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Research article

Stabilization of stochastic systems driven by G-Lévy process via discrete-time feedback control

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Abstract: In this paper, we establish sufficient conditions for the mean square and quasi-sure exponential stability of stochastic differential equations driven by the G-Lévy process under discrete-time feedback control in the drift term. Departing from the conventional Lyapunov-based approach, we propose a comparative method to analyze the stabilization effect of the designed control. An example validates the effectiveness of the proposed control strategy.

Keywords: G-Lévy process; discrete-time feedback; stability
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1. Introduction

The G-expectation framework, pioneered by Peng (see [1–3]), has emerged as a powerful tool for modeling uncertainty in finance and stochastic systems. Unlike classical probability theory, G-expectation accommodates sublinearity and positive homogeneity, enabling robust formulations in risk management and stochastic control. Within this framework, G-Brownian motion and its associated stochastic calculus have been extensively studied. On that basis, stochastic differential equations driven by G-Brownian motion have been studied by many scholars. For foundational studies on such equations, we refer the readers to [4–6]. Further developments are discussed in [7, 8].

An important problem in studying stochastic differential equations is the stability. In G-expectation framework, many scholars have studied the stability of stochastic differential equations driven by G-Brownian motion. In particular, by constructing a special Lyapunov function, Hu et al. [9] derived sufficient conditions for the p-th moment stability of solutions to such equations. Zhang and Chen [10] established sufficient conditions for exponential stability and quasi-sure exponential stability for a specific class of stochastic differential equations driven by G-Brownian motion. The primary stabilization strategy employed in these studies is the G-Lyapunov method. Recently, Mao et al. [11, 12] applied discrete-time feedback control to stabilize stochastic differential equations driven by classical Brownian motion. From Mao's perspective, for an unstable stochastic system, discrete-time state feedback is introduced into the drift or diffusion term to enhance stability. Notably, Mao adopted a comparative approach rather than relying on Lyapunov function arguments. Inspired by Mao's work, Yin et al. [13] studied quasi-sure exponential stabilization of G-Brownian systems via discrete-time feedback control in the stochastic term. Subsequently, Yin et al. [14] provided verifiable stabilization criteria for stochastic differential equations driven by G-Brownian motion based on discrete-time observations.

However, both G-Brownian motion and standard Brownian motion are continuous-path processes, which limits their applicability in modeling real-world phenomena involving jumps. Then Hu and Peng [15] introduced the G-Lévy process, which has a jump part. However, due to the complexity of dependence structures, there has been little systematic development on the G-Lévy process in the literature, which is in contrast to the extensive studies on G-Brownian motion. Ren [16] provided a representation of the sublinear expectation as an upper expectation, while Paczka [17,18] developed an integration theory for G-Lévy processes and derived a corresponding Itô's formula. Wang and Gao [19] proved the existence and uniqueness of solutions to the following stochastic differential equations driven by a G-Lévy process:

$$dY(t) = b(t, Y(t))dt + h(t, Y(t))d\langle B\rangle_t + \sigma(t, Y(t))dB_t + \int_{R_0^d} K(t, Y(t), z)L(dt, dz), \tag{1.1}$$

where $Y(0) = Y_0$ is the initial value with $\hat{\mathbb{E}}[|Y_0|^2] < \infty$, $(\langle B\rangle_t)_{t\geq 0}$ is the mutual variation process of the G-Brownian motion $(B_t)_{t\geq 0}$, $L(\cdot, \cdot)$ is a Poisson random measure associated with a G-Lévy process X, and the coefficients b, h, σ, K do not need to satisfy the Lipschitz condition.

When (1.1) exhibits instability, a critical challenge is designing a discrete-time feedback control $u(t, Y([t/\tau]\tau))$ in the drift term, such that the stochastic system under control

$$dY(t) = [b(t, Y(t)) + u(t, Y([t/\tau]\tau))]dt + h(t, Y(t))d\langle B\rangle_t + \sigma(t, Y(t))dB_t + \int_{R_0^d} K(t, Y(t), z)L(dt, dz) \tag{1.2}$$

achieves stability. Here, $[t/\tau]$ is the integer part of t/τ. This approach is significantly different from the stabilization by a continuous-time feedback control, which relies on the current or delayed state, $u(t, Y(t))$ or $u(t, Y(t - \tau))$. The discrete-time strategy is not only more practical but also more cost-effective. Several significant contributions have been made in this research direction: Shen et al. [20] studied the mean square exponential stability and quasi-sure exponential stability for a stochastic system driven by a G-Lévy process. Yuan and Zhu [21] discussed the exponential stability of neutral stochastic functional differential equations driven by a G-Lévy process with discrete-time feedback control. These studies primarily relied on the G-Lyapunov method. However, as noted in [14], for the stability of a stochastic system with discrete-time feedback control, the Lyapunov method may not work in the G-expectation framework sometimes, so we should resort to other methods to deal with it.

Inspired by Mao's comparative approach for classical Brownian systems [11, 12] and Yin's extension to G-Brownian motion [14], this work considers the stabilization of a stochastic differential equation driven by a G-Lévy process (1.2) with discrete-time feedback. The main contributions of this paper are as follows: First, we develop a comparative analysis framework that circumvents the technical difficulties associated with traditional Lyapunov methods in the G-expectation setting. While Lyapunov-based approaches face challenges due to the sublinear nature and positive homogeneity of G-expectations, our comparative method provides an alternative pathway for stability analysis. Second, we establish mean-square and quasi-sure exponential stability for system (1.2), generalizing the results of Yin et al. [14] from G-Brownian motion to G-Lévy processes. A G-Lévy process makes the analysis more difficult owing to the complexity of dependence structures.

The rest of this paper is organized as follows. Section 2 presents essential preliminaries and notations within the G-expectation framework. In Section 3, we establish key lemmas and rigorously prove the main stability theorem. Section 4 provides a numerical example to validate the theoretical results. Finally, Section 5 concludes the paper.

2. Preliminaries

This section outlines essential notations and foundational concepts required for subsequent analysis. For G-Brownian motion and its calculus, see [3,22,23]. The theory of G-Lévy processes is developed in [15,17,18].

Let $\Omega := \mathbb{D}_0(R^+, R^d)$ denote the canonical space of all càdlàg functions valued in R^d endowed with the Skorohod topology. Define $\Omega_T := \{X_\cdot = w_{\cdot\wedge T} : w \in \Omega\}$. We consider the space $L^p_G(\Omega_T)$ of random variables on Ω_T equipped with a sublinear G-expectation $\hat{\mathbb{E}}$, under which the canonical process is a G-Lévy process. The norm on $L^p_G(\Omega_T)$ is given by $\|\cdot\|_p^p := \hat{\mathbb{E}}[|\cdot|^p]$. Crucially, G-expectation $\hat{\mathbb{E}}$ is characterized by two key deviations from classical expectation: (i) Sub-additivity, $\hat{\mathbb{E}}[X+Y] \le \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y], X, Y \in L^p_G(\Omega_T)$(replaces linearity); (ii) positive homogeneity, $\hat{\mathbb{E}}[\lambda X] = \lambda\hat{\mathbb{E}}[X], X \in L^p_G(\Omega_T)$ holds only for $\lambda \ge 0$, unlike classical homogeneity over all $\lambda \in R$.

