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Research article

Multiple sums for cyclically symmetric products of binomial coefficients

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Abstract: Carlitz' multiple sums involving cyclically symmetric products of binomial coefficients are extended by introducing weight monomials of m-variables. By making use of recursive reductions and the Lagrange expansion formula, we determine, in closed form, the related rational generating functions that significantly generalize the classical result discovered by Carlitz in 1965. Several novel applications are presented as consequences.

Keywords: circular sum; generating function; binomial coefficient; Lucas number; Fibonacci number; Lagrange expansion formula
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1. Introduction and outline

In mathematics, physics, and applied sciences, Fibonacci and Lucas numbers play important roles (cf. Koshy [14]). They are defined as follows:

- Initial conditions

$$\mathrm{F}_0 = 0,\ \mathrm{F}_1 = 1 \quad \text{and} \quad \mathrm{L}_0 = 2,\ \mathrm{L}_1 = 1.$$

- Recurrence relations ($n \geq 2$)

$$\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2} \quad \text{and} \quad \mathrm{L}_n = \mathrm{L}_{n-1} + \mathrm{L}_{n-2}.$$

- Generating functions

$$\sum_{n=0}^{\infty} y^n \mathrm{F}_n = \frac{y}{1-y-y^2} \quad \text{and} \quad \sum_{n=0}^{\infty} y^n \mathrm{L}_n = \frac{2-y}{1-y-y^2}.$$

- Binet formulae ($\alpha, \beta = \frac{1\pm\sqrt{5}}{2}$)

$$\mathrm{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{L}_n = \alpha^n + \beta^n.$$

- Explicit formulae

$$\mathrm{F}_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k-1}{k} \quad \text{and} \quad \mathrm{L}_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k}.$$

Here and forth, $\lfloor x \rfloor$ and $\lceil x \rceil$ stand, respectively, for the greatest integer $\leq x$ and the smallest integer $\geq x$ for a real number x.

It is classically well-known (cf. Koshy [14, Example 3.1]) that F_{m+2} (and L_m) counts the number of subsets (including the null set) of the set $[1, m]$ (consisting of the first m natural numbers) such that consecutive numbers in $[1, m]$ are not allowed if these m elements are arranged in a line (and in a circle).

There exist numerous binomial sums involving Fibonacci and Lucas numbers in the mathematical literature (for example, [1,4,15] on closed formulae, and [2,13] about k-recursive sums). However, the related multiple sums are rare.

Denote by $\mathbb{N}$ the set of natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, consider the binomial matrix $\left[\binom{i}{n-j}\right]_{0\leq i,j\leq n}$. When determining its characteristic polynomial

$$\sum_{k=0}^{n+1} (-1)^{\binom{k+1}{2}} \binom{n+1}{k}_F x^{1+n-k}, \text{ where } \binom{m}{k}_F = \frac{\mathrm{F}_m \mathrm{F}_{m-1} \cdots \mathrm{F}_{m-k+1}}{\mathrm{F}_1 \mathrm{F}_2 \cdots \mathrm{F}_k}.$$

Carlitz [6] discovered the following beautiful formula for the circular sums:

$$\sum_{\mathbf{k}\in[0,n]^m} \binom{n-k_1}{k_m} \prod_{i=1}^{m-1} \binom{n-k_{i+1}}{k_i} = \frac{\mathrm{F}_{mn+m}}{\mathrm{F}_m}.$$

For different proofs and related works, the reader can consult [3,5,17] for the matrix approach, [18] for the induction principle, and [7,10,19] for "the recursive construction method".

The objective of the present paper is to investigate its generalization weighted by characterizing monomials (which serve as encoding the summands)

$$\Omega_n^m(x_\iota | [1, m]) = \sum_{\mathbf{k}\in[0,n]^m} x_m^{k_m} \binom{n-k_1}{k_m} \prod_{i=1}^{m-1} \binom{n-k_{i+1}}{k_i} x_i^{k_i},$$

where for $i, j \in \mathbb{N}_0$ subject to $i \leq j$, we use notations, for brevity, $[i, j]$ for the integers from i to j and $(x_\iota | [i, j])$ for the variables $\{x_\iota\}_{\iota=i}^{j}$. The generating function is shown to be related to Fibonacci and Lucas polynomials through subset enumerations without consecutive elements. This provides a deep generalization of the aforementioned formula due to Carlitz [6].

As preliminaries, we introduce, in the next section, the enumerative functions of the Fibonacci and Lucas subsets by identifying each subset with its enumerator (a multivariate monomial). Then in Section 3, the rational generating function for multiple sums $\Omega_n^m(x_\iota | [1, m])$ will be established by making use of recursive reductions and the Lagrange expansion formula. When the m variables $(x_\iota|[1,m])$ are cyclically generated by cycles $\langle x\rangle$, $\langle u,v\rangle$ and $\langle u,v,w\rangle$ of lengths "1", "2" and "3", respectively, we shall examine, in Section 4, three classes of corresponding multiples sums, that yield several remarkable summation formulae, including a few known ones. Finally, the paper will end with Section 5, where further challenging problems are proposed.

2. Subsets of $\boxed{\mathfrak{M} = (x_\iota|[1,m])}$

Let $\mathfrak{M}$ be the finite set $(x_\iota|[1,m])$. We are going to enumerate the subsets (including the empty one) of $\mathfrak{M}$, when the elements of $\mathfrak{M}$ are arranged in a line and in a circle. For any subset $\mathrm{S}\subset\mathfrak{M}$, denote its enumerator by

$$\sigma(\mathrm{S}) = \begin{cases} 1, & \mathrm{S}=\emptyset; \\ \prod_{x\in\mathrm{S}} x, & \mathrm{S}\neq\emptyset. \end{cases}$$

2.1. Subsets of linear $\boxed{\mathfrak{M} = (x_\iota|[1,m])}$

When the elements of $\mathfrak{M}$ are arranged in a line, consider the subsets $\mathfrak{F}(\mathfrak{M})$ (including the null set) such that in each given subset $\mathrm{S}\subset\mathfrak{F}(\mathfrak{M})$, the consecutive elements in the linear $\mathfrak{M}$ are not allowed. Let $\mathcal{F}_m$ stand for the enumerative function

$$\mathcal{F}_m := \mathcal{F}_m(\mathfrak{M}) = \mathcal{F}_m(x_\iota|[1,m]) = \sum_{\mathrm{S}\in\mathfrak{F}(\mathfrak{M})}\sigma(\mathrm{S}).$$

Then it follows directly from the definition that

$$\begin{aligned}
\mathcal{F}_0 &= 1,\\
\mathcal{F}_1 &= 1+x_1,\\
\mathcal{F}_2 &= 1+x_1+x_2,\\
\mathcal{F}_3 &= 1+x_1+x_2+x_3+x_1x_3,\\
\mathcal{F}_4 &= 1+x_1+x_2+x_3+x_4+x_1x_3+x_1x_4+x_2x_4,\\
\mathcal{F}_5 &= 1+x_1+x_2+x_3+x_4+x_5+x_1x_3+x_1x_4+x_1x_5+x_2x_4+x_2x_5+x_3x_5+x_1x_3x_5,\\
\mathcal{F}_6 &= 1+x_1+x_2+x_3+x_4+x_5+x_6+x_1x_3+x_1x_4+x_1x_5+x_1x_6+x_2x_4\\
&\quad +x_2x_5+x_2x_6+x_3x_5+x_3x_6+x_4x_6+x_1x_3x_5+x_1x_3x_6+x_1x_4x_6+x_2x_4x_6.
\end{aligned}$$

For each subset $\mathrm{S}\in\mathfrak{F}(\mathfrak{M})$, we have $x_m\in\mathrm{S}$ or $x_m\notin\mathrm{S}$. Then all the subsets $\mathfrak{F}(\mathfrak{M})$ subject to $x_m\notin\mathrm{S}$ are enumerated by $\mathcal{F}_{m-1} := \mathcal{F}_{m-1}(x_\iota|[1,m-1])$ and the remaining subsets of $\mathfrak{F}(\mathfrak{M})$ with $x_m\in\mathrm{S}$ by $x_m\mathcal{F}_{m-2} := x_m\mathcal{F}_{m-2}(x_\iota|[1,m-2])$. Consequently, we find the following recurrence relation:

$$\mathcal{F}_m = \mathcal{F}_{m-1} + x_m\mathcal{F}_{m-2} \quad \text{for} \quad m\geq 2. \tag{2.1}$$

This corresponds exactly to the recursion of the classical Fibonacci sequence.

2.2. Subsets of circular $\mathfrak{M} = (x_\iota|[1, m])$

When the elements of $\mathfrak{M}$ are arranged in a circle, consider the subsets $\mathfrak{L}(\mathfrak{M})$ (including the empty set) such that for each given subset $\mathrm{S} \in \mathfrak{L}(\mathfrak{M})$, any pair of its elements are not consecutive in the circular $\mathfrak{M}$. Let $\mathcal{L}_m$ stand for the enumerative function

$$\mathcal{L}_m := \mathcal{L}_m(\mathfrak{M}) = \mathcal{L}_m(x_\iota|[1, m]) = \sum_{\mathrm{S} \in \mathfrak{L}(\mathfrak{M})} \sigma(\mathrm{S}).$$

Define for convention the initial two functions:

$$\mathcal{L}_0 = 2 \quad \text{and} \quad \mathcal{L}_1 = 1.$$

Then it is not hard to determine the next five functions:

$$\begin{aligned}
\mathcal{L}_2 &= 1 + x_1 + x_2, \\
\mathcal{L}_3 &= 1 + x_1 + x_2 + x_3, \\
\mathcal{L}_4 &= 1 + x_1 + x_2 + x_3 + x_4 + x_1x_3 + x_2x_4, \\
\mathcal{L}_5 &= 1 + x_1 + x_2 + x_3 + x_4 + x_5 + x_1x_3 + x_1x_4 + x_2x_4 + x_2x_5 + x_3x_5, \\
\mathcal{L}_6 &= 1 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_4 \\
&\quad + x_2x_5 + x_2x_6 + x_3x_5 + x_3x_6 + x_4x_6 + x_1x_3x_5 + x_2x_4x_6.
\end{aligned}$$

For each subset $\mathrm{S} \in \mathfrak{L}(\mathfrak{M})$, it holds that $x_m \in \mathrm{S}$ or $x_m \notin \mathrm{S}$. Then all the subsets $\mathfrak{L}(\mathfrak{M})$ subject to $x_m \notin \mathrm{S}$ are enumerated by $\mathcal{F}_{m-1}(x_\iota|[1, m-1])$ and the remaining subsets of $\mathfrak{L}(\mathfrak{M})$ with $x_m \in \mathrm{S}$ by $x_m\mathcal{F}_{m-3}(x_\iota|[2, m-2])$. Consequently, we deduce the following expression:

$$\mathcal{L}_m(x_\iota|[1, m]) = \mathcal{F}_{m-1}(x_\iota|[1, m-1]) + x_m\mathcal{F}_{m-3}(x_\iota|[2, m-2]). \tag{2.2}$$

