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Research article

Chaoticity of time-varying discrete dynamic systems via equicontinuity

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Abstract: This paper studies the preservation of chaotic properties in time-varying discrete dynamic systems, including various forms of expansiveness, sensitivity, and $\mathcal{M}^{\alpha}$-shadowing properties, and explores the interrelations among these properties. Specifically, it is first proved that positive ep-expansiveness and positive continuum-wise expansiveness are preserved under product operators, topological conjugacy, or composite operators. Subsequently, it is shown that a system characterized by positive ep-expansiveness has only a finite number of periodic points. Furthermore, an equivalence relation is established between positive expansiveness (or positive ep-expansiveness, or positive continuum-wise expansiveness) and sensitivity. Finally, it is noted that the $\mathcal{M}^{\alpha}$-shadowing property is incompatible with equicontinuity. An equivalent condition is presented to determine whether the limit map of a given system retains the $\mathcal{M}^{\alpha}$-shadowing property.

Keywords: time-varying discrete dynamic systems; expansiveness; positively expansive; shadowing property
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1. Introduction

In the development of science and technology, there is a desire to find mathematical models that can accurately characterize complex problems in the real world. However, the ever-changing nature of the real world makes it difficult to describe using a single equation. For instance, the population of a species constantly fluctuates due to factors such as temperature and season, which illustrates that the world we inhabit is a complex system that evolves over time. Autonomous discrete dynamic systems (ADDSs, also called classical dynamic systems) are generated by iterating a single map f, and their ability to describe certain complex systems is limited. Time-varying discrete dynamic systems (T-VDDSs, also called nonautonomous discrete dynamic systems), which are generated by a sequence of maps $f_{1,\infty} = \{f_1, f_2, ...\}$ that change over time, can describe more evolutionary states in the real world. If $f_i = f$ ($i \in \mathbb{N}$, $\mathbb{N}$ is the set of positive integers), then the T-VDDS degenerates into an autonomous one.

The foundation of research on T-VDDSs was laid by Kolyada and Snoha [1]. Subsequently, many scholars have conducted studies on the chaotic properties of T-VDDSs. For example, in 2002, Kempf [2] researched the Ω-limit sets in T-VDDSs. Krabs [3] investigated the stability and attractivity of T-VDDSs using Lyapunov functions. In 2004, Kolyada et al. [4] studied the minimality of T-VDDSs defined by sequences of continuous self-maps in compact Hausdorff spaces. In 2006, Canovas [5] discussed the relationship between the Ω-limit set of $f_{1,\infty}$ and that of f. The theoretical research related to T-VDDSs saw further development in 2009 when Shi and Chen [6] proposed concepts such as periodic points in time-varying cases and introduced three definitions of chaos. In 2013, Wu and Zhu [7] proved the preservation of distributed chaos and sensitivity within composite operators based on the work in [6]. In 2018, Radhika and Ruchi [8] introduced stronger forms of sensitivity, such as thick sensitivity, ergodic sensitivity, and multi-sensitivity, proving that if f_i ($i \in \mathbb{N}$) converge to f at a "sufficiently fast" rate, then various forms of sensitivity and transitivity for T-VDDSs and ADDSs coincide. Further research literature related to T-VDDSs can be found in [9–12].

As the theory of dynamic systems is gradually integrated into computer science, expansiveness and shadowing properties, as important concepts in the study of topological dynamic systems, have garnered significant attention in the autonomous case. Classical expansiveness is widely recognized for its importance in the study of dynamic systems. The connections between various derived types of expansiveness and shadowing properties are also taken seriously. For example, in 2003, Sakai [13] discussed various shadowing properties of positively expansive maps on compact metric spaces and proved that a positively expansive map has the classical shadowing property is equivalent to it having the s-limit shadowing property or the Lipschitz shadowing property. Subsequently, the definition of positive expansiveness was extended. In 2007, Richeson and Wiseman [14] introduced the concepts of weakly positive expansive maps and strongly positive expansive maps, proving that these concepts are equivalent on compact metric spaces. In 2012, Barwell et al. [15] discussed the concepts of the shadowing property and expansiveness for continuous maps restricted to subsets of a domain, and provided new equivalences between the shadowing property and expansiveness. In 2016, Morales [16] studied shadowable points by first investigating the relationship between the classical shadowing property and shadowable points, and noted that the shadowing properties of points can reflect certain characteristics of the entire system. The connection between expansiveness and shadowing properties was explored further in 2018 when Carvalho and Cordeiro [17] examined the relationship between positively n-expansive maps and the L-shadowing property in finite metric spaces. Furthermore, in 2021, Sakai [18] proved that positively expansive open maps on compact metric spaces exhibit the two-sided limit shadowing property. In connected spaces, this property is equivalent to the average shadowing property. In recent years, there have been significant breakthroughs in the study of expansiveness. In 2023, Lee and Morales [19] introduced a definition for positively expansive semigroups of linear operators in Banach spaces and proved that positively expansive semigroups are neither uniformly bounded nor equicontinuous. In 2024, Estaremi and Huang [20] conducted research on the expansiveness and positive expansiveness of composition operators in Orlicz spaces, discussing the conditions under which a composition operator can be expansive, positively expansive, and uniformly expansive. Further research findings can be found in [21–23].

The aforementioned research primarily focuses on the autonomous case. However, as mentioned earlier, time-varying conditions are the norm in nature. Thus, it is worth exploring expansiveness and shadowing properties under time-varying conditions. In fact, there is relatively little content addressing this aspect. In recent years, several new concepts of expansiveness and related results have emerged in time-varying systems. For example, in 2022, Vasisht and Das [24] studied the relationship between $\aleph_0$-expansiveness and meagre-expansiveness, proving some results regarding the non-existence of time-varying systems with $\aleph_0$-expansive and meagre expansive properties. Liu et al. [25] introduced positively expansive systems and positively expansive measures, demonstrating that for every T-VDDS with a positively expansive measure, there is a Borel probability measure such that the measure of the set of stable points is zero. In 2024, Makrooni and Abbasi [26] defined orbit shift continuum-wise expansivity, orbit shift persistence, and orbit shift α-persistence, proving that every continuum-wise expansive T-VDDS is orbit shift continuum-wise expansive.

Inspired by these studies, this paper focuses on the positively ep-expansive property and the positively continuum-wise expansive property, deriving results about their preservation in three contexts and exploring the close relationship between these properties and sensitivity. Additionally, the concept of $\mathcal{M}^{\alpha}$-shadowable points is introduced, leading to the result that the $\mathcal{M}^{\alpha}$-shadowing property and equicontinuity are mutually exclusive, along with a necessary and sufficient condition for the limit map of a system to exhibit the $\mathcal{M}^{\alpha}$-shadowing property.

The remaining structure of this paper is as follows. In Section 2, the concept of $\mathcal{M}^{\alpha}$-shadowable points is introduced, along with other basic concepts needed in the paper. In Section 3, the properties of positively ep-expansive and positively continuum-wise expansive systems are presented, along with their specific proofs. In Section 4, a detailed proof is provided showing that the $\mathcal{M}^{\alpha}$-shadowing property and equicontinuity are mutually exclusive. Finally, an equivalence condition is obtained for the limit map f of the sequence $f_{1,\infty}$ to possess the $\mathcal{M}^{\alpha}$-shadowing property.

2. Preliminaries

Assume that (X, d_1) is a compact metric space, where d_1 is the metric on X. Let $f_{1,\infty} = \{f_n\}_{n=1}^{\infty}$ denote a sequence of continuous maps. $(X, d_1, f_{1,\infty})$ is called a time-varying discrete dynamic system (T-VDDSs). In this paper, it is denoted as $(X, f_{1,\infty})$ or simply X without confusion. Let $\mathbb{N}_0$ denote the set of non-negative integers. $Orb(x, f_{1,\infty}) = \{f_1^n(x) : n \in \mathbb{N}_0\}$ is defined to represent the orbit of a point $x \in X$. Furthermore, f_n^i is defined as $f_n^i = f_{n+i-1} \circ \cdots \circ f_n$ for $n \in \mathbb{N}$ and f_1^0 denotes the identity map. For any $k \in \mathbb{N}$ and $f_{1,\infty}^{[k]} = \{f_{k(n-1)+1}^k\}_{n=1}^{\infty}$, $(X, f_{1,\infty}^{[k]})$ is referred to as the kth iteration of the system.