There exists a subset $\mathfrak{B} \subset \mathcal{M}(\Omega_T)$ of probability measures satisfying

$$\hat{\mathbb{E}}[\xi] = \max_{P\in\mathfrak{B}} E_P(\xi) \ \ for\ all\ \xi \in L^1_G(\Omega_T).$$

The associated capacity $\mathbb{C}$ is defined as:

$$\mathbb{C}(A) := \sup_{P\in\mathfrak{B}} P(A), \ \ A \in \mathcal{B}(\Omega),$$

where $\mathcal{B}(\Omega)$ denotes the Borel σ-algebra on Ω. A is called a polar set if $\mathbb{C}(A) = 0$. A property holding outside a polar set is said to hold quasi-surely (q.s.).

The following spaces of stochastic processes will be useful.

• $M^{p,0}_G(0,T)$: Space of a simple process $\eta_t(w) = \sum_{k=0}^{N-1} \xi_k(w) I_{[t_k,t_{k+1})}(t)$, with $\xi_k \in L^p_G(\Omega_{t_k})$, $0 = t_0 \le t_1 \le \cdots t_{N-1} \le t_N = T, p \ge 1$.

• $M^p_G(0,T)$: Completion of $M^{p,0}_G(0,T)$ under the norm $\|\eta\|_{M^p_G(0,T)} = [\int_0^T \hat{\mathbb{E}}[|\eta_t|^p]dt]^{\frac{1}{p}}, \quad p \ge 1$.

• $H^S_G([0,T] \times R^d_0)$: Elementary random fields $K(r,z)(w) = \sum_{k=1}^{n-1}\sum_{l=1}^m F_{k,l}(w)\mathbb{I}_{]t_k,t_{k+1}]}(r)\psi_l(z)$, where $R^d_0 := R^d\backslash\{0\}$, $\{\psi_l\}_{l=1}^m \subset C_{b,lip}(R^d)$ have disjoint supports with $\psi_l(0) = 0$, and $F_{k,l} = \phi_{k,l}(X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}})$, $\phi_{k,l} \in C_{b,lip}(R^{d\times k})$. Here $C_{b,Lip}(R^n)$ denotes the set of bounded Lipschitz functions on R^n.

• $H^p_G([0,T] \times R^d_0)$: Completion of $H^S_G([0,T] \times R^d_0)$ under the norm

$$\|K\|^p_{H^p_G([0,T]\times R^d_0)} := \hat{\mathbb{E}}[\int_0^T \sup_{v\in\mathcal{V}} \int_{R^d_0} |K(r,z)|^p v(dz)dr], \quad p \ge 1,$$

where $\mathcal{V}$ is a set of Borel measures on $(R_0^d, \mathcal{B}(R_0^d))$ with $\sup_{v\in\mathcal{V}} v(R_0^d) < \infty$, and $\mathcal{B}(R_0^d)$ denotes the Borel σ-algebra on R_0^d.

For the G-Lévy process X, $\triangle X_u = X_u - X_{u^-}$ denote its jump; here, X_{u^-} is the left limit of at point u. The Poisson random measure is defined by $L(]s,t], A) := \sum_{s<u\leq t} \mathbb{I}_A(\Delta X_u)$, $0 < s < t < \infty$, $A \in \mathcal{B}(R_0^d)$.

For $\eta \in M_G^p(0,T)$, $K \in H_G^p([0,T] \times R_0^d)$, $p \geq 2$, the stochastic integrals $\int_0^t \eta_s dB_s$ (with respect to G-Brownian motion B_s) and $\int_0^T \int_{R_0^d} K(r,z)L(dr,dz)$ (with respect to Poisson random measure $L(\cdot,\cdot)$) are well-defined. Similarly, for $\eta \in M_G^p(0,T)$, $p \geq 1$, integrals $\int_0^t \eta_s ds$ and $\int_0^t \eta_s d\langle B\rangle_s$ can also be defined, respectively, where $(\langle B\rangle_t)_{t\geq 0}$ is the quadratic variation process of G-Brownian motion $(B_t)_{t\geq 0}$. All of these integrals belong to $L_G^p(\Omega_T)$ for $p \geq 1$.

Lemma 2.1. *[17] For $K(r,z) \in H_G^2([0,T] \times R_0^d)$, then*:

$$\hat{\mathbb{E}}[|\int_0^T \int_{R_0^d} K(r,z)L(dr,dz)|^2] \leq C_T \hat{\mathbb{E}}[\int_0^T \sup_{v\in\mathcal{V}} \int_{R_0^d} K^2(r,z)v(dz)dr],$$

where $C_T = 2(T+1)$.

For $K(r,z) \in H_G^2([0,T] \times R_0^d)$, by Theorem 13 in [18], we know that $P_t^0 := \int_0^t \int_{R_0^d} K(r,z)L(dr,dz) - \int_0^t \sup_{v\in\mathcal{V}} \int_{R_0^d} K(r,z)v(dz)dr$ is a G-martingale. Moreover, by [19], we have the following BDG-type inequality:

$$\hat{\mathbb{E}}[\sup_{0\leq t\leq T} |\int_0^t \int_{R_0^d} K(r,z)L(dr,dz)|^2] \leq 2(C+T)\hat{\mathbb{E}}[\int_0^T \sup_{v\in\mathcal{V}} \int_{R_0^d} K^2(r,z)v(dz)dr], \tag{2.1}$$

where C is a positive constant.

Lemma 2.2. *(Markov's inequality) Let $X \in L_G^p(\Omega_T)$, $p \geq 1$. Then, for each constant $M > 0$*:

$$\mathbb{C}(|X| > M) \leq \frac{\hat{\mathbb{E}}[|X|^p]}{M^p}.$$

Lemma 2.3. *[3,4] For $p \geq 2$, $\eta \in M_G^p(0,T)$. Then*:

$$\underline{\sigma}^2 \hat{\mathbb{E}}[\int_0^T |\eta_r|^2 dr] \leq \hat{\mathbb{E}}[|\int_0^T \eta_r dB_r|^2] = \hat{\mathbb{E}}[\int_0^T |\eta_r|^2 d\langle B\rangle_r] \leq \bar{\sigma}^2 \hat{\mathbb{E}}[\int_0^T |\eta_r|^2 dr],$$

$$\hat{\mathbb{E}}[\sup_{0\leq u\leq T} |\int_0^u \eta_r dB_r|^p] \leq C_p T^{\frac{p}{2}-1} \int_0^T \hat{\mathbb{E}}|\eta_r|^p dr,$$

where constants $\underline{\sigma}^2 \leq \bar{\sigma}^2$, $C_p > 0$ is a constant that is only dependent on p.