Alternatively, the subsets of $\mathfrak{L}(\mathfrak{M})$ subject to $x_m \notin \mathrm{S}$ are enumerated by

$$\mathcal{L}_{m-1}(x_\iota|[1, m-1]) + x_1x_{m-1}\mathcal{F}_{m-5}(x_\iota|[3, m-3]),$$

and the remaining subsets of $\mathfrak{L}(\mathfrak{M})$ with $x_m \in \mathrm{S}$ by

$$x_m\mathcal{L}_{m-2}(x_\iota|[1, m-2]) - x_1x_m\mathcal{F}_{m-5}(x_\iota|[3, m-3]).$$

Consequently, we find the following relation, which differs from the usual recursion of the classical Lucas sequence:

$$\begin{aligned}
\mathcal{L}_m &= \mathcal{L}_{m-1} + x_m\mathcal{L}_{m-2} \qquad\qquad\qquad\qquad (m > 2) \\
&+ \begin{cases} 0, & m = 3; \\ x_1(x_3 - x_4), & m = 4; \\ x_1(x_{m-1} - x_m)\mathcal{F}_{m-5}(x_\iota|[3, m-3]), & m > 4. \end{cases}
\end{aligned} \tag{2.3}$$

These polynomials $\mathcal{F}_m$ and $\mathcal{L}_m$ will serve as natural combinatorial tools in the recursive construction for the generating function of multiple circular sums $\Omega_n^m(\mathfrak{M})$. When $x_1 = x_2 = \cdots = x_m = 1$, it is almost obvious that $\mathcal{F}_m$ and $\mathcal{L}_m$ become the usual Fibonacci and Lucas numbers F_{m+2} and L_m, respectively.

3. Generating functions of multiple sums $\Omega_n^m(\mathfrak{M})$

By making use of recursive reductions and the Lagrange expansion formula, we shall establish the rational generating function for multiple circular sums $\Omega_n^m(\mathfrak{M})$. We observe that there exists an unexpected hidden connection between the multiple circular sums $\Omega_n^m(\mathfrak{M})$ and the enumerative functions $\mathcal{L}_m$, which is crucial in our recursive construction of the generating function (Theorem 2).

3.1. Another Fibonacci-like sequence

In order to proceed smoothly with our derivation, we introduce another Fibonacci-like sequence $\mathbf{F}_m$ (a variant of $\mathcal{F}_m$) defined by the recurrence relation

$$\mathbf{F}_m = \mathbf{F}_{m-1} + x_m\mathbf{F}_{m-2} : \quad \mathbf{F}_m = \mathcal{F}_m(x_1 \to Tx_1)$$

with the initial values below:

$$\mathbf{F}_0 = 1 \quad \text{and} \quad \mathbf{F}_1 = 1 + Tx_1.$$

The next five terms are given by

$$\begin{aligned}
\mathbf{F}_2 &= 1 + x_2 + Tx_1,\\
\mathbf{F}_3 &= 1 + x_2 + x_3 + Tx_1(1 + x_3),\\
\mathbf{F}_4 &= 1 + x_2 + x_3 + x_4 + x_2x_4 + Tx_1(1 + x_3 + x_4),\\
\mathbf{F}_5 &= 1 + x_2 + x_3 + x_4 + x_5 + x_2x_4 + x_2x_5 + x_3x_5 + Tx_1(1 + x_3 + x_4 + x_5 + x_3x_5),\\
\mathbf{F}_6 &= 1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_5 + x_3x_6 + x_4x_6\\
&\quad + x_2x_4x_6 + Tx_1(1 + x_3 + x_4 + x_5 + x_6 + x_3x_5 + x_3x_6 + x_4x_6).
\end{aligned}$$

We explicitly write $\mathbf{F}_m$ as a linear function of T:

$$\mathbf{F}_m = \mathcal{F}_m(x_1 \to Tx_1) = U_m + Tx_1V_m,$$

where U_m and V_m are given by

$$\begin{aligned}
U_0 &= 1, & V_0 &= 0,\\
U_1 &= 1, & V_1 &= 1,\\
U_2 &= 1 + x_2, & V_2 &= 1,\\
U_3 &= 1 + x_2 + x_3, & V_3 &= 1 + x_3,\\
U_4 &= 1 + x_2 + x_3 + x_4 + x_2x_4, & V_4 &= 1 + x_3 + x_4,\\
U_5 &= 1 + x_2 + x_3 + x_4 + x_5 + x_2x_4 + x_2x_5 + x_3x_5, & V_5 &= 1 + x_3 + x_4 + x_5 + x_3x_5,\\
U_6 &= 1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_2x_4 + x_2x_5 & V_6 &= 1 + x_3 + x_4 + x_5 + x_6\\
&\quad + x_2x_6 + x_3x_5 + x_3x_6 + x_4x_6 + x_2x_4x_6; & &\quad + x_3x_5 + x_3x_6 + x_4x_6.
\end{aligned}$$

Then it is not difficult to show that

$$\begin{aligned}
U_m &= U_{m-1} + x_mU_{m-2} = \mathcal{F}_{m-1}(x_\iota|[2,m]) : \; U_0 = 1, U_1 = 1;\\
V_m &= V_{m-1} + x_mV_{m-2} = \mathcal{F}_{m-2}(x_\iota|[3,m]) : \quad V_0 = 0, \; V_1 = 1.
\end{aligned}$$

Therefore, we find that

$$\mathbf{F}_m = \mathcal{F}_m(x_1 \to Tx_1) = \mathcal{F}_{m-1}(x_\iota|[2,m]) + Tx_1\mathcal{F}_{m-2}(x_\iota|[3,m]).$$

3.2. Recursive construction for the generating function

Recall the binomial relations

$$x_1^{k_1}\binom{n-k_2}{k_1} = [T^{k_1}](1+x_1T)^{n-k_2},$$
$$\binom{n-k_1}{k_m} = [T^{n-k_1}]\frac{T^{k_m}}{(1-T)^{k_m+1}},$$

where $[T^k]\phi(T)$ denotes the coefficient of T^k in the formal power series $\phi(T)$. We can first deal with the binomial sum with respect to k_1:

$$\begin{aligned}
\Delta_n^1 &= \sum_{k_1=0}^{n} x_1^{k_1}\binom{n-k_2}{k_1}\binom{n-k_1}{k_m}\\
&= \sum_{k_1=0}^{n}[T^{k_1}](1+x_1T)^{n-k_2}[T^{n-k_1}]\frac{T^{k_m}}{(1-T)^{k_m+1}}\\
&= [T^n]\frac{T^{k_m}(1+x_1T)^n}{(1-T)^{k_m+1}(1+x_1T)^{k_2}}\\
&= [T^n]\frac{T^{k_m}\mathbf{F}_1^n}{(1-T)^{k_m+1}}\times\frac{\mathbf{F}_0^{k_2}}{\mathbf{F}_1^{k_2}}.
\end{aligned}$$

Next, we can treat the binomial sum with respect to k_2:

$$\begin{aligned}
\Delta_n^2 &= \sum_{k_2=0}^{n} x_2^{k_2}\binom{n-k_3}{k_2}\Delta_n^1\\
&= [T^n]\frac{T^{k_m}(1+x_1T)^n}{(1-T)^{k_m+1}}\sum_{k_2=0}^{n}\binom{n-k_3}{k_2}\frac{x_2^{k_2}}{(1+x_1T)^{k_2}}\\
&= [T^n]\frac{T^{k_m}(1+x_1T)^n}{(1-T)^{k_m+1}}\left\{\frac{1+x_1T+x_2}{1+x_1T}\right\}^{n-k_3}\\
&= [T^n]\frac{T^{k_m}\mathbf{F}_2^n}{(1-T)^{k_m+1}}\times\frac{\mathbf{F}_1^{k_3}}{\mathbf{F}_2^{k_3}}.
\end{aligned}$$

Analogously, the sum with respect to k_3 reads as

$$\begin{aligned}
\Delta_n^3 &= \sum_{k_3=0}^{n} x_3^{k_3}\binom{n-k_4}{k_3}\Delta_n^2\\
&= [T^n]\frac{T^{k_m}(1+x_1T+x_2)^n}{(1-T)^{k_m+1}}\sum_{k_3=0}^{n}\binom{n-k_4}{k_3}\left\{\frac{x_3(1+x_1T)}{1+x_1T+x_2}\right\}^{k_3}\\
&= [T^n]\frac{T^{k_m}(1+x_1T+x_2)^n}{(1-T)^{k_m+1}}\left\{\frac{1+x_2+x_3+x_1T+x_1x_3T}{1+x_1T+x_2}\right\}^{n-k_4}\\
&= [T^n]\frac{T^{k_m}\mathbf{F}_3^n}{(1-T)^{k_m+1}}\times\frac{\mathbf{F}_2^{k_4}}{\mathbf{F}_3^{k_4}}.
\end{aligned}$$

By the induction principle, we can proceed with summing over k_ℓ ($1 < \ell < m$):

$$\Delta_n^\ell = \sum_{k_\ell=0}^{n} x_\ell^{k_\ell}\binom{n-k_{\ell+1}}{k_\ell}\Delta_n^{\ell-1} = [T^n]\frac{T^{k_m}\mathbf{F}_\ell^n}{(1-T)^{k_m+1}} \times \frac{\mathbf{F}_{\ell-1}^{k_{\ell+1}}}{\mathbf{F}_\ell^{k_{\ell+1}}}.$$

Finally, we can determine the sum with respect to k_m:

$$\Omega_n^m(\mathfrak{M}) = \Delta_n^m = \sum_{k_m=0}^{n} x_m^{k_m}\Delta_n^{m-1} = [T^n]\frac{\mathbf{F}_{m-1}^n}{1-T}\sum_{k_m=0}^{\infty}\left\{\frac{Tx_m\mathbf{F}_{m-2}}{(1-T)\mathbf{F}_{m-1}}\right\}^{k_m},$$

which simplifies into the algebraic expression in terms of coefficient T^n.