Assume that (Y, d_2) is a compact metric space and let $g_{1,\infty} = \{g_n\}_{n=1}^{\infty}$. Then $(Y, d_2, g_{1,\infty})$, similar to $(X, d_1, f_{1,\infty})$, is also a T-VDDS. The product system of these two systems is denoted as $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$, where the metric ρ on $X \times Y$ is defined as

$$\rho((j, k), (u, v)) = \max\{d_1(j, u), d_2(k, v)\}$$

for any pairs $(j, k), (u, v) \in X \times Y$. The sequence of continuous maps in this product system is constructed as $f_{1,\infty} \times g_{1,\infty} = \{f_n \times g_n\}_{n=1}^{\infty}$. For any $(j, k) \in X \times Y$ and $n \in \mathbb{N}_0$, $(f_1^n \times g_1^n)((j, k))$ is defined as

$$(f_1^n \times g_1^n)((j, k)) = (f_n \times g_n) \circ \cdots \circ (f_2 \times g_2) \circ (f_1 \times g_1)(j, k) = (f_1^n(j), g_1^n(k)).$$

If there is a homeomorphism $\sigma : X \to Y$ satisfying $g_n \circ \sigma = \sigma \circ f_n$ for all $n \in \mathbb{N}$, then $f_{1,\infty}$ and $g_{1,\infty}$ are called topologically conjugate to each other.

Definition 2.1 ([6]) Assume that $X, f_{1,\infty})$ is a T-VDDS.

(1) Let $r \in X$ and $k \in \mathbb{N}$. If $f_n^k(r) = r$ for all $n \in \mathbb{N}$, then r is called a periodic point of $f_{1,\infty}$. If $k = 1$, then r is called a fixed point of $f_{1,\infty}$.

(2) If there is a point $r \in X$ such that, for some $t \in \mathbb{N}_0$ and any $n \in \mathbb{N}$, $f_n^t(r)$ is a periodic point of $f_{1,\infty}$, then r is called an eventually periodic point of $f_{1,\infty}$.

The sets of periodic points, fixed points, and eventually periodic points of $f_{1,\infty}$ are denoted by $P(f_{1,\infty})$, $Fix(f_{1,\infty})$, and $EP(f_{1,\infty})$, respectively.

Definition 2.2 Assume that $f_{1,\infty}$ is a homeomorphism map sequence on X. For any $p, q \in X$, if an $\varepsilon > 0$ exists such that $d(f_1^i(p), f_1^i(q)) \leq \varepsilon$ for all $i \in \mathbb{N}$, then $p = q$. In this case, $f_{1,\infty}$ is said to be expansive, and ε is called an expansivity constant.

This is the classical definition of expansiveness. Next, we will introduce some derived related definitions.

Definition 2.3 Assume that M is a subset of X and that $f_{1,\infty}$ is a map sequence on X.

(1) For any $r, s \in M$, if an $\varepsilon > 0$ exists such that $d(f_1^i(r), f_1^i(s)) \leq \varepsilon$ ($i \in \mathbb{N}_0$) implies $r = s$, then $f_{1,\infty}$ is said to be positively expansive (PE) on M. If $M = X$, then $f_{1,\infty}$ is said to be positively expansive (PE).

(2) Let M be a dense subset of X that contains $EP(f_{1,\infty})$. If $f_{1,\infty}$ is positively expansive on M, then $f_{1,\infty}$ is said to be positively ep-expansive (PEX).

Here, some basic concepts are introduced. A space is called perfect if it has no isolated points. A nonempty compact, connected metric space is typically referred to as a continuum. If the continuum is not a single point set, it is called nondegenerate. If a subset M of a continuum also satisfies the conditions of the continuum, then M is referred to as a subcontinuum. Define $\operatorname{diam} M = \sup\{d(x, y) \mid x, y \in M\}$.

Definition 2.4 ([21]) Assume that $f_{1,\infty}$ is a homeomorphism map sequence on X, and let M be any nondegenerate subcontinuum of X. If an $\varepsilon > 0$ exists such that for each M, one can find an $n \in \mathbb{N}_0$ satisfying $\operatorname{diam} f_1^n(M) > \varepsilon$, then $f_{1,\infty}$ is said to be continuum-wise expansive (CE). If these conditions remain unchanged, and $f_{1,\infty}$ is replaced by a map sequence on X, then $f_{1,\infty}$ is said to be positively continuum-wise expansive (PCE).

Definition 2.5 ([6]) Assume that $(X, f_{1,\infty})$ is a T-VDDS. $(X, f_{1,\infty})$ is said to be sensitive if an $\varepsilon > 0$ exists such that, for any $r \in X$ and any neighborhood U of r, an $s \in U$ and an $i \in \mathbb{N}_0$ exist with $d(f_1^i(r), f_1^i(s)) > \varepsilon$.

Definition 2.6 ([27]) Let $\{a_i\}_{i=0}^{+\infty}$ be a sequence of points in X. For any $\delta > 0$, the sequence $\{a_i\}_{i=0}^{+\infty}$ is called a δ ergodic pseudo-orbit of $f_{1,\infty}$ if

$$\lim_{n\to\infty} \frac{1}{n} |\{0 \leq i < n : d(f_{i+1}(a_i), a_{i+1}) < \delta\}| = 1.$$

Definition 2.7 ([27]) Assume that $(X, f_{1,\infty})$ is a T-VDDS. If for every $\varepsilon > 0$, a $\delta > 0$ and a point $p \in X$ exist such that any δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \left|\left\{0 \leq i < n : d(f_1^i(p), a_i) < \varepsilon\right\}\right| > \alpha, \quad \alpha \in [0, 1),$$

then, the system $(X, f_{1,\infty})$ is said to have the $\mathcal{M}^{\alpha}$-shadowing property ($\mathcal{M}^{\alpha}$-SP).

Definition 2.8 If for any $\varepsilon > 0$, a $\delta > 0$ and a point $p \in X$ exist such that any δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ through q (i.e., $a_0 = q$) satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \left|\left\{0 \le i < n : d(f_1^i(p), a_i) < \varepsilon\right\}\right| > \alpha, \quad \alpha \in [0, 1), \tag{2.1}$$

then the point $q \in X$ is said to be an $\mathcal{M}^{\alpha}$-shadowable point of $f_{1,\infty}$.

3. Positively ep-expansive and positively continuum-wise expansive

This section primarily investigated that if $f_{1,\infty}$ is PEX, then $|P(f_{1,\infty})| \le \aleph_0$. Then, the properties of PEX and PCE in T-VDDSs are discussed. The results showed that these two properties were preserved under topological conjugacy and system composition. Moreover, if both subsystems were PEX (or PCE), then the product system also possessed the same property. Furthermore, under time-varying conditions, both PEX systems and PCE systems were shown to be equivalent to sensitive systems.

Theorem 3.1 Let $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ be two T-VDDSs.

(1) If $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ both have PEX, then so does $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$.

(2) If $f_{1,\infty}$ is topologically conjugate to $g_{1,\infty}$, then $(X, d_1, f_{1,\infty})$ is PEX if and only if $(Y, d_2, g_{1,\infty})$ has PEX.

(3) If $f_{1,\infty}$ is equicontinuous, then $(X, d_1, f_{1,\infty})$ are PEX if and only if $(X, d_1, f_{1,\infty}^{[m]})$ is PEX, where $m \in \mathbb{N}$.

Proof. (1) By the assumption, $f_{1,\infty}$ and $g_{1,\infty}$ are PEX. Let M be a dense subset of X that contains $EP(f_{1,\infty})$. Then $f_{1,\infty}$ is PE on M. So, for any $p, q \in M$, if there is an $\varepsilon > 0$ such that $d_1(f_1^i(p), f_1^i(q)) \le \varepsilon$ for all $i \in \mathbb{N}_0$, then $p = q$. Similarly, let N be a dense subset of Y that contains $EP(g_{1,\infty})$. Then $g_{1,\infty}$ is PE on N. So, for any $r, s \in N$, if there is an $\varepsilon > 0$ such that $d_2(g_1^i(r), g_1^i(s)) \le \varepsilon$ for all $i \in \mathbb{N}_0$, then $r = s$.

It is easy to see that $M \times N$ is a dense subset of $X \times Y$, as both M and N are dense in X and Y, respectively. Since $EP(f_{1,\infty}) \subseteq M$, $EP(g_{1,\infty}) \subseteq N$, and $(f_1^n \times g_1^n)(p, q) = (f_1^n(p), g_1^n(q))$ for any $(p, q) \in X \times Y$ and $n \in \mathbb{N}_0$, then, $EP(f_{1,\infty} \times g_{1,\infty}) \subseteq M \times N$.