Lemma 2.4. *[4] For $p \geq 1$, $\eta \in M_G^p(0,T)$. Then there exists a positive constant C_p' depending on p, such that*:

$$\hat{\mathbb{E}}[\sup_{0\leq u\leq T} |\int_0^u \eta_r d\langle B\rangle_r|^p] \leq C_p' T^{p-1} \int_0^T \hat{\mathbb{E}}|\eta_r|^p dr.$$

Remark 2.1. *When there exists a unique probability measure $P \in \mathfrak{B}$, the G-expectation $\hat{\mathbb{E}}$ collapses to the classical linear expectation E_P. As a result, the G-Brownian motion coincides with the classical Brownian motion under P, and the G-Lévy process reduces to a classical Lévy process with the Lévy measure given by the unique measure $v \in \mathcal{V}$. In this case, the volatility parameters satisfy $\underline{\sigma}^2 = \bar{\sigma}^2$, and the quadratic variation of the Brownian motion becomes $\langle B\rangle_t = t$.*

3. Stability of controlled stochastic systems

Consider an n-dimensional unstable system:

$$dX(t) = b(t, X(t))dt + h_{ij}(t, X(t))d\langle B^i, B^j\rangle_t + \sigma_i(t, X(t))dB^i_t + \int_{R^d_0} K(t, X(t), z)L(dt, dz), \tag{3.1}$$

with initial condition $X(0) = X_0 \in R^n$. Here, $(\langle B^i, B^j\rangle_t)_{t\geq 0}$ represents the mutual variation process of d-dimensional G-Brownian motion $(B_t)_{t\geq 0}$, $i, j = 1, 2, \cdots, d$. For $x \in R^n$, $b(\cdot, x), h_{ij}(\cdot, x), \sigma_i(\cdot, x) \in M^2_G([0, T]; R^n), K(\cdot, x, \cdot) \in H^2_G([0, T] \times R^d_0; R^n)$, where $M^2_G([0, T]; R^n)$ and $H^2_G([0, T] \times R^d_0; R^n)$ denote the spaces of n-dimension stochastic processes where each element belongs to $M^2_G(0, T)$ and $H^2_G([0, T] \times R^d_0)$, respectively. In this paper, we use the Einstein convention, i.e., the above repeated indices of i and j within one term imply the summation form 1 to d,

$$\int_0^t h_{ij}(s, Y_s)d\langle B^i, B^j\rangle_s := \sum_{i,j=1}^{d} \int_0^t h_{ij}(s, Y_s)d\langle B^i, B^j\rangle_s,$$

$$\int_0^t \sigma_i(s, Y_s)dB^i_s := \sum_{i=1}^{d} \int_0^t \sigma_i(s, Y_s)dB^i_s.$$

Remark 3.1. *Under standard regularity conditions from Wang and Gao [19], Eq (3.1) admits a unique solution X. Furthermore, if there exists a Lyapunov-type function $V(t, x) \in C^{1,2}([0, +\infty] \times R^n, R^+)$ satisfying $C_1|x|^2 \leq V(t, x) \leq C_2|x|^2$ and the differential inequality $V_t(t, x) + V_x^T(t, x)b(t, x) + \sup_{Q\in\mathcal{Q}} tr[(V_x^T(t, x)h_{ij}(t, x) + \frac{1}{2}\sigma_i^T(t, x)V_{xx}(t, X_t)\sigma_j(t, x))QQ^T] + \sup_{v\in\mathcal{V}} \int_{R^d_0}(V(t, x + K(t, x, z)) - V(t, x))v(dz) \geq \lambda V(t, x)$, then [20, Theorem 3.4] establishes mean-square exponential instability. Here, $C^{1,2}([0, +\infty] \times R^n, R^+)$ is the space of positive functions such that V_t, V_x, V_{xx} are continuous on $[0, +\infty) \times R^n$, V_{xx} satisfies local Lipschitz condition, C_1, C_2, λ are positive constants, $\mathcal{Q}$ is a set of all $d\times d$-dimensional positive definite symmetric matrices in the space of all $d\times d$-dimensional symmetric matrices, and x^T denote the transpose of x.*

To counteract system instability, we introduce discrete-time feedback control $u(t, [t/\tau]\tau)$ in the drift term, yielding the controlled dynamics:

$$\begin{aligned} dX(t) = {} & [b(t, X(t)) + u(t, X(\delta_t))]dt + h_{ij}(t, X(t))d\langle B^i, B^j\rangle_t \\ & + \sigma_i(t, X(t))dB^i_t + \int_{R^d_0} K(t, X(t), z)L(dt, dz), \end{aligned} \tag{3.2}$$

with the initial value $X(0) = X_0 \in R^n$ such that $\hat{\mathbb{E}}[|X_0|^2] < \infty$. Here, $\delta_t = [t/\tau]\tau$, $\tau > 0$ is a constant which stands for the duration between two consecutive state observations, and $[t/\tau] = \sup\{k \in Z; k \leq t/\tau\}$ denotes the integer part.

For comparative analysis, we simultaneously examine the reference system with continuous feedback:

$$dY(t) = [b(t, Y(t)) + u(t, Y(t))]dt + h_{ij}(t, Y(t))d\langle B^i, B^j\rangle_t$$

$$+ \sigma_i(t, Y(t))dB_t^i + \int_{R_0^d} K(t, Y(t), z)L(dt, dz), \tag{3.3}$$

with the same initial value $Y(0) = X_0 \in R^n$ such that $\hat{\mathbb{E}}[|X_0|^2] < \infty$.

The analysis proceeds under the following assumptions:

(H1) For $x, y \in R^n, i, j = 1, 2, \cdots, d$, there exists a constant $M > 0$ such that

$$|b(t,x) - b(t,y)| \vee |h_{ij}(t,x) - h_{ij}(t,y)| \vee |\sigma_i(t,x) - \sigma_i(t,y)| \vee |u(t,x) - u(t,y)| \leq M|x-y|$$

and

$$\sup_{v \in \mathcal{V}} \int_{R_0^d} |K(t,x,z) - K(t,y,z)|^2 v(dz) \leq M|x-y|^2.$$

(H2) For a constant $\eta > 0$ and $x, y \in R^n, i, j = 1, 2, \cdots, d$,

$$\sup_{Q \in \mathcal{Q}} tr[(2x^T h_{ij}(t,x) + \sigma_j^T(t,x)\sigma_i(t,x))QQ^T]$$
$$+ x^T[b(t,x) + u(t,x)] + \sup_{v \in \mathcal{V}} \int_{R_0^d} [|x + K(t,x,z)|^2 - |x|^2] v(dz) \leq -\eta |x|^2. \tag{3.4}$$