Lemma 1 (Algebraic generating function)**.**

$$\Omega_n^m(\mathfrak{M}) = [T^n]\frac{\mathbf{F}_{m-1}^{n+1}}{(1-T)\mathbf{F}_{m-1} - Tx_m\mathbf{F}_{m-2}}.$$

3.3. Generating function via the Lagrange expansion formula

To evaluate $\Omega_n^m(\mathfrak{M})$ explicitly, the Lagrange expansion formula will be crucial, which is reproduced as follows (cf. Chu [8, 9], Comtet [12, §3.8] and Wilf [20, §5.1]). For a formal power series $\varphi(T)$ subject to the condition $\varphi(0) \neq 0$, the functional equation $y = T/\varphi(T)$ determines T as an implicit function of y. Then, for another formal power series $\Phi(T)$ in the variable T, the following expansions hold for both composite series:

$$\Phi(T(y)) = \Phi(0) + \sum_{n=1}^{\infty}\frac{y^n}{n}[T^{n-1}]\{\Phi'(T)\varphi^n(T)\}, \tag{3.1}$$

$$\frac{\Phi(T(y))}{1 - T\varphi'(T)/\varphi(T)} = \sum_{n=0}^{\infty} y^n[T^n]\{\Phi(T)\varphi^n(T)\}. \tag{3.2}$$

By specifying the functions

$$\begin{aligned}
\varphi(T) &= \mathbf{F}_{m-1} = \mathcal{F}_{m-2}(x_\iota\|[2,m-1]) + x_1T\mathcal{F}_{m-3}(x_\iota\|[3,m-1]),\\
\Phi(T) &= \frac{\mathbf{F}_{m-1}}{(1-T)\mathbf{F}_{m-1} - x_mT\mathbf{F}_{m-2}}\\
&= \frac{\mathcal{F}_{m-2}(x_\iota\|[2,m-1]) + x_1T\mathcal{F}_{m-3}(x_\iota\|[3,m-1])}{\left\{\begin{array}{l}(1-T)[\mathcal{F}_{m-2}(x_\iota\|[2,m-1]) + x_1T\mathcal{F}_{m-3}(x_\iota\|[3,m-1])]\\ -x_mT[\mathcal{F}_{m-3}(x_\iota\|[2,m-2]) + x_1T\mathcal{F}_{m-4}(x_\iota\|[3,m-2])]\end{array}\right\}};
\end{aligned}$$

we can deduce that

$$y = y(T) = T/\varphi(T) \quad\Longrightarrow\quad T = T(y) = \frac{y\mathcal{F}_{m-2}(x_\iota\|[2,m-1])}{1 - x_1y\mathcal{F}_{m-3}(x_\iota\|[3,m-1])}.$$

According to the Lagrange expansion formula displayed in (3.2), we can express

$$\Omega_n^m(\mathfrak{M}) = [T^n]\frac{\mathbf{F}_{m-1}^{n+1}}{(1-T)\mathbf{F}_{m-1} - x_mT\mathbf{F}_{m-2}} = [T^n]\Phi(T)\varphi^n(T) = [y^n]\frac{\Phi(T)}{1 - T\varphi'(T)/\varphi(T)}.$$

After some simplifications, we arrive at the following rational generating function:

$$\Omega_n^m(\mathfrak{M}) = [y^n]\frac{1}{1 - yA_m + y^2B_m},$$

where the two coefficients A_m and B_m are given explicitly by

$$\begin{aligned}
A_m &= \mathcal{F}_{m-2}(x_\iota\|[2, m-1]) + x_1\mathcal{F}_{m-3}(x_\iota\|[3, m-1]) + x_m\mathcal{F}_{m-3}(x_\iota\|[2, m-2]),\\
B_m &= x_1x_m\Big\{\mathcal{F}_{m-3}(x_\iota\|[3, m-1])\mathcal{F}_{m-3}(x_\iota\|[2, m-2])\\
&\qquad - \mathcal{F}_{m-2}(x_\iota\|[2, m-1])\mathcal{F}_{m-4}(x_\iota\|[3, m-2])\Big\}.
\end{aligned}$$

Now, it remains to simplify the coefficients A_m and B_m. By examining the positions of x_1 and then x_m, we can first reduce A_m as below:

$$A_m = \mathcal{F}_{m-1}(x_\iota\|[1, m-1]) + x_m\mathcal{F}_{m-3}(x_\iota\|[2, m-2]) = \mathcal{L}_m(\mathfrak{M}).$$

Then by looking at the position of x_2, we can reformulate B_m as

$$\begin{aligned}
\frac{B_m}{x_1x_m} &= \mathcal{F}_{m-3}(x_\iota\|[3, m-1])\left\{\begin{matrix} \mathcal{F}_{m-4}(x_\iota\|[3, m-2]) \\ +x_2\,\mathcal{F}_{m-5}(x_\iota\|[4, m-2]) \end{matrix}\right\}\\
&\quad - \mathcal{F}_{m-4}(x_\iota\|[3, m-2])\left\{\begin{matrix} \mathcal{F}_{m-3}(x_\iota\|[3, m-1]) \\ +x_2\,\mathcal{F}_{m-4}(x_\iota\|[4, m-1]) \end{matrix}\right\}\\
&= x_2\mathcal{F}_{m-3}(x_\iota\|[3, m-1])\mathcal{F}_{m-5}(x_\iota\|[4, m-2])\\
&\quad - x_2\mathcal{F}_{m-4}(x_\iota\|[4, m-1])\mathcal{F}_{m-4}(x_\iota\|[3, m-2]).
\end{aligned}$$

Analogously, by considering the position of x_3, we obtain

$$\begin{aligned}
\frac{B_m}{x_1x_m} &= x_2x_3\mathcal{F}_{m-5}(x_\iota\|[5, m-1])\mathcal{F}_{m-5}(x_\iota\|[4, m-2])\\
&\quad - x_2x_3\mathcal{F}_{m-4}(x_\iota\|[4, m-1])\mathcal{F}_{m-6}(x_\iota\|[5, m-2]).
\end{aligned}$$

Repeating this process $(m-4)$ times, we arrive at

$$\begin{aligned}
\frac{B_m}{x_1x_m} &= \prod_{k=2}^{m-3}(-x_k) \times \mathcal{F}_1(x_\iota\|[m-1, m-1])\mathcal{F}_1(x_\iota\|[m-2, m-2])\\
&\quad - \prod_{k=2}^{m-3}(-x_k) \times \mathcal{F}_0(\emptyset)\mathcal{F}_2(x_\iota\|[m-1, m-2])\\
&= \prod_{k=2}^{m-3}(-x_k)\Big\{(1 + x_{m-1})(1 + x_{m-2}) - (1 + x_{m-1} + x_{m-2})\Big\},
\end{aligned}$$

which leads us to the simpler formula

$$B_m = \prod_{k=1}^{m}(-x_k) = (-1)^m\prod_{k=1}^{m}x_k.$$

This is a deep generalization of Cassini's formula for Fibonacci numbers

$$\mathrm{F}_{n+1}^2 - \mathrm{F}_n\mathrm{F}_{n+2} = (-1)^n.$$

Summing up, the following important theorem has been established.

Theorem 2 (Generating function)**.**

$$\Omega_n^m(\mathfrak{M}) = \Omega_n^m(x_\iota|[1,m]) = [y^n]\frac{1}{1 - y\mathcal{L}_m(\mathfrak{M}) + (-1)^m y^2 \prod_{k=1}^m x_k}.$$

In particular, when $n = 1$, we have the following interesting fact:

$$\mathcal{L}_m(\mathfrak{M}) = \Omega_1^m(\mathfrak{M}) \quad \text{for} \quad m \in \mathbb{N}.$$

The first five generating functions are recorded below:

$$\Omega_n^1(x_\iota|[1,1]) = [y^n]\frac{1}{1 - y - y^2(x_1)},$$
$$\Omega_n^2(x_\iota|[1,2]) = [y^n]\frac{1}{1 - y(1 + x_1 + x_2) + y^2(x_1x_2)},$$
$$\Omega_n^3(x_\iota|[1,3]) = [y^n]\frac{1}{1 - y(1 + x_1 + x_2 + x_3) - y^2(x_1x_2x_3)},$$
$$\Omega_n^4(x_\iota|[1,4]) = [y^n]\frac{1}{1 - y(1 + x_1 + x_2 + x_3 + x_4 + x_1x_3 + x_2x_4) + y^2(x_1x_2x_3x_4)},$$
$$\Omega_n^5(x_\iota|[1,5]) = [y^n]\frac{1}{\left\{\begin{aligned}1 - y^2(x_1x_2x_3x_4x_5) - y(1 + x_1 + x_2 + x_3 + x_4 + x_5\\ + x_1x_3 + x_1x_4 + x_2x_4 + x_2x_5 + x_3x_5)\end{aligned}\right\}}.$$

4. Closed formulae of multiple sums $\Omega_n^m(\mathfrak{M})$

After having presented the theoretical basis, we shall concretely evaluate, in this section, $\Omega_n^m(\mathfrak{M})$ by examining three particular cases when $\mathfrak{M}$ is generated by $\langle x\rangle$, $\langle u, v\rangle$ and $\langle u, v, w\rangle$. Several unusual properties (such as algebraic structures, symmetries and periodicities) will emerge from the outcome of our computations, which may serve as inspirations for further investigations, particularly from the combinatorial point of view.

In order to shorten lengthy expressions, we denote, for $m \in \mathbb{N}$ and $\mathbf{k} = (k_1, k_2, \cdots, k_m) \in \mathbb{N}_0^m$, by $\lfloor\mathbf{k}|$, $|\mathbf{k}\rceil$ and $|\mathbf{k}|$ the three linear sums

$$\lfloor\mathbf{k}| = \sum_{i=1}^{\lceil\frac{m}{2}\rceil} k_{2i-1}, \quad |\mathbf{k}\rceil = \sum_{i=1}^{\lfloor\frac{m}{2}\rfloor} k_{2i} \quad \text{and} \quad |\mathbf{k}| = \sum_{i=1}^{m} k_i = \lfloor\mathbf{k}| + |\mathbf{k}\rceil.$$

4.1. $\boxed{\mathfrak{M} = \{\langle x\rangle\}}$

Let $x_\iota \to x$ for $1 \le \iota \le m$, equivalently, $\mathfrak{M} = \{\langle x\rangle\}$. In this case, write $\mathcal{F}_m(\mathfrak{M})$ and $\mathcal{L}_m(\mathfrak{M})$ shortly by $\mathcal{F}_m(x)$ and $\mathcal{L}_m(x)$, respectively. The corresponding generating function for Fibonacci polynomials becomes

$$\sum_{m=0}^{\infty} y^m\mathcal{F}_m(x) = \frac{1 + xy}{1 - y - xy^2} = \frac{1}{\alpha_1(x) - \beta_1(x)}\Big\{\frac{\alpha_1^2(x)}{1 - y\alpha_1(x)} - \frac{\beta_1^2(x)}{1 - y\beta_1(x)}\Big\},$$

where

$$\alpha_1(x) = \frac{1+\sqrt{1+4x}}{2} \qquad \text{and} \qquad \beta_1(x) = \frac{1-\sqrt{1+4x}}{2} \tag{4.1}$$

subject to the conditions

$$\alpha_1(x) + \beta_1(x) = 1 \qquad \text{and} \qquad \alpha_1(x)\beta_1(x) = -x.$$

This leads us to the Binet formula

$$\mathcal{F}_m(x) = \frac{\alpha_1^{m+2}(x) - \beta_1^{m+2}(x)}{\alpha_1(x) - \beta_1(x)},$$

as well as the binomial expression

$$\mathcal{F}_m(x) = \sum_{k=0}^{\lceil \frac{m}{2} \rceil} \binom{m-k+1}{k} x^k.$$