The following proves that $f_{1,\infty} \times g_{1,\infty}$ is PE on $M \times N$. In fact, for any $(p, r), (q, s) \in M \times N$ and $i \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\rho((f_1^i \times g_1^i)(p, r), (f_1^i \times g_1^i)(q, s)) \\
&= \rho((f_1^i(p), g_1^i(r)), (f_1^i(q), g_1^i(s))) \\
&= \max\{d_1(f_1^i(p), f_1^i(q)), d_2(g_1^i(r), g_1^i(s))\}.
\end{aligned}$$

If an $\varepsilon > 0$ exists such that $\rho((f_1^i \times g_1^i)(p, r), (f_1^i \times g_1^i)(q, s)) \le \varepsilon$ for all $i \in \mathbb{N}_0$, then $\max\{d_1(f_1^i(p), f_1^i(q)), d_2(g_1^i(r), g_1^i(s))\} \le \varepsilon$. That is, $d_1(f_1^i(p), f_1^i(q)) \le \varepsilon$ and $d_2(g_1^i(r), g_1^i(s)) \le \varepsilon$ for all $i \in \mathbb{N}_0$. Since $f_{1,\infty}$ and $g_{1,\infty}$ are PE, then $p = q$, $r = s$. So, $(p, r) = (q, s)$. Therefore, $f_{1,\infty} \times g_{1,\infty}$ is PEX.

(2) (Necessity) Let $f_{1,\infty}$ be PEX. That is, $f_{1,\infty}$ is PE on the set M, where M is a dense subset of X that contains $EP(f_{1,\infty})$. This means that, for any $p, q \in M$, if an $\varepsilon > 0$ exists such that $d_1(f_1^i(p), f_1^i(q)) \le \varepsilon$ for all $i \in \mathbb{N}_0$, then $p = q$.

Let $N = \sigma(M) \subset Y$. Since $\overline{M} = X$ and σ is a continuous bijection, $\overline{N} = \overline{\sigma(M)} = \sigma(\overline{M}) = \sigma(X) = Y$. So, N is dense in Y. The following proves that $EP(g_{1,\infty}) \subseteq N$.

Let $u \in X$ be an eventually periodic point of $f_{1,\infty}$. Then $f_n^t(u)$ is a periodic point for some $t \in \mathbb{N}_0$ and any $n \in \mathbb{N}$. So, one can find a $k \in \mathbb{N}$ such that $f_m^k(f_n^t(u)) = f_n^t(u)$ for any $m \in \mathbb{N}$.

Since σ is a homeomorphism, then for any $m \in \mathbb{N}$

$$f_m^k(\sigma(f_n^t(u))) = \sigma(f_m^k(f_n^t(u))) = \sigma(f_n^t(u)).$$

Thus $\sigma(f_n^t(u))$ is a periodic point of $g_{1,\infty}$, and because

$$\sigma(f_n^t(u)) = \sigma(f_{n+t-1} \circ ... \circ f_n(u)) = (g_{n+t-1} \circ ... \circ g_n)(\sigma(u)) = g_n^t(\sigma(u))$$

for any $n \in \mathbb{N}$, then $\sigma(u)$ is an eventually periodic point of $g_{1,\infty}$.

Similarly, if $v \in Y$ is an eventually periodic point of $g_{1,\infty}$, it can also be inferred that $\sigma^{-1}(v)$ is an eventually periodic point of $f_{1,\infty}$. Hence, $EP(g_{1,\infty}) \subseteq \sigma(M) = N$.

Now it only remains to prove that $g_{1,\infty}$ is PE on N. In fact, for any $r, s \in N$, if there is an $\varepsilon' > 0$ such that $d_2(g_1^i(r), g_1^i(s)) \leq \varepsilon'$ for any $i \in \mathbb{N}$. Put $p = \sigma^{-1}(r)$, $q = \sigma^{-1}(s)$, then $p, q \in X$ and

$$d_2(g_1^i(r), g_1^i(s)) = d_2(g_1^i(\sigma(p)), g_1^i(\sigma(q))) = d_2(\sigma(f_1^i(p)), \sigma(f_1^i(q))) \leq \varepsilon'$$

for any $i \in \mathbb{N}$. Since X and Y are compact spaces and σ^{-1} is continuous, then an $\varepsilon^* > 0$ exists such that $d_1(f_1^i(p), f_1^i(q)) \leq \varepsilon^*$ for any $i \in \mathbb{N}$. By the PE of $f_{1,\infty}$, $p = q$ holds. Then $r = \sigma(p) = \sigma(q) = s$. Thus, $g_{1,\infty}$ is PE.

Sufficiency can be proven using a similar method.

(3) (Necessity) Since $f_{1,\infty}$ is equicontinuous, for any $\varepsilon > 0$, a $\delta > 0$ exists such that for any $u, v \in X$ with $d(u, v) < \delta$, $d(f_1^i(u), f_1^i(v)) < \varepsilon$ for all $i \in \mathbb{N}_0$. Hence, for any $i \in \mathbb{N}_0$ and every $j \in \mathbb{N}$ with $im + 1 \leq j \leq (i + 1)m$, the inequality $d(f_{im+1}^{j-im}(u), f_{im+1}^{j-im}(v)) < \varepsilon$ holds whenever $d(u, v) < \delta$.

Let $f_{1,\infty}$ be PEX. Assume that A is a dense subset of X containing $EP(f_{1,\infty})$, and that $f_{1,\infty}$ is PE on A. Then for any $u, v \in A$, if an $\varepsilon > 0$ exists such that $d_1(f_1^i(u), f_1^i(v)) \leq \varepsilon$ for all $i \in \mathbb{N}_0$, then $u = v$. Denote $h_{1,\infty} = f_{1,\infty}^{[m]}$. If $f_n^t(x_0) = x_0$ for some $t \in \mathbb{N}_0$ and any $n \in \mathbb{N}$, then $h_n^t(x_0) = f_{(n-1)m+1}^{tm}(x_0) = x_0$ for some $t \in \mathbb{N}_0$ and any $n \in \mathbb{N}$. By the definition of eventually periodic point, one has $EP(f_{1,\infty}^{[m]}) = EP(f_{1,\infty})$. The following proves that $f_{1,\infty}^{[m]}$ is positively expansive on A.

It can be observed that for any $u, v \in A$, any $i \in \mathbb{N}_0$, and any $j \in \mathbb{N}$ with $im + 1 \leq j \leq (i + 1)m$, if $d(f_1^{im}(u), f_1^{im}(v)) < \delta$, then

$$d(f_{im+1}^{j-im}(f_1^{im}(u)), f_{im+1}^{j-im}(f_1^{im}(v))) < \varepsilon.$$

That is

$$d(f_1^j(u), f_1^j(v)) < \varepsilon.$$

By the arbitrariness of i, it follows that $d(f_1^j(u), f_1^j(v)) < \varepsilon$ for all $j \in \mathbb{N}_0$. According to PEX of $f_{1,\infty}$, one has $u = v$.

In summary, for any $u, v \in A$, a $\delta > 0$ exists such that if $d(f_1^{im}(u), f_1^{im}(v)) < \delta$ holds for all $i \in \mathbb{N}_0$, then it must follow that $u = v$. Thus, $f_{1,\infty}^{[m]}$ is PEX.

(Sufficiency) Let $f_{1,\infty}^{[m]}$ be PEX. Assume that $f_{1,\infty}^{[m]}$ is PE on a dense subset A of X containing $EP(f_{1,\infty}^{[m]})$ (i.e., $EP(f_{1,\infty})$). For any $u, v \in A$, if there is an $\varepsilon > 0$ such that $d(f_1^i(u), f_1^i(v)) \leq \varepsilon$ for all $i \in \mathbb{N}_0$, then $d(f_1^{im}(u), f_1^{im}(v)) < \varepsilon$ for all $i \in \mathbb{N}_0$. By the PEX of $f_{1,\infty}^{[m]}$, one has $u = v$. This means that $f_{1,\infty}$ is PEX.

The following two examples are provided to confirm Theorem 3.1 (1) and, in contrast, to show that the converse of this theorem does not always hold.

Example 3.1 Let $X = Y = [0, +\infty]$. Define the maps $\psi(x) = 2x$ for each $x \in X$ and $\omega(r) = 3r$ for each $r \in Y$. Consider two T-VDDSs $f_{1,\infty}$ and $g_{1,\infty}$, where $f_{1,\infty} = \{f_1 = \psi, f_2 = \psi, f_3 = \psi, \ldots\}$ and $g_{1,\infty} = \{g_1 = \omega, g_2 = \omega, g_3 = \omega, \ldots\}$.