Remark 3.2. *For $t \geq 0$, we suppose that $b(t,0) = u(t,0) = h_{ij}(t,0) = \sigma_i(t,0) = K(t,0,z) = 0$ to ensure the existence of the zero solution. Under assumption (H1), it follows from [19] that the stochastic systems (3.2) and (3.3) admit unique solutions, denoted by X and Y respectively, with finite second moments. The Lipschitz constant M in (H1) bounds the system's sensitivity to state variations, thereby ensuring well-posedness. The decay rate η in (H2) defines the minimum guaranteed stabilization strength and the exponential convergence rate of the feedback control.*

Lemma 3.1. *Under (H1) and (H2), the solution Y of the reference system satisfies*

$$\hat{\mathbb{E}}[|Y(t)|^2] \leq \hat{\mathbb{E}}[|X_0|^2] e^{-\eta t}.$$

Proof. Consider the application of G-Itô's formula to $e^{\eta t}|Y(t)|^2$, we have:

$$\begin{aligned}
e^{\eta t}|Y(t)|^2 &= |X_0|^2 + \int_0^t \eta e^{\eta r}|Y(r)|^2 dr + \int_0^t 2e^{\eta r} Y^T(r)[b(r,Y(r)) + u(r,Y(r))]dr \\
&\quad + \int_0^t e^{\eta r}[2Y^T(r)h_{ij}(r,Y(r)) + \sigma_j^T(r,Y(r))\sigma_i(r,Y(r))]d\langle B^i, B^j\rangle_r \\
&\quad + \int_0^t 2Y^T(r)\sigma_i(r,Y(r))dB_r^i + \int_0^t \int_{R_0^d} [|Y(r^-) + K(r,Y(r),z)|^2 - |Y(r^-)|^2]L(dr,dz) \\
&= |X_0|^2 + \int_0^t \eta e^{\eta r}|Y(r)|^2 dr + \int_0^t 2e^{\eta r} Y^T(r)[b(r,Y(r)) + u(r,Y(r))]dr \\
&\quad + \int_0^t e^{\eta r} \sup_{Q \in \mathcal{Q}} tr[(2Y^T(r)h_{ij}(r,Y(r)) + \sigma_j^T(r,Y(r))\sigma_i(r,Y(r)))QQ^T]dr \\
&\quad + \int_0^t \sup_{v \in \mathcal{V}} \int_{R_0^d} e^{\eta r}[|Y(r^-) + K(r,Y(r),z)|^2 - |Y(r^-)|^2]v(dz)dr
\end{aligned}$$

$$+\int_0^t 2e^{\eta r}Y^T(r)\sigma_i(r,Y(r))dB_r^i + M_t^0 + P_t^0, \tag{3.5}$$

where

$$\begin{aligned} M_t^s =& \int_s^t e^{\eta r}[2Y^T(r)h_{ij}(r,Y(r)) + \sigma_j^T(r,Y(r))\sigma_i(r,Y(r))]\, d\langle B^i,B^j\rangle_r \\ &- \int_s^t \sup_{Q\in\mathcal{Q}} e^{\eta r}\,\mathrm{tr}[(2Y^T(r)h_{ij}(r,Y(r)) + \sigma_j^T(r,Y(r))\sigma_i(r,Y(r)))QQ^T]\, dr, \\ P_t^s =& \int_s^t \int_{R_0^d} e^{\eta r}[|Y(r^-)+K(r,Y(r),z)|^2 - |Y(r^-)|^2]L(dr,dz) \\ &- \int_s^t \sup_{v\in\mathcal{V}} \int_{R_0^d} e^{\eta r}[|Y(r^-)+K(r,Y(r),z)|^2 - |Y(r^-)|^2]v(dz)\, dr. \end{aligned}$$

It is known that $\{M_t^0\}_{t\geq 0}$ is a G-martingale. Moreover, by [18, Theorem 13], $\{P_t^0\}_{t\geq 0}$ is also a G-martingale.

By assumption (H2) and taking G-expectation on both side of (3.5), we obtain:

$$\hat{\mathbb{E}}[|Y(t)|^2] \leq \hat{\mathbb{E}}[|X_0|^2]e^{-\eta t}.$$

Lemma 3.2. *Let assumption (H1) hold, then the solution X of stochastic system (3.2) satisfies*

$$\hat{\mathbb{E}}[|X(t)|^2] \leq \hat{\mathbb{E}}[|X_0|^2]e^{(6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}} v(R_0^d))t}$$

and

$$\begin{aligned} &\hat{\mathbb{E}}[|X(t)-X(\delta_t)|^2] \\ &\leq 4M\tau[4M\tau + M(C_2^{'}\tau+\bar{\sigma}^2) + 2\tau + 2]\hat{\mathbb{E}}[|X_0|^2]e^{(6M+2M\bar{\sigma}^2+\sup_{v\in\mathcal{V}} v(R_0^d)+\bar{\sigma}^2M^2)t}. \end{aligned}$$

Proof. Applying G-Itô's formula to $|X(t)|^2$, we derive that:

$$\begin{aligned} |X(t)|^2 =& |X_0|^2 + \int_0^t 2X^T(r)[b(r,X(r)) + u(r,X(\delta_r))]dr + \int_0^t 2X^T(r)\sigma_i(r,X(r))dB_r^i \\ &+ \int_0^t [2X^T(r)h_{ij}(r,X(r)) + \sigma_j^T(r,X(r))\sigma_i(r,X(r))]d\langle B^i,B^j\rangle_r \\ &+ P_t^0 + \int_0^t \sup_{v\in\mathcal{V}} \int_{R_0^d} [|X(r^-)+K(r,X(r),z)|^2 - |X(r^-)|^2]v(dz)dt, \end{aligned} \tag{3.6}$$

where $P_t^s = \int_s^t \int_{R_0^d}[|X(r^-)+K(r,X(r),z)|^2 - |X(r^-)|^2]L(dr,dz) - \int_s^t \sup_{v\in\mathcal{V}}\int_{R_0^d}[|X(r^-)+K(r,X(r),z)|^2 - |X(r^-)|^2]v(dz)dr$ is a G-martingale.