For the corresponding Lucas polynomials, they satisfy the recurrence relation

$$\mathcal{L}_m(x) = \mathcal{L}_{m-1}(x) + x\mathcal{L}_{m-2}(x), \quad m \geq 2;$$

and the generating function

$$\sum_{m=0}^{\infty} y^m \mathcal{L}_m(x) = \frac{2-y}{1-y-xy^2} = \frac{1}{1-y\alpha_1(x)} + \frac{1}{1-y\beta_1(x)}.$$

This leads us to the Binet formula

$$\mathcal{L}_m(x) = \alpha_1^m(x) + \beta_1^m(x),$$

and the binomial expression

$$\mathcal{L}_m(x) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{m-k} \binom{m-k}{k} x^k.$$

The initial terms of the Fibonacci and Lucas polynomials are recorded below:

$$\begin{aligned}
&\mathcal{F}_0(x) = 1, && \mathcal{L}_0(x) = 2,\\
&\mathcal{F}_1(x) = 1 + x, && \mathcal{L}_1(x) = 1,\\
&\mathcal{F}_2(x) = 1 + 2x, && \mathcal{L}_2(x) = 1 + 2x,\\
&\mathcal{F}_3(x) = 1 + 3x + x^2, && \mathcal{L}_3(x) = 1 + 3x,\\
&\mathcal{F}_4(x) = 1 + 4x + 3x^2, && \mathcal{L}_4(x) = 1 + 4x + 2x^2,\\
&\mathcal{F}_5(x) = 1 + 5x + 6x^2 + x^3, && \mathcal{L}_5(x) = 1 + 5x + 5x^2,\\
&\mathcal{F}_6(x) = 1 + 6x + 10x^2 + 4x^3; && \mathcal{L}_6(x) = 1 + 6x + 9x^2 + 2x^3.
\end{aligned}$$

According to Theorem 2, we have the formulae as in the theorem below, where the Binet form expression follows by the partial fraction decomposition:

$$\begin{aligned}
\frac{1}{1-y\mathcal{L}_m(x)+(-1)^m y^2 x^m} =& \frac{1}{1-y(\alpha_1^m(x)+\beta_1^m(x))+y^2\alpha_1^m(x)\beta_1^m(x)}\\
=& \frac{1}{\alpha_1^m(x)-\beta_1^m(x)}\left\{\frac{\alpha_1^m(x)}{1-y\alpha_1^m(x)} - \frac{\beta_1^m(x)}{1-y\beta_1^m(x)}\right\}.
\end{aligned}$$

Theorem 3. *Assume $\alpha_1(x)$ and $\beta_1(x)$ as in* (4.1). *Then the multiple sums $\Omega_n^m(x)$ admit the rational generating function*

$$\Omega_n^m(x) = \sum_{\mathbf{k}\in[0,n]^m} x^{|\mathbf{k}|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i} = [y^n]\frac{1}{1-y\mathcal{L}_m(x)+(-1)^m y^2 x^m},$$

as well as the Binet form expression

$$\Omega_n^m(x) = \frac{\alpha_1^{mn+m}(x)-\beta_1^{mn+m}(x)}{\alpha_1^m(x)-\beta_1^m(x)}.$$

When $n = 1$, this theorem results in the following curious fact:

$$\Omega_1^m(x) = \mathcal{L}_m(x) \quad \text{for} \quad m \in \mathbb{N}.$$

Theorem 3 itself extends and unifies substantially the known formulae due to Carlitz [6] and Chu [10] as shown in the following interesting particular cases.

4.1.1. $\boxed{x \to 1}$

In this case, the golden ratio comes out

$$\alpha_1(1) = \alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta_1(1) = \beta = \frac{1-\sqrt{5}}{2}.$$

Then we recover from Theorem 3 the following important formula.

Corollary 4 (Carlitz [6])**.**

$$\sum_{\mathbf{k}\in[0,n]^m}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i} = \frac{\alpha^{mn+m}-\beta^{mn+m}}{\alpha^m-\beta^m} = \frac{\mathrm{F}_{mn+m}}{\mathrm{F}_m}.$$

4.1.2. $\boxed{x \to -1}$

Analogously, we can write explicitly

$$\alpha_1(-1) = e^{\frac{\pi \mathbf{i}}{3}} = \frac{1+\mathbf{i}\sqrt{3}}{2} = \cos\Big(\frac{\pi}{3}\Big) + \mathbf{i}\sin\Big(\frac{\pi}{3}\Big),$$
$$\beta_1(-1) = e^{-\frac{\pi \mathbf{i}}{3}} = \frac{1-\mathbf{i}\sqrt{3}}{2} = \cos\Big(\frac{\pi}{3}\Big) - \mathbf{i}\sin\Big(\frac{\pi}{3}\Big).$$

Denoting by $U_n(x)$ the Chebyshev polynomial (cf. [16]) of the second kind, we deduce from Theorem 3 another formula.

Corollary 5 (Chu [11])**.**

$$\sum_{\mathbf{k}\in[0,n]^m}(-1)^{|\mathbf{k}|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i} = U_n\Big(\cos\frac{m\pi}{3}\Big) = \frac{\alpha_1^{mn+m}(-1)-\beta_1^{mn+m}(-1)}{\alpha_1^m(-1)-\beta_1^m(-1)}.$$

4.1.3. $\boxed{x \to \tau(1+\tau)}$

Furthermore, by using parametric expressions

$$\alpha_1(\tau(1+\tau)) = 1+\tau \qquad \text{and} \qquad \beta_1(\tau(1+\tau)) = -\tau,$$

we deduce the following more general formula with a free variable τ.

Corollary 6.

$$\sum_{\mathbf{k}\in[0,n]^m} \{\tau(1+\tau)\}^{|\mathbf{k}|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i} = \frac{(1+\tau)^{mn+m}-(-\tau)^{mn+m}}{(1+\tau)^m-(-\tau)^m}.$$

4.2. $\boxed{\mathfrak{M} = \{\langle u, v\rangle\}}$

When $\mathfrak{M} = \{\langle u, v\rangle\}$ is given by

$$\begin{aligned} x_{2\imath-1} &\to u \text{ for } 1 \le \imath \le \lceil \tfrac{m}{2} \rceil, \\ x_{2\jmath} &\to v \quad \text{for } 1 \le \jmath \le \lfloor \tfrac{m}{2} \rfloor; \end{aligned}$$

the corresponding $\mathcal{F}_m(\mathfrak{M})$ and $\mathcal{L}_m(\mathfrak{M})$ are abbreviated to $\mathcal{F}_m(\langle u, v\rangle)$ and $\mathcal{L}_m(\langle u, v\rangle)$, respectively. When the elements of $\mathfrak{M} = \{\langle u, v\rangle\}$ are arranged in a line, the first enumerative functions $\mathcal{F}_m(\langle u, v\rangle)$ read as

$$\begin{aligned}
\mathcal{F}_0(\langle u, v\rangle) &= 1, \\
\mathcal{F}_1(\langle u, v\rangle) &= 1+u, \\
\mathcal{F}_2(\langle u, v\rangle) &= 1+u+v, \\
\mathcal{F}_3(\langle u, v\rangle) &= 1+2u+v+u^2, \\
\mathcal{F}_4(\langle u, v\rangle) &= 1+2u+2v+u^2+uv+v^2, \\
\mathcal{F}_5(\langle u, v\rangle) &= 1+3u+2v+3u^2+2uv+v^2+u^3, \\
\mathcal{F}_6(\langle u, v\rangle) &= 1+3u+3v+3u^2+4uv+3v^2+u^3+u^2v+uv^2+v^3.
\end{aligned}$$

By classifying the subsets of the linear $\mathfrak{M} = \{\langle u, v\rangle\}$ with respect to the ultimate element v and u, we can derive the recurrence relations

$$\begin{aligned}
\mathcal{F}_{2n}(\langle u, v\rangle) &= \mathcal{F}_{2n-1}(\langle u, v\rangle) + v\mathcal{F}_{2n-2}(\langle u, v\rangle), \\
\mathcal{F}_{2n+1}(\langle u, v\rangle) &= \mathcal{F}_{2n}(\langle u, v\rangle) + u\mathcal{F}_{2n-1}(\langle u, v\rangle).
\end{aligned}$$

For the generating functions defined by

$$\mathrm{P}(y) = \sum_{n=0}^{\infty} y^n \mathcal{F}_{2n}(\langle u, v\rangle) \quad \text{and} \quad \mathrm{Q}(y) = \sum_{n=0}^{\infty} y^n \mathcal{F}_{2n+1}(\langle u, v\rangle),$$

they can be manipulated as follows:

$$\begin{aligned}
\mathrm{P}(y) &= 1 + y\mathrm{Q}(y) + yv\mathrm{P}(y), \\
\mathrm{Q}(y) &= u + \mathrm{P}(y) + yu\mathrm{Q}(y).
\end{aligned}$$

Resolving this linear system gives explicit generating functions:

$$\mathrm{P}(y)=\frac{1}{1-y(1+u+v)+y^2uv}\quad\text{and}\quad \mathrm{Q}(y)=\frac{1+u-uvy}{1-y(1+u+v)+y^2uv}.$$

Both P and Q can be decomposed into partial fractions:

$$\begin{aligned}\mathrm{P}(y)&=\frac{1}{(1-y\alpha_2)(1-y\beta_2)}=\frac{1}{\alpha_2-\beta_2}\Big\{\frac{\alpha_2}{1-y\alpha_2}-\frac{\beta_2}{1-y\beta_2}\Big\},\\ \mathrm{Q}(y)&=\frac{1+u-uvy}{(1-y\alpha_2)(1-y\beta_2)}=\frac{1}{\alpha_2-\beta_2}\Big\{\frac{(\alpha_2-v)\alpha_2}{1-y\alpha_2}-\frac{(\beta_2-v)\beta_2}{1-y\beta_2}\Big\};\end{aligned}$$

where

$$\begin{aligned}\alpha_2&:=\alpha_2(u,v)=\frac{1}{2}\Big\{1+u+v+\sqrt{(1+u+v)^2-4uv}\Big\},\\ \beta_2&:=\beta_2(u,v)=\frac{1}{2}\Big\{1+u+v-\sqrt{(1+u+v)^2-4uv}\Big\};\end{aligned}\tag{4.2}$$

subject to the conditions

$$\alpha_2+\beta_2=1+u+v\qquad\text{and}\qquad\alpha_2\beta_2=uv.$$

Therefore, we deduce the following Binet form expressions:

$$\begin{aligned}\mathcal{F}_{2n}(\langle u,v\rangle)&=[y^n]\mathrm{P}(y)=\frac{\alpha_2^{n+1}-\beta_2^{n+1}}{\alpha_2-\beta_2},\\ \mathcal{F}_{2n+1}(\langle u,v\rangle)&=[y^n]\mathrm{Q}(y)=\frac{(\alpha_2-v)\alpha_2^{n+1}-(\beta_2-v)\beta_2^{n+1}}{\alpha_2-\beta_2}.\end{aligned}$$