For any $x, y \in X$ and any $r, s \in Y$, define the metrics on X and Y by $d_1(x, y) = \left|x - y\right|$ and $d_2(r, s) = \left|r - s\right|$, respectively. Let M denotes the set of all rational numbers in X. Since $\mathrm{EP}(f_{1,\infty}) = \emptyset$, then M is a dense subset of X containing $\mathrm{EP}(f_{1,\infty})$. Similarly, let N denote the set of all rational numbers in Y. Since $\mathrm{EP}(g_{1,\infty}) = \emptyset$, then N is a dense subset of Y containing $\mathrm{EP}(g_{1,\infty})$. For any $p, q \in M$, suppose that an $\varepsilon > 0$ exists such that

$$d_1(f_1^i(p), f_1^i(q)) = 2^i\left|p - q\right| < \varepsilon$$

for all $i \in \mathbb{N}_0$. Letting $i \to \infty$, it follows that $p = q$. Therefore, $f_{1,\infty}$ is PEX. Similarly, it can be verified that $g_{1,\infty}$ is also PEX. For the product system $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$, the metric ρ on $X \times Y$ is defined for any $(j, k), (u, v) \in X \times Y$ by

$$\rho((j, k), (u, v)) = \max\{d_1(j, u), d_2(k, v)\}.$$

The sequence of maps $f_{1,\infty} \times g_{1,\infty}$ on $X \times Y$ is defined as

$$f_{1,\infty} \times g_{1,\infty} = \{f_1 \times g_1 = \psi \times \omega, f_2 \times g_2 = \psi \times \omega, f_3 \times g_3 = \psi \times \omega, \ldots\}.$$

It is clear that the set $M \times N$ is a dense subset of $X \times Y$ containing $\mathrm{EP}(f_{1,\infty} \times g_{1,\infty})$, where $\mathrm{EP}(f_{1,\infty} \times g_{1,\infty}) = \emptyset$. So, $f_{1,\infty} \times g_{1,\infty}$ is PE on $M \times N$, and hence $f_{1,\infty} \times g_{1,\infty}$ is PEX.

Example 3.2 Let $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ be two T-VDDSs, and let $M \times N$ be a dense subset of $X \times Y$. Assume that (a, c) is a k-periodic point of $f_{1,\infty} \times g_{1,\infty}$, where $(a, c) \in M \times N$. That is, for any n, $(f_n^k \times g_n^k)(a, c) = (a, c)$. Then, for any n, $(f_n^k \times g_n^k)((f_n^k \times g_n^k)(a, c)) = (f_n^k \times g_n^k)(a, c)$. It is easy to see that (a, c) is a special eventually periodic point of $f_{1,\infty} \times g_{1,\infty}$. Since $(f_n^k \times g_n^k)((f_n^k \times g_n^k)(a, c)) = (f_n^k \times g_n^k)(a, c)$, then

$$(f_n^k(f_n^k(a)), g_n^k(g_n^k(c))) = (f_n^k(a), g_n^k(c)).$$

That is

$$f_n^k(f_n^k(a)) = f_n^k(a) \quad \text{and} \quad g_n^k(g_n^k(c)) = g_n^k(c),$$

where a and c belong to the dense subsets M and N, respectively.

The argument above implies that points a and c are eventually periodic points of $f_{1,\infty}$ and $g_{1,\infty}$, respectively. However, it does not necessarily imply that $EP(f_{1,\infty}) \subseteq M$ and $EP(g_{1,\infty}) \subseteq N$. In fact, assuming that $g_{1,\infty}$ has another k^*-periodic point b, similar to the previous discussion, b is also an eventually periodic point of $g_{1,\infty}$. However, (a, b) is not an eventually periodic point of $f_{1,\infty} \times g_{1,\infty}$. Hence, it cannot be concluded that b necessarily belongs to N, leading to a potential omission. Thus, $EP(g_{1,\infty}) \subseteq N$ is not necessarily true. This shows that the product system $f_{1,\infty} \times g_{1,\infty}$ is PCE does not necessarily imply 'the two subsystems $f_{1,\infty}$ and $g_{1,\infty}$ are also PCE', thereby demonstrating that the converse of Theorem 3.1 (1) does not hold.

By [21], one can see that PCE exhibits favorable properties under autonomous conditions. But how about the case under time-varying conditions? Specifically, Theorems 3.2 - 3.6 below give some results.

Lemma 3.2 ([7]) If $f_{1,\infty}$ converges uniformly to f, then for any $\varepsilon > 0$ and any $m \in \mathbb{N}$, a $\delta(\varepsilon) > 0$ and an $N(m) \in \mathbb{N}$ exist such that for any pair $u, v \in X$ with $d(u, v) < \delta(\varepsilon)$, $d(f_n^m(u), f_n^m(v)) < \frac{\varepsilon}{2}$ for all $n \geq N(m)$.

Theorem 3.2 Let $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ be two T-VDDSs.

(1) If $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ both are PCE, then so does $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$.

(2) If $f_{1,\infty}$ is topologically conjugate to $g_{1,\infty}$, then $(X, d_1, f_{1,\infty})$ is PCE if and only if $(Y, d_2, g_{1,\infty})$ has PCE.

(3) If $f_{1,\infty}$ is equicontinuous and converges uniformly to f, then $(X, d_1, f_{1,\infty})$ is PCE if and only if $(X, d_1, f_{1,\infty}^{[m]})$ is PCE for any $m \in \mathbb{N}$.

Proof. (1) In this proof, the sets that are PCE should be invariant. By assumption, $f_{1,\infty}$ and $g_{1,\infty}$ are PCE. Let M be any nondegenerate subcontinuum of X. One can find an $\varepsilon_1 > 0$ such that for each M, there is an $n_1 \in \mathbb{N}_0$ satisfying

$$\operatorname{diam} f_1^{n_1}(M) = \sup\{d_1(x_1, y_1) \mid x_1, y_1 \in f_1^{n_1}(M)\} > \varepsilon_1.$$

Similarly, let N be any nondegenerate subcontinuum of Y. One can find an $\varepsilon_2 > 0$ such that for each N, there is an $n_2 \in \mathbb{N}_0$ satisfying

$$\operatorname{diam} g_1^{n_2}(N) = \sup\{d_2(x_2, y_2) \mid x_2, y_2 \in g_1^{n_2}(N)\} > \varepsilon_2.$$

First, it is easy to see that $M \times N$ is a nondegenerate subcontinuum of $X \times Y$, because M and N are nondegenerate subcontinua of X and Y, respectively. Choose $n = \max\{n_1, n_2\}$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then, for any $(x_1, x_2), (y_1, y_2) \in (f_1^n \times g_1^n)(M \times N)$ and $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\operatorname{diam}(f_1^n \times g_1^n)(M \times N) \\
&= \sup\{\rho((p, q), (r, s)) \mid (p, q), (r, s) \in (f_1^n \times g_1^n)(M \times N)\} \\
&= \sup\{\max\{d_1(p, r), d_2(q, s)\} \mid p, r \in f_1^n(M), q, s \in g_1^n(N)\} \\
&> \varepsilon.
\end{aligned}$$

Therefore, by the arbitrariness of M and N, one can find an $\varepsilon > 0$ such that for each $M \times N$, there is an $n \in \mathbb{N}_0$ satisfying $\operatorname{diam}(f_1^n \times g_1^n)(M \times N) > \varepsilon$. Hence, $f_{1,\infty} \times g_{1,\infty}$ is PCE.

(2) (Necessity) Assume that $f_{1,\infty}$ be PCE. Let M be any non-degenerate subcontinuum of X. One can find an $\varepsilon > 0$ such that for each M, there is an $n \in \mathbb{N}_0$ satisfying $\operatorname{diam} f_1^n(M) = \sup\{d_1(u, v) \mid u, v \in f_1^n(M)\} > \varepsilon$. That is, $\operatorname{diam} f_1^n(M) = \sup\{d_1(f_1^n(u), f_1^n(v)) \mid u, v \in M\} > \varepsilon$. Since $f_{1,\infty}$ is topologically conjugate to $g_{1,\infty}$, a homeomorphism $\sigma : X \to Y$ exists such that $\sigma \circ f_n = g_n \circ \sigma$ for all $n \in \mathbb{N}$. Let $N = \sigma(M)$. It can be easily deduced from the fact that σ is a continuous bijection that N is a non-degenerate subcontinuum of Y.

By the uniform continuity of σ^{-1}, for any $\varepsilon > 0$ and $u_2, v_2 \in N$, an $\varepsilon^* > 0$ exists such that $d_2(u_2, v_2) \leq \varepsilon^*$ implies $d_1(\sigma^{-1}(u_2), \sigma^{-1}(v_2)) \leq \varepsilon$. This means that for any $u_1, v_1 \in M$, if $d_1(u_1, v_1) > \varepsilon$, then $d_2(\sigma(u_1), \sigma(v_1)) > \varepsilon^*$. Since $\sigma \circ f_n = g_n \circ \sigma$ for all $n \in \mathbb{N}$, it follows that $f_1^n = \sigma^{-1} \circ g_1^n \circ \sigma$. So, for any $u, v \in M$ and $d_1(f_1^n(u), f_1^n(v)) > \varepsilon$, one has

$$\begin{aligned}
&d_2(\sigma(f_1^n(u)), \sigma(f_1^n(v))) \\
&= d_2(\sigma(\sigma^{-1} \circ g_1^n \circ \sigma(u)), \sigma(\sigma^{-1} \circ g_1^n \circ \sigma(v))) \\
&= d_2(g_1^n(\sigma(u)), g_1^n(\sigma(v))) > \varepsilon^*.
\end{aligned}$$

Let $\sigma(u) = r, \sigma(v) = s$. Then

$$\begin{aligned}\operatorname{diam} g_1^n(N) &= \operatorname{diam} g_1^n(\sigma(M))\\ &= \sup\{d_2(g_1^n(\sigma(u)), g_1^n(\sigma(v))) \mid \sigma(u), \sigma(v) \in \sigma(M)\}\\ &= \sup\{d_2(g_1^n(r), g_1^n(s)) \mid r, s \in N\}\\ &= \sup\{d_2(r, s) \mid r, s \in g_1^n(N)\} > \varepsilon^*.\end{aligned}$$

That is, one can find an $\varepsilon^* > 0$ such that, for any nondegenerate subcontinuum N of Y, there is an $n \in \mathbb{N}_0$ satisfying $\operatorname{diam} g_1^n(N) > \varepsilon^*$. Thus, $g_{1,\infty}$ is PCE.