Taking G-expectation on both sides of (3.6), by assumption (H1) and Lemma 2.3, we have:

$$\hat{\mathbb{E}}[|X(t)|^2] \leq \hat{\mathbb{E}}[|X_0|^2] + (3M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2)\int_0^t \hat{\mathbb{E}}[|X(r)|^2]dr + M\int_0^t \hat{\mathbb{E}}[|X(\delta_r)|^2]dr$$

$$
\begin{aligned}
&+ \hat{\mathbb{E}}\Big[\int_0^t \sup_{v\in\mathcal{V}} \int_{R_0^d} [2X(r^-)K(r, X(r), z) + K^2(r, X(r), z)]v(dz)dr\Big] \\
&\le \hat{\mathbb{E}}[|X_0|^2] + (3M + 2M\bar{\sigma}^2 + M^2\bar{\sigma}^2)\int_0^t \hat{\mathbb{E}}[|X(r)|^2]dr + M\int_0^t \hat{\mathbb{E}}[|X(\delta_r)|^2]dr \\
&+ \sup_{v\in\mathcal{V}} v(R_0^d)\int_0^t \hat{\mathbb{E}}[|X(r)|^2]dr + 2M\int_0^t \hat{\mathbb{E}}[|X(r)|^2]dr \\
&= \hat{\mathbb{E}}[|X_0|^2] + (5M + 2M\bar{\sigma}^2 + M^2\bar{\sigma}^2 + \sup_{v\in\mathcal{V}} v(R_0^d))\int_0^t \hat{\mathbb{E}}[|X(r)|^2]dr \\
&+ M\int_0^t \hat{\mathbb{E}}[|X(\delta_r)|^2]dr \\
&\le \hat{\mathbb{E}}[|X_0|^2] + (6M + 2M\bar{\sigma}^2 + M^2\bar{\sigma}^2 + \sup_{v\in\mathcal{V}} v(R_0^d))\int_0^t \sup_{0\le s\le r} \hat{\mathbb{E}}[|X(s)|^2]dr.
\end{aligned}
\tag{3.7}
$$

Noting that $\int_0^t \sup_{0\le s\le r} \hat{\mathbb{E}}[|X(s)|^2]dr$ is increasing in t, we obtain that:

$$
\sup_{0\le s\le t} \hat{\mathbb{E}}[|X(s)|^2] \le \hat{\mathbb{E}}[|X_0|^2] + (6M + 2M\bar{\sigma}^2 + M^2\bar{\sigma}^2 + \sup_{v\in\mathcal{V}} v(R_0^d))\int_0^t \sup_{0\le s\le r} \hat{\mathbb{E}}[|X(s)|^2]dr. \tag{3.8}
$$

Then, by Gronwall's inequality, we have:

$$
\hat{\mathbb{E}}[|X(t)|^2] \le \hat{\mathbb{E}}[|X_0|^2]e^{(6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}} v(R_0^d))t}. \tag{3.9}
$$

For $t \in [l\tau, (l+1)\tau)$, where $l \ge 0$ is an integer, we have $\delta_t = [t/\tau]\tau = l\tau$. According to Eq (3.2), we obtain:

$$
\begin{aligned}
|X(t) - X(\delta_t)|^2 &= |X(t) - X(l\tau)|^2 \\
&= |\int_{l\tau}^t [b(r, X(r)) + u(r, X(\delta_r))]dr + \int_{l\tau}^t h_{ij}(r, X(r))d\langle B^i, B^j\rangle_r \\
&\quad + \int_{l\tau}^t \sigma_i(r, X(r))dB_r^i + \int_{l\tau}^t \int_{R_0^d} K(r, X(r), z)L(dr, dz)|^2.
\end{aligned}
\tag{3.10}
$$

By elementary inequality, it holds that:

$$
\begin{aligned}
&|X(t) - X(\delta_t)|^2 \\
&\le 4|\int_{l\tau}^t [b(r, X(r)) + u(r, X(\delta_r))]dr|^2 + 4|\int_{l\tau}^t h_{ij}(r, X(r))d\langle B^i, B^j\rangle_r|^2 \\
&\quad + 4|\int_{l\tau}^t \sigma_i(r, X(r))dB_r^i|^2 + 4|\int_{l\tau}^t \int_{R_0^d} K(r, X(r), z)L(dr, dz)|^2.
\end{aligned}
\tag{3.11}
$$

Taking G-expectation on both sides, by (H1), Lemmas 2.1, 2.3, and 2.4, we have:

$$
\begin{aligned}
&\hat{\mathbb{E}}[|X(t) - X(\delta_t)|^2] \\
&\le [8M^2\tau + 4M^2(C_2'\tau + \bar{\sigma}^2) + 8M\tau + 8M]\int_{l\tau}^t \hat{\mathbb{E}}[|X(r)|^2]dr + 8M^2\tau^2\hat{\mathbb{E}}[|X(l\tau)|^2],
\end{aligned}
\tag{3.12}
$$

where $C_2^{'}$ is the constant in Lemma 2.4.

Together with (3.9), this implies that:

$$\begin{aligned}&\hat{\mathbb{E}}[|X(t)-X(\delta_t)|^2]\\&\leq[16M^2\tau^2+4M^2(C_2^{'}\tau+\bar{\sigma}^2)\tau+8M\tau^2+8M\tau]\hat{\mathbb{E}}[|X_0|^2]e^{(6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}}v(R_0^d))t}.\end{aligned}\tag{3.13}$$

Lemma 3.3. *Let assumptions (H1) and (H2) hold, then*:

$$\begin{aligned}&\hat{\mathbb{E}}[|X(t)-Y(t)|^2]\\&\leq J(\tau)\hat{\mathbb{E}}[|X_0|^2](e^{(13M+4M\bar{\sigma}^2+2\sup_{v\in\mathcal{V}}v(R_0^d)+2\bar{\sigma}^2M^2)t}-1),\end{aligned}\tag{3.14}$$

where $J(\tau)=\frac{2M[16M^2\tau^2+4M^2(C_2^{'}\tau+\bar{\sigma}^2)\tau+8M\tau^2+8M\tau]}{6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}}v(R_0^d)}$.

Proof. Applying G-Itô's formula to $|X(t)-Y(t)|^2$, we have:

$$\begin{aligned}&|X(t)-Y(t)|^2\\&=\int_0^t 2(X(r)-Y(r))^T[b(r,X(r))-b(r,Y(r))+u(r,X(\delta_r))-u(r,Y(r))]dr\\&\quad+\int_0^t 2(X(r)-Y(r))^T(h_{ij}(r,X(r))-h_{ij}(r,Y(r)))d\langle B^i,B^j\rangle_r\\&\quad+\int_0^t(\sigma_j(r,X(r))-\sigma_j(r,Y(r)))^T(\sigma_i(r,X(r))-\sigma_i(r,Y(r)))d\langle B^i,B^j\rangle_r\\&\quad+\int_0^t 2(X(r)-Y(r))^T(\sigma_i(r,X(r))-\sigma_i(r,Y(r)))dB_r^i\\&\quad+\int_0^t\int_{R_0^d}[|X(r^-)-Y(r^-)+K(r,X(r),z)-K(r,Y(r),z)|^2-|X(r^-)-Y(r^-)|^2]L(dr,dz).\end{aligned}\tag{3.15}$$