Alternatively, by means of binomial expansions, we can also derive polynomial expressions

$$\begin{aligned}\mathcal{F}_{2n}(\langle u,v\rangle)&=[y^n]\mathrm{P}(y)=[y^n]\frac{1}{1-y(1+u+v)+y^2uv}\\ &=[y^n]\sum_{k=0}^{n}y^{n-k}(1+u+v-uvy)^{n-k}\\ &=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}(-uv)^k(1+u+v)^{n-2k},\\ \mathcal{F}_{2n+1}(\langle u,v\rangle)&=[y^n]\mathrm{Q}(y)=[y^n]\frac{1+u-uvy}{1-y(1+u+v)+y^2uv}\\ &=[y^n](1+u)\sum_{k=0}^{n}y^{n-k}(1+u+v-uvy)^{n-k}\\ &-[y^n](uv)\sum_{k=0}^{n}y^{1+n-k}(1+u+v-uvy)^{n-k}\\ &=\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\frac{(1+u)(1+n-k)+kv}{1+n-k}\binom{1+n-k}{k}(-uv)^k(1+u+v)^{n-2k}.\end{aligned}$$

When the elements of $\mathfrak{M} = \{\langle u, v\rangle\}$ are arranged in a circle, the enumerative function $\mathcal{L}_m(\langle u, v\rangle)$ can also be calculated. The initial terms read as

$$\begin{aligned}
&\mathcal{L}_0(\langle u, v\rangle) = 2,\\
&\mathcal{L}_1(\langle u, v\rangle) = 1,\\
&\mathcal{L}_2(\langle u, v\rangle) = 1 + u + v,\\
&\mathcal{L}_3(\langle u, v\rangle) = 1 + 2u + v,\\
&\mathcal{L}_4(\langle u, v\rangle) = 1 + 2u + 2v + u^2 + v^2,\\
&\mathcal{L}_5(\langle u, v\rangle) = 1 + 3u + 2v + 2u^2 + 2uv + v^2,\\
&\mathcal{L}_6(\langle u, v\rangle) = 1 + 3u + 3v + 3u^2 + 3uv + 3v^2 + u^3 + v^3.
\end{aligned}$$

Recalling (2.2), we can write

$$\begin{aligned}
\mathcal{L}_{2n}(\langle u, v\rangle) &= \mathcal{F}_{2n-1}(\langle u, v\rangle) + v\mathcal{F}_{2n-3}(u \rightleftharpoons v),\\
\mathcal{L}_{2n+1}(\langle u, v\rangle) &= \mathcal{F}_{2n}(\langle u, v\rangle) + u\mathcal{F}_{2n-2}(u \rightleftharpoons v).
\end{aligned}$$

Then the generating functions can be determined by

$$\begin{aligned}
\widetilde{P}(y) = \sum_{n=0}^{\infty} y^n \mathcal{L}_{2n}(\langle u, v\rangle) &= 2 + y + y\mathrm{Q}(y) + y^2 v\mathrm{Q}(y|u \rightleftharpoons v)\\
&= 2 + yv + \frac{y(1 + u - uvy)}{1 - y(1 + u + v) + uvy^2} + \frac{y^2 v(1 + v - uvy)}{1 - y(1 + u + v) + uvy^2}\\
&= \frac{2 - (1 + u + v)y}{1 - y(1 + u + v) + uvy^2},\\
\widetilde{Q}(y) = \sum_{n=0}^{\infty} y^n \mathcal{L}_{2n+1}(\langle u, v\rangle) &= \mathrm{P}(y) + yu\mathrm{P}(y|u \rightleftharpoons v)\\
&= \frac{1}{1 - y(1 + u + v) + uvy^2} + \frac{yu}{1 - y(1 + u + v) + uvy^2}\\
&= \frac{1 + yu}{1 - y(1 + u + v) + uvy^2}.
\end{aligned}$$

Since $\alpha_2(u, v)$, $\beta_2(u, v)$ and $\mathcal{F}_{2n}(\langle u, v\rangle)$ are symmetric in u and v, the following Binet form formulae hold:

$$\begin{aligned}
\mathcal{L}_{2n}(\langle u, v\rangle) &= \mathcal{F}_{2n}(\langle u, v\rangle) - uv\mathcal{F}_{2n-4}(\langle u, v\rangle)\\
&= \frac{\alpha_2^{n+1} - \beta_2^{n+1}}{\alpha_2 - \beta_2} - \alpha_2\beta_2\frac{\alpha_2^{n-1} - \beta_2^{n-1}}{\alpha_2 - \beta_2}\\
&= \alpha_2^n + \beta_2^n,\\
\mathcal{L}_{2n+1}(\langle u, v\rangle) &= \mathcal{F}_{2n}(\langle u, v\rangle) + u\mathcal{F}_{2n-2}(\langle u, v\rangle)\\
&= \frac{\alpha_2^{n+1} - \beta_2^{n+1}}{\alpha_2 - \beta_2} + u\frac{\alpha_2^n - \beta_2^n}{\alpha_2 - \beta_2};
\end{aligned}$$

as well as the binomial sum expressions

$$\begin{aligned}\mathcal{L}_{2n}(\langle u,v\rangle) &= \mathcal{F}_{2n}(\langle u,v\rangle) - uv\mathcal{F}_{2n-4}(\langle u,v\rangle)\\ &= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \frac{n}{n-k}\binom{n-k}{k}(-uv)^k(1+u+v)^{n-2k},\\ \mathcal{L}_{2n+1}(\langle u,v\rangle) &= \mathcal{F}_{2n}(\langle u,v\rangle) + u\mathcal{F}_{2n-2}(\langle u,v\rangle)\\ &= \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \frac{(n-k)(1+2u+v)-ku}{(n-k)(1+u+v)}\binom{n-k}{k}(-uv)^k(1+u+v)^{n-2k}.\end{aligned}$$

For even m, taking into account also the partial fractions

$$\begin{aligned}\frac{1}{1-y\mathcal{L}_{2m}(\langle u,v\rangle)+y^2u^mv^m} &= \frac{1}{1-y(\alpha_2^m+\beta_2^m)+y^2\alpha_2^m\beta_2^m}\\ &= \frac{1}{\alpha_2^m-\beta_2^m}\left\{\frac{\alpha_2^m}{1-y\alpha_2^m}-\frac{\beta_2^m}{1-y\beta_2^m}\right\},\end{aligned}$$

and then applying Theorem 2, we derive the following formulae.

Theorem 7. *Assume $\alpha_2(u,v)$ and $\beta_2(u,v)$ as in* (4.2). *Then the multiple sums $\Omega_n^m(\langle u,v\rangle)$ admit the rational generating function*

$$\begin{aligned}\Omega_n^m(\langle u,v\rangle) &= \sum_{\mathbf{k}\in[0,n]^m} u^{\lfloor \mathbf{k}|}v^{|\mathbf{k}\rceil}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i}\\ &= [y^n]\frac{1}{1-y\mathcal{L}_m(\langle u,v\rangle)+(-1)^m y^2u^{\lceil \frac{m}{2}\rceil}v^{\lfloor \frac{m}{2}\rfloor}},\end{aligned}$$

as well as the Binet form expression

$$\Omega_n^{2m}(\langle u,v\rangle) = \frac{\alpha_2^{mn+m}(u,v)-\beta_2^{mn+m}(u,v)}{\alpha_2^m(u,v)-\beta_2^m(u,v)}.$$

We remark that there is an interesting coincidence:

$$\begin{aligned}\Omega_n^2(\langle u,v\rangle) &= [y^n]\frac{1}{1-y\mathcal{L}_2(\langle u,v\rangle)+y^2uv}\\ &= [y^n]\frac{1}{1-y(1+u+v)+y^2uv}\\ &= [y^n]\mathrm{P}(y) = \mathcal{F}_{2n}(\langle u,v\rangle).\end{aligned}$$

For odd m, three initial formulae are shown below as examples:

$$\begin{aligned}\Omega_n^1(\langle u,v\rangle) &= [y^n]\frac{1}{1-y-y^2u},\\ \Omega_n^3(\langle u,v\rangle) &= [y^n]\frac{1}{1-y(1+2u+v)-y^2u^2v},\end{aligned}$$

$$\Omega_n^5(\langle u,v\rangle) = [y^n]\frac{1}{1-y(1+3u+2v+2u^2+2uv+v^2)-y^2u^3v^2}.$$

When $u = v = x$, it is not hard to check that

$$\alpha_2(x,x) = \alpha_1^2(x) \quad \text{and} \quad \beta_2(x,x) = \beta_1^2(x),$$

as well as

$$\mathcal{L}_m(\langle x,x\rangle) = \alpha_1^m(x) + \beta_1^m(x) = \mathcal{L}_m(x).$$

From this, we recover the formulae in Theorem 3.

4.2.1. $\boxed{u = -v = x}$

In this case, the multiple sums $\Omega_n^m(\langle x,-x\rangle)$ become

$$\Omega_n^m(\langle x,-x\rangle) = \sum_{\mathbf{k}\in[0,n]^m} (-1)^{\lceil \mathbf{k} \rceil} x^{|\mathbf{k}|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i}.$$

Writing explicitly

$$\mathcal{L}_{2m}(\langle x,-x\rangle) = [y^m]\frac{2-y}{1-y-x^2y^2},$$
$$\mathcal{L}_{2m+1}(\langle x,-x\rangle) = [y^m]\frac{1+xy}{1-y-x^2y^2};$$

we can show, from Theorem 7, the generating function.

Proposition 8 (Generating function)**.**

$$\Omega_n^m(\langle x,-x\rangle) = [y^n]\frac{1}{1-y\mathcal{L}_m(\langle x,-x\rangle)+(-1)^{\lceil \frac{m}{2}\rceil}y^2x^m}.$$

For even m, observe that

$$\mathcal{L}_{2m}(\langle x,-x\rangle) = \frac{[y^m]}{\alpha_2-\beta_2}\Big\{\frac{2\alpha_2-1}{1-y\alpha_2} - \frac{2\beta_2-1}{1-y\beta_2}\Big\} = \alpha_2^m + \beta_2^m,$$

where

$$\alpha_2(x,-x) = \frac{1+\sqrt{1+4x^2}}{2} \quad \text{and} \quad \beta_2(x,-x) = \frac{1-\sqrt{1+4x^2}}{2}.$$

We derive the following formulae of Binet form.