The proof of sufficiency is similar to that of necessity.

(3) (Necessity) Assume that $f_{1,\infty}$ is PCE and that m is fixed. Let M be any non-degenerate subcontinuum of X. That is, one can find an $\varepsilon > 0$ such that there is an $n \in \mathbb{N}_0$ satisfying $\operatorname{diam} f_1^n(M) = \sup\{d(r, s) \mid r, s \in f_1^n(M)\} = \sup\{d(f_1^n(r), f_1^n(s)) \mid r, s \in M\} > \varepsilon$.

The following will prove that there is an $\varepsilon^* > 0$ such that, for any non-degenerate subcontinuum M of X, there is an $l_0 \in \mathbb{N}$ satisfying

$$\operatorname{diam} f_1^{l_0 m}(M) = \sup\{d(r, s) \mid r, s \in f_1^{l_0 m}(M)\} > \varepsilon^*.$$

Using Lemma 3.1, a positive number $\varepsilon^* < \varepsilon$ and an $N_0 = \max\{N(1), ..., N(m)\}$ exist such that for any $r, s \in X : d(r, s) < \varepsilon^*$, whenever $n > N_0$, the inequality $d(f_n^i(r), f_n^i(s)) < \varepsilon$ holds for all $1 \le i \le m$. For any $n > N_0$, an $i_0 \in \{0, 1, ..., m-1\}$ and an $l \in \mathbb{N}_0$ exist such that $n - i_0 = ml$. For $n - i_0 + 1 > N_0$, since

$$d(f_1^n(r), f_1^n(s)) = d(f_{n-i_0+1}^{i_0}(f_1^{n-i_0}(r)), f_{n-i_0+1}^{i_0}(f_1^{n-i_0}(s))) > \varepsilon,$$

then $d(f_1^{n-i_0}(r), f_1^{n-i_0}(s)) > \varepsilon^*$. So, one can find an l_0 satisfying $l_0 m = n - i_0$, i.e., $d(f_1^{l_0 m}(r), f_1^{l_0 m}(s)) > \varepsilon^*$. Thus, there is an $\varepsilon^* > 0$ such that for any nondegenerate subcontinuum M of X, there is an $l_0 \in \mathbb{N}$ satisfying $\operatorname{diam} f_1^{l_0 m}(M) = \sup\{d(x, y) \mid x, y \in f_1^{l_0 m}(M)\} > \varepsilon^*$. Hence, $f_{1,\infty}^{[m]}$ is PCE.

(Sufficiency) This point is evident.

Now, an example is provided to illustrate Theorem 3.2 (1).

Example 3.3 Let $X = [0, 1]$. The metric d on X is defined by $d(x, y) = \left|x - y\right|$. Then X is a continuum. For any interval $M = [a, b] \subset X$, it is clear that M is a non-degenerate subcontinuum. For each $x \in X$, define a map $\psi(x) = 2x$. Let $f_{1,\infty} = \{f_1 = \psi, f_2 = \psi, f_3 = \psi, \ldots\}$. It is easy to verify that an $\varepsilon > 0$ and a sufficiently large $n \in \mathbb{N}$ exist satisfying

$$\operatorname{diam} f_1^n(M) = \sup\{d(f_1^n(r), f_1^n(s)) \mid r, s \in M\} = \sup\{2^n\left|r - s\right| \mid r, s \in M\} > \varepsilon.$$

So, it follows that $f_{1,\infty}$ is PCE. Let $Y = X$ and $g_{1,\infty} = f_{1,\infty}$. Then $g_{1,\infty}$ is also PCE. Combining the previous arguments, it is not difficult to verify that $f_{1,\infty} \times g_{1,\infty}$ is PCE.

The following give a counterexample to show that the converse of Theorem 3.2 (1) does not hold.

Example 3.4 Let $X = [0, 1]$ and $Y = \{0, 1\}$. $f_{1,\infty}$ and $g_{1,\infty}$ are defined on X and Y, respectively. It is clear that $g_{1,\infty}$ can never be PCE in Y. The reason is that Y fails to satisfy the basic condition of connectedness and therefore contains no nondegenerate subcontinuum. Take $M = [0, 1] \subset X$ and $N = \{0\} \subset Y$. Then $M \times N = [0, 1] \times \{0\}$, which is a nondegenerate subcontinuum of $X \times Y$. Consequently, Here, $f_{1,\infty} \times g_{1,\infty}$ may be PCE. This indicates that the product system is PCE, but this does not imply that the two subsystems also are PCE. Consequently, the converse of Theorem 3.2 (1) does not hold.

Theorem 3.3 Let $(X, f_{1,\infty})$ be a T-VDDS. If $f_{1,\infty}$ is PEX, then $|P(f_{1,\infty})| \leq \aleph_0$, where $\aleph_0$ denotes the cardinality of the set of all natural numbers.

Proof. Assume that $f_{1,\infty}$ is PEX. Let A be a dense subset of X that contains $EP(f_{1,\infty})$, on which $f_{1,\infty}$ is PE. One can then find an $\varepsilon > 0$, for any $p, q \in A$ with $p \neq q$, there is some $i \in \mathbb{N}_0$ such that $d(f_1^i(p), f_1^i(q)) \geq \varepsilon$.

The following proves that $Fix(f_{1,\infty})$ is finite. Suppose that $Fix(f_{1,\infty})$ is not finite. Then there are an accumulation point $x \in Fix(f_{1,\infty})$ and a sequence $\{a_i\}_{i=0}^{+\infty}$ in $Fix(f_{1,\infty})$ which converges to x, where $a_m \neq a_n$ for any $m, n \in \mathbb{N}_0$ with $m \neq n$. There is an $a_t \in \{a_i\}_{i=0}^{+\infty}$ such that $d(a_t, x) < \varepsilon$. From the definition, it is evident that $Fix(f_{1,\infty}) \subseteq EP(f_{1,\infty})$. Consequently

$$a_t, x \in Fix(f_{1,\infty}) \subseteq EP(f_{1,\infty}) \subseteq A \subseteq X.$$

Since $a_t, x \in Fix(f_{1,\infty})$, then $f_n(a_t) = a_t$ and $f_n(x) = x$ for any $n \in \mathbb{N}$. Then $d(f_1^i(a_t), f_1^i(x)) = d(a_t, x) < \varepsilon$ for any $i \in \mathbb{N}_0$. This contradicts the PEX of $f_{1,\infty}$, thereby $Fix(f_{1,\infty})$ must be finite. By Theorem 3.1(3), the PEX of $f_{1,\infty}$ is equivalent to that of $f_{1,\infty}^{[m]}$, where $m \in \mathbb{N}$. Then, $Fix(f_{1,\infty}^{[m]})$ is finite. Because $P(f_{1,\infty}) = \bigcup_{m\in\mathbb{N}} Fix(f_{1,\infty}^{[m]})$, therefore $|P(f_{1,\infty})| \leq \aleph_0$.

We now establish the equivalence of PE and sensitivity under time-varying conditions.

Theorem 3.4 Let $(X, f_{1,\infty})$ be a T-VDDS with a perfect phase space X, and let M be an uncountable dense subset of X. Then $f_{1,\infty}$ is sensitive if and only if $f_{1,\infty}$ is PE on M.

Proof. (Necessity) Let $f_{1,\infty}$ be sensitive. There is an $\varepsilon^* > 0$ such that for any $p \in X$ and any neighborhood $U(p)$ of p, a $q \in U(p)$ and an $i \in \mathbb{N}_0$ with $d(f_1^i(p), f_1^i(q)) > \varepsilon^*$ both exist.

Let $\{U_i\}_{i=1}^{+\infty}$ be a countable base of X. Choose $\varepsilon = \frac{\varepsilon^*}{2}$. Suppose that $f_{1,\infty}$ is PE on a countable dense set $S = \{p_i\}_{i=1}^{+\infty}$ with an expansive constant ε. Choose a point $p \in U_{i_0}$ for an $i_0 \in \mathbb{N}$.

