partial differential equations, *Computing*, **33** (1984), 297–313. https://doi.org/10.1007/BF02242274
20. A. Zafarullah, Application of the method of lines to parabolic partial differential equations with error estimates, *JACM*, **17** (1970), 294–302. https://doi.org/10.1145/321574.321583
21. D. S. Dzhumabaev, About one method of investigating ordinary differential equations, *AN News KAZSSR, Phys. Math. Ser.*, **3** (1982), 1–4.

22. D. S. Dzhumabaev, Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation, *USSR Comput. Math. Math. Phys.*, **29** (1989), 34–36. https://doi.org/10.1016/0041-5553(89)90038-4

23. Z. M. Kadirbayeva, On an algorithm for solving a problem with parameter for the essentially loaded differential equations, *Lobachevskii J. Math.*, **43** (2022), 3183–3191. https://doi.org/10.1134/S1995080222140177

24. Z. M. Kadirbayeva, D. S. Dzhumabaev, Numerical implementation of solving a control problem for loaded differential equations with multi-point condition, *Bull. Karaganda Uni. Math. Ser.*, **100** (2020), 81–91. https://doi.org/10.31489/2020m4/81-91

25. Z. M. Kadirbayeva, R. I. Kadirbayeva, Numerical solution of second-order impulsive differential equations with loadings subject to integral boundary conditions, *Math. Methods Appl. Sci.*, **48** (2025), 6269–6277. https://doi.org/10.1002/mma.10670

26. Z. F. Khankishiyev, On solution of a nonlocal problem with dynamic boundary conditions for a loaded linear parabolic equation by straight-line methods, *Bull. Comput. Appl. Math.*, **45** (2017), 75–96.

27. V. M. Abdullaev, Numerical solution of a boundary value problem for a loaded parabolic equation with nonlocal boundary conditions, *Vestnik KRAUNC. Fiz. Mat. Nauki.*, **32** (2020), 15–28. https://doi.org/10.26117/2079-6641-2020-32-3-15-28

28. S. K. Kuanysh, A. T. Assanova, Z. M. Kadirbayeva, Method of lines for a loaded parabolic equation, *J. Math. Mech. Comput. Sci.*, **125** (2025), 3–17. https://doi.org/10.26577/JMMCS2025125101

29. A. T. Assanova, Z. M. Kadirbayeva, S. K. Kuanysh, Numerical-analytical method for solving initial-boundary value problem for loaded parabolic equation, *Bull. Karaganda Uni. Math. Ser.*, **117** (2025), 34–45. https://doi.org/10.31489/2025m1/34-45

30. A. A. Samarskii, *The Theory of Difference Schemes*, New York: Marcel Dekker, 2001.

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