Proposition 9. *Assume $\alpha_2(x,-x)$ and $\beta_2(x,-x)$ as above. Then*

$$\Omega_n^{2m}(\langle x,-x\rangle) = \frac{\alpha_2^{mn+m}(x,-x)-\beta_2^{mn+m}(x,-x)}{\alpha_2^m(x,-x)-\beta_2^m(x,-x)}.$$

Instead, for odd m, three initial generating functions are displayed below:

$$\Omega_n^1(\langle x,-x\rangle) = [y^n]\frac{1}{1-y-y^2x},$$
$$\Omega_n^3(\langle x,-x\rangle) = [y^n]\frac{1}{1-y(1+x)+y^2x^3},$$
$$\Omega_n^5(\langle x,-x\rangle) = [y^n]\frac{1}{1-y(1+x+x^2)-y^2x^5}.$$

4.2.2. $\boxed{u = -v = 1}$

In this case, $\Omega_n^m(\langle 1,-1\rangle)$ becomes

$$\Omega_n^m(\langle 1,-1\rangle) = \sum_{\mathbf{k}\in[0,n]^m} (-1)^{|\mathbf{k}]}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i} = [y^n]\frac{1}{1-y\mathcal{L}_m(\langle 1,-1\rangle)+(-1)^{\lceil \frac{m}{2}\rceil}y^2}.$$

In view of

$$\begin{aligned}\mathcal{L}_{2m}(\langle 1,-1\rangle) &= [y^m]\frac{2-y}{1-y-y^2} = \mathrm{L}_m,\\ \mathcal{L}_{2m+1}(\langle 1,-1\rangle) &= [y^m]\frac{1+y}{1-y-y^2} = \mathrm{F}_m + \mathrm{F}_{m+1} = \mathrm{F}_{m+2},\\ \alpha_2(1,-1) = \alpha &\quad\text{and}\quad \beta_2(1,-1) = \beta,\end{aligned}$$

then from Propositions 8 and 9, the multiple sums $\Omega_n^m(\langle 1,-1\rangle)$ are evaluated by the corollary below.

Corollary 10 (Chu [11])**.**

$$\begin{aligned}\Omega_n^{2m}(\langle 1,-1\rangle) &= [y^n]\frac{1}{1-y\mathrm{L}_m+(-1)^m y^2} = \frac{\alpha^{mn+m}-\beta^{mn+m}}{\alpha^m-\beta^m} = \frac{\mathrm{F}_{mn+m}}{\mathrm{F}_m},\\ \Omega_n^{2m+1}(\langle 1,-1\rangle) &= [y^n]\frac{1}{1-y\mathrm{F}_{m+2}-(-1)^m y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^{mk}\binom{n-k}{k}\mathrm{F}_{m+2}^{n-2k}.\end{aligned}$$

The first five formulae for odd m are displayed as examples:

$$\begin{aligned}\Omega_n^1(\langle 1,-1\rangle) &= [y^n]\frac{1}{1-y-y^2} = \mathrm{F}_{n+1},\\ \Omega_n^3(\langle 1,-1\rangle) &= [y^n]\frac{1}{1-2y+y^2} = n+1,\\ \Omega_n^5(\langle 1,-1\rangle) &= [y^n]\frac{1}{1-3y-y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-k}{k}3^{n-2k},\\ \Omega_n^7(\langle 1,-1\rangle) &= [y^n]\frac{1}{1-5y+y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k\binom{n-k}{k}5^{n-2k},\\ \Omega_n^9(\langle 1,-1\rangle) &= [y^n]\frac{1}{1-8y-y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-k}{k}8^{n-2k}.\end{aligned}$$

4.2.3. $\boxed{v = -u = 1}$

In this case, $\Omega_n^m(\langle -1,1\rangle)$ becomes

$$\Omega_n^m(\langle -1,1\rangle) = \sum_{\mathbf{k}\in[0,n]^m} (-1)^{\lfloor\mathbf{k}|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i} = [y^n]\frac{1}{1-y\mathcal{L}_m(\langle -1,1\rangle)+(-1)^{\lfloor \frac{m}{2}\rfloor}y^2}.$$

In view of

$$\mathcal{L}_{2m}(\langle -1,1\rangle) = [y^m]\frac{2-y}{1-y-y^2} = \mathrm{L}_m,$$
$$\mathcal{L}_{2m+1}(\langle -1,1\rangle) = [y^m]\frac{1-y}{1-y-y^2} = \mathrm{F}_{m+1}-\mathrm{F}_m = \mathrm{F}_{m-1},$$
$$\alpha_2(-1,1)=\alpha \quad \text{and} \quad \beta_2(-1,1)=\beta,$$

then from Propositions 8 and 9, the multiple sums $\Omega_n^m(\langle -1,1\rangle)$ are evaluated by the corollary below.

Corollary 11 (Chu [11])**.**

$$\Omega_n^{2m}(\langle -1,1\rangle) = [y^n]\frac{1}{1-y\mathrm{L}_m+(-1)^m y^2} = \frac{\mathrm{F}_{mn+m}}{\mathrm{F}_m},$$
$$\Omega_n^{2m+1}(\langle -1,1\rangle) = [y^n]\frac{1}{1-y\mathrm{F}_{m-1}+(-1)^m y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^{mk+k}\binom{n-k}{k}\mathrm{F}_{m-1}^{n-2k}.$$

The first five formulae for odd m are displayed as examples:

$$\Omega_n^{1}(\langle -1,1\rangle) = [y^n]\frac{1}{1-y+y^2} = (-1)^{\lfloor \frac{n+1}{3}\rfloor}\chi(n\not\equiv_3 2),$$
$$\Omega_n^{3}(\langle -1,1\rangle) = [y^n]\frac{1}{1-y^2} = \frac{1+(-1)^n}{2},$$
$$\Omega_n^{5}(\langle -1,1\rangle) = [y^n]\frac{1}{1-y+y^2} = (-1)^{\lfloor \frac{n+1}{3}\rfloor}\chi(n\not\equiv_3 2),$$
$$\Omega_n^{7}(\langle -1,1\rangle) = [y^n]\frac{1}{1-y-y^2} = \mathrm{F}_{n+1},$$
$$\Omega_n^{9}(\langle -1,1\rangle) = [y^n]\frac{1}{1-2y+y^2} = n+1.$$

4.2.4. $\boxed{u\to\omega, v\to\omega^2:\ \omega=e^{2\pi\mathbf{i}/3}}$

In this case, $\Omega_n^m(\langle \omega,\omega^2\rangle)$ becomes

$$\Omega_n^m(\langle \omega,\omega^2\rangle) = \sum_{\mathbf{k}\in[0,n]^m}\omega^{|\mathbf{k}|+|\mathbf{k}|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i}$$
$$= [y^n]\frac{1}{1-y\mathcal{L}_m(\langle \omega,\omega^2\rangle)+(-1)^m\omega^{\lfloor \frac{3m}{2}\rfloor}y^2}.$$

Writing explicitly

$$\mathcal{L}_{2m}(\langle \omega,\omega^2\rangle) = [y^m]\frac{2}{1+y^2} = 2(-1)^{\lfloor \frac{m}{2}\rfloor}\chi(m\equiv_2 0),$$
$$\mathcal{L}_{2m+1}(\langle \omega,\omega^2\rangle) = [y^m]\frac{1+y\omega}{1+y^2} = (-1)^{\lfloor \frac{m}{2}\rfloor}\chi(m\equiv_2 0)+\omega(-1)^{\lfloor \frac{m}{2}\rfloor}\chi(m\equiv_2 1);$$

we evaluate the multiple sums $\Omega_n^m(\langle \omega,\omega^2\rangle)$ as in the following corollary.

Corollary 12 (Generating functions and explicit formulae)**.**

$$\begin{aligned}
\Omega_n^{4m}(\langle\omega,\omega^2\rangle) &= [y^n]\frac{1}{1-y\mathcal{L}_{4m}(\langle\omega,\omega^2\rangle)+(-1)^{4m}\omega^{6m}y^2}\\
&= [y^n]\frac{1}{1-2y(-1)^m+y^2} = (-1)^{mn}(n+1),\\
\Omega_n^{4m+1}(\langle\omega,\omega^2\rangle) &= [y^n]\frac{1}{1-y\mathcal{L}_{4m+1}(\langle\omega,\omega^2\rangle)+(-1)^{4m+1}\omega^{6m+1}y^2}\\
&= [y^n]\frac{1}{1-y(-1)^m-\omega y^2} = (-1)^{mn}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\omega^k,\\
\Omega_n^{4m+2}(\langle\omega,\omega^2\rangle) &= [y^n]\frac{1}{1-y\mathcal{L}_{4m+2}(\langle\omega,\omega^2\rangle)+(-1)^{4m+2}\omega^{6m+3}y^2}\\
&= [y^n]\frac{1}{1+y^2} = (-1)^{\lfloor\frac{n}{2}\rfloor}\chi(n\equiv_2 0),\\
\Omega_n^{4m+3}(\langle\omega,\omega^2\rangle) &= [y^n]\frac{1}{1-y\mathcal{L}_{4m+3}(\langle\omega,\omega^2\rangle)+(-1)^{4m+3}\omega^{6m+4}y^2}\\
&= [y^n]\frac{1}{1+\omega y(-1)^m-\omega y^2} = (-1)^{mn}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\omega^{n-k}.
\end{aligned}$$

4.3. $\boxed{\mathfrak{M}=\{\langle u,v,w\rangle\}}$

In this case, $\mathfrak{M}=\{\langle u,v,w\rangle\}$ is determined by

$$\begin{aligned}
&x_{3\imath-2}\to u \text{ for } 1\le\imath\le\lfloor\tfrac{m+2}{3}\rfloor,\\
&x_{3\jmath-1}\to v \text{ for } 1\le\jmath\le\lfloor\tfrac{m+1}{3}\rfloor,\\
&x_{3\kappa}\to w \quad\text{for } 1\le\kappa\le\lfloor\tfrac{m}{3}\rfloor.
\end{aligned}$$

When the elements of $\mathfrak{M}=\{\langle u,v,w\rangle\}$ are arranged in a line, the first enumerative functions $\mathcal{F}_m(\langle u,v,w\rangle)$ read as

$$\begin{aligned}
\mathcal{F}_0(\langle u,v,w\rangle) &= 1,\\
\mathcal{F}_1(\langle u,v,w\rangle) &= 1+u,\\
\mathcal{F}_2(\langle u,v,w\rangle) &= 1+u+v,\\
\mathcal{F}_3(\langle u,v,w\rangle) &= 1+u+v+w+uw,\\
\mathcal{F}_4(\langle u,v,w\rangle) &= 1+2u+v+w+uw+u^2+uv,\\
\mathcal{F}_5(\langle u,v,w\rangle) &= 1+2u+2v+w+u^2+v^2+2uv+uw+vw+uvw,\\
\mathcal{F}_6(\langle u,v,w\rangle) &= 1+2u+2v+2w+u^2+v^2+w^2\\
&\quad+2uv+3uw+2vw+2uvw+uw^2+u^2w.
\end{aligned}$$