On one hand, if $d(f_1^i(p_1), f_1^i(p)) \leq \varepsilon$ for all $i > 0$. In this situation, one can choose a point $p^{(1)} \in U_{i_0}$. Since $f_{1,\infty}$ is sensitive, then there is an $m_1 \in \mathbb{N}$ such that $d(f_1^{m_1}(p), f_1^{m_1}(p^{(1)})) > \varepsilon^* = 2\varepsilon$. It can be readily shown that $d(f_1^{m_1}(p_1), f_1^{m_1}(p^{(1)})) > \varepsilon$. Choose a closed neighborhood W_1 of $p^{(1)}$ in U_{i_0} such that

$$W_1 \subset U_{i_0} \quad \text{and} \quad d(f_1^{m_1}(W_1), f_1^{m_1}(p_1)) > \varepsilon.$$

On the other hand, if $d(f_1^i(p_1), f_1^i(p)) \leq \varepsilon$ for every $i > 0$ does not hold, then some integer m_1 exists such that $d(f_1^{m_1}(p_1), f_1^{m_1}(p)) > \varepsilon$. Choose the point $p^{(1)} = p$, then $d(f_1^{m_1}(p_1), f_1^{m_1}(p^{(1)})) > \varepsilon$. One can also choose a closed neighborhood W_1 of $p^{(1)}$ in U_{i_0} such that

$$W_1 \subset U_{i_0} \quad \text{and} \quad d(f_1^{m_1}(W_1), f_1^{m_1}(p_1)) > \varepsilon.$$

Now choose $p^{(2)} \in W_1$. Since $f_{1,\infty}$ is sensitive, then there is an $m_2 \in \mathbb{N}$ such that $d(f_1^{m_2}(p^{(2)}), f_1^{m_2}(p_2)) > \varepsilon$. Choose a closed neighborhood W_2 of $p^{(2)}$ such that

$$W_2 \subset W_1 \subset U_{i_0} \quad \text{and} \quad d(f_1^{m_2}(W_2), f_1^{m_2}(p_2)) > \varepsilon$$

for some $m_2 > 0$.

Similarly, choose $p^{(3)} \in W_2$. Since $f_{1,\infty}$ is sensitive, then there is an $m_3 \in \mathbb{N}$ such that $d(f_1^{m_3}(p^{(3)}), f_1^{m_3}(p_3)) > \varepsilon$. Choose a closed neighborhood W_3 of $p^{(3)}$ such that

$$W_3 \subset W_2 \subset W_1 \subset U_{i_0} \quad \text{and} \quad d(f_1^{m_3}(W_3), f_1^{m_3}(p_3)) > \varepsilon$$

for some $m_3 > 0$. In fact, for any $t \in \mathbb{N}$, one can choose a point $p^{(t)}$ in W_t such that

$$W_t \subset W_{t-1} \subset \ldots \subset W_1 \subset U_{i_0} \quad \text{and} \quad d(f_1^{m_t}(W_t), f_1^{m_t}(p_t)) > \varepsilon$$

for some $m_t > 0$. Inductively, a set sequence $U_{i_0} \supset W_1 \supset W_2 \supset \cdots$ is obtained. Since the space X is perfect, one can find a point $q_1 \in \cap_{n=1}^{+\infty} W_n$ such that, for every $j \in \mathbb{N}_0$, there is a positive integer $m_j > 0$ satisfying $d(f_1^{m_j}(q_1), f_1^{m_j}(p_j)) > \varepsilon$. Then $f_{1,\infty}$ is positively expansive on $S \cup \{q_1\}$ with an expansive constant ε. By repeating the procedure, it can be concluded that $f_{1,\infty}$ is PE on a dense subset $M = S \cup \{q_1\} \cup \{q_2\} \cup ....$

(Sufficiency) Assume that M is an uncountable dense subset of X, and that $f_{1,\infty}$ is PE on M. One can then find an $\varepsilon > 0$ such that, for any $p, q \in M$ $(p \neq q)$, some $i \in \mathbb{N}_0$ exists such that $d(f_1^i(p), f_1^i(q)) \geq \varepsilon$. Given the fact that $(X, f_{1,\infty})$ is a perfect space, one can always find a point a in $U(p)$, where $U(p)$ denotes a neighborhood of p. By the PE of $f_{1,\infty}$, there is an $i^* \in \mathbb{N}$ for which $d(f_1^{i^*}(p), f_1^{i^*}(a)) > \varepsilon$. Therefore, $f_{1,\infty}$ is sensitive.

Naturally, we next consider whether the equivalence between PEX and sensitivity can be established under time-varying conditions. Ultimately, the following result is obtained.

Theorem 3.5 Let $(X, f_{1,\infty})$ be a T-VDDS with a perfect phase space X. Then $f_{1,\infty}$ is PEX if and only if $f_{1,\infty}$ is sensitive.

Proof. (Necessity) By assumption, $f_{1,\infty}$ is PEX. Let M be a dense subset of X that contains $EP(f_{1,\infty})$ and $f_{1,\infty}$ is PE on M. Then there is an $\varepsilon > 0$ such that, for any $u, v \in M \subseteq X(u \neq v)$, there is some $i \in \mathbb{N}_0$ that satisfies $d(f_1^i(u), f_1^i(v)) > \varepsilon$. Given the fact that $(X, f_{1,\infty})$ is a perfect space, one can always find a point r in $M \cap U(u)$, where $U(u)$ denotes a neighborhood of u. By the PE of $f_{1,\infty}$, there is some $i^* \in \mathbb{N}_0$ that satisfies $d(f_1^{i^*}(u), f_1^{i^*}(r)) > \varepsilon$. Therefore, $f_{1,\infty}$ is sensitive.

(Sufficiency) The proof can be obtained by combining Theorem 3.1 with the conditions that $EP(f) = \emptyset$ and X is infinite.

Theorem 3.6 Let $(X, f_{1,\infty})$ be a T-VDDS. If $f_{1,\infty}$ is PCE, then $f_{1,\infty}$ is sensitive.

Proof. Let $f_{1,\infty}$ be PCE. There is an $\varepsilon > 0$ such that for any non-degenerate subcontinuum M of X, there is an $n \in \mathbb{N}_0$ satisfying

$$\text{diam}\, f_1^n(M) = sup\{d(p, q) \mid p, q \in f_1^n(M)\} = sup\{d(f_1^n(p), f_1^n(q)) \mid p, q \in M\} > \varepsilon.$$

Let u be arbitrary point in X, and let U be an open set containing u. There is then a non-degenerate subcontinuum N such that $u \in N \subset U$. Since $f_{1,\infty}$ is PCE, then an $n^* \in \mathbb{N}_0$ exists such that

$$\text{diam}\, f_1^{n^*}(N) = sup\{d(f_1^{n^*}(r), f_1^{n^*}(s)) \mid r, s \in N\} > \varepsilon.$$

Let $\varepsilon^* = \frac{\varepsilon}{2}$. One can choose a point $v \in N \subset U$ such that $d(f_1^{n^*}(u), f_1^{n^*}(v)) < \frac{\varepsilon}{2} = \varepsilon^*$. Therefore, $f_{1,\infty}$ is sensitive.

By the definitions, if $f_{1,\infty}$ is a homeomorphism map sequence on X, the concept of PCE for $f_{1,\infty}$ becomes equivalent to CE. Combining this with Theorem 3.6, the following corollary can be readily derived.

Corollary 3.1 Let $(X, f_{1,\infty})$ be a T-VDDS, where $f_{1,\infty}$ is a sequence of homeomorphisms. If $f_{1,\infty}$ is CE, then $f_{1,\infty}$ is sensitive.

4. The $\mathcal{M}^{\alpha}$-shadowing property and equicontinuity

Sakai [13] first introduced the definition of the $\underline{d}$-shadowing property, which was later extended by Oprocha et al. [27] in 2014 through the introduction of the $\mathcal{M}^{\alpha}$-shadowing property($\mathcal{M}^{\alpha}$-SP). In this section, we discuss the relationship between the $\mathcal{M}^{\alpha}$-SP and equicontinuity.

Define $\mathcal{M}^{\alpha}Sh(f_{1,\infty})$ as the set of $\mathcal{M}^{\alpha}$-shadowable points of $f_{1,\infty}$. Meanwhile, for a given ε, define $S(f_{1,\infty},\varepsilon,q)$ as the set of all δ corresponding to ε via the relation defined by (2.1), namely that
$S(f_{1,\infty},\varepsilon,q)=\{\delta>0\,|$ for the $\varepsilon>0$, and the δ ergodic pseudo-orbit of $f_{1,\infty}$ through q satisfies (2.1)$\}$.

Before starting the proof, it is necessary to prove the following lemma.