Taking the G-expectation on both sides and using assumption (H1), similarly to that in (3.7), we have:

$$\begin{aligned}&\hat{\mathbb{E}}[|X(t)-Y(t)|^2]\\&\leq(3M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2)\int_0^t\hat{\mathbb{E}}[|X(r)-Y(r)|^2]dr+M\int_0^t\hat{\mathbb{E}}[|X(\delta_r)-Y(r)|^2]dr\\&\quad+\hat{\mathbb{E}}[\int_0^t\sup_{v\in\mathcal{V}}\int_{R_0^d}[|X(r^-)-Y(r^-)+K(r,X(r),z)-K(r,Y(r),z)|^2-|X(r^-)-Y(r^-)|^2]v(dz)dr]\\&\leq(5M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2)\int_0^t\hat{\mathbb{E}}[|X(r)-Y(r)|^2]dr+2M\int_0^t\hat{\mathbb{E}}[|X(\delta_r)-X(r)|^2]dr\\&\quad+(\sup_{v\in\mathcal{V}}v(R_0^d)+2M)\int_0^t\hat{\mathbb{E}}[|X(r)-Y(r)|^2]dr\\&=(7M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}}v(R_0^d))\int_0^t\hat{\mathbb{E}}[|X(r)-Y(r)|^2]dr+2M\int_0^t\hat{\mathbb{E}}[|X(\delta_r)-X(r)|^2]dr.\end{aligned}\tag{3.16}$$

By Gronwall's inequality, we deduce:

$$\hat{\mathbb{E}}[|X(t)-Y(t)|^2]\leq 2Me^{(7M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}}v(R_0^d))t}\int_0^t\hat{\mathbb{E}}[|X(\delta_r)-X(r)|^2]dr.\tag{3.17}$$

Substituting the bound from Lemma 3.2 yields the desired comparison estimate:

$$\begin{aligned}
&\hat{\mathbb{E}}[|X(t)-Y(t)|^2]\\
&\leq \frac{2M[16M^2\tau^2+4M^2(C_2'\tau+\bar{\sigma}^2)\tau+8M\tau^2+8M\tau]}{6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}}v(R_0^d)}\\
&\cdot\hat{\mathbb{E}}[|X_0|^2](e^{(13M+4M\bar{\sigma}^2+2\sup_{v\in\mathcal{V}}v(R_0^d)+2\bar{\sigma}^2M^2)t}-1).
\end{aligned}\tag{3.18}$$

Let $J(\tau)=\frac{2M[16M^2\tau^2+4M^2(C_2'\tau+\bar{\sigma}^2)\tau+8M\tau^2+8M\tau]}{6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}}v(R_0^d)}$, then the proof is complete.

Lemma 3.4. *Given a suitable constant $\theta\in(0,1)$ and assuming the conditions of Lemma 3.1 hold, then for $0<\tau<\bar{\tau}$, the system (3.2) satisfies*

$$\hat{\mathbb{E}}[|X(l\kappa\tau)|^2]\leq\hat{\mathbb{E}}[|X_0|^2]e^{-\gamma l\kappa\tau}.\tag{3.19}$$

Here, $\bar{\tau}$ is the unique root to the following equation:

$$2J(\tau)(e^{(13M+4M\bar{\sigma}^2+2\sup_{v\in\mathcal{V}}v(R_0^d)+2\bar{\sigma}^2M^2)(\tau+\frac{\ln(\frac{2}{\theta})}{\eta})}-1)=1-\theta,\tag{3.20}$$

where $\gamma>0$, l and κ are positive integers and η is the constant in Lemma 3.1.

Proof. It is obvious that $J(\tau)$ is continuous and increasing on $\tau>0$, $J(0)=0$, and $\lim_{\tau\to\infty}J(\tau)=+\infty$. Then it is easy to see that the left side of (3.20) is continuous and increasing on $\tau>0$. Therefore, Eq (3.20) admits a unique solution $\bar{\tau}$. Let $X(l\tau)=X_l$ for $l=0,1,2,\cdots$, where $\tau\in(0,\bar{\tau})$. It is clear that

$$e^{-\eta\kappa\tau}\leq\frac{\theta}{2},$$

if the following inequality

$$\frac{\ln(\frac{2}{\theta})}{\eta\tau}\leq\kappa<1+\frac{\ln(\frac{2}{\theta})}{\eta\tau}\tag{3.21}$$

holds for a positive integer κ.

By Lemma 3.1, we know that:

$$\hat{\mathbb{E}}[|Y_\kappa|^2]\leq\hat{\mathbb{E}}[|X_0|^2]e^{-\eta\kappa\tau}\leq\frac{\theta}{2}\hat{\mathbb{E}}[|X_0|^2],\tag{3.22}$$

where $Y_\kappa=Y(\kappa\tau)$.

Moreover, by elementary inequality, Lemma 3.3, and (3.22), we have:

$$\begin{aligned}
&\hat{\mathbb{E}}[|X_\kappa|^2]\\
&\leq 2\hat{\mathbb{E}}[|X_\kappa-Y_\kappa|^2]+2\hat{\mathbb{E}}[|Y_\kappa|^2]\\
&\leq[2J(\tau)(e^{(13M+4M\bar{\sigma}^2+2\sup_{v\in\mathcal{V}}v(R_0^d)+2\bar{\sigma}^2M^2)\kappa\tau}-1)+\theta]\hat{\mathbb{E}}[|X_0|^2].
\end{aligned}\tag{3.23}$$

By (3.21), it is easy to show that:

$$e^{(13M+4M\bar{\sigma}^2+2\sup_{v\in\mathcal{V}}v(R_0^d)+2\bar{\sigma}^2M^2)\kappa\tau}\leq e^{(13M+4M\bar{\sigma}^2+2\sup_{v\in\mathcal{V}}v(R_0^d)+2\bar{\sigma}^2M^2)(\tau+\frac{\ln(\frac{2}{\theta})}{\eta})}.$$

Then it follows from (3.20), we have:

$$\theta + 2J(\tau)(e^{(13M+4M\bar{\sigma}^2+2\sup_{v\in\mathcal{V}} v(R_0^d)+2\bar{\sigma}^2M^2)\kappa\tau} - 1) \leq 1.$$

Hence, for some positive constant γ, we obtain:

$$\theta + 2J(\tau)(e^{(13M+4M\bar{\sigma}^2+2\sup_{v\in\mathcal{V}} v(R_0^d)+2\bar{\sigma}^2M^2)\kappa\tau} - 1) = e^{-\gamma\kappa\tau}.$$

Consequently, we get:

$$\hat{\mathbb{E}}[|X_\kappa|^2] \leq \hat{\mathbb{E}}[|X_0|^2]e^{-\gamma\kappa\tau}. \tag{3.24}$$

In the following, we analyze the solution of (3.2) on $t \geq \kappa\tau$. $X(\kappa\tau)$ can be regarded as the initial value at time $t = \kappa\tau$. It follows from (3.24) that:

$$\hat{\mathbb{E}}[|X_{2\kappa}|^2] \leq \hat{\mathbb{E}}[|X_\kappa|^2]e^{-\gamma\kappa\tau} \leq \hat{\mathbb{E}}[|X_0|^2]e^{-2\gamma\kappa\tau}. \tag{3.25}$$

Repeating the same process as above, we derive that:

$$\hat{\mathbb{E}}[|X_{l\kappa}|^2] \leq \hat{\mathbb{E}}[|X_{(l-1)\kappa}|^2]e^{-\gamma\kappa\tau} \leq \hat{\mathbb{E}}[|X_0|^2]e^{-l\gamma\kappa\tau}. \tag{3.26}$$

Now, we will show the main result of our work.