By classifying the subsets of $\mathfrak{M}=\{\langle u,v,w\rangle\}$ with respect to the ultimate elements w, u and v, we can derive the recurrence relations

$$\mathcal{F}_{3n}(\langle u,v,w\rangle) = \mathcal{F}_{3n-1}(\langle u,v,w\rangle) + w\mathcal{F}_{3n-2}(\langle u,v,w\rangle),$$

$$\mathcal{F}_{3n+1}(\langle u,v,w\rangle) = \mathcal{F}_{3n}(\langle u,v,w\rangle) + u\mathcal{F}_{3n-1}(\langle u,v,w\rangle),$$
$$\mathcal{F}_{3n+2}(\langle u,v,w\rangle) = \mathcal{F}_{3n+1}(\langle u,v,w\rangle) + v\mathcal{F}_{3n}(\langle u,v,w\rangle).$$

For the generating functions defined by

$$\mathcal{P}(y) = \sum_{n=0}^{\infty} y^n \mathcal{F}_{3n}(\langle u,v,w\rangle),$$
$$\mathcal{Q}(y) = \sum_{n=0}^{\infty} y^n \mathcal{F}_{3n+1}(\langle u,v,w\rangle),$$
$$\mathcal{R}(y) = \sum_{n=0}^{\infty} y^n \mathcal{F}_{3n+2}(\langle u,v,w\rangle),$$

they can be manipulated as follows:

$$\mathcal{P}(y) = 1 + y\mathcal{R}(y) + wy\mathcal{Q}(y),$$
$$\mathcal{Q}(y) = u + \mathcal{P}(y) + uy\mathcal{R}(y),$$
$$\mathcal{R}(y) = \mathcal{Q}(y) + v\mathcal{P}(y).$$

Resolving this linear system gives explicit generating functions:

$$\mathcal{P}(y) = \frac{1 + uwy}{1 - (1+u+v+w)y - uvwy^2},$$
$$\mathcal{Q}(y) = \frac{1 + u}{1 - (1+u+v+w)y - uvwy^2},$$
$$\mathcal{R}(y) = \frac{1 + u + v + uvwy}{1 - (1+u+v+w)y - uvwy^2}.$$

All three $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ can be decomposed into partial fractions:

$$\mathcal{P}(y) = \frac{1 + uwy}{(1-\alpha_3 y)(1-\beta_3 y)} = \frac{1}{\alpha_3 - \beta_3}\left\{\frac{\alpha_3 + uw}{1-\alpha_3 y} - \frac{\beta_3 + uw}{1-\beta_3 y}\right\},$$
$$\mathcal{Q}(y) = \frac{1 + u}{(1-\alpha_3 y)(1-\beta_3 y)} = \frac{1+u}{\alpha_3 - \beta_3}\left\{\frac{\alpha_3}{1-\alpha_3 y} - \frac{\beta_3}{1-\beta_3 y}\right\},$$
$$\mathcal{R}(y) = \frac{1 + u + v + uvwy}{(1-\alpha_3 y)(1-\beta_3 y)} = \frac{1}{\alpha_3 - \beta_3}\left\{\frac{(\alpha_3 - w)\alpha_3}{1-\alpha_3 y} - \frac{(\beta_3 - w)\beta_3}{1-\beta_3 y}\right\};$$

where

$$\alpha_3 := \alpha_3(u,v,w) = \frac{1}{2}\left\{1 + u + v + w + \sqrt{(1+u+v+w)^2 + 4uvw}\right\},$$
$$\beta_3 := \beta_3(u,v,w) = \frac{1}{2}\left\{1 + u + v + w - \sqrt{(1+u+v+w)^2 + 4uvw}\right\};$$

with

$$\alpha_3 + \beta_3 = 1 + u + v + w \qquad \text{and} \qquad \alpha_3\beta_3 = -uvw.$$

Therefore, we deduce the following Binet form expressions:

$$\begin{aligned}\mathcal{F}_{3n}(\langle u,v,w\rangle) &= [y^n]\mathcal{P}(y) = \frac{\alpha_3^{n+1}-\beta_3^{n+1}}{\alpha_3-\beta_3} + \frac{uw(\alpha_3^n-\beta_3^n)}{\alpha_3-\beta_3},\\ \mathcal{F}_{3n+1}(\langle u,v,w\rangle) &= [y^n]\mathcal{Q}(y) = \frac{1+u}{\alpha_3-\beta_3}(\alpha_3^{n+1}-\beta_3^{n+1}),\\ \mathcal{F}_{3n+2}(\langle u,v,w\rangle) &= [y^n]\mathcal{R}(y) = \frac{(\alpha-w)\alpha_3^{n+1}-(\beta-w)\beta_3^{n+1}}{\alpha_3-\beta_3}.\end{aligned}$$

When the elements of $\mathfrak{M} = \{\langle u,v,w\rangle\}$ are arranged in a circle, the enumerative function $\mathcal{L}_m(\langle u,v,w\rangle)$ can also be determined. Its initial functions read as

$$\begin{aligned}&\mathcal{L}_0(\langle u,v,w\rangle) = 2,\\ &\mathcal{L}_1(\langle u,v,w\rangle) = 1,\\ &\mathcal{L}_2(\langle u,v,w\rangle) = 1+u+v,\\ &\mathcal{L}_3(\langle u,v,w\rangle) = 1+u+v+w,\\ &\mathcal{L}_4(\langle u,v,w\rangle) = 1+2u+v+w+uv+uw,\\ &\mathcal{L}_5(\langle u,v,w\rangle) = 1+2u+2v+w+u^2+v^2+uv+uw+vw,\\ &\mathcal{L}_6(\langle u,v,w\rangle) = 1+2u+2v+2w+u^2+v^2+w^2+2uv+2uw+2vw+2uvw.\end{aligned}$$

Recalling (2.2), we can write

$$\begin{aligned}\mathcal{L}_{3n}(\langle u,v,w\rangle) &= \mathcal{F}_{3n-1}(\langle u,v,w\rangle) + w\mathcal{F}_{3n-3}(u\to v, v\to w, w\to u),\\ \mathcal{L}_{3n+1}(\langle u,v,w\rangle) &= \mathcal{F}_{3n}(\langle u,v,w\rangle) + u\mathcal{F}_{3n-2}(u\to v, v\to w, w\to u),\\ \mathcal{L}_{3n+2}(\langle u,v,w\rangle) &= \mathcal{F}_{3n+1}(\langle u,v,w\rangle) + v\mathcal{F}_{3n-1}(u\to v, v\to w, w\to u).\end{aligned}$$

Then the generating functions can be determined by

$$\begin{aligned}\widetilde{\mathcal{P}}(y) &= \sum_{n=0}^{\infty} y^n\mathcal{L}_{3n} = 2 + y\mathcal{R}(y) + yw\mathcal{P}(y|u\to v, v\to w, w\to u)\\ &= \frac{2-(1+u+v+w)y}{1-(1+u+v+w)y-uvwy^2},\\ \widetilde{\mathcal{Q}}(y) &= \sum_{n=0}^{\infty} y^n\mathcal{L}_{3n+1} = \mathcal{P}(y) + yu\mathcal{Q}(y|u\to v, v\to w, w\to u)\\ &= \frac{1+(1+v+w)uy}{1-(1+u+v+w)y-uvwy^2},\\ \widetilde{\mathcal{R}}(y) &= \sum_{n=0}^{\infty} y^n\mathcal{L}_{3n+2} = v + \mathcal{Q}(y) + yv\mathcal{R}(y|u\to v, v\to w, w\to u)\\ &= \frac{1+u+v-uvy}{1-(1+u+v+w)y-uvwy^2}.\end{aligned}$$

By decomposing $\widetilde{\mathcal{P}}$, $\widetilde{\mathcal{Q}}$, and $\widetilde{\mathcal{R}}$ into partial fractions, we derive the Binet form formulae:

$$\begin{aligned}\mathcal{L}_{3n}(\langle u,v,w\rangle)=[y^n]\widetilde{\mathcal{P}}(y) &= \frac{[y^n]}{\alpha_3-\beta_3}\Big\{\frac{2\alpha_3-(1+u+v+w)}{1-y\alpha_3}-\frac{2\beta_3-(1+u+v+w)}{1-y\beta_3}\Big\}\\ &= \frac{[y^n]}{\alpha_3-\beta_3}\Big\{\frac{\alpha_3-\beta_3}{1-y\alpha_3}-\frac{\beta_3-\alpha_3}{1-y\beta_3}\Big\}=\alpha_3^n+\beta_3^n,\\ \mathcal{L}_{3n+1}(\langle u,v,w\rangle)=[y^n]\widetilde{\mathcal{Q}}(y) &= \frac{[y^n]}{\alpha_3-\beta_3}\Big\{\frac{\alpha_3+u(1+v+w)}{1-y\alpha_3}-\frac{\beta_3+u(1+v+w)}{1-y\beta_3}\Big\}\\ &= \frac{\alpha_3^{n+1}-\beta_3^{n+1}}{\alpha_3-\beta_3}+\frac{u(1+v+w)}{\alpha_3-\beta_3}(\alpha_3^n-\beta_3^n),\\ \mathcal{L}_{3n+2}(\langle u,v,w\rangle)=[y^n]\widetilde{\mathcal{R}}(y) &= \frac{[y^n]}{\alpha_3-\beta_3}\Big\{\frac{\alpha_3(1+u+v)-uv}{1-y\alpha_3}-\frac{\beta_3(1+u+v)-uv}{1-y\beta_3}\Big\}\\ &= \frac{1+u+v}{\alpha_3-\beta_3}(\alpha_3^{n+1}-\beta_3^{n+1})-uv\frac{\alpha_3^n-\beta_3^n}{\alpha_3-\beta_3}.\end{aligned}$$

Consequently, the multiple sums $\Omega_n^m(\langle u,v,w\rangle)$ can be evaluated as below.

Theorem 13. *Defining for simplicity*

$$\lambda(\mathbf{k}):=\sum_{\imath=1}^{\lfloor\frac{m+2}{3}\rfloor}k_{3\imath-2},\quad \mu(\mathbf{k}):=\sum_{\jmath=1}^{\lfloor\frac{m+1}{3}\rfloor}k_{3\jmath-1},\quad \nu(\mathbf{k}):=\sum_{\kappa=1}^{\lfloor\frac{m}{3}\rfloor}k_{3\kappa-2},$$

we have the following generating function:

$$\begin{aligned}\Omega_n^m(\langle u,v,w\rangle) &= \sum_{\mathbf{k}\in[0,n]^m}u^{\lambda(\mathbf{k})}v^{\mu(\mathbf{k})}w^{\nu(\mathbf{k})}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i}\\ &= [y^n]\frac{1}{1-y\mathcal{L}_m(\langle u,v,w\rangle)+(-1)^m y^2u^{\lfloor\frac{m+2}{3}\rfloor}v^{\lfloor\frac{m+1}{3}\rfloor}w^{\lfloor\frac{m}{3}\rfloor}}.\end{aligned}$$

In particular, there is the Binet form expression

$$\begin{aligned}\Omega_n^{3m}(\langle u,v,w\rangle) &= [y^n]\frac{1}{1-y(\alpha_3^m+\beta_3^m)+y^2\alpha_3^m\beta_3^m}\\ &= \frac{\alpha_3^{mn+m}(u,v,w)-\beta_3^{mn+m}(u,v,w)}{\alpha_3^m(u,v,w)-\beta_3^m(u,v,w)}.\end{aligned}$$

There is an interesting coincidence:

$$\begin{aligned}(1+u)\Omega_n^3(\langle u,v,w\rangle) &= [y^n]\frac{1+u}{1-y\mathcal{L}_3(\langle u,v,w\rangle)-y^2uvw}\\ &= [y^n]\frac{1+u}{1-y(1+u+v+w)-y^2uvw}\\ &= [y^n]\mathcal{Q}(y)=\mathcal{F}_{3n+1}(\langle u,v,w\rangle).\end{aligned}$$

By specifying u,v,w with concrete values, we deduce the following remarkable formulae.