Lemma 4.1 Let $(X,f_{1,\infty})$ be a T-VDDS. Then $f_{1,\infty}$ has the $\mathcal{M}^{\alpha}$-SP if and only if $\mathcal{M}^{\alpha}Sh(f_{1,\infty})\neq\emptyset$.

Proof. (Sufficiency) Assume that $\mathcal{M}^{\alpha}Sh(f_{1,\infty})\neq\emptyset$. Take a point $p\in\mathcal{M}^{\alpha}Sh(f_{1,\infty})$. For a given $\varepsilon>0$, one can choose a $\delta>0$ and a point $a\in X$ such that any δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ through p satisfies

$$\limsup_{n\to\infty}\frac{1}{n}|\{0\le i<n: d(f_1^i(a),a_i)<\varepsilon\}|>\alpha,\ \ \alpha\in[0,1).$$

Let q be an arbitrary point in X that is distinct from p, and let $\{u_i\}_{i=0}^{+\infty}$ be a δ ergodic pseudo-orbit of $f_{1,\infty}$ through q (i.e., $u_0=q$). Then

$$\lim_{n\to\infty}\frac{1}{n}|\{0\le i<n: d(f_{i+1}(u_i),u_{i+1})<\delta\}|=1.$$

Construct a sequence $\{v_i\}_{i=0}^{+\infty}$, where $v_0=p, v_1=u_1,\cdots,v_i=u_i,\cdots$. It follows that $\{v_i\}_{i=0}^{+\infty}$ is also a δ ergodic pseudo-orbit of $f_{1,\infty}$ through p. Since $p\in\mathcal{M}^{\alpha}Sh(f_{1,\infty})$, then a point $r\in X$ exists such that the δ ergodic pseudo-orbit $\{v_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ through p satisfies

$$\limsup_{n\to\infty}\frac{1}{n}|\{0\le i<n: d(f_1^i(r),v_i)<\varepsilon\}|>\alpha,\ \ \alpha\in[0,1).$$

Therefore

$$\limsup_{n\to\infty}\frac{1}{n}|\{0\le i<n: d(f_1^i(r),u_i)<\varepsilon\}|>\alpha,\ \ \alpha\in[0,1).$$

Since q and ε are chosen arbitrarily, it follows that for any $\varepsilon>0$, a $\delta>0$ and a point $r\in X$ exists such that any δ ergodic pseudo-orbit $\{u_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ satisfies

$$\limsup_{n\to\infty}\frac{1}{n}|\{0\le i<n: d(f_1^i(r),u_i)<\varepsilon\}|>\alpha,\ \ \alpha\in[0,1).$$

Therefore, $f_{1,\infty}$ has the $\mathcal{M}^{\alpha}$-SP.

(Necessity) This conclusion is evident.

Theorem 4.1 Let $(X,f_{1,\infty})$ be a T-VDDS, where $f_{1,\infty}$ is equicontinuous. Then, $f_{1,\infty}$ cannot possess the $\mathcal{M}^{\alpha}$-SP.

Proof. Suppose that $f_{1,\infty}$ has the $\mathcal{M}^{\alpha}$-SP. By Lemma 4.1, $\mathcal{M}^{\alpha}Sh(f_{1,\infty})\neq\emptyset$. Choose a point $x\in\mathcal{M}^{\alpha}Sh(f_{1,\infty})$ and a point $y\in X$.

Let $\varepsilon\in(0,\frac{1}{2})$. For $\frac{\varepsilon}{2}>0$, a $\delta>0$ and a point $p\in X$ exist such that any δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ through x satisfies

$$\limsup_{n\to\infty}\frac{1}{n}|\{0\le i<n: d(f_1^i(p),a_i)<\frac{\varepsilon}{2}\}|>\alpha,\ \ \alpha\in[0,1).$$

Let $\{t_i\}_{i=0}^{+\infty}$ be an increasing sequence of natural numbers, where $t_0 = 0$ and $t_{i+1} \gg t_i$ for all $i \in \mathbb{N}_0$. This sequence partitions the set of non-negative integers into two subsets N_1 and N_2, where

$$N_1 = \{0, 1, ..., t_1\} \cup \{t_2 + 1, t_2 + 2, ..., t_3\} \cup \cdots$$

and

$$N_2 = \{t_1 + 1, t_1 + 2, ..., t_2\} \cup \{t_3 + 1, t_3 + 2, ..., t_4\} \cup \cdots.$$

For all $i \in \mathbb{N}_0$, let

$$a_i = \begin{cases} f_1^i(x) & i \in N_1; \\ f_1^i(y) & i \in N_2. \end{cases}$$

It is straightforward to see that the sequence

$$\{a_i\}_{i=0}^{+\infty} = \{f_1^0(x), f_1^1(x), ..., f_1^{t_1}(x), f_1^{t_1+1}(y), ..., f_1^{t_2}(y), f_1^{t_2+1}(x), ...\}$$

is a δ ergodic pseudo-orbit of $f_{1,\infty}$ through x. A point $p \in X$ exists such that the δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ through x satisfies

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f_1^i(p), a_i) < \frac{\varepsilon}{2}\}| > \alpha, \quad \alpha \in [0, 1).$$

Therefore, it can be concluded that there are infinitely many indices $i \in N_1$ and $j \in N_2$ such that

$$d(f_1^i(p), a_i) < \frac{\varepsilon}{2} \quad \text{and} \quad d(f_1^j(p), a_j) < \frac{\varepsilon}{2},$$

respectively. For each $i \in \mathbb{N}_0$, define $\varepsilon_i = \frac{1}{2+i}$. For each $\varepsilon_i > 0$, there are a $p_i \in X$ and strictly increasing sequences $\{r_n^i\}_{n=0}^{+\infty}$ and $\{s_n^i\}_{n=0}^{+\infty}$ such that the δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ satisfies

$$d(f_1^{r_n^i}(p_i), a_{r_n^i}) < \frac{\varepsilon_i}{2} \quad \text{and} \quad d(f_1^{s_n^i}(p_i), a_{s_n^i}) < \frac{\varepsilon_i}{2},$$

where $a_{r_n^i} = f_1^{r_n^i}(x)$ and $a_{s_n^i} = f_1^{s_n^i}(y)$ for all $i \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$. Without loss of generality, assume that $r_n^i < s_n^i$ for all $i \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$.

On the basis of the construction of $\{p_i\}_{i=1}^{\infty}$, $p_i \to p$ for some $p \in X$. Fix an $m \ge 1$. By the equicontinuity of $f_{1,\infty}$, a $t_m \ge m$ exists such that $d(f_1^n(p), f_1^n(p_i)) < \frac{\varepsilon_m}{2}$ for all $i \ge t_m$ and all $n \in \mathbb{N}$. Choose an $n \in \mathbb{N}_0$ such that $r_n^{t_m} \ge t_m$, then

$$d(f_1^{r_n^{t_m}}(p), a_{r_n^{t_m}}) \le d(f_1^{r_n^{t_m}}(p), f_1^{r_n^{t_m}}(p_{t_m})) + d(f_1^{r_n^{t_m}}(p_{t_m}), a_{r_n^{t_m}}) < \frac{\varepsilon_m}{2} + \frac{\varepsilon_m}{2} = \varepsilon_m.$$

By the arbitrariness of m, there is an increasing sequence $\{k_n\}_{n=0}^{+\infty}$ such that $d(f_1^{k_n}(p), f_1^{k_n}(x)) < \varepsilon_m$. Using a similar method, one can likewise construct an increasing sequence $\{l_n\}_{n=0}^{+\infty}$ satisfying $d(f_1^{l_n}(p), f_1^{l_n}(y)) < \varepsilon_m$.

Note that $k_n \le l_n$ for all $n \in \mathbb{N}_0$. Since $x \in \mathcal{M}^\alpha Sh(f_{1,\infty})$, for any $\varepsilon > 0$, a $\delta > 0$ and a point $p \in X$ exist such that the δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ through x satisfies

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f_1^i(p), a_i) < \frac{\varepsilon}{2}\}| > \alpha, \quad \alpha \in [0, 1).$$

Then, there is an $\varepsilon^* > 0$ such that $d(f_1^i(p), a_i) < \varepsilon^*$ holds for all $i \in \mathbb{N}_0$. Specifically, taking $i = 0$, one has $d(f_1^0(p), a_0) = d(p, x) < \varepsilon^*$. By the equicontinuity of $f_{1,\infty}$, for any $\varepsilon_m > 0$, there is an $\varepsilon^* > 0$ such that for any $x, p \in X$: $d(p, x) < \varepsilon^*$, $d(f_1^n(p), f_1^n(x)) < \varepsilon_m$ holds for each $n \in \mathbb{N}_0$. Then

$$d(f_1^{l_n}(x), f_1^{l_n}(y)) \leq d(f_1^{l_n}(x), f_1^{l_n}(p)) + d(f_1^{l_n}(p), f_1^{l_n}(y)) < \varepsilon_m + \varepsilon_m < 2\varepsilon_m.$$

By the arbitrariness of m

$$\liminf_{n\to\infty} d(f_1^n(x), f_1^n(y)) = 0.$$

Thus (x, y) is a non-trivial proximal pair. Since a sequence of equicontinuous maps cannot admit non-trivial proximal pairs, this leads to a contradiction of the assumption. Therefore, $f_{1,\infty}$ does not possess the $\mathcal{M}^\alpha$-SP.