Theorem 3.1. *Let the conditions of Lemma 3.1 hold and for $\tau < \bar{\tau}$, then the solution of stochastic system (3.2) is mean square exponentially stable*:

$$\hat{\mathbb{E}}[|X(t)|^2] \leq \bar{C}\hat{\mathbb{E}}[|X_0|^2]e^{-\gamma t},$$

where $\bar{C} > 0$ is a constant.

Moreover, the solution is quasi-sure exponentially stable:

$$\limsup_{t\to\infty} \frac{\ln|X(t)|}{t} \leq -\frac{\gamma}{4}, \quad q.s.$$

where $\bar{\tau}$ and γ are defined in Lemma 3.4.

Proof. Let $\tau \in (0, \bar{\tau})$. For $t \in [l\kappa\tau, (l+1)\kappa\tau), l = 1, 2, \cdots$, it follows from Lemma 3.2 that:

$$\begin{aligned}\hat{\mathbb{E}}[|X(t)|^2] &\leq \hat{\mathbb{E}}[|X_{l\kappa}|^2]e^{(6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}} v(R_0^d))(t-l\kappa\tau)} \\ &\leq \hat{\mathbb{E}}[|X_{l\kappa}|^2]e^{(6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}} v(R_0^d))\kappa\tau}.\end{aligned} \tag{3.27}$$

Combining with Lemma 3.4, we have:

$$\begin{aligned}\hat{\mathbb{E}}[|X(t)|^2] &\leq \hat{\mathbb{E}}[|X_0|^2]e^{(6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}} v(R_0^d))\kappa\tau}e^{-\gamma l\kappa\tau} \\ &\leq \hat{\mathbb{E}}[|X_0|^2]e^{(6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}} v(R_0^d)+\gamma)\kappa\tau}e^{-\gamma t} \\ &\leq \bar{C}\hat{\mathbb{E}}[|X_0|^2]e^{-\gamma t},\end{aligned} \tag{3.28}$$

where $\bar{C} = e^{(6M+2M\bar{\sigma}^2+M^2\bar{\sigma}^2+\sup_{v\in\mathcal{V}} v(R_0^d)+\gamma)\kappa\tau}$. Therefore, the solution of (3.2) is mean square exponentially stable.

In the following, we prove that the solution of (3.2) is quasi-sure exponentially stable. For $l\kappa\tau \le t \le u \le (l+1)\kappa\tau$, by using assumption (H1), Lemmas 2.3, 2.4, and BDG-type inequality (2.1), we have:

$$\begin{aligned}
&\hat{\mathbb{E}}[\sup_{l\kappa\tau\le t\le u}|X(t)|^2] \\
&\le 5\hat{\mathbb{E}}[|X(l\kappa\tau)|^2] + 5\hat{\mathbb{E}}[\sup_{l\kappa\tau\le t\le u}|\int_{l\kappa\tau}^t [b(r,X(r)) + u(r,X(\delta_r))]dr|^2] \\
&\quad + 5\hat{\mathbb{E}}[\sup_{l\kappa\tau\le t\le u}|\int_{l\kappa\tau}^t h_{ij}(r,X(r))d\langle B^i,B^j\rangle_r|^2] + 5\hat{\mathbb{E}}[\sup_{l\kappa\tau\le t\le u}|\int_{l\kappa\tau}^t \sigma_i(r,X(r))dB_r^i|^2] \\
&\quad + 5\hat{\mathbb{E}}[\sup_{l\kappa\tau\le t\le u}|\int_{l\kappa\tau}^t\int_{R_0^d} K(t,X(r),z)L(dr,dz)|^2] \\
&\le 5\hat{\mathbb{E}}[|X(l\kappa\tau)|^2] + [20M^2\kappa\tau + 5M^2\kappa\tau C_2' + 5M^2C_2 + 10M(C+\kappa\tau)]\int_{l\kappa\tau}^u \hat{\mathbb{E}}[\sup_{l\kappa\tau\le t\le s}|X(t)|^2]ds,
\end{aligned} \tag{3.29}$$

where C_2', C_2, and C are constants in Lemmas 2.3, 2.4, and inequality (2.1), respectively.

By the Gronwall's inequality, we have:

$$\hat{\mathbb{E}}[\sup_{l\kappa\tau\le t\le (l+1)\kappa\tau}|X(t)|^2] \le \tilde{C}\hat{\mathbb{E}}[|X(l\kappa\tau)|^2], \tag{3.30}$$

where $\tilde{C} = 5e^{[20M^2\kappa\tau+5M^2\kappa\tau C_2'+5M^2C_2+10M(C+\kappa\tau)]\kappa\tau}$.

By Lemma 3.4, for all $l = 0, 1, 2, \cdots$, we have:

$$\hat{\mathbb{E}}[\sup_{l\kappa\tau\le t\le (l+1)\kappa\tau}|X(t)|^2] \le \tilde{C}\hat{\mathbb{E}}[|X_0|^2]e^{-l\gamma\kappa\tau}. \tag{3.31}$$

In view of Markov's inequality, we obtain:

$$\mathbb{C}(\sup_{l\kappa\tau\le t\le (l+1)\kappa\tau}|X(t)|^2 \ge e^{-\frac{1}{2}\gamma l\kappa\tau}) \le \tilde{C}\hat{\mathbb{E}}[|X_0|^2]e^{-\frac{1}{2}l\kappa\tau\gamma}.$$

Using the Borel-Cantelli lemma, we deduce that there exists a $l_0(w)$, such that for $l > l_0(w)$,

$$\sup_{l\kappa\tau\le t\le (l+1)\kappa\tau}|X(t)|^2 \le e^{-\frac{1}{2}\gamma l\kappa\tau},\ q.s.$$

Then for $l\kappa\tau \le t \le (l+1)\kappa\tau$,

$$\frac{\ln|X(t)|}{t} = \frac{\ln|X(t)|^2}{2t} \le \frac{\ln|X(t)|^2}{2l\kappa\tau} \le \frac{\ln\sup_{l\kappa\tau\le t\le (l+1)\kappa\tau}|X(t)|^2}{2l\kappa\tau} \le -\frac{\gamma}{4}. \tag{3.32}$$

Taking limsup in (3.32) results in a quasi-sure exponential estimate, i.e.,

$$\limsup_{t\to\infty}\frac{\ln|X(t)|}{t} \le -\frac{\gamma}{4},\ \ q.s..$$

4. Example

To illustrate the theoretical framework, we analyze the following stochastic differential equation driven by a G-Lévy process with time-dependent coefficients:

$$\begin{aligned} dX(t) =& b(t, X(t))dt + h_{ij}(t, X(t))d\langle B^i, B^j\rangle_t \\ &+ \sigma_i(t, X(t))dB_t^i + \int_{R_0^d} K(t, X(t), z)L(dt, dz), \end{aligned} \tag{4.1}$$

with initial value $X(0) = X_0$ such that $\hat{\mathbb{E}}[|X_0|^2] < \infty$.