4.3.1. $\boxed{u = 1, v = \omega \ \& \ w = \omega^2 : \ \omega = e^{2\pi\mathbf{i}/3}}$

It is not hard to determine

$$\begin{aligned}\mathcal{L}_{3m}(\langle 1, \omega, \omega^2\rangle) &= [y^m]\frac{2-y}{1-y-y^2} = \mathrm{L}_m,\\ \mathcal{L}_{3m+1}(\langle 1, \omega, \omega^2\rangle) &= [y^m]\frac{1}{1-y-y^2} = \mathrm{F}_{m+1}.\end{aligned}$$

Then we have the following summation formulae.

Corollary 14.

$$\begin{aligned}\Omega_n^{3m}(\langle 1, \omega, \omega^2\rangle) &= [y^n]\frac{1}{1-y\mathrm{L}_m+(-1)^m y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^{mk+k}\binom{n-k}{k}\mathrm{L}_m^{n-2k},\\ \Omega_n^{3m+1}(\langle 1, \omega, \omega^2\rangle) &= [y^n]\frac{1}{1-y\mathrm{F}_{m+1}-(-1)^m y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^{mk}\binom{n-k}{k}\mathrm{F}_{m+1}^{n-2k}.\end{aligned}$$

4.3.2. $\boxed{u = \omega, v = \omega^2 \ \& \ w = 1 : \ \omega = e^{2\pi\mathbf{i}/3}}$

It is routine to check that

$$\begin{aligned}\mathcal{L}_{3m}(\langle \omega, \omega^2, 1\rangle) &= [y^m]\frac{2-y}{1-y-y^2} = \mathrm{L}_m,\\ \mathcal{L}_{3m+2}(\langle \omega, \omega^2, 1\rangle) &= [y^m]\frac{-y}{1-y-y^2} = -\mathrm{F}_m.\end{aligned}$$

From them, we deduce the two summation formulae below.

Corollary 15.

$$\begin{aligned}\Omega_n^{3m}(\langle \omega, \omega^2, 1\rangle) &= [y^n]\frac{1}{1-y\mathrm{L}_m+(-1)^m y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^{mk+k}\binom{n-k}{k}\mathrm{L}_m^{n-2k},\\ \Omega_n^{3m+2}(\langle \omega, \omega^2, 1\rangle) &= [y^n]\frac{1}{1+y\mathrm{F}_m+(-1)^m y^2} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^{n+mk+k}\binom{n-k}{k}\mathrm{F}_m^{n-2k}.\end{aligned}$$

We remark that there is also a common Binet form expression

$$\Omega_n^{3m}(\langle 1, \omega, \omega^2\rangle) = \Omega_n^{3m}(\langle \omega, \omega^2, 1\rangle) = \frac{\alpha^{mn+n}-\beta^{mn+n}}{\alpha^m-\beta^m}.$$

4.3.3. $\boxed{u = \mathbf{i}, v = \mathbf{i}^2, w = \mathbf{i}^3 : \ \alpha_3, \beta_3 = \pm\mathbf{i}}$

It is not difficult to compute

$$\mathcal{L}_{3m}(\langle \mathbf{i}, \mathbf{i}^2, \mathbf{i}^3\rangle) = [y^m]\widetilde{\mathcal{P}}(y) = [y^m]\frac{2}{1+y^2} = 2(-1)^{\lfloor \frac{m}{2}\rfloor}\chi(m \equiv_2 0),$$

$$\begin{aligned}\mathcal{L}_{3m+1}(\langle \mathbf{i},\mathbf{i}^2,\mathbf{i}^3\rangle) &= [y^m]\widetilde{Q}(y) = [y^m]\frac{1+y}{1+y^2} = (-1)^{\lfloor\frac{m}{2}\rfloor},\\ \mathcal{L}_{3m+2}(\langle \mathbf{i},\mathbf{i}^2,\mathbf{i}^3\rangle) &= [y^m]\widetilde{\mathcal{R}}(y) = [y^m]\frac{\mathbf{i}(1+y)}{1+y^2} = (-1)^{\lfloor\frac{m}{2}\rfloor}\mathbf{i}.\end{aligned}$$

Thus, we deduce six interesting summation formulae, suggesting that the related sums possess the periodicity with respect to m of the same parity.

Corollary 16.

$$\begin{aligned}\Omega_n^{6m}(\langle \mathbf{i},\mathbf{i}^2,\mathbf{i}^3\rangle) &= [y^n]\frac{1}{1-2(-1)^m y+y^2} = (-1)^{mn}(n+1),\\ \Omega_n^{6m+3}(\langle \mathbf{i},\mathbf{i}^2,\mathbf{i}^3\rangle) &= [y^n]\frac{1}{1+y^2} = (-1)^{\lfloor\frac{n}{2}\rfloor}\chi(n\equiv_2 0),\\ \Omega_n^{6m+1}(\langle \mathbf{i},\mathbf{i}^2,\mathbf{i}^3\rangle) &= [y^n]\frac{1}{1-(-1)^m y-\mathbf{i}y^2} = (-1)^{mn}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\mathbf{i}^k,\\ \Omega_n^{6m+4}(\langle \mathbf{i},\mathbf{i}^2,\mathbf{i}^3\rangle) &= [y^n]\frac{1}{1-(-1)^m y-\mathbf{i}y^2} = (-1)^{mn}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\mathbf{i}^k,\\ \Omega_n^{6m+2}(\langle \mathbf{i},\mathbf{i}^2,\mathbf{i}^3\rangle) &= [y^n]\frac{1}{1-(-1)^m\mathbf{i}y-\mathbf{i}y^2} = (-1)^{mn}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\mathbf{i}^{n-k},\\ \Omega_n^{6m+5}(\langle \mathbf{i},\mathbf{i}^2,\mathbf{i}^3\rangle) &= [y^n]\frac{1}{1-(-1)^m\mathbf{i}y-\mathbf{i}y^2} = (-1)^{mn}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\mathbf{i}^{n-k}.\end{aligned}$$

5. Conclusions and further problems

By introducing the enumerative functions for the subsets (without consecutive elements) of the given linear and circular set $\mathfrak{M}$, we examined Carlitz' multiple sums weighted by characterizing monomials. The rational generating function is established (Theorem 2) which provides a deep generalization of Carlitz' classical result. During the course of studying related applications, we came across the following remarkable phenomena, where the underlying relations hidden behind these coincidences have not been well-understood till now.

- **Observation 1.** According to Corollaries 10 and 11, the following two equalities hold, which simply interchange two cyclic parameters $\{1,-1\}$:

$$\Omega_n^{2m}(\langle 1,-1\rangle) = \Omega_n^{2m}(\langle -1,1\rangle) = \frac{\mathrm{F}_{mn+m}}{\mathrm{F}_m},$$
$$\Omega_n^{2m+1}(\langle 1,-1\rangle) = \Omega_n^{2m+7}(\langle -1,1\rangle) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^{mk}\binom{n-k}{k}\mathrm{F}_{m+2}^{n-2k}.$$

- **Observation 2.** By comparing the formulae stated in Corollaries 14 and 15, we found two identities that permute three cyclic parameters $\{1, \omega, \omega^2\}$:

$$\Omega_n^{3m}(\langle 1, \omega, \omega^2 \rangle) = \Omega_n^{3m}(\langle \omega, \omega^2, 1 \rangle) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{mk+k} \binom{n-k}{k} \mathrm{L}_m^{n-2k},$$

$$\Omega_n^{3m+1}(\langle 1, \omega, \omega^2 \rangle) = (-1)^n \Omega_n^{3m+5}(\langle \omega, \omega^2, 1 \rangle) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{mk} \binom{n-k}{k} \mathrm{F}_{m+1}^{n-2k}.$$

- **Observation 3.** In view of Corollaries 12 and 16, we deduce the following two surprising equalities, which relate expressions involving the third roots of unity $\{\omega, \omega^2\}$ to those involving the fourth roots $\{\mathbf{i}, \mathbf{i}^2, \mathbf{i}^3\}$:

$$\Omega_n^{4m}(\langle \omega, \omega^2 \rangle) = \Omega_n^{6m}(\langle \mathbf{i}, \mathbf{i}^2, \mathbf{i}^3 \rangle) = (-1)^{mn}(n+1),$$
$$\Omega_n^{4m+2}(\langle \omega, \omega^2 \rangle) = \Omega_n^{6m+3}(\langle \mathbf{i}, \mathbf{i}^2, \mathbf{i}^3 \rangle) = (-1)^{\lfloor \frac{n}{2} \rfloor} \chi(n \equiv_2 0).$$

  It is noteworthy that the values for the two multiple sums on the second line are independent upon m, while those on the first line depend on the parity of m only.

- **Observation 4.** Finally, Corollary 16 immediately implies two further equalities with their resulting expressions being independent upon m:

$$(-1)^{mn} \Omega_n^{6m+1}(\langle \mathbf{i}, \mathbf{i}^2, \mathbf{i}^3 \rangle) = (-1)^{mn} \Omega_n^{6m+4}(\langle \mathbf{i}, \mathbf{i}^2, \mathbf{i}^3 \rangle) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \mathbf{i}^k,$$

$$(-1)^{mn} \Omega_n^{6m+2}(\langle \mathbf{i}, \mathbf{i}^2, \mathbf{i}^3 \rangle) = (-1)^{mn} \Omega_n^{6m+5}(\langle \mathbf{i}, \mathbf{i}^2, \mathbf{i}^3 \rangle) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \mathbf{i}^{n-k}.$$

Among these eight equalities, the first one can easily be justified by reversing the order of multiple sums. However, it is difficult to show the remaining seven identities by directly manipulating the sums or through combinatorial construction.

For Carlitz' original identity, there is a combinatorial proof by Benjamin and Rouse [4] through the domino tiling and an inductive proof. It would be interesting to construct similar proofs for the four pairs of just-mentioned identities.

Author contributions

Marta Na Chen: Computation, Writing, and Editing; Wenchang Chu: Original draft, Review, and Supervision. Both authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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