Next, a necessary and sufficient condition will be given for the limit map f of $f_{1,\infty}$ to have the $\mathcal{M}^\alpha$-SP. A lemma needs to be introduced first to assist in the proof.

Lemma 4.2 Assume that $f_{1,\infty}$ converges uniformly to f. For each $l \in \mathbb{N}$, define $(f_n)^l = \underbrace{f_n \circ \cdots \circ f_n}_{l \text{ times}}$. Then the sequence $\{(f_n)^l\}_{n=1}^\infty$ converges uniformly to f^l.

Proof. By the uniform convergence of $f_{1,\infty}$, for every $\frac{\varepsilon}{2} > 0$, an $n_0 \in \mathbb{N}$ exists such that $d(f_n(r), f(r)) < \frac{\varepsilon}{2}$ holds for all $r \in X$ and $n \geq n_0$. Since $f_{1,\infty}$ is uniformly continuous, one has

$$\begin{aligned} d((f_n)^2(r), f^2(r)) &= d(f_n(f_n(r)), f(f(r))) \\ &\leq d(f_n(f_n(r)), f(f_n(r))) + d(f(f_n(r)), f(f(r))) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that $\{(f_n)^2\}_{n=1}^\infty$ converges uniformly to f^2. Assume that the sequence $\{(f_n)^l\}_{n=1}^\infty$ converges uniformly to f^l. Then

$$\begin{aligned} d((f_n)^{l+1}(r), f^{l+1}(r)) &= d(f_n((f_n)^l(r)), f(f^l(r))) \\ &\leq d(f_n((f_n)^l(r)), f((f_n)^l(r))) + d(f((f_n)^l(r)), f(f^l(r))) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $\{(f_n)^{l+1}\}_{n=1}^\infty$ converges uniformly to f^{l+1}.

Therefore, by mathematical induction, the conclusion holds.

Theorem 4.2 Let $(X, d, f_{1,\infty})$ be a T-VDDS, where $f_{1,\infty}$ converges uniformly to f. Then f has the $\mathcal{M}^\alpha$-SP if and only if for any $\varepsilon > 0$, there is a $q \in X$ such that

$$\cup_{t\geq 1} \cap_{n\geq t} S(f_n, \varepsilon, q) \neq \emptyset.$$

Proof. (Necessity) Assume that f possesses the $\mathcal{M}^\alpha$-SP. Choose an $\varepsilon > 0$. A δ with $0 < \delta < \frac{\varepsilon}{2}$ and a point $p \in X$ exist such that any δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of f satisfies

$$\limsup_{n\to\infty} \frac{1}{n} |\{0 \leq i < n : d(f^i(p), a_i) < \frac{\varepsilon}{2}\}| > \alpha, \quad \alpha \in [0, 1).$$

Since $f_{1,\infty}$ converges uniformly to f, it follows from Lemma 4.2 that for any $l \in \mathbb{N}$, the sequence $\{(f_t)^l\}_{t=1}^{+\infty}$ converges uniformly to f^l. Hence, for any $\frac{\delta}{2} > 0$, there is a t_0 such that $d((f_t)^l(x), f^l(x)) < \frac{\delta}{2}$ for all $t > t_0$ and $x \in X$. Fix $t > t_0$ and a point $q \in X$. Let $\{q_i\}_{i=0}^{+\infty}$ be a $\frac{\delta}{2}$ ergodic pseudo-orbit of f_t through q. Then

$$\lim_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f_t(q_i), q_{i+1}) < \frac{\delta}{2}\}| = 1.$$

Since

$$d(f(q_i), q_{i+1}) \le d(f(q_i), f_t(q_i)) + d(f_t(q_i), q_{i+1}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

then it is easily concluded that $\{q_i\}_{i=0}^{+\infty}$ is a δ ergodic pseudo-orbit of f through q. Since f possesses the $\mathcal{M}^\alpha$-SP, then a point $c \in X$ exists such that $\{q_i\}_{i=0}^{+\infty}$ satisfies

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f^i(c), q_i) < \frac{\varepsilon}{2}\}| > \alpha, \ \ \alpha \in [0, 1).$$

Since

$$d((f_t)^i(c), q_i) \le d((f_t)^i(c), f^i(c)) + d(f^i(c), q_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

it follows that for any $\varepsilon > 0$, a $\frac{\delta}{2} > 0$ and a point $c \in X$ exist such that the $\frac{\delta}{2}$ ergodic pseudo-orbit $\{q_i\}_{i=0}^{+\infty}$ of f_t through q satisfies

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d((f_t)^i(c), q_i) < \varepsilon\}| > \alpha, \ \ \alpha \in [0, 1).$$

Note that t is chosen arbitrarily, and thus one can get

$$\frac{\delta}{2} \in \bigcup_{t\ge 1} \bigcap_{n\ge t} S(f_n, \varepsilon, q) \neq \varnothing.$$

(Sufficiency) Assume that for any $\varepsilon > 0$, there is a point $q \in X$ such that $\bigcup_{t\ge1} \bigcap_{n\ge t} S(f_n, \frac{\varepsilon}{2}, q) \neq \varnothing$. Take any $T \in \mathbb{N}$ such that $\bigcap_{n\ge T} S(f_n, \frac{\varepsilon}{2}, q) \neq \varnothing$. Arbitrarily choose a $\delta \in S(f_n, \frac{\varepsilon}{2}, q)$. Since $f_{1,\infty}$ converges uniformly to f, it follows from Lemma 4.2 that for any $l \in \mathbb{N}$, the sequence $\{(f_t)^l\}_{t=1}^{+\infty}$ converges uniformly to f^l. Hence, for any $\frac{\delta}{2} > 0$, there is a $t_0 > T$ such that $d((f_t)^l(x), f^l(x)) < \frac{\delta}{2}$ for all $t \ge t_0$ and $x \in X$. Let $\{q_i\}_{i=0}^{+\infty}$ be a $\frac{\delta}{2}$ ergodic pseudo-orbit of f through q. Then

$$\lim_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f(q_i), q_{i+1}) < \frac{\delta}{2}\}| = 1.$$

Since

$$d(f_{t_0}(q_i), q_{i+1}) \le d(f_{t_0}(q_i), f(q_i)) + d(f(q_i), q_{i+1}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

then $\{q_i\}_{i=0}^{+\infty}$ is a δ ergodic pseudo-orbit of f_{t_0} through q. Since $\delta \in \bigcap_{n\ge T} S(f_n, \frac{\varepsilon}{2}, q)$, then there is a point $c \in X$ such that $\{q_i\}_{i=0}^{+\infty}$ satisfies

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d((f_{t_0})^i(c), q_i) < \frac{\varepsilon}{2}\}| > \alpha, \ \ \alpha \in [0, 1).$$

Since

$$d(f^i(c), q_i) \le d(f^i(c), (f_{t_0})^i(c)) + d((f_{t_0})^i(c), q_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

the point q is an $\mathcal{M}^\alpha$-shadowable point of f. Therefore, $\mathcal{M}^\alpha Sh(f) \neq \varnothing$. By Lemma 4.1 (the case where $f_n = f$ for all $n \in \mathbb{N}$), it can be concluded that f possesses the $\mathcal{M}^\alpha$-SP.

5. Conclusions

The preservation of chaotic properties in T-VDDSs is investigated. The focus is on the behavior of PEX and PCE systems under product operators, topological conjugacy, and composite operators. The results show that if both subsystems are PEX (or PCE), then their product system is also. PEX (or PCE) properties are preserved under topological conjugacy and composite operators. PEX systems are proven to have only finitely many periodic points. Additionally, an equivalence between PE (PEX, PCE) and sensitivity is established. Futhermore, it is demonstrated that the $\mathcal{M}^{\alpha}$-SP and equicontinuity are mutually exclusive. A necessary and sufficient condition is provided for the limit map f of a system $f_{1,\infty}$ to possess the $\mathcal{M}^{\alpha}$-SP. It can be noted that the discussion of the preservation of PEX and PCE under product operators is under the restrictions of equicontinuity and uniform convergence. In future research, similar results under weaker conditions or chaotic properties in non-compact metric spaces will be achieved.

Author contributions

J.Z.Zhao: Conceptualization, validation, writing - original draft; T.L.Lu: Writing - review and editing, supervision, funding acquisition; Y. Z.Zhang: Formal analysis, investigation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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