Let $b(t, X(t)) = (0.1 + 0.05\sin t)X(t), h_{ij}(t, X(t)) = (0.08 + 0.02\cos t)X(t), \sigma_i(t, x) = (0.12 + 0.03\sin 2t)X(t)$, and $K(t, X(t), z) = (0.15 + 0.05\cos t)X(t)R(z)$.

In the G-expectation framework, the uncertainty parameters $\sup_{Q\in\mathcal{Q}} \operatorname{tr}(QQ^T)$ and $\sup_{v\in\mathcal{V}} \int_{R_0^d}(|R(z)| + |R(z)|^2)v(dz)$ characterize the volatility and jump uncertainty, respectively. We choose:

$$\sup_{Q\in\mathcal{Q}} \operatorname{tr}(QQ^T) = 0.9, \quad \sup_{v\in\mathcal{V}} \int_{R_0^d}(|R(z)| + |R(z)|^2)v(dz) = 0.7.$$

These values represent a system with moderate uncertainty in both the diffusion and jump components. Then assumption (H1) is satisfied with $M = 2.5$.

Consider the Lyapunov functional $V(t, x) = |x|^2$. For the uncontrolled system:

$$\begin{aligned} & V_t(t, x) + V_x^T(t, x)b(t, x) \\ &\quad + \sup_{Q\in\mathcal{Q}} \operatorname{tr}\left[(V_x^T(t, x)h_{ij}(t, x) + \frac{1}{2}\sigma_j^T(t, x)V_{xx}(t, x)\sigma_i(t, x))QQ^T\right] \\ &\quad + \sup_{v\in\mathcal{V}} \int_{R_0^d}(V(t, x + K(t, x, z)) - V(t, x))v(dz) \\ &= (0.2 + 0.1\sin t)|x|^2 + (0.16 + 0.04\cos t + (0.12 + 0.03\sin 2t)^2) \times 0.9|x|^2 \\ &\quad + ((0.15 + 0.05\cos t)^2 + 2(0.15 + 0.05\cos t)) \times 0.7|x|^2 \\ &\geq (0.1 + 0.1153 + 0.147)|x|^2 = 0.3623|x|^2. \end{aligned}$$

Since the total expression is bounded below by $0.3623|x|^2$, the uncontrolled system is mean-square exponentially unstable according to Remark 3.1.

Introducing the feedback control $u(t, x) = -2.5x$, we obtain the controlled system:

$$\begin{aligned} dX(t) =& [b(t, X(t)) + u(t, X([t/\tau]\tau))]dt + h_{ij}(t, X(t))d\langle B^i, B^j\rangle_t \\ &+ \sigma_i(t, X(t))dB_t^i + \int_{R_0^d} K(t, X(t), z)L(dt, dz), \quad t \geq 0. \end{aligned} \tag{4.2}$$

For the controlled system, we verify assumption (H2):

$$\begin{aligned} & x^T[b(t, x) + u(t, x)] + \sup_{Q\in\mathcal{Q}} \operatorname{tr}[(2x^T h_{ij}(t, x) + \sigma_j^T(t, x)\sigma_i(t, x))QQ^T] \\ &\quad + \sup_{v\in\mathcal{V}} \int_{R_0^d} [|x + K(t, x, z)|^2 - |x|^2]v(dz) \end{aligned}$$

$$
\begin{aligned}
&= [(0.1 + 0.05\sin t - 2.5) + (0.16 + 0.04\cos t + (0.12 + 0.03\sin 2t)^2) \times 0.9 \\
&\quad + ((0.15 + 0.05\cos t)^2 + 2(0.15 + 0.05\cos t)) \times 0.7]|x|^2 \\
&\leq (-2.35 + 0.2003 + 0.308)|x|^2 = -1.8417|x|^2.
\end{aligned}
$$

Thus, assumption (H2) is satisfied with $\eta = 1.8417$.

We compute the maximum allowable sampling interval $\bar{\tau}$ using:

$$
\begin{aligned}
M = 2.5, \quad \eta = 1.8417, \quad \bar{\sigma}^2 = 0.9, \\
\sup_{v\in\mathcal{V}} v(R_0^d) = 0.7, \quad C_2' = 0.8, \quad \theta = 0.25.
\end{aligned}
$$

The function $J(\tau)$ becomes:

$$
\begin{aligned}
J(\tau) &= \frac{2M[16M^2\tau^2 + 4M^2(C_2'\tau + \bar{\sigma}^2)\tau + 8M\tau^2 + 8M\tau]}{6M + 2M\bar{\sigma}^2 + M^2\bar{\sigma}^2 + \sup_{v\in\mathcal{V}} v(R_0^d)} \\
&= 27.11\tau^2 + 8.23\tau.
\end{aligned}
$$

The critical equation is:

$$
2J(\tau)\left(e^{A(\tau+B)} - 1\right) = 1 - \theta,
$$

where

$$
\begin{aligned}
A &= 13M + 4M\bar{\sigma}^2 + 2\sup_{v\in\mathcal{V}} v(R_0^d) + 2\bar{\sigma}^2 M^2 = 54.15, \\
B &= \frac{\ln(2/\theta)}{\eta} = \frac{\ln(8)}{1.8417} \approx 1.129.
\end{aligned}
$$

Thus,

$$
2(27.11\tau^2 + 8.23\tau)\left(e^{54.15(\tau+1.129)} - 1\right) = 0.75.
$$

The numerical solution yields $\bar{\tau} \approx 0.0021$. Therefore, for any sampling interval $\tau < 0.0021$, the controlled system is both mean-square and quasi-sure exponentially stable according to Theorem 3.1.

5. Conclusions

In this paper, we consider the stabilization problem for stochastic systems driven by a G-Lévy process via discrete-time feedback control. The main contribution lies in establishing verifiable criteria under which the controlled system achieves both mean-square and quasi-sure exponential stability. We prove that there exists a maximum allowable sampling interval $\bar{\tau} > 0$ such that the stochastic system is stabilized when the discrete step size $\tau < \bar{\tau}$, extending existing results from G-Brownian motion to the more general G-Lévy framework. The proposed methodology demonstrates potential applicability in systems subject to discontinuous dynamics, such as financial models addressing sudden market shocks or networked control systems with intermittent observations.

Author contributions

Guanghua Wei and Bingjun Wang: Writing–original draft, writing–review and editing, Conceptualization, methodology, validation, supervision; Mingxia Yuan: Investigation, validation. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could appear to influence the work reported in this paper